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# BLOWUP AND ILL-POSEDNESS FOR THE COMPLEX, PERIODIC KDV EQUATION

J. L. BONA AND F. B. WEISSLER

ABSTRACT. The present essay is concerned with complex-valued solutions of the Korteweg-de Vries equation. Interest will be focussed upon the initial-value problem with initial data that is periodic in space. Derived here are results of local and global well posedness, singularity formation in finite time and, perhaps surprisingly, results of non-existence. The overall picture is notably different from the situation that obtains for real-valued solutions.

*Keywords:* Korteweg-de Vries equation, blowup, well posed and ill posed initial-value problems, complex-valued solutions, Gevrey spaces

*2010 AMS Classification:* 30D10, 30D15, 30H99, 35B44, 35B65, 35E15, 35Q51, 35Q53

## 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Studied here are complex-valued solutions  $u(t, x)$  of the Korteweg-de Vries equation written in the form

$$u_t + u_{xxx} + (u^2)_x = 0, \tag{1.1}$$

which are  $2\pi$ -periodic in the spatial variable,  $x \in \mathbb{R}$ . Functions periodic of period  $2\pi$  are identified with functions defined on the unit circle  $\mathbb{S}^1$ .

There are two principal results in the present essay. The first shows that unlike their real-valued counterparts, complex-valued solutions to the initial-value problem for (1.1) can blow up in finite time, even for smooth data. The second main line of development is comprised of theorems showing that the initial-value problem for (1.1) is not even locally well posed in complex-valued  $L^2(\mathbb{S}^1)$ , or even in high order  $H^s(\mathbb{S}^1)$  spaces. Indeed, it turns out that the well-posedness theory that obtains in the space of  $L^2(\mathbb{S}^1)$  functions with zero mean value does not extend, as it does in the case of real valued functions, to the full space  $L^2(\mathbb{S}^1)$ .

The first blowup results for complex-valued solutions of (1.1) are due to Birnir [2, 3]. His solutions were defined on the circle, just as those discussed here. Birnir's results [2, 3] and their relation to the theory developed in this paper and in [10] will be discussed in more detail presently (see Remark 1.10). Later, the present authors established in [9] a blowup result of the sort put forward here for certain periodic solutions of a much broader class of nonlinear dispersive evolution equations. For complex-valued solutions of (1.1) posed on the entire real axis, finite time blow up was established in [10, 23, 28] by various means, all of which are quite different from those employed here.

The theory developed here is mostly specific to (1.1). It provides two different, but related, criteria connected to initial data which imply finite-time blowup (see Theorems 1.3 and 1.4 below). The proof of the first of these is a simplified version of the proof of blowup offered in [9].

The ill-posedness and non-existence results are contained in Theorems 1.5, 1.6 and 1.7. Of course, these results depend crucially on the fact that it is complex-valued solutions that are in question, and make particular use of the fact that the solution need not have zero mean value.

A secondary, technical outcome of our development is showing existence and uniqueness of local solutions to (1.1) for initial values in certain Gevrey regularity classes (see Definition 3.2 and Theorem 1.1). A specific estimate for the minimum existence time of a solution in this class is obtained, and for small data, solutions are shown to be global. These local existence results are useful both in the proof of blowup in Theorem 1.4 and in the ill-posedness results described in Theorem 1.6.

**1.1. Function Classes.** For future reference, the function spaces used in this paper are now set forth. Throughout, all functions are assumed to be complex valued. The Fourier coefficients  $\hat{f}(k)$  of a function  $f \in L^2(\mathbb{S}^1)$  are

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$$

for  $k \in \mathbb{Z}$ , so that  $f \in L^2(\mathbb{S}^1)$  is given almost everywhere by its Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}$$

with  $a_k = \hat{f}(k)$ . In terms of its Fourier coefficients, the norm of  $f$  is

$$\|f\|_{L^2(\mathbb{S}^1)}^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

The following closed subspaces of  $L^2(\mathbb{S}^1)$  will find use:

$$L_0^2(\mathbb{S}^1) = \left\{ f \in L^2(\mathbb{S}^1) : \int_0^{2\pi} f(x) dx = 0 \right\} = \left\{ f \in L^2(\mathbb{S}^1) : \hat{f}(0) = 0 \right\}; \quad (1.2)$$

$$L_+^2(\mathbb{S}^1) = \left\{ f \in L^2(\mathbb{S}^1) : \hat{f}(k) = 0, \forall k < 0 \right\} = \left\{ f \in L^2(\mathbb{S}^1) : f(x) = \sum_{k=0}^{\infty} a_k e^{ikx} \right\}; \quad (1.3)$$

$$L_{0,+}^2(\mathbb{S}^1) = \left\{ f \in L^2(\mathbb{S}^1) : \hat{f}(k) = 0, \forall k \leq 0 \right\} = \left\{ f \in L^2(\mathbb{S}^1) : f(x) = \sum_{k=1}^{\infty} a_k e^{ikx} \right\}. \quad (1.4)$$

For  $s \in \mathbb{R}$ , the norm of a function  $u \in H^s(\mathbb{S}^1)$  is taken to be

$$\|u\|_{H^s(\mathbb{S}^1)}^2 = \sum_{k=0}^{\infty} (1+k^2)^s |a_k|^2. \quad (1.5)$$

For  $s \geq 0$ , the subspace  $H_+^s(\mathbb{S}^1)$  of  $H^s(\mathbb{S}^1)$  consists of elements whose negative Fourier coefficients vanish, which is to say,

$$H_+^s(\mathbb{S}^1) = \left\{ f \in H^s(\mathbb{S}^1) : f(x) = \sum_{k=0}^{\infty} a_k e^{ikx} \right\} = H^s(\mathbb{S}^1) \cap L_+^2(\mathbb{S}^1). \quad (1.6)$$

**1.2. Preliminaries.** With only moderate regularity, any complex-valued, spatially  $2\pi$ -periodic solution of (1.1) admits a Fourier series development

$$u(t, x) = \sum_{k=-\infty}^{\infty} a_k(t) e^{ikx}.$$

The usual Duhamel representation of the initial-value problem for (1.1) with initial data  $u_0$  is

$$u(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-\tau)(u^2)_x(\tau) d\tau, \quad (1.7)$$

where  $\mathcal{S}(t)$  is the unitary group defined on all the spaces  $H^s(\mathbb{S}^1)$ ,  $s \in \mathbb{R}$ , by

$$\mathcal{S}(t) \sum_{k=-\infty}^{\infty} a_k e^{ikx} = \sum_{k=-\infty}^{\infty} a_k e^{ik^3 t} e^{ikx}.$$

It is straightforward to ascertain, for example, that  $u \in C([-T, T]; H^3(\mathbb{S}^1))$  is a solution of (1.7) if and only if  $u \in C([-T, T]; H^3(\mathbb{S}^1)) \cap C^1([-T, T]; L^2(\mathbb{S}^1))$  is a solution of (1.1) with  $u(0, \cdot) = u_0$ . If instead,  $u \in C([-T, T]; L^2(\mathbb{S}^1))$  is a solution of the integral equation (1.7), then it is a weak solution of (1.1). More precisely, if  $u \in C([-T, T]; L^2(\mathbb{S}^1))$ , then the two terms on the right hand-side of (1.7) are, respectively, in  $C^1([-T, T]; H^{-3}(\mathbb{S}^1))$  and  $C^1([-T, T]; W^{1,-1}(\mathbb{S}^1))$ , and so  $u_t \in C([-T, T]; H^{-3}(\mathbb{S}^1) + W^{1,-1}(\mathbb{S}^1))$ .

There is considerable previous literature dealing with the KdV equation (1.1) posed on the torus. We mention particularly the works [11], [22], [26] for local well posedness and [18] for both local and global well posedness in the  $L^2(\mathbb{S}^1)$ -based Sobolev spaces  $H^s(\mathbb{S}^1)$ . The latter article has a very good summary of the preceding work. The theory proceeds by first showing well posedness for solutions lying in the subspace  $L_0^2(\mathbb{S}^1)$  of functions with mean value zero. (Note that the mean value of a solution of (1.1) is constant in time; simply integrate (1.1) in  $x$  over  $\mathbb{S}^1$ .) The analysis, which uses at several points the mean-zero assumption, shows that for initial values  $u_0 \in L_0^2(\mathbb{S}^1)$ , there is a solution  $u$  of (1.7) lying in the space  $C([-T, T]; L_0^2(\mathbb{S}^1))$  where the time  $T > 0$  and  $\sup_{[-T, T]} \|u(t)\|_{L^2}$  depend only on  $\|u_0\|_{L^2}$ . The contraction mapping argument used to prove this result is carried out in a Bourgain space strictly smaller than  $C([-T, T]; L_0^2(\mathbb{S}^1))$ , and so the solution is more regular than indicated above. Uniqueness is proved in this smaller space. To obtain the same result without the restriction of zero mean value, a straightforward Galilean transformation

$$\tilde{u}(t, x) = \frac{1}{2}a_0 + u(t, x - a_0 t). \quad (1.8)$$

[11, page 219] is used to reduce the problem to the mean-zero case.

A delicate point arises here, because the authors of [11] and [26] do not make explicit whether or not they are considering only real-valued solutions, or are allowing complex-valued solutions when establishing local existence. On the other hand, [22] and [18] are more careful in this respect, and [22, Theorem 1.6] explicitly deals with complex-valued solutions, again of mean-value zero. However, the extension of well-posedness to solutions of non-zero mean value via the above Galilean transformation applies only to solutions having a real mean value. This point will be elaborated presently.

For real-valued solutions, conservation laws may be used to show that the solutions are global in time (and this even goes down to  $H^s(\mathbb{S}^1)$  for  $s \geq -1/2$ , see [18]). Conservation laws are no longer so obviously helpful in the complex-valued case. The equation (1.1) is still locally well-posed in  $L_0^2(\mathbb{S}^1)$ , even if complex-valued solutions are allowed. More precisely, each zero mean-value initial data  $u_0$  in  $L_0^2(\mathbb{S}^1)$  gives rise to a maximal solution

$$u \in C((-T_{\max}^-, T_{\max}^+); L_0^2(\mathbb{S}^1)), \quad (1.9)$$

where  $0 < T_{\max}^-, T_{\max}^+ \leq \infty$ . If either  $T_{\max}^-$  or  $T_{\max}^+$  is finite, then  $\|u(t)\|_{L^2(\mathbb{S}^1)}$  goes to infinity as  $t \rightarrow -T_{\max}^-$  or  $t \rightarrow T_{\max}^+$ , respectively. Moreover, if  $[-T_1, T_2] \subset (-T_{\max}^-, T_{\max}^+)$  for some  $T_1, T_2 \geq 0$ , and if  $u_0^n \rightarrow u_0$  in  $L_0^2(\mathbb{S}^1)$ , the maximal solution  $u^n$  with initial value  $u_0^n$  is defined on  $[-T_1, T_2]$  for sufficiently large  $n$  and converges uniformly to the solution  $u$  on  $[-T_1, T_2]$ . These facts all follow by standard methods directly from [22, Theorem 1.6]. Of course, as noted above, uniqueness of local solutions is proved in an appropriate Bourgain space. In what follows, we will cite [22, Theorem 1.6] to mean the local well-posedness of (1.1) in  $L_0^2(\mathbb{S}^1)$  as just described. Of course the main interest of [22], and in particular [22, Theorem 1.6] is to prove local well-posedness in negative order Sobolev spaces as well, but here we only need the result in  $L_0^2(\mathbb{S}^1)$ .

The analysis in the present article is based on the observation that the closed subspace  $L_{0,+}^2(\mathbb{S}^1)$  of  $L_0^2(\mathbb{S}^1)$  is invariant under the action of (1.1). More precisely, if  $u_0 \in L_{0,+}^2(\mathbb{S}^1)$ , that is, if  $u_0 \in L_0^2(\mathbb{S}^1)$  is of the form

$$u_0(x) = \sum_{k=1}^{\infty} a_k(0) e^{ikx}, \quad (1.10)$$

where  $a_k(0) \in \mathbb{C}$  for  $k = 1, 2, 3, \dots$ , then the resulting solution  $u(t)$  remains in  $L_{0,+}^2(\mathbb{S}^1)$  for all  $t \in (-T_{\max}^-, T_{\max}^+)$ , i.e.  $u(t) \in L_0^2(\mathbb{S}^1)$  is of the form

$$u(t, x) = \sum_{k=1}^{\infty} a_k(t) e^{ikx}. \quad (1.11)$$

This is obvious formally, just from the fact that both the linear and nonlinear terms in the equation preserve this subspace. Rigorously, it is a consequence of the fact that the set of functions in  $L_0^2(\mathbb{S}^1)$  whose Fourier series contains only positive modes is preserved by the iterative process based on the integral equation, used to construct solutions. As  $L_{0,+}^2(\mathbb{S}^1)$  is closed in  $L^2(\mathbb{S}^1)$ , the solution must be in the same subspace.

With this housekeeping out of the way, the main results of the paper can be stated. We begin with two results about local and global existence. As mentioned already, there is local well-posedness theory already in existence for (1.1) on the torus, so this part of the paper is of secondary interest. Our theory is set in Gevrey-type spaces. Not only the

results, but the methods used in the proofs will be needed for later developments. The theory provides lower bounds on  $T_{\max}^{\pm}$  which in certain cases imply global well posedness. There are also conditions on the initial data that imply finite time singularity formation.

**Theorem 1.1.** *Let  $u_0 \in L_{0,+}^2(\mathbb{S}^1)$  be of the form (1.10), and let  $u \in C((-T_{\max}^-, T_{\max}^+); L_0^2(\mathbb{S}^1))$  be the resulting maximal solution of (1.7), which is necessarily of the form (1.11). Suppose that there exist  $\lambda$  with  $0 < \lambda < 1$ ,  $\sigma > 1$ , and  $M > 0$  such that*

$$|a_k(0)| \leq Mk^{-\sigma}\lambda^k, \quad k = 1, 2, 3, \dots \quad (1.12)$$

Define  $L = L(\sigma)$  by

$$L(\sigma) = 2^{1+\sigma} \sum_{m=1}^{\infty} m^{-\sigma} \quad (1.13)$$

and let

$$T = -\frac{\log \lambda}{ML(\sigma)}. \quad (1.14)$$

It follows that  $T_{\max}^- > T$  and  $T_{\max}^+ > T$ . Moreover,

$$|a_k(t)| \leq Mk^{-\sigma}\lambda^k e^{ML(\sigma)k|t|}, \quad (1.15)$$

for all  $k = 1, 2, 3, \dots$  and all  $t \in [-T, T]$ . In particular,  $u \in C^1((-T, T); H^m(\mathbb{S}^1))$  for all  $m = 1, 2, 3, \dots$  and so certainly  $u \in C([-T, T]; C(\mathbb{S}^1))$ .

**Remark:** Note that the Gevrey-type hypothesis (1.12) entails that  $u_0 \in C^\infty(\mathbb{S}^1)$ .

**Theorem 1.2.** *Let  $u_0 \in L_{0,+}^2(\mathbb{S}^1)$  be of the form (1.10), and let*

$$u \in C((-T_{\max}^-, T_{\max}^+); L_0^2(\mathbb{S}^1))$$

be the resulting maximal solution of (1.7), which is necessarily of the form (1.11). Then, the two values  $T_{\max}^-$  and  $T_{\max}^+$  are either both finite or both infinite. In the case where they are finite, it must be that  $T_{\max}^- + T_{\max}^+ \leq 2\pi$ . If  $T_{\max}^- + T_{\max}^+ > 2\pi$ , then the solution is global in both directions, i.e.  $T_{\max}^- = T_{\max}^+ = \infty$ , and time periodic with period  $2\pi$ . In particular, global existence is an open condition in  $L_{0,+}^2(\mathbb{S}^1)$  on the initial value while finite time blowup is a closed condition in the same space.

Next, two results are presented that give easily verified criteria on an initial value  $u_0 \in L_{0,+}^2(\mathbb{S}^1)$  which imply that the resulting solution of (1.1) blows up in finite time.

**Theorem 1.3.** *Let  $u_0 \in L_{0,+}^2(\mathbb{S}^1)$  be of the form (1.10), and let  $u \in C((-T_{\max}^-, T_{\max}^+); L_0^2(\mathbb{S}^1))$  be the resulting maximal solution of (1.7). If  $|a_1(0)| \geq 6$ , then  $T_{\max}^- < \infty$  and  $T_{\max}^+ < \infty$ . If instead, the integer  $\mu \geq 2$  is such that  $a_1(0) = a_2(0) = \dots = a_{\mu-1}(0) = 0$  and  $|a_\mu(0)| \geq 6\mu^2$ , then  $T_{\max}^- < \infty$  and  $T_{\max}^+ < \infty$ .*

*On the other hand, given any  $\gamma \in \mathbb{C}$  with  $0 < |\gamma| < 1$ , there exists  $u_0 \in L_{0,+}^2(\mathbb{S}^1)$  of the form (1.10) with  $|a_1(0)| = 6\gamma$  such that the resulting maximal solution of (1.7) is global, i.e.  $T_{\max}^- = T_{\max}^+ = \infty$ . Likewise, for every integer  $\mu \geq 2$ , there exists  $u_0 \in L_{0,+}^2(\mathbb{S}^1)$  of the form (1.10) with  $a_1(0) = a_2(0) = \dots = a_{\mu-1}(0) = 0$  and  $|a_\mu(0)| = 6\mu^2\gamma$  such that the resulting maximal solution of (1.7) is global.*

**Theorem 1.4.** *Let  $u_0 \in L^2_{0,+}(\mathbb{S}^1)$  of the form (1.10) be a trigonometric polynomial, viz.*

$$u_0(x) = \sum_{k=1}^m a_k(0) e^{ikx}, \quad (1.16)$$

where  $a_k(0) \in \mathbb{C}$  for  $k = 1, 2, 3, \dots, m$ , for some integer  $m \geq 1$ . For every  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ , let

$$u_{0,\lambda}(x) = \sum_{k=1}^m a_k(0) \lambda^k e^{ikx}, \quad (1.17)$$

and let  $u_\lambda \in C((-T_{\max}^{\lambda,-}, T_{\max}^{\lambda,+}); L^2_0(\mathbb{S}^1))$  be the maximal solution of (1.7) with initial value  $u_{0,\lambda}$ . There exists a  $\lambda_0$  with  $0 < \lambda_0 < \infty$  such that:

- if  $|\lambda| \geq \lambda_0$ , then the resulting solution blows up in finite time in both time directions, i.e.  $T_{\max}^{\lambda,\pm} < \infty$ ,
- while if  $0 < |\lambda| < \lambda_0$ , then the resulting solution is global in both time directions, i.e.  $T_{\max}^{\lambda,\pm} = \infty$ .

It follows from Theorem 1.3 that the value of  $\lambda_0$  in Theorem 1.4 satisfies  $\lambda_0 |a_\mu(0)| \leq 6\mu^2$ , where  $a_\mu(0)$  is the first nonzero coefficient in (1.16). However, even though Theorem 1.3 is best possible in the sense given by the statement, it is not likely that it predicts the exact value of  $\lambda_0$  in Theorem 1.4 in general. This will be elaborated in remarks following the proof of Theorem 1.3 in Section 5.

Attention is now turned to three theorems which give different ways of seeing that (1.1) is not even locally well-posed on  $L^2(S^1)$  or  $H^s(S^1)$  if complex-valued initial data with nonzero mean value are allowed. The first two show that no reasonable contraction mapping argument can be made in a small ball around the zero-solution.

**Theorem 1.5.** *Let  $\epsilon > 0$ ,  $M > 0$ ,  $T > 0$  and  $s \geq 0$  be given. There exists an initial value  $\tilde{u}_0 \in H^s_+(\mathbb{S}^1)$  with  $\|\tilde{u}_0\|_{H^s_+(\mathbb{S}^1)} \leq \epsilon$  such that there does not exist any solution  $\tilde{u} \in C([0, T]; H^s_+(\mathbb{S}^1))$  of (1.1), with initial value  $\tilde{u}_0$  for which  $\sup_{0 \leq t \leq T} \|\tilde{u}(t)\|_{H^s(\mathbb{S}^1)} \leq M$ .*

The statement and proof of Theorem 1.5 were inspired by Christ's work in [17].<sup>1</sup> This is a so called *norm inflation* result (see, for example, [12] in the context of the Navier-Stokes system, [1] for Schrödinger equations, [20] for the Degasperis-Procesi equation, [6] for the BBM equation and [19] for the MHD equations; this is only a sample of the extant results). What sets this result apart from those just mentioned is that the norm inflation can take place in arbitrarily smooth  $H^s(\mathbb{S}^1)$  spaces, whereas the usual norm inflation theory is set in very weak spaces. The methods used to prove Theorem 1.5 can also be used to prove the following stronger version of ill-posedness. It establishes the existence of arbitrarily small initial values which give rise to local solutions of arbitrarily large norm, in an arbitrarily short period of time, thus completely destroying any possibility of continuity at zero. Unfortunately, our proof of it seems to work only in  $H^s_+(\mathbb{S}^1)$  for  $s > 0$ .

<sup>1</sup>The authors thank Tsukasa Iwabuchi for bringing this paper to their attention.

**Theorem 1.6.** *Let  $s > 0$ . There exists a sequence  $\varphi_n \in H_+^s(\mathbb{S}^1)$  and numerical sequences  $\{M_n\}_{n>0}$  and  $\{t_n\}_{n>0}$  with the following properties:*

- (i) *For each  $\varphi_n$  there is a unique solution of (1.1)  $\tilde{u}_n : [0, T_{max}^n] \rightarrow H_+^s(\mathbb{S}^1)$  with initial value  $\varphi_n$ .*
- (ii)  *$0 < t_n < T_{max}^n$  and  $\|\tilde{u}(t_n)\|_{H^s(\mathbb{S}^1)} \geq M_n$ .*
- (iii)  *$\varphi_n \rightarrow 0$  in  $H^s(\mathbb{S}^1)$ ,*
- (iv)  *$t_n \rightarrow 0$ ,*
- (v) *and  $M_n \rightarrow \infty$ .*

See [21] for similar results for the nonlinear Schrödinger equation.

The final ill-posedness result is that certain initial values in  $L_+^2(\mathbb{S}^1)$  can not give rise to *any* local solution in  $L_+^2(\mathbb{S}^1)$ . In fact, the proof shows this to be the case for a class of initial values slightly larger than announced here.

**Theorem 1.7.** *Let  $\tilde{u}_0 \in L_+^2(\mathbb{S}^1)$  not be of class  $C^\infty(\mathbb{S}^1)$  and have mean value  $\frac{1}{2\pi} \int_0^{2\pi} \tilde{u}_0(x) dx$  with non-zero imaginary part. Then there does not exist a solution  $\tilde{u} \in C((-T, T); L_+^2(\mathbb{S}^1))$  of (1.1), with initial value  $\tilde{u}_0$ , for any  $T > 0$ .*

The approach taken here to proving these results is an analysis of the Fourier coefficients of the solution  $u(t, x)$  of (1.1) in the form (1.11). Indeed, if  $u(t, x)$  is an  $L_{0,+}^2(\mathbb{S})$ -solution, then the coefficients must satisfy the coupled system

$$a'_k(t) - ik^3 a_k(t) + ikb_k(t) = 0, \quad k = 1, 2, 3, \dots, \quad (1.18)$$

of ordinary differential equations, where  $b_1(t) \equiv 0$  and, for  $k \geq 2$ ,

$$b_k(t) = \sum_{m=1}^{k-1} a_m(t) a_{k-m}(t). \quad (1.19)$$

The system (1.18)–(1.19) is equivalent to the system of integral equations

$$a_k(t) = e^{ik^3 t} a_k(0) - ik e^{ik^3 t} \int_0^t e^{-ik^3 s} b_k(s) ds, \quad k = 1, 2, 3, \dots. \quad (1.20)$$

Since each  $b_k(t)$  depends only on the  $a_m(t)$  for  $1 \leq m < k$ , this is in fact a system of non-homogeneous linear equations which can be solved explicitly by recursive calculations. For future reference, this conclusion is formalized here.

**Proposition 1.8.** *Given  $a_k(0) \in \mathbb{C}, k = 1, 2, 3, \dots$ , there exist unique functions  $a_k \in C^\infty(\mathbb{R}; \mathbb{C})$  which are solutions to the system (1.18)–(1.19) with these initial values  $a_k(0)$ , or equivalently solutions to the integral equations (1.20).*

*In particular, given an initial value  $u_0$  of the form (1.10), there is at most one possible solution  $u$  of (1.1) of the form (1.11) with this initial value  $u_0$ .*

However, while the  $a_k(t)$  exist for all  $t \in \mathbb{R}$ , it is not at all certain that the formal Fourier series displayed in (1.11) converges and provides a solution of (1.1). This remark motivates the following definition.



**Definition 1.9.** *A formal solution of (1.1) is a Fourier series*

$$u(t, x) = \sum_{k=1}^{\infty} a_k(t) e^{ikx}, \quad (1.21)$$

where the  $a_k \in C^\infty(\mathbb{R}; \mathbb{C})$ ,  $k = 1, 2, 3, \dots$  constitute a solution of the system (1.18)–(1.19), or equivalently (1.20). We emphasize that there is no requirement of convergence of the Fourier series (1.21).

It is clear that if  $u_0 \in L^2_{0,+}(\mathbb{S}^1)$  is of the form (1.10) and  $u \in C((-T_{\max}^-, T_{\max}^+); L^2_0(\mathbb{S}^1))$  is the resulting maximal  $L^2$  solution of (1.1), then  $u$  can be extended (uniquely) as a formal solution for all  $t \in \mathbb{R}$ .

The idea behind the proof of Theorem 1.3 is as follows. It emerges from the analysis in Section 4 that the coefficients  $a_k(t)$  are *time* periodic. This enables one to consider the solution  $u$  as a double Fourier series, in space and time. The key point is that a lower estimate on the space-time  $L^2$ -norm of a global solution can be obtained in terms of the initial data, which implies finite-time blowup when this lower bound is infinite, a point explored in Section 5.

**Remark 1.10.** It is instructive at this point to compare the blowup results and methods used in [3] with those of the current paper and of the authors' previous work [10]. In fact, there is a striking similarity of methods and results in [3] and in [10],

In both [3] and [10], an explicit solution  $u(t, z)$  of (1.1) is studied, where  $t \in \mathbb{R}$ , and  $z, u \in \mathbb{C}$ . If the space variable  $z$  is restricted to a horizontal line  $\Im z = y_0$ , the resulting function  $u(t, x + iy_0)$  is a complex-valued solution of (1.1) on the real line. In both articles, the solution  $u(t, z)$  exhibits poles which are initially located away from the horizontal line  $z = x + iy_0$ , so that the initial value  $u(0, x + iy_0)$  is a smooth function of  $x \in \mathbb{R}$ ; and in both cases, the initial value  $u(0, x + iy_0)$  gives rise to either a global (in time) regular solution or one that blows up in finite time, depending on whether or not  $y_0$  belongs to a clearly defined subset of  $\mathbb{R}$ , given by the union of intervals. In both [3] and [10], in the case where the solution does blow up in finite time, blowup occurs at a time  $t = t_0$  when one of the poles moves across the horizontal line  $\Im z = y_0$ , so that  $u(t_0, x + iy_0)$  exhibits a singularity. Furthermore, the solution becomes regular again for  $t > t_0$  as the pole moves away from the line  $\Im z = y_0$ .

In the case of [10] the explicit solution used is the two-soliton solution of (1.1), which decays in space as  $\Re z \rightarrow \pm\infty$ .

In [3], the explicit solution used is based on the doubly periodic Weierstrass elliptic functions. In particular, for a fixed  $y = y_0$ , the function  $u(t, x + iy_0)$  is periodic in  $x \in \mathbb{R}$ . Consequently, these solutions live on  $\mathbb{S}^1$ , as do the solutions in the present paper. However, the results and methods put forth here can not be directly applied to the solutions in [3] since these latter do not live in  $L^2_{0,+}(\mathbb{S}^1)$  or even  $L^2_+(\mathbb{S}^1)$ . Nonetheless, there is a structural similarity between the construction in the present paper and that in [3] which is further explicated in Remark 5.10 below.

All three papers, moreover, do share certain features. First of all, and somewhat superficially, the alternative between global existence of a regular solution and finite time

blowup is determined by the value of a parameter present in the initial value. More significantly, perhaps, in all three papers, there is an underlying solution which is global, but not always regular. In the present paper, while not an explicit solution as in [3] and [10], it is given by a formal Fourier series whose coefficients constitute the solution of an infinite system of ODEs, the formal solution of Definition 1.9, which exists for all  $t \in \mathbb{R}$ . Time periodicity of the solution to this system implies that it returns as a regular solution to (1.1) at some point after blowup, but it is not clear if regularity returns immediately after the blowup time, as it does in [3] and [10].

Finally, it is observed that the single equation (1.1) with complex-valued solutions is equivalent, by considering real and imaginary parts, to a coupled system of the form

$$\begin{cases} u_t + u_{xxx} + P(u, v)_x = 0, \\ v_t + v_{xxx} + Q(u, v)_x = 0, \end{cases} \quad (1.22)$$

where  $u$  and  $v$  here are real-valued functions of the two variables  $x$  and  $t$ . In the case of (1.1), the choice  $P(u, v) = u^2 - v^2$  and  $Q(u, v) = 2uv$  is required. Consequently, the main results of this paper immediately give corresponding finite-time blowup and ill-posedness results for this specific system. It is natural to ask if the methods used here could extend to a more general class of systems of the form (1.22) where, say,  $P$  and  $Q$  are arbitrary homogeneous quadratic polynomials. Unfortunately, the answer to this question turns out to be negative, as will be seen in the last section of this paper. This suggests that the local illposedness results presented in this paper might not extend to more general systems. See Remark 7.1 at the end of Section 7.

In a little more detail, the paper is organized as follows. Section 2 is devoted to some basic properties of the system (1.18)–(1.19) of ODEs. These lead to a proof of Theorem 1.1 in Section 3. The method is to study a related ode system where the complex oscillations have been eliminated. Time periodicity of the coefficients  $a_k(t)$  is established in Section 4, and this is used to prove Theorem 1.2. The two blowup theorems are proved in Section 5, while Section 6 contains the proofs of the ill-posedness results. Section 7 shows that the methods used here do not in any natural way extend to a larger class of systems of real-valued functions of the form (1.22), as mentioned just above. In an Appendix, a number of remarks pertinent to the development in Section 3 are provided.

We close the introduction by recalling that our interest in studying finite time blowup of complex-valued solutions of (1.1) is motivated by a desire to understand what might be the mechanism of finite-time blowup of real valued solutions of the generalized Korteweg-de Vries equation  $u_t + u_{xxx} + (u^p)_x = 0$  in the supercritical case  $p > 5$ . In [8, 10], it was observed that finite time blowup of complex valued solutions on the line of both the Korteweg-de Vries equation and the modified Korteweg-de Vries equation occurs when a singularity in the solution, initially located off the real axis, moves onto the real axis. This is also the case in [3], as already noted. (See Remark 5.10 and the related work of Grujic and his collaborators detailed in [7] and in the references contained therein.) This suggests that an approach to proving the existence of (real valued) solutions to the super-critical generalized Korteweg-de Vries equation on the real line which blowup in finite time is to study extensions of those solutions to a region in the complex plane, and determine how the singularities in those solutions, which are initially off the real axis, evolve in time.

The relevant conjecture is that blowup occurs when such a singularity moves onto the real axis.

## 2. BASIC PROPERTIES OF THE ODE SYSTEM

In this section, solutions to the system (1.18)–(1.19) are studied. The discussion begins with the following proposition, whose proof is immediate.

**Proposition 2.1.** *Let  $a_k \in C^\infty(\mathbb{R}; \mathbb{C})$ ,  $k = 1, 2, 3, \dots$ , be a solution of the system (1.18)–(1.19), or equivalently (1.20). It follows that  $a_1(t) = e^{it}a_1(0)$ . Furthermore, if  $\mu \geq 2$  is an integer such that*

$$a_1(0) = a_2(0) = \dots = a_{\mu-1}(0) = 0, \quad a_\mu(0) \neq 0, \quad (2.1)$$

it follows that

$$a_1(t) = a_2(t) = \dots = a_{\mu-1}(t) = 0, \quad t \in \mathbb{R}, \quad (2.2)$$

$$b_1(t) = b_2(t) = \dots = b_{2\mu-1}(t) = 0, \quad t \in \mathbb{R}, \quad (2.3)$$

and

$$a_k(t) = e^{ik^3t}a_k(0), \quad \mu \leq k \leq 2\mu - 1. \quad (2.4)$$

An important parameter that arises in connection with the study of solutions to (1.1) of the form (1.11) is the radius of convergence of an associated power series.

**Definition 2.2.** *Let  $a_k \in C^\infty(\mathbb{R}; \mathbb{C})$ ,  $k = 1, 2, 3, \dots$ , be a solution of the system (1.18)–(1.19). Denote by  $\rho(t)$  the radius of convergence of the power series*

$$\sum_{k=1}^{\infty} a_k(t)z^k. \quad (2.5)$$

For all  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ ,  $\rho_\lambda(t)$  denotes the radius of convergence of the related power series

$$\sum_{k=1}^{\infty} a_k(t)\lambda^k z^k. \quad (2.6)$$

It is clear that

$$\rho_\lambda(t) = |\lambda|^{-1}\rho(t). \quad (2.7)$$

Note that  $0 \leq \rho(t) \leq \infty$ , and one cannot *a priori* exclude the possibility that either  $\rho(t) = 0$  or  $\rho(t) = \infty$  (see Corollaries 3.11, 3.12, and 4.5 below, as well as Proposition 5.7, which develop further properties of  $\rho(t)$ ).

The following proposition gives a condition under which a formal solution of equation (1.1), determined by its Fourier coefficients  $a_k(t)$  (as in Definition 1.9), is in fact a smooth, and therefore a classical solution of (1.1).

**Proposition 2.3.** *Let  $a_k \in C^\infty(\mathbb{R}; \mathbb{C})$ ,  $k = 1, 2, 3, \dots$  be a solution of the system (1.18)–(1.19), or equivalently (1.20), and let  $\rho(t)$  be as in Definition 2.2 for these  $a_k$ 's. If  $\rho(t) > 1$  on some time interval  $t_1 < t < t_2$ , then  $u(t, x)$  given by (1.11) is a  $C^\infty$  solution of (1.1) on that interval. On the other hand, if  $\rho(t) < 1$  for some fixed value of  $t \in \mathbb{R}$ , then the series in (1.11) does not converge in  $\mathcal{D}'(\mathbb{S}^1)$ .*

*Proof.* Suppose first that  $\rho(t) > 1$  on some interval  $t_1 < t < t_2$ . It follows immediately that the function  $u(t, \cdot)$  given by the Fourier series (1.11) is in  $C^\infty(\mathbb{S}^1)$  for every  $t \in (t_1, t_2)$ . Of course,  $u(t, \cdot)^2$  is also in  $C^\infty(\mathbb{S}^1)$  and is given by the Fourier series  $\sum_{k=2}^{\infty} b_k(t)e^{ikx}$ , where the  $b_k(t)$  are the series of convolutions defined in (1.19). Indeed, if  $\sum_{k=1}^{\infty} a_k(t)z^k$  is an analytic function in the disc of radius  $\rho(t)$ , then so must be its square  $\sum_{k=2}^{\infty} b_k(t)z^k$ .

It then follows from the system (1.18)-(1.19) that the formal Fourier series for  $u_t(t, \cdot)$  is likewise in  $C^\infty(\mathbb{S}^1)$  for every  $t \in (t_1, t_2)$  and that (1.1) is verified. Successive differentiation of equation (1.1) shows that all space-time derivatives of  $u$  are in  $C^\infty(\mathbb{S}^1)$  for every  $t \in (t_1, t_2)$ , and thus  $u \in C^\infty((t_1, t_2) \times \mathbb{S}^1)$ .

Next, suppose that there is a  $t \in \mathbb{R}$  with  $\rho(t) < 1$ , and let

$$u_m(t, x) = \sum_{k=1}^m a_k(t)e^{ikx}. \quad (2.8)$$

For this value of  $t$ , denote  $f_m(\cdot) = u_m(t, \cdot)$  and argue by contradiction. Suppose there exists  $f \in \mathcal{D}'(\mathbb{S}^1)$  such that  $f_m \rightarrow f$  in the sense of  $\mathcal{D}'(\mathbb{S}^1)$ . It follows that for each  $k = 1, 2, 3, \dots$ ,

$$\langle f, e^{-ikx} \rangle = \lim_{m \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} f_m(x)e^{-ikx} dx = a_k(t). \quad (2.9)$$

Furthermore, since  $\mathcal{D}(\mathbb{S}^1) = C^\infty(\mathbb{S}^1) = \bigcap_{s \in \mathbb{R}} H^s(\mathbb{S}^1)$ , it follows that

$$\mathcal{D}'(\mathbb{S}^1) = \bigcup_{s \in \mathbb{R}} H^s(\mathbb{S}^1). \quad (2.10)$$

Hence  $f \in H^s(\mathbb{S}^1)$  for some  $s \in \mathbb{R}$ , so

$$\sum_{k=1}^{\infty} (1+k^2)^s |a_k(t)|^2 < \infty. \quad (2.11)$$

By the root test, this last condition implies that  $\rho(t) \geq 1$ , contradicting the hypothesis that  $\rho(t) < 1$ . □

**Remark 2.4.** If the functions  $a_k \in C^\infty(\mathbb{R}; \mathbb{C})$ ,  $k = 1, 2, 3, \dots$ , constitute a solution of the system (1.18)–(1.19), then for every  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ , the functions

$$a_{k,\lambda}(t) = \lambda^k a_k(t), \quad k = 1, 2, 3, \dots, \quad (2.12)$$

likewise constitute a solution. This is to say, if  $u$  of the form (1.11) is a formal solution of (1.1), then so is

$$u_\lambda(t, x) = \sum_{k=1}^{\infty} a_{k,\lambda}(t)e^{ikx} = \sum_{k=1}^{\infty} a_k(t)\lambda^k e^{ikx} \quad (2.13)$$

for any fixed  $\lambda \in \mathbb{C}$ . Observe that formally,  $u_\lambda(t, x) = u(t, x - i \log \lambda)$  since

$$u(t, x - i \log \lambda) = \sum_{k=1}^{\infty} a_k(t)e^{ik(x - i \log \lambda)} = \sum_{k=1}^{\infty} a_k(t)e^{ikx}\lambda^k. \quad (2.14)$$

In terms of initial values and maximal regular solutions, if  $u_0 \in L_0^2(\mathbb{S}^1)$  is of the form (1.10), and if  $u \in C((-T_{\max}^-, T_{\max}^+); L_0^2(\mathbb{S}^1))$  is the resulting maximal solution, then  $u_\lambda$  has initial value

$$u_{0,\lambda}(x) = \sum_{k=1}^{\infty} a_k(0) \lambda^k e^{ikx}. \quad (2.15)$$

If  $u_{0,\lambda} \in L_0^2(\mathbb{S}^1)$ , it gives rise to a maximal solution

$$u_\lambda \in C((-T_{\max,\lambda}^-, T_{\max,\lambda}^+); L_0^2(\mathbb{S}^1)),$$

which coincides with the formal solution on the interval of existence. Suppose further that  $|\lambda| > 1$ . It follows by (2.7) that  $\rho_\lambda(t) < \rho(t)$  for all  $t \in \mathbb{R}$  where  $\rho(t) < \infty$ , with equality if  $\rho(t) = \infty$ . It can thereby be deduced from Proposition 2.3 that

$$(-T_{\max,\lambda}^-, T_{\max,\lambda}^+) \subset (-T_{\max}^-, T_{\max}^+).$$

The following proposition shows that the above line of reasoning can be pushed quite a bit further.

**Proposition 2.5.** *Let  $u_0 \in L_0^2(\mathbb{S}^1)$  be of the form (1.10), and let*

$$u \in C((-T_{\max}^-, T_{\max}^+); L_0^2(\mathbb{S}^1))$$

*be the resulting maximal solution of (1.1), which is necessarily of the form (1.11), i.e.*

$$u(t, x) = \sum_{k=1}^{\infty} a_k(t) e^{ikx}.$$

*Let  $\rho(t)$  be given by Definition 2.2. The following assertions hold.*

- (i) *Either  $\rho(t) \equiv 1$  on  $(-T_{\max}^-, T_{\max}^+)$  or  $1 < \rho(t) \leq \infty$  for all  $t \in (-T_{\max}^-, T_{\max}^+)$ .*
- (ii) *The case  $\rho(t) \equiv 1$  on  $(-T_{\max}^-, T_{\max}^+)$  prevails if  $u_0 \in L_0^2(\mathbb{S}^1)$  is not  $C^\infty$ .*
- (iii) *If  $1 < \rho(t) \leq \infty$  for all  $t \in (-T_{\max}^-, T_{\max}^+)$ , then  $u \in C^\infty((-T_{\max}^-, T_{\max}^+) \times \mathbb{S}^1)$ .*
- (iv) *If  $\rho(0) > 1$  and  $T_{\max}^+ < \infty$ , then  $T_{\max}^+ = \inf\{t > 0; \rho(t) \leq 1\}$ .*
- (v) *If  $\rho(0) > 1$  and  $T_{\max}^- < \infty$ , then  $-T_{\max}^- = \sup\{t < 0; \rho(t) \leq 1\}$ .*

*In the last two statements,  $\rho(t)$  for  $t \notin (-T_{\max}^-, T_{\max}^+)$  is the radius of convergence in Definition 2.2 associated to the coefficients  $a_k(t)$  of the formal solution which extends the solution  $u \in C((-T_{\max}^-, T_{\max}^+); L_0^2(\mathbb{S}^1))$  to all  $t \in \mathbb{R}$ , as per Definition 1.9.*

*Proof.* It follows from Proposition 2.3 that  $\rho(t) \geq 1$  for all  $t \in (-T_{\max}^-, T_{\max}^+)$ . For statement (i) in the proposition, it thus suffices to show that  $\rho(t)$  can not take on both a value bigger than 1 and the value 1 in the interval  $(-T_{\max}^-, T_{\max}^+)$ .

We argue by contradiction, and suppose that (say)  $\rho(0) > 1$  and  $\rho(t_0) = 1$  for some  $0 < t_0 < T_{\max}^+$ . For  $\lambda > 0$ , consider the formal solution,  $u_\lambda(t, x)$  given just above in Remark 2.4, and let  $\rho_\lambda(t)$  be given by Definition 2.2. It follows from formula (2.7) that if  $\lambda < \rho(0)$ , then  $\rho_\lambda(0) > 1$  and so  $u_{0,\lambda} \in L_0^2(\mathbb{S}^1)$ . In this case,  $u_{0,\lambda}$  gives rise to a maximal solution

$$u_\lambda \in C((-T_{\max,\lambda}^-, T_{\max,\lambda}^+); L_0^2(\mathbb{S}^1)),$$

which coincides with the formal solution on the interval of existence. Since  $\rho(t_0) = 1$ , it follows from formula (2.7) that  $\rho_\lambda(t_0) < 1$  for  $\lambda > 1$ . Consequently, by Proposition 2.3 it must be that  $T_{\max,\lambda}^+ \leq t_0$  for  $1 < \lambda < \rho(0)$ . Furthermore, since  $\rho(0) > 1$ , it follows easily that  $u_{0,\lambda} \rightarrow u_0$  in  $L^2(\mathbb{S}^1)$  as  $\lambda \rightarrow 1$ . By continuous dependence of solutions in  $L^2_0(\mathbb{S}^1)$  (see [22, Theorem 1.6]), it must be that

$$T_{\max}^+ \leq \liminf_{\lambda \rightarrow 1} T_{\max,\lambda}^+ \leq t_0 < T_{\max}^+, \tag{2.16}$$

which gives the desired contradiction. Statement (i) is thereby established.

The remaining statements are now straightforward consequences of Proposition 2.3 and statement (i).  $\square$

This result is analogous to, but much weaker than, the fact that a real-valued  $u_0 \in L^2(\mathbb{S}^1)$  which is not in  $H^1(\mathbb{S}^1)$  gives rise to a (global) solution which can never be in  $H^1(\mathbb{S}^1)$ . Indeed, because of global well-posedness of (1.1) in  $H^1_{\mathbb{R}}(\mathbb{S}^1)$ , such a solution would have to be in  $H^1(\mathbb{S}^1)$  throughout its trajectory. In the present situation, we see that an initial value which is not analytic on  $\mathbb{R}$  (considered as a periodic function) can not yield a solution to (1.1) which becomes analytic at any point during its existence. This observation will be essential to the proof of Theorem 1.7.

Also, in light of Proposition 2.5, it might seem reasonable to conjecture that  $\rho(t)$  is always a constant function. This turns out not to be true. We will see that if  $u_0 \in L^2_0(\mathbb{S}^1)$  is a trigonometric polynomial, then  $\rho(t)$  is definitely not a constant function. For such an initial value,  $\rho(0) = \infty$  and Proposition 5.7 excludes the possibility that  $\rho(t) \equiv \infty$ . See Remark 5.9 for additional commentary on this issue.

**Remark 2.6.** In thinking about the results in this section, it could be helpful to observe that for  $\lambda > 0$ , the solutions  $u_\lambda(t, x)$  defined by (2.13) can be interpreted as follows. If one replaces  $\lambda > 0$  by  $r > 0$  and  $x \in \mathbb{R}$  by  $\theta \in \mathbb{R}$ , and sets  $z = re^{i\theta}$ , then

$$u_\lambda(t, x) = \sum_{k=1}^{\infty} a_k(t) z^k$$

While this transformation is not used explicitly in the main body of this paper, it underlies many of the arguments. In the last section, on systems, this transformation appears in a fundamental role.

In addition, we warn the reader that this transformation is completely different from the one used in Remark 5.10 below. In particular, the expression for  $u_\lambda$  here is not at all the same as the one in formula (5.27). The two different expressions for  $u_\lambda$  give rise to different interpretations of the solutions.

### 3. LOCAL EXISTENCE IN A CLASS OF GEVREY SPACES

To prove that (1.1) admits local and, in some cases, global solutions with initial values of the form (1.10), estimates on the behavior of the solutions  $\{a_k(t)\}_{k=1,2,3,\dots}$  of the equations (1.20) for a given  $t \in \mathbb{R}$  in terms of the behavior of the initial sequence  $\{a_k(0)\}_{k=1,2,3,\dots}$  are needed.

One helpful observation is that if the functions  $a_k(t)$  satisfy the system (1.20), it follows that, for  $t > 0$ ,

$$|a_k(t)| \leq |a_k(0)| + k \int_0^t |b_k(s)| ds. \quad (3.1)$$

This suggests studying the system obtained by replacing the inequalities in (3.1) by equations. Taking up this idea, consider the infinite system

$$c'_k(t) - kd_k(t) = 0, \quad k = 1, 2, 3, \dots \quad (3.2)$$

of ordinary differential equations with given initial values  $c_k(0) \in \mathbb{C}, k = 1, 2, 3, \dots$ , where  $c_k : \mathbb{R} \rightarrow \mathbb{C}$ ,

$$d_1(t) \equiv 0 \quad \text{and} \quad (3.3)$$

$$d_k(t) = \sum_{m=1}^{k-1} c_m(t)c_{k-m}(t), \quad k \geq 2. \quad (3.4)$$

Ultimately, interest will focus on the case  $c_k(0) \geq 0$  for all  $k \geq 1$ , but this restriction is not in force yet. The system (3.2) - (3.4) is equivalent to the infinite sequence

$$c_k(t) = c_k(0) + k \int_0^t d_k(s) ds, \quad k = 1, 2, \dots, \quad (3.5)$$

of integral equations with the side conditions (3.3) and (3.4).

Since each  $d_k(t)$  depends only on the  $c_m(t)$  with  $1 \leq m < k$ , this system consists of linear, non-homogeneous equations, which admit global smooth solutions. In principle, these solutions can be calculated explicitly in terms of the initial conditions. It is straightforward to show by induction that  $c_k$  is a polynomial of degree  $k - 1$ , and so has the form

$$c_k(t) = \sum_{m=0}^{k-1} c_{k,m} t^m. \quad (3.6)$$

If  $c_k(0) \geq 0, k = 1, 2, 3, \dots$ , then  $c_k(t) \geq 0, k = 1, 2, 3, \dots$ , for all  $t \geq 0$ . Moreover, in this case  $c_{k,m} \geq 0$ , for all  $k = 1, 2, 3, \dots$  and  $0 \leq m \leq k - 1$ .

The following proposition is straightforwardly established by induction.

**Proposition 3.1.** *Let  $a_k(0) \in \mathbb{C}$  and  $c_k(0) \geq 0$  be such that  $|a_k(0)| \leq c_k(0)$  for all  $k = 1, 2, 3, \dots$ . Let  $a_k(t)$  and  $c_k(t)$  be the resulting solutions, respectively, of (1.20) and (3.5). It follows that  $|a_k(t)| \leq c_k(|t|)$  for all  $t \in \mathbb{R}$ . Likewise,  $|b_k(t)| \leq d_k(|t|)$  for all  $k = 1, 2, 3, \dots$  and for all  $t \in \mathbb{R}$ , where  $b_k(t)$  and  $d_k(t)$  are given by (1.19) and (3.4).*

Attention is turned to the task of obtaining estimates on the behavior of the sequence  $\{c_k(t)\}_{k=1,2,3,\dots}$  for a given  $t \geq 0$  in terms of the behavior of the initial sequence  $\{c_k(0)\}_{k=1,2,3,\dots}$ . The following definition will be helpful.

**Definition 3.2.** *Given  $\lambda > 0$  and  $\sigma > 1$ , denote by  $\mathcal{W}_{\lambda,\sigma}$  the set of sequences  $\{\gamma_k\}_{k=1,2,3,\dots} \subset \mathbb{C}$  such that*

$$|\gamma_k| \leq M k^{-\sigma} \lambda^k, \quad k = 1, 2, 3, \dots, \quad (3.7)$$

for some  $M > 0$ .

**Remark 3.3.** Clearly  $\mathcal{W}_{\lambda,\sigma}$  is a Banach space, where the norm of a sequence is the smallest  $M > 0$  for which (3.7) is verified. There is an important relationship between these spaces and the radius of convergence  $\rho$  of the related power series

$$f(z) = \sum_{k=1}^{\infty} \gamma_k z^k. \quad (3.8)$$

On the one hand, if  $\{\gamma_k\} \in \mathcal{W}_{\lambda,\sigma}$ , then  $\rho \geq 1/\lambda$ , and so the function  $f$  given by (3.8) is holomorphic, at least in the open ball of radius  $1/\lambda$  around the origin. Since  $\sigma > 1$  the series converges uniformly if  $|z| = 1/\lambda$ , and so  $f$  is in fact a continuous function on the closed ball of radius  $1/\lambda$  around the origin.

On the other hand, given a the radius of convergence  $\rho$  for  $f$ , then  $\{\gamma_k\} \in \mathcal{W}_{\lambda,\sigma}$  for all  $\lambda > 1/\rho$  and all  $\sigma > 1$ .

It is clear therefore that the spaces  $\mathcal{W}_{\lambda,\sigma}$  give a classification of functions holomorphic in a neighborhood of zero which is strongly related to the radius of convergence of the associated power series. Moreover, the value of  $\sigma > 1$  corresponds to a measure of regularity on the critical circle. For example, if  $\{\gamma_k\} \in \mathcal{W}_{\lambda,\sigma}$  and  $\sigma > m$ , then  $f$  is of class  $C^m$  as a function on the closed ball of radius  $1/\lambda$  around the origin. These matters are not pursued further in this essay.

The analysis commences with the next lemma, which is the key to estimating the functions  $c_k(t)$ .

**Lemma 3.4.** *If  $\sigma > 1$ , then*

$$\sum_{m=1}^{k-1} m^{-\sigma} (k-m)^{-\sigma} \leq L(\sigma) k^{-\sigma}, \quad (3.9)$$

where

$$L(\sigma) = 2^{1+\sigma} \sum_{m=1}^{\infty} m^{-\sigma}. \quad (3.10)$$

*Proof.* Using the facts that the sum  $\sum_{m=1}^{k-1} m^{-\sigma} (k-m)^{-\sigma}$  is symmetric in  $m$  and  $k-m$  and that if  $m \leq k/2$  then  $(k-m)^{-\sigma} \leq (k/2)^{-\sigma}$ , it follows that

$$\begin{aligned} \sum_{m=1}^{k-1} m^{-\sigma} (k-m)^{-\sigma} &\leq 2 \sum_{m=1}^{\lfloor k/2 \rfloor} m^{-\sigma} (k-m)^{-\sigma} \\ &\leq 2 \left(\frac{k}{2}\right)^{-\sigma} \sum_{m=1}^{\lfloor k/2 \rfloor} m^{-\sigma} \\ &\leq 2^{1+\sigma} k^{-\sigma} \sum_{m=1}^{\infty} m^{-\sigma}, \end{aligned}$$

where  $\lfloor b \rfloor$  is the usual greatest integer function of a real number  $b$ . □

**Remark 3.5.** The sum appearing on the right-hand side of the last formula is of course the Riemann zeta function, which appeared unexpectedly in this calculation (see also the last paragraph of Remark (4) in the Appendix).



**Remark 3.6.** Note that  $L(\sigma) \rightarrow \infty$  both as  $\sigma \rightarrow 1$  and as  $\sigma \rightarrow \infty$ . Also,  $L(2) = 8 \sum_{m=1}^{\infty} m^{-2} = \frac{4}{3}\pi^2$ .

**Proposition 3.7.** For  $M > 0$ ,  $\lambda > 0$  and  $\sigma > 1$ , suppose that  $c_k(0) \in \mathbb{R}$  is such that

$$|c_k(0)| \leq Mk^{-\sigma} \lambda^k$$

for all  $k \geq 1$ . Let the functions  $c_k(t)$ ,  $t \in \mathbb{R}$  be the resulting solutions of (3.5). It follows that

$$|c_k(t)| \leq Mk^{-\sigma} \lambda^k e^{MLk|t|}, \quad (3.11)$$

for all  $k = 1, 2, 3, \dots$  and all  $t \in \mathbb{R}$ , where  $L = L(\sigma)$  is as in (3.10).

In other words, if  $\{c_k(0)\} \in \mathcal{W}_{\lambda, \sigma}$ , then  $\{c_k(t)\} \in \mathcal{W}_{\lambda e^{ML|t|}, \sigma}$  for all  $t \in \mathbb{R}$ .

*Proof.* The formula is certainly true for  $k = 1$ . Indeed  $c_1(t)$  is constant, so information is being lost. Let  $k \geq 2$ . Assume inductively that (3.11) holds for all values up through and including  $k - 1$ . The proof is written for  $t \geq 0$ ; the proof for  $t \leq 0$  is similar. The inductive step is carried out via the following string of inequalities:

$$\begin{aligned} |c_k(t)| &\leq Mk^{-\sigma} \lambda^k + k \int_0^t \sum_{m=1}^{k-1} |c_m(s) c_{k-m}(s)| ds \\ &\leq Mk^{-\sigma} \lambda^k + k \int_0^t \sum_{m=1}^{k-1} M m^{-\sigma} \lambda^m e^{MLms} M (k-m)^{-\sigma} \lambda^{k-m} e^{ML(k-m)s} ds \\ &= Mk^{-\sigma} \lambda^k + k \lambda^k M^2 \int_0^t e^{MLks} ds \sum_{m=1}^{k-1} m^{-\sigma} (k-m)^{-\sigma} \\ &= Mk^{-\sigma} \lambda^k + k \lambda^k M^2 \left( \frac{e^{MLkt} - 1}{MLk} \right) \sum_{m=1}^{k-1} m^{-\sigma} (k-m)^{-\sigma} \\ &\leq Mk^{-\sigma} \lambda^k + k \lambda^k M^2 \left( \frac{e^{MLkt} - 1}{MLk} \right) L k^{-\sigma} \\ &= Mk^{-\sigma} \lambda^k + \lambda^k M (e^{MLkt} - 1) k^{-\sigma} = M \lambda^k e^{MLkt} k^{-\sigma}, \end{aligned}$$

where the penultimate step made use of Lemma 3.4.  $\square$

**Remark 3.8.** Since  $d_k(t) = \sum_{m=1}^{k-1} c_m(t) c_{k-m}(t)$  and  $c'_k(t) = k d_k(t)$ , the above calculation provides the inequality

$$|c'_k(t)| = k |d_k(t)| \leq M^2 L(\sigma) k^{-\sigma+1} \lambda^k e^{MLk|t|}. \quad (3.12)$$

which will be needed later.

The estimate (3.11) is not best possible since, as noted above, information is lost even for  $k = 1$ . In the case where  $c_k(0) = 0$  for all  $k \geq 2$ , one can obtain, not surprisingly, a much stronger bound.

**Proposition 3.9.** *Let  $c_k(0) \in \mathbb{R}$  be such that  $c_k(0) = 0$  for all  $k \geq 2$ , and let the functions  $c_k(t)$ ,  $t \in \mathbb{R}$  be the resulting solutions of (3.5). It follows that for all  $\sigma > 1$ , the functions  $c_k(t)$  satisfy*

$$|c_k(t)| \leq (2L)^{k-1} |c_1(0)|^k k^{-\sigma} |t|^{k-1}, \quad (3.13)$$

for all  $k = 1, 2, 3, \dots$  and all  $t \in \mathbb{R}$ , where  $L = L(\sigma)$  is as in (3.10).

*Proof.* The formula is certainly true for  $k = 1$ . Let  $k \geq 2$ . Assume (3.13) holds for all values up to and including  $k - 1$ . Again, the the proof is presented for  $t \geq 0$ , the case  $t \leq 0$  being entirely analogous. Since  $c_k(0) = 0$  for all  $k \geq 2$ , (3.5), (3.4) and the induction hypothesis imply that

$$\begin{aligned} |c_k(t)| &\leq k \int_0^t \sum_{m=1}^{k-1} |c_m(s) c_{k-m}(s)| ds \\ &\leq k \int_0^t \sum_{m=1}^{k-1} (2L)^{m-1} |c_1(0)|^m m^{-\sigma} s^{m-1} (2L)^{k-m-1} |c_1(0)|^{k-m} (k-m)^{-\sigma} s^{k-m-1} ds \\ &= k(2L)^{k-2} |c_1(0)|^k \int_0^t s^{k-2} ds \sum_{m=1}^{k-1} m^{-\sigma} (k-m)^{-\sigma} \\ &\leq \frac{k}{k-1} (2L)^{k-2} |c_1(0)|^k t^{k-1} L k^{-\sigma} \leq (2L)^{k-1} |c_1(0)|^k k^{-\sigma} t^{k-1}, \end{aligned}$$

where Lemma 3.4 has been used again.  $\square$

The preceding results, along with the relation (3.1), lead to the following omnibus theorem about existence of solutions of (1.1).

**Proposition 3.10.** *Fix  $\lambda > 0$  and  $\sigma > 1$ . Suppose that the sequence  $\{a_k(0)\}_{k=1,2,3,\dots}$  belongs to  $\mathcal{W}_{\lambda,\sigma}$ , and let  $M > 0$  be such that*

$$|a_k(0)| \leq M k^{-\sigma} \lambda^k, \quad k = 1, 2, 3, \dots \quad (3.14)$$

Let  $a_k(t)$ ,  $t \in \mathbb{R}$ ,  $k = 1, 2, 3, \dots$  be the corresponding solution of the system (1.18)-(1.19), or equivalently (1.20). It follows that

$$|a_k(t)| \leq M k^{-\sigma} \lambda^k e^{MLk|t|}, \quad (3.15)$$

for all  $k = 1, 2, 3, \dots$  and all  $t \in \mathbb{R}$ , where  $L = L(\sigma)$  is defined in (3.10). In other words,  $\{a_k(t)\}_{k=1,2,3,\dots} \in \mathcal{W}_{\lambda e^{ML|t|}, \sigma}$  for  $t \in \mathbb{R}$ . In particular,

$$\rho(t) \geq \lambda^{-1} e^{-ML|t|} \quad (3.16)$$

for all  $t \in \mathbb{R}$ , where  $\rho(t)$  is as in Definition 2.2.

Now suppose further that  $\lambda \in (0, 1)$  and let

$$u_0(x) = \sum_{k=1}^{\infty} a_k(0) e^{ikx} \quad (3.17)$$

where, again, the  $\{a_k(0)\}_{k=1,2,3,\dots} \in \mathcal{W}_{\lambda,\sigma}$ , so that  $u_0 \in C^\infty(\mathbb{S}^1)$ , and take

$$T = -\frac{\log \lambda}{ML(\sigma)} > 0. \quad (3.18)$$

There is a unique solution  $u \in C((-T, T); H^3(\mathbb{S}^1)) \cap C^1((-T, T); L^2(\mathbb{S}^1))$  of (1.1) of the form  $u(t, x) = \sum_{k=1}^{\infty} a_k(t) e^{ikx}$ , for all  $t \in (-T, T)$ , starting at  $u(0, \cdot) = u_0$ . Furthermore,  $u(t, \cdot) \in C^\infty(\mathbb{S}^1)$  for all  $t \in (-T, T)$ ,  $u \in C^1((-T, T); H^m(\mathbb{S}^1))$  for all  $m = 1, 2, 3, \dots$  and so a fortiori  $u \in C([-T, T]; C(\mathbb{S}^1))$ . In particular,  $T_{\max}^- > T$  and  $T_{\max}^+ > T$ , where  $(-T_{\max}^-, T_{\max}^+)$  is the interval of existence of the maximal  $L^2$ -solution of (1.1) with initial value  $u_0$ .

In the special case where  $a_k(0) = 0$  for all  $k \geq 2$ , the lower bound  $T$  on the time of existence of a solution of (1.1) can be taken as

$$T = \frac{1}{2|a_1(0)|L(\sigma)}, \quad (3.19)$$

for any choice of  $\sigma > 1$ , and the absolute value of the coefficients  $a_k(t)$  can be bounded above by

$$|a_k(t)| \leq (2L)^{k-1} |a_1(0)|^k k^{-\sigma} |t|^{k-1}, \quad (3.20)$$

for  $k = 1, 2, 3, \dots$ , where  $L = L(\sigma)$  is as before. In particular,

$$\rho(t) \geq \frac{1}{2L|a_1(0)||t|} \quad (3.21)$$

for all  $t \in \mathbb{R}$ , where  $\rho(t)$  is the radius of convergence discussed in Definition 2.2.

*Proof.* Given the sequence  $\{a_k(0)\}_{k=1,2,3,\dots} \in \mathcal{W}_{\lambda,\sigma}$ , let  $a_k(t)$ ,  $t \in \mathbb{R}$ ,  $k = 1, 2, 3, \dots$  be the resulting solution of the system (1.20). Next, set  $c_k(0) = |a_k(0)|$ , and let  $c_k(t)$  be the resulting solutions of (3.5). It follows from Propositions 3.1 and 3.7 that (3.15) holds for all  $t \in \mathbb{R}$ . Of course, (3.16) and (3.18) now follow.

Similarly, in the special case where  $a_k(0) = 0$  for all  $k \geq 2$ , Propositions 3.1 and 3.9 imply (3.20).

Turning to the solution of (1.1), we first dispense with uniqueness. If

$$u \in C((-T, T); H^3(\mathbb{S}^1)) \cap C^1((-T, T); L^2(\mathbb{S}^1))$$

is a solution of (1.1) of the form  $u(t, x) = \sum_{k=1}^{\infty} a_k(t) e^{ikx}$ , then the coefficients  $a_k(t)$  must satisfy the system (1.18)-(1.19), or equivalently the integral equations (1.20). This uniquely determines the coefficients  $a_k(t)$ , given the values of  $a_k(0)$ ,  $k = 1, 2, 3, \dots$ . Of course uniqueness of the coefficients holds on any interval  $(-\tau, \tau)$  for any  $\tau > 0$ .

As for existence, since  $\rho(t) > 1$  for every  $t \in (-T, T)$ , where  $T > 0$  is given respectively by (3.18) or (3.19), it follows from Proposition 2.3 that the series  $u(t, x) = \sum_{k=1}^{\infty} a_k(t) e^{ikx}$  converges and  $u \in C^\infty((-T, T) \times \mathbb{S}^1)$  is a solution of (1.1). Finally, the assertion  $u(\pm T, \cdot) \in C(\mathbb{S}^1)$  is a consequence of (3.15) and the requirement that  $\sigma > 1$ .  $\square$

*Proof of Theorem 1.1.* Theorem 1.1 is an immediate consequence of Proposition 3.10 along with the observation, made in the introduction, that  $L_{0,+}^2(\mathbb{S}^1)$  is preserved by the local flow on  $L_0^2(\mathbb{S}^1)$  defined by (1.1).  $\square$

**Corollary 3.11.** *Let  $\{a_k(0)\}_{k=1,2,3,\dots}$  be a sequence of complex numbers and let  $\{a_k(t) : \mathbb{R} \rightarrow \mathbb{C}\}_{k=1,2,3,\dots}$  be the corresponding solutions of the system (1.18)-(1.19), or equivalently (1.20). Let  $\rho(t)$  be the corresponding radius of convergence as in Definition 2.2. Then  $\rho : \mathbb{R} \rightarrow [0, \infty]$  is lower semi-continuous and either  $\rho(t) \equiv 0$  on  $\mathbb{R}$ , or  $0 < \rho(t) \leq \infty$  for all  $t \in \mathbb{R}$ .*

*Proof.* If  $\rho(0) > 0$ , it follows from the considerations in Remark 3.3 that the sequence  $\{a_k(0)\}_{k=1,2,3,\dots}$  is in the space  $\mathcal{W}_{\lambda,\sigma}$  for all  $\lambda > 1/\rho(0)$  and all  $\sigma > 1$ . (If  $\rho(0) = \infty$ , this is true for all  $\lambda > 0$ .) Hence by Proposition 3.10, and in particular formula (3.16), it follows that for any  $\lambda > 1/\rho(0)$ , and all  $\sigma > 1$ ,

$$\rho(t) \geq \frac{1}{\lambda} e^{-ML|t|} \tag{3.22}$$

for some  $M > 0$  and  $L = L(\sigma)$  as defined in (3.10). It follows immediately that  $\rho(t) > 0$  for all  $t \in \mathbb{R}$ .

By the temporal translation invariance of the system (1.18)-(1.19), it follows similarly that if  $\rho(t_0) > 0$  for some  $t_0 \in \mathbb{R}$ , then  $\rho(t) > 0$  for all  $t \in \mathbb{R}$ . This establishes the second part of the assertion.

From (3.22) it is clear that

$$\liminf_{t \rightarrow 0} \rho(t) \geq \frac{1}{\lambda}$$

for all  $\lambda > 1/\rho(0)$ . In consequence, it transpires that

$$\liminf_{t \rightarrow 0} \rho(t) \geq \rho(0). \tag{3.23}$$

This proves lower-semicontinuity at  $t = 0$ . Again calling upon the temporal translation invariance of the system (1.18)-(1.19), lower semi-continuity of  $\rho : \mathbb{R} \rightarrow [0, \infty]$  is established at all  $t \in \mathbb{R}$ . □

The special case of a trigonometric polynomial is singled out next, as it will appear later in our commentary.

**Corollary 3.12.** *Suppose  $u_0 \in L^2_{0,+}(\mathbb{S}^1)$  is a trigonometric polynomial. If  $\rho(t)$  is the radius of convergence in Definition 2.2, where the  $a_k(t)$  are the solutions of the system (1.18)-(1.19), then  $\rho(t) \rightarrow \infty$  as  $|t| \rightarrow 0$ .*

*Proof.* Since  $u_0$  has only finitely many nonzero modes, it follows that for every  $\lambda > 0$  and  $\sigma > 1$ , there exists  $M > 0$  such that (3.14) is satisfied. It then follows from (3.16) that

$$\liminf_{t \rightarrow 0} \rho(t) \geq \frac{1}{\lambda}$$

for all  $\lambda > 0$ . □

Corollaries 3.11 and 3.12 leave open the possibility that  $\rho(t) \equiv \infty$  for a given sequence  $\{a_k(0)\}_{k=1,2,3,\dots}$ . It will appear later (Proposition 5.7) that this can never happen.

There is a set of what the authors thought to be interesting remarks and conjectures centered around the foregoing analysis that would naturally appear next. Some of these indicate that the methods used thus far have a wider range of applicability than one might

imagine at first blush. These are postponed until the Appendix, however, as they interrupt the flow toward our primary objectives.

#### 4. TIME PERIODICITY OF SOLUTIONS AND GLOBAL EXISTENCE

**Proposition 4.1.** *Let  $a_k(0) \in \mathbb{C}$ , for  $k = 1, 2, 3 \dots$ , and let the functions  $a_k(t)$ ,  $k \geq 1$ , be the resulting solutions of the system (1.18)-(1.19), or equivalently the system of integral equations (1.20). It follows that the  $a_k(t)$ ,  $k \geq 1$  are periodic with period  $2\pi$ . More precisely, each  $a_k(t)$  is of the form*

$$a_k(t) = \sum_{h=k}^{k^3} \alpha_{k,h} e^{iht} \quad (4.1)$$

where the  $\alpha_{k,h}$  are numerical constants. Furthermore,  $\alpha_{k,h} = 0$  whenever  $h - k$  is not a multiple of 6.

*Proof.* Formula (4.1) will be proved by induction. Since  $b_1(t) \equiv 0$ , it is immediately clear that

$$a_1(t) = e^{it} a_1(0), \quad (4.2)$$

(see Proposition 2.1) which establishes the result for  $k = 1$ .

Suppose (4.1) has been proved up through  $k - 1$ , for some  $k \geq 2$ . Consider the term  $b_k(t)$ . By formula (1.19),  $b_k(t)$  is the sum of terms of the form  $a_m(t)a_{k-m}(t)$ , where  $1 \leq m \leq k - 1$ . Applying the induction hypothesis to  $a_m(t)$  and  $a_{k-m}(t)$ , it is seen that  $a_m(t)a_{k-m}(t)$  is a sum of terms of the form

$$\alpha_{m,h} \alpha_{k-m,h'} e^{i(h+h')t}$$

where  $h - m$  is a multiple of 6, as is  $h' - (k - m)$ , and

$$\begin{aligned} 1 &\leq m \leq k - 1, \\ m &\leq h \leq m^3, \\ k - m &\leq h' \leq (k - m)^3. \end{aligned}$$

It follows that  $h + h' - k$  is a multiple of 6, and

$$k \leq h + h' \leq m^3 + (k - m)^3 < k^3. \quad (4.3)$$

It is this last inequality which is the key to the proof. It thus transpires that each  $b_k(t)$  is the sum of terms of the form  $\beta_{k,h} e^{iht}$ , where  $k \leq h < k^3$ , and  $h - k$  is a multiple of 6.

When formula (1.20) is used to calculate  $a_k(t)$ , there obtains a linear combination of terms of the form

$$e^{ik^3 t} \int_0^t e^{-ik^3 s} e^{ihs} ds = \frac{e^{ik^3 t} - e^{iht}}{i(k^3 - h)}.$$

To expressions of this form must be added the linear term  $e^{ik^3 t} a_k(0)$  from formula (1.20). Since  $k^3 - k = k(k + 1)(k - 1)$  is always a multiple of 6, this shows that  $a_k(t)$  has the desired form, and completes the proof of the proposition.  $\square$

*Proof of Theorem 1.2.* The solution is necessarily of the form (1.11), and the coefficients  $a_k(t)$  satisfy the system of integral equations (1.20). By Proposition 4.1, if the solution exists for a temporal interval of length more than  $2\pi$ , i.e. if  $T_{\max}^- + T_{\max}^+ > 2\pi$ , it must be global in both time directions. Thus, if either  $T_{\max}^-$  or  $T_{\max}^+$  is infinite, the solution must be global in both time directions, so both are infinite. The last statement follows from continuous dependence [22, Theorem 1.6]: if  $T_{\max}^- + T_{\max}^+ > 2\pi$ , then that is also true for nearby initial values.  $\square$

**Corollary 4.2.** *Under the hypotheses of Proposition 3.10, suppose that  $T \geq \pi$ , where  $T$  is given by (3.18), respectively by (3.19) in the special case where  $a_k(0) = 0$  for all  $k \geq 2$ . Then, the resulting solution  $u \in C((-T, T); H^3(\mathbb{S}^1)) \cap C^1((-T, T); L^2(\mathbb{S}^1))$  of (1.1), shown to exist in Proposition 3.10, is in fact global in time, for both positive and negative time, i.e.  $u \in C^1((-\infty, \infty); H^3(\mathbb{S}^1))$ , and is  $2\pi$ -periodic in time.*

*Proof.* This results from the  $2\pi$ -periodicity of the coefficients together with the fact that  $2T \geq 2\pi$  and that the solution is continuous on  $[-T, T]$ . Put another way, the formal solution is in fact a global solution of the equation since it is a solution on a period.  $\square$

**Corollary 4.3.** *Let  $u_0 \in L^2_{0,+}(\mathbb{S}^1)$  be of the form (1.10), and let*

$$u \in C((-T_{\max}^-, T_{\max}^+); L^2_0(\mathbb{S}^1))$$

*be the resulting maximal solution of (1.7). If the solution  $u$  is global, it follows that the solution  $u(t, x)$  is given by the double Fourier series*

$$u(t, x) = \sum_{k=1}^{\infty} \sum_{h=k}^{k^3} \alpha_{k,h} e^{iht} e^{ikx} \quad (4.4)$$

*where the convergence in (4.4) is in (at least)  $L^2([0, 2\pi] \times [0, 2\pi])$ .*

It is important to note that, in addition to formula (4.1), we've shown that each  $b_k(t)$ , for  $k \geq 2$ , can be expressed as

$$b_k(t) = \sum_{h=k}^{k^3-1} \beta_{k,h} e^{iht}, \quad (4.5)$$

where the  $\beta_{k,h}$ ,  $k \geq 2$ ,  $k \leq h < k^3$ , are numerical constants. In particular, recalling the origin (1.18)-(1.19) of the  $b_j$ 's,

$$u(t, x)^2 = \sum_{k=1}^{\infty} \sum_{h=k}^{k^3-1} \beta_{k,h} e^{iht} e^{ikx}. \quad (4.6)$$

Substituting formulas (4.1) and (4.5) into the system (1.18), the relationship

$$h\alpha_{k,h} - k^3\alpha_{k,h} + k\beta_{k,h} = 0, \quad \forall k \geq 2, k \leq h < k^3. \quad (4.7)$$

among the various coefficients emerges. Indeed, equation (4.7) is formally equivalent to equation (1.1), written for the coefficients of the double Fourier series of the solution, for solutions of the form (1.11). Equation (4.7) implies that

$$\alpha_{k,h} = \frac{k}{k^3 - h} \beta_{k,h}, \quad k \geq 2, k \leq h < k^3. \quad (4.8)$$

**Remark 4.4.** It is interesting to examine the relationship between the initial value  $u_0(x)$  given by (1.10) and the coefficients  $\alpha_{k,k^3}$  in the double sum (4.4). By (4.1), one sees that

$$a_k(0) = \sum_{h=k}^{k^3-1} \alpha_{k,h} + \alpha_{k,k^3}. \quad (4.9)$$

On the one hand, given an initial value of the form (1.10), that is, given all the  $a_k(0)$ ,  $k = 1, 2, 3, \dots$ , the resulting solutions  $a_k(t)$  of the system (1.18)–(1.19) are determined and in principle calculable. Likewise, (4.1) allows the same conclusion for all the coefficients  $\alpha_{k,h}$ .

On the other hand, it follows from a recurrence argument that the collection  $a_k(0)$ ,  $k = 1, 2, 3, \dots$  is determined by the collection  $\alpha_{k,k^3}$ ,  $k = 1, 2, 3, \dots$ . More precisely, each  $a_k(0)$  is determined by the collection  $\alpha_{m,m^3}$ ,  $m = 1, 2, 3, \dots, k$ . Indeed, for  $k = 1$  this is clear since  $a_1(0) = \alpha_{1,1}$  by (4.9). Next, assume the assertion is valid up through  $k - 1$ , for some  $k \geq 2$ . To see that  $a_k(0)$  is determined by the collection  $\alpha_{m,m^3}$ ,  $m = 1, 2, 3, \dots, k$ , refer to (4.9) to see that it suffices to show that each  $\alpha_{k,h}$  with  $h < k$  is determined by the collection  $\alpha_{m,m^3}$ ,  $m = 1, 2, 3, \dots, k - 1$ . For that, it suffices by (4.8) to show that each  $\beta_{k,h}$  with  $h < k$  is determined by the collection  $\alpha_{m,m^3}$ ,  $m = 1, 2, 3, \dots, k - 1$ . We know from (1.19) that each  $b_k(t)$  is determined by the  $a_m(t)$ ,  $m = 1, 2, 3, \dots, k - 1$ , and each  $a_m(t)$  is determined by the  $a_\ell(0)$ ,  $1 \leq \ell \leq m$ . By the induction hypothesis, the  $a_\ell(0)$ ,  $1 \leq \ell \leq m < k$  are determined by the  $\alpha_{\ell,\ell^3}$ ,  $\ell = 1, 2, 3, \dots, m$ . In other words,  $b_k(t)$  is determined by the  $\alpha_{\ell,\ell^3}$ ,  $\ell = 1, 2, 3, \dots, k - 1$ , and, because of (4.5), so are all the  $\beta_{k,h}$  with  $h < k$ . This completes the induction argument.

To summarize the above, *either* the collection of all the  $a_k(0)$ ,  $k = 1, 2, 3, \dots$ , *or* the collection of all the  $\alpha_{k,k^3}$ ,  $k = 1, 2, 3, \dots$ , uniquely determine the functions  $a_k(t)$ ,  $k = 1, 2, 3, \dots$ , and all the coefficients  $\alpha_{k,h}$ ,  $k = 1, 2, 3, \dots$ ,  $k \leq h \leq k^3$ .

While the collection  $a_k(0)$ ,  $k = 1, 2, 3, \dots$  has a clear interpretation as explicitly giving the initial value for the solution of (1.1), it does not seem immediately obvious what is the interpretation of the collection  $\alpha_{k,k^3}$ ,  $k = 1, 2, 3, \dots$ .

Finally, in light Proposition 4.1, there obtains the following improvement of Corollary 3.11.

**Corollary 4.5.** *Let  $\{a_k(0)\}_{k=1,2,3,\dots}$  be a sequence of complex numbers and let  $\{a_k(t) : \mathbb{R} \rightarrow \mathbb{C}\}_{k=1,2,3,\dots}$  be the corresponding solutions of the system (1.18)–(1.19), or equivalently (1.20). Let  $\rho(t)$  be the corresponding radius of convergence as in Definition 2.2. Either  $\rho(t) \equiv 0$  on  $\mathbb{R}$ , or  $\inf_{t \in \mathbb{R}} \rho(t) > 0$  for all  $t \in \mathbb{R}$ .*

*Proof.* The proof is the same as for Corollary 3.11 with the additional observation that in the case where  $\rho(t)$  is not identically zero, then  $\inf_{t \in \mathbb{R}} \rho(t) = \inf_{t \in [0, 2\pi]} \rho(t) > 0$ , by formula (3.16).  $\square$

## 5. FINITE TIME BLOWUP

It turns out that, without too much effort, all the “diagonal” coefficients  $\alpha_{k,k}$ , and in the case where (2.1) holds, all the coefficients  $\alpha_{k\mu,k\mu^3}$  can be computed explicitly. We begin with a lemma.

**Lemma 5.1.** *Let  $a_k(0) \in \mathbb{C}$ , for  $k = 1, 2, 3, \dots$ , and let the functions  $a_k(t)$ ,  $k \geq 1$ , and  $b_k(t)$ ,  $k \geq 2$ , be the resulting solutions of the system (1.18)–(1.19), or equivalently the system of integral equations (1.20). Let  $\alpha_{k,h}$ , for  $k \leq h \leq k^3$  be such that (4.1) holds, and let  $\beta_{k,h}$ , for  $k \leq h \leq k^3 - 1$  be such that (4.5) holds. It follows that*

$$\beta_{k,k} = \sum_{m=1}^{k-1} \alpha_{m,m} \alpha_{k-m,k-m}, \quad (5.1)$$

for all  $k \geq 2$ .

If, in addition,  $\mu \geq 2$  is an integer such that (2.1) holds, then

$$\alpha_{k,h} = \beta_{k,h} = 0, \quad k \leq h < k\mu^2, \quad (5.2)$$

and

$$\beta_{k,k\mu^2} = \sum_{m=1}^{k-1} \alpha_{m,m\mu^2} \alpha_{k-m,(k-m)\mu^2}, \quad (5.3)$$

for all  $k \geq \mu$ . Moreover, for  $k \not\equiv 0 \pmod{\mu}$ , we have

$$\alpha_{k,k\mu^2} = \beta_{k,k\mu^2} = 0. \quad (5.4)$$

*Proof.* By formulas (1.19) and (4.1), it is deduced that

$$\begin{aligned} b_k(t) &= \sum_{m=1}^{k-1} \left\{ \left[ \alpha_{m,m} e^{imt} + \sum_{h=m+1}^{m^3} \alpha_{m,h} e^{iht} \right] \left[ \alpha_{k-m,k-m} e^{i(k-m)t} + \sum_{h'=k-m+1}^{(k-m)^3} \alpha_{m,h'} e^{ih't} \right] \right\} \\ &= \left[ \sum_{m=1}^{k-1} \alpha_{m,m} \alpha_{k-m,k-m} \right] e^{ikt} + \sum_{h>k} \gamma_h e^{iht}, \end{aligned}$$

for some constants  $\gamma_h$ , and where the last sum above is finite. The relation (5.1) now follows from (4.5).

Next, suppose that (2.1) holds. Use induction to establish (5.2) and (5.3) simultaneously. It follows immediately from Proposition 2.1 that (5.2) and (5.3) are true for all  $1 \leq k \leq 2\mu - 1$ . Further, fix  $k \geq 2\mu$  and suppose that (5.2) and (5.3) are true for all values up through  $k - 1$ . Formulas (1.19) and (4.1) along with the induction hypothesis imply that

$$\begin{aligned} b_k(t) &= \sum_{m=1}^{k-1} \left\{ \left[ \alpha_{m,m\mu^2} e^{im\mu^2 t} + \sum_{h=m\mu^2+1}^{m^3} \alpha_{m,h} e^{iht} \right] \right. \\ &\quad \left. \times \left[ \alpha_{k-m,(k-m)\mu^2} e^{i(k-m)\mu^2 t} + \sum_{h'=(k-m)\mu^2+1}^{(k-m)^3} \alpha_{m,h'} e^{ih't} \right] \right\} \\ &= \left[ \sum_{m=1}^{k-1} \alpha_{m,m\mu^2} \alpha_{k-m,(k-m)\mu^2} \right] e^{ik\mu^2 t} + \sum_{h>k\mu^2} \gamma_h e^{iht}, \end{aligned}$$

which shows (5.3) and the second part of (5.2) for  $k$ . The first part of (5.2) follows from (4.8).



Finally, (5.4) is proved by induction using (5.3) and (4.8). Indeed, if  $k \not\equiv 0 \pmod{\mu}$  and  $1 \leq m \leq k-1$ , then either  $m \not\equiv 0 \pmod{\mu}$  or  $k-m \not\equiv 0 \pmod{\mu}$  (or both).  $\square$

**Proposition 5.2.** *In the situation specified in Lemma 5.1, the “diagonal” coefficients are*

$$\alpha_{k,k} = 6k \left( \frac{a_1(0)}{6} \right)^k, \quad (5.5)$$

for all  $k \geq 1$ . If the integer  $\mu \geq 2$  is such that (2.1) holds, then the coefficients  $\alpha_{k\mu, k\mu^3}$  are given by

$$\alpha_{k\mu, k\mu^3} = 6\mu^2 k \left( \frac{a_\mu(0)}{6\mu^2} \right)^k. \quad (5.6)$$

In particular, the  $\alpha_{k,k}$  are determined solely by  $a_1(0)$ , independent of the values of  $a_k(0)$  for  $k > 1$ . Likewise, in the case where  $a_\mu(0)$  is the first nonzero coefficient at  $t = 0$ , then the  $\alpha_{k\mu, k\mu^3}$  are determined solely by  $a_\mu(0)$ .

*Proof.* Formula (5.5) falls out from another induction argument. A special case of (4.8) is

$$\alpha_{k,k} = \frac{1}{k^2 - 1} \beta_{k,k}, \quad (5.7)$$

for all  $k \geq 2$ . We know that  $a_1(t) = e^{it}a_1(0)$  by (4.2), so it follows that  $\alpha_{1,1} = a_1(0)$ . To get an idea of the further argument, consider the next two values,  $k = 2$  and  $k = 3$ . To compute  $\alpha_{2,2}$ , first determine  $\beta_{2,2}$ . Since  $a_1(t) = a_1(0)e^{it}$ , it follows from (1.19) that  $b_2(t) = a_1(0)^2 e^{2it}$ , i.e.  $\beta_{2,2} = a_1(0)^2$ . It then follows from (5.7) that  $\alpha_{2,2} = a_1(0)^2/3$ . To calculate  $\alpha_{3,3}$ , use (4.1) to write

$$a_2(t) = \sum_{h=2}^8 \alpha_{2,h} e^{iht}.$$

It then transpires that

$$b_3(t) = 2a_1(t)a_2(t) = \sum_{h=2}^8 2a_1(0)\alpha_{2,h} e^{i(h+1)t},$$

from which one deduces that

$$\beta_{3,3} = 2a_1(0)\alpha_{2,2} = 2a_1(0)^3/3.$$

Finally, equation (5.7) yields

$$\alpha_{3,3} = \frac{1}{8}\beta_{3,3} = \frac{1}{12}a_1(0)^3.$$

The cases  $k = 1, 2, 3$  having been established, assume the formula to be correct up through  $k-1$ , and attempt to show it holds for  $k$ . First let us calculate  $\beta_{k,k}$ . By formula

(5.1) and the induction hypothesis, it is seen that

$$\begin{aligned}
\beta_{k,k} &= \sum_{m=1}^{k-1} 6m \left( \frac{a_1(0)}{6} \right)^m 6(k-m) \left( \frac{a_1(0)}{6} \right)^{k-m} \\
&= 36 \left( \frac{a_1(0)}{6} \right)^k \left[ \sum_{m=1}^{k-1} m(k-m) \right] \\
&= 6 \left( \frac{a_1(0)}{6} \right)^k k(k^2 - 1),
\end{aligned}$$

and the inductive step now follows from (5.7).

In the case where (2.1) holds, formula (5.6) is also established by induction. The case  $k = 1$ , i.e.  $\alpha_{\mu,\mu^3} = a_\mu(0)$ , follows from Proposition 2.1. For the induction step, use (5.3) with  $k$  replaced by  $k\mu$ , and apply (5.4) to see that

$$\begin{aligned}
\beta_{k\mu,k\mu^3} &= \sum_{m=1}^{k-1} \alpha_{m\mu,m\mu^3} \alpha_{(k-m)\mu,(k-m)\mu^3} \\
&= \sum_{m=1}^{k-1} 6\mu^2 m \left( \frac{a_\mu(0)}{6\mu^2} \right)^m 6\mu^2 (k-m) \left( \frac{a_\mu(0)}{6\mu^2} \right)^{k-m} \\
&= 36\mu^4 \left( \frac{a_\mu(0)}{6\mu^2} \right)^k \left[ \sum_{m=1}^{k-1} m(k-m) \right] \\
&= 6\mu^4 \left( \frac{a_\mu(0)}{6\mu^2} \right)^k k(k^2 - 1).
\end{aligned}$$

Finally, it follows from (4.8) that

$$\alpha_{k\mu,k\mu^3} = \frac{k\mu}{(k\mu)^3 - k\mu^3} \beta_{k\mu,k\mu^3} = \frac{1}{\mu^2(k^2 - 1)} \beta_{k\mu,k\mu^3} = 6\mu^2 k \left( \frac{a_\mu(0)}{6\mu^2} \right)^k. \quad (5.8)$$

□

*Proof of Theorem 1.3.* Using (4.4) and (5.5), a lower bound on the space-time  $L^2$ -norm of the solution may be obtained by the following simple observation:

$$\|u\|_{L^2([0,2\pi] \times [0,2\pi])}^2 = \sum_{k=1}^{\infty} \sum_{h=k}^{k^3} |\alpha_{k,h}|^2 \geq \sum_{k=1}^{\infty} |\alpha_{k,k}|^2 = \sum_{k=1}^{\infty} 36k^2 \left( \frac{|a_1(0)|}{6} \right)^{2k}.$$

Since this last sum diverges if  $|a_1(0)| \geq 6$ , the putative global solution of (1.1) can only exist if  $|a_1(0)| < 6$ . The second assertion of the theorem is proved in the same way, but using (5.6) instead of (5.5). The last part of the theorem, the sharpness of the conditions in the first part, is established in Remarks 5.4 and 5.6(ii) below. □

**Corollary 5.3.** *Let  $u_0 \in L^2_{0,+}(\mathbb{S}^1)$  be of the form (1.10), and suppose that  $|a_1(0)| > 6$  so that the resulting maximal solution  $u \in C((-T_{\max}^-, T_{\max}^+); L^2_0(\mathbb{S}^1))$  of (1.1) blows up in*

finite time. Then the formal solution given by (1.21) of (1.1) with initial value  $u_0$ , where the coefficients  $a_k(t)$  satisfy the system (1.18)–(1.19), is not in  $\mathcal{D}'(\mathbb{S}^1 \times \mathbb{S}^1)$ .

The same is true if  $a_1(0) = a_2(0) = \dots = a_{\mu-1}(0) = 0$  and  $|a_\mu(0)| > 6\mu^2$ .

*Proof.* We prove only the first statement. For positive integers  $m$ , consider the  $C^\infty$  functions on  $\mathbb{S}^1 \times \mathbb{S}^1$ ,

$$u_m(t, x) = \sum_{k=1}^{k=m} \sum_{h=k}^{h=k^3} \alpha_{k,h} e^{iht} e^{ikx}. \quad (5.9)$$

so that  $\langle u_m, e^{-iht} e^{-ikx} \rangle = \alpha_{k,h}$  for all sufficiently large  $m$ . If there is a distribution  $u \in \mathcal{D}'(\mathbb{S}^1 \times \mathbb{S}^1)$  such that  $u_m \rightarrow u$  as distributions, then

$$\langle u, e^{-iht} e^{-ikx} \rangle = \alpha_{k,h}.$$

From the calculations in the proof of Theorem 1.3, if  $\alpha_{1,1} = a$ , then  $\alpha_{k,k} = 6k(a/6)^k$ , so that

$$\langle u, e^{-ikt} e^{-ikx} \rangle = 6k \left( \frac{a}{6} \right)^k.$$

Suppose now that  $|a| > 6$ , and fix  $1 < b < \frac{|a|}{6}$ , so that  $\frac{|a|}{6b} > 1$ . Consider the following sequence of test functions:

$$\varphi_k(t, x) = b^{-k} e^{-ikt} e^{-ikx}, \quad k = 1, 2, \dots.$$

It is clear that  $\varphi_k$  and all its partial derivatives in  $t$  and  $x$  of any order converge uniformly to 0 on  $\mathbb{S}^1 \times \mathbb{S}^1$ , so in particular,

$$\varphi_k \rightarrow 0 \quad \text{in} \quad \mathcal{D}(\mathbb{S}^1 \times \mathbb{S}^1).$$

On the other hand, if  $u$  is evaluated on  $\phi_k$ , there obtains

$$\langle u, \varphi_k \rangle = b^{-k} \langle u, e^{-ikt} e^{-ikx} \rangle = 6k \left( \frac{a}{6b} \right)^k \not\rightarrow 0.$$

From this, it is concluded that  $u$  is not an element of  $\mathcal{D}'(\mathbb{S}^1 \times \mathbb{S}^1)$ . □

**Remark 5.4.** The following example shows that the condition  $|a_1(0)| \geq 6$  in Theorem 1.3 cannot in general be improved. If  $|\gamma| < 1$ , then

$$u(t, x) = \sum_{k=1}^{\infty} 6k\gamma^k e^{ik(x+t)} = \frac{6\gamma e^{i(x+t)}}{(1 - \gamma e^{i(x+t)})^2} \quad (5.10)$$

is a global, smooth, traveling-wave solution of (1.1) on the circle. Note that

$$u(0, x) = \sum_{k=1}^{\infty} 6k\gamma^k e^{ikx} \quad (5.11)$$

and so  $a_1(0) = 6\gamma$  can take any value with modulus less than 6. Also, in the notation of Proposition 5.2

$$u(t, x) = \sum_{k=1}^{\infty} \alpha_{k,k} e^{ikt} e^{ikx}. \quad (5.12)$$

**Remark 5.5.** Consider the initial value  $u_0(x) = a_1(0)e^{ix}$  and the resulting maximal solution  $u \in C((-T_{\max}^-, T_{\max}^+); L_0^2(\mathbb{S}^1))$  of (1.7), which is of the form (1.11). By Theorem 1.3, if  $|a_1(0)| \geq 6$ , then  $T_{\max}^- < \infty$  and  $T_{\max}^+ < \infty$ , and by Theorem (1.2),  $T_{\max}^- + T_{\max}^+ \leq 2\pi$ . On the other hand, applying Proposition 3.10, formula (3.19), Corollary 4.2, and Remark 3.6, it is ascertained that if  $|a_1(0)| \leq \frac{3}{4\pi^3}$ , then the resulting solution is global. Of course this last condition can not be sharp, since global existence is an open condition. Unpublished numerical simulations indicate that the condition for blowup is likewise not sharp for this example.

**Remark 5.6.** (i) It is well-known that for  $\mu > 0$ , the self-similar transformation

$$u_\mu(t, x) = \mu^2 u(\mu^3 t, \mu x), \quad (5.13)$$

leaves invariant solutions of (1.1). However, in the present, periodic context, one requires  $\mu > 0$  to be an integer in order that  $u_\mu$  be  $2\pi$ -periodic in space whenever  $u$  is  $2\pi$ -periodic in space. Thus, if  $u$  is a formal solution of (1.1) of the form (1.11) and  $\mu \in \mathbb{N}$ , then so is

$$u_\mu(t, x) = \mu^2 \sum_{k=1}^{\infty} a_k(\mu^3 t) e^{ik\mu x} = \sum_{k=1}^{\infty} a_{k,\mu}(t) e^{ikx}, \quad (5.14)$$

where

$$a_{k,\mu}(t) = \begin{cases} \mu^2 a_{k/\mu}(\mu^3 t), & k \equiv 0 \pmod{\mu}, \\ 0, & k \not\equiv 0 \pmod{\mu}. \end{cases} \quad (5.15)$$

In other words, as can be checked directly, if the functions  $a_k \in C^\infty(\mathbb{R}; \mathbb{C})$ ,  $k = 1, 2, 3, \dots$ , constitute a solution of the system (1.18)–(1.19), or equivalently (1.20), then for every positive integer  $\mu$ , so do the functions  $a_{k,\mu} \in C^\infty(\mathbb{R}; \mathbb{C})$ ,  $k = 1, 2, 3, \dots$ , given by (5.15). Furthermore, if  $\rho^\mu(t)$  denotes the radius of convergence of the power series

$$\sum_{k=1}^{\infty} a_{k,\mu}(t) z^k, \quad (5.16)$$

then clearly

$$\rho^\mu(t) = \rho(\mu^3 t)^{1/\mu}, \quad (5.17)$$

where  $\rho(t)$  is as in Definition 2.2.

In the specific case of regular solutions, if  $u_0 \in L_0^2(\mathbb{S}^1)$  is of the form (1.10), and if  $u \in C((-T_{\max}^-, T_{\max}^+); L_0^2(\mathbb{S}^1))$  is the resulting maximal solution, then the existence times for  $u_\mu$ , which has initial value given by  $u_{0,\mu}(x) = \mu^2 u_0(\mu x)$ , are

$$T_{\max,\mu}^- = \frac{T_{\max}^-}{\mu^3} \quad \text{and} \quad T_{\max,\mu}^+ = \frac{T_{\max}^+}{\mu^3}. \quad (5.18)$$

(ii) The transformation (5.13) may be applied to the examples in the two previous remarks. First, in Remark 5.4, if  $|\gamma| < 1$  and  $\mu \geq 2$  is an integer, then

$$u_\mu(t, x) = \sum_{k=1}^{\infty} 6k\mu^2 \gamma^k e^{ik\mu(x+\mu^2 t)} = \frac{6\mu^2 \gamma e^{i\mu(x+\mu^2 t)}}{(1 - \gamma e^{i\mu(x+\mu^2 t)})^2} \quad (5.19)$$

is a global, smooth, traveling-wave solution of (1.1) on the circle. Note that

$$u_\mu(0, x) = \sum_{k=1}^{\infty} 6k\mu^2\gamma^k e^{ik\mu x}, \quad (5.20)$$

so  $a_{1,\mu}(0) = a_{2,\mu}(0) = \dots = a_{\mu-1,\mu}(0) = 0$ , and  $a_{\mu,\mu}(0) = 6\mu^2\gamma$  can take any value with modulus less than  $6\mu^2$ . Also, for this solution,

$$u_\mu(t, x) = \sum_{k=1}^{\infty} \alpha_{k\mu, k\mu^3} e^{ik\mu^3 t} e^{ik\mu x}. \quad (5.21)$$

(iii) Next, for the example in Remark 5.5, if the initial value is taken to be  $a_\mu(0)e^{i\mu x}$ , the resulting solution  $u_\mu(t, x)$  of (1.1) is given by (5.13) where  $u(t, x)$  is the solution with initial value  $a_1(0)e^{ix}$  and  $a_1(0) = a_\mu(0)/\mu^2$ . It follows that  $u_\mu$  blows up in finite time (in both directions) if  $|a_\mu(0)| \geq 6\mu^2$ , which we also know directly from Theorem 1.3, and is global if  $|a_\mu(0)| \leq \frac{3\mu^2}{4\pi^3}$ . In particular, this shows that there exist initial values with arbitrarily large norm in  $L^2(\mathbb{S}^1)$  which give rise to global solutions. For future reference, it will be helpful to obtain a lower bound on the radius of convergence  $\rho^\mu(t)$  of the power series (5.16) for this example. Putting formula (5.17) together with the estimate (3.21), and using the fact that  $a_1(0) = a_\mu(0)/\mu^2$ , the estimate

$$\rho^\mu(t) \geq \frac{1}{[2L|a_\mu(0)|\mu|t|]^{1/\mu}}, \quad (5.22)$$

emerges, where  $L = L(\sigma)$  is as in (3.10) and  $\sigma > 1$ .

(iv) The previous example illustrates another interesting point. Consider the initial value  $u_0(x) = ae^{i\mu x}$ , where now the parameter  $a \in \mathbb{C}$  is not necessarily related to the positive integer  $\mu$ . Note that  $\|u_0\|_{L^2(\mathbb{S}^1)} = \sqrt{|a|}$ . The above analysis shows that for a given value of  $\|u_0\|_{L^2(\mathbb{S}^1)}$ , where  $u_0$  is of the specified form, if the positive integer  $\mu$  is sufficiently large, then  $\mu \geq \sqrt{\frac{4\pi^3}{3}}\|u_0\|_{L^2(\mathbb{S}^1)}$ , and the resulting  $L^2$  solution of (1.1) must be global. In other words, the more highly oscillatory is the initial value, as indicated in this specific context by a large value of  $\mu$ , the more likely it is that the resulting solution will be global. This phenomenon has already been observed in the contexts of the Navier-Stokes system [13, Section 2] and of the nonlinear Schrödinger equation [16, Corollary 2.5].

The following proposition, which in some ways complements Corollary 4.5, is needed for the proof of Theorem 1.4.

**Proposition 5.7.** *Let  $\{a_k(0)\}_{k=1,2,3,\dots}$  be a sequence of complex numbers (not all equal to zero) and let  $\{a_k(t) : \mathbb{R} \rightarrow \mathbb{C}\}_{k=1,2,3,\dots}$  be the corresponding solution of the system (1.18)-(1.19), or equivalently (1.20). Let  $\rho(t)$  be the corresponding radius of convergence as in Definition 2.2. It follows that  $\rho(t)$  is not identically infinite. In particular,  $\rho(t)$  is finite for some value of  $t \in [0, 2\pi)$ .*

*Proof.* We argue by contradiction and suppose that  $\rho(t) \equiv \infty$ . It follows from Proposition 2.3 that  $u$  given by (1.11) is a  $C^\infty$  solution of (1.1) on  $\mathbb{R} \times \mathbb{S}^1$ . Let  $a_{k,\lambda}(t)$  be given by (2.12), for nonzero  $\lambda \in \mathbb{C}$  and  $\rho_\lambda(t)$  be as in Definition 2.2. By formula (2.7), it must also

be true that  $\rho_\lambda(t) \equiv \infty$  and that, again by Proposition 2.3,  $u_\lambda$  given by (2.13) is likewise a  $C^\infty$  solution of (1.1) on  $\mathbb{R} \times \mathbb{S}^1$  for all  $\lambda \in \mathbb{C}$ . Since the initial value of  $u_\lambda$  is given by (2.15), this contradicts Theorem 1.3 for sufficiently large  $|\lambda|$ .

It follows therefore that  $\rho(t)$  cannot be identically infinite.  $\square$

The following theorem includes Theorem 1.4 as a special case.

**Theorem 5.8.** *Let  $\{a_k(0)\}_{k=1,2,3,\dots}$  be a sequence of complex numbers (not all equal to zero) and let  $\{a_k(t) : \mathbb{R} \rightarrow \mathbb{C}\}_{k=1,2,3,\dots}$  be the solution of the system (1.18)-(1.19), or equivalently (1.20), with initial data  $\{a_k(0)\}_{k=1,2,3,\dots}$ . Let  $\rho(t)$  be the corresponding radius of convergence as in Definition 2.2. Suppose further that  $\rho(0) > 0$ . Let  $0 \leq \lambda_0 \leq \lambda_1 \leq \infty$  be defined by*

$$\lambda_0 = \inf_{[0,2\pi]} \rho(t) \leq \rho(0) = \lambda_1. \quad (5.23)$$

It follows that  $0 < \lambda_0 < \infty$ .

Let  $a_{k,\lambda}(t)$  be given by (2.12), for nonzero  $\lambda \in \mathbb{C}$  and  $\rho_\lambda(t)$  be as in Definition 2.2. If  $0 < |\lambda| < \lambda_1$  then  $u_{0,\lambda}$  given by the series (2.15) is  $C^\infty$ , hence in  $L_0^2(\mathbb{S}^1)$ , and therefore gives rise to a maximal solution

$$u_\lambda \in C((-T_{\max,\lambda}^-, T_{\max,\lambda}^+); L_0^2(\mathbb{S}^1))$$

of (1.1), which is necessarily of the form (2.13). Furthermore,

- (i) if  $0 < |\lambda| < \lambda_0 \leq \lambda_1$ , then  $u_\lambda$  is global and  $u_\lambda \in C^\infty(\mathbb{R} \times \mathbb{S}^1)$  ;
- (ii) if  $0 < \lambda_0 \leq |\lambda| < \lambda_1$ , then  $u_\lambda$  blows up in finite time ;
- (iii) in the case  $0 < \lambda_0 < |\lambda| < \lambda_1$ ,  $u_\lambda \in C^\infty((-T_{\max,\lambda}^-, T_{\max,\lambda}^+) \times \mathbb{S}^1)$  and the positive and negative blowup times are given by Proposition 2.5.

Moreover,  $\inf_{[0,2\pi]} \rho(t)$  is realized, i.e. there exists  $t \in [0, 2\pi]$  such that  $\rho(t) = \lambda_0$

*Proof.* Recall that  $\rho(t)$  is  $2\pi$ -periodic by Proposition 4.1. It follows from Proposition 5.7 and Corollary 4.5, with the hypothesis  $\rho(0) > 0$ , that  $0 < \inf_{[0,2\pi]} \rho(t) < \infty$ , and so  $0 < \lambda_0 < \infty$ . Furthermore, since  $0 < \rho(0) \leq \infty$  it must be that  $0 < \lambda_1 \leq \infty$ . Lastly, it is clear that  $\lambda_0 \leq \lambda_1$  with equality only if  $\rho(t)$  achieves its minimum at  $t = 0$ .

Recall that by (2.7)  $\rho_\lambda(t) = |\lambda|^{-1}\rho(t)$  for all nonzero  $\lambda \in \mathbb{C}$ . In particular, if  $|\lambda| > \lambda_1$ , so that  $\rho_\lambda(0) = |\lambda|^{-1}\rho(0) < 1$ , then by Proposition 2.3 the series (2.15) does not converge in  $\mathcal{D}'(\mathbb{S}^1)$ .

On the other hand, if  $0 < |\lambda| < \lambda_1$ , so that  $\rho_\lambda(0) = |\lambda|^{-1}\rho(0) > 1$ , then again by Proposition 2.3 the function  $u_{0,\lambda}$  given by the series (2.15) is  $C^\infty$ , and so in  $L_0^2(\mathbb{S}^1)$ . It therefore gives rise to a maximal solution

$$u_\lambda \in C((-T_{\max,\lambda}^-, T_{\max,\lambda}^+); L_0^2(\mathbb{S}^1)),$$

which is necessarily of the form (2.13). If in addition  $0 < |\lambda| < \lambda_0 \leq \lambda_1$ , so that  $\rho_\lambda(t) > 1$  for all  $t \in [0, 2\pi]$ , then by Proposition 2.3 the resulting solution  $u_\lambda$  is global and also of class  $C^\infty$  on  $\mathbb{R} \times \mathbb{S}^1$ . If instead,  $\lambda_0 < |\lambda| < \lambda_1$ , then  $\rho_\lambda(t)$  must assume a value less than 1 for some  $t \in [0, 2\pi)$ , and so again by Proposition 2.3 the solution can not be global. The positive and negative existence times are given by Proposition 2.5.

In the case  $0 < |\lambda| = \lambda_0 < \lambda_1$ , then the solution blows up in finite time. Indeed, since (for suitable small  $|\epsilon|$ )  $0 < \lambda_0 < |\lambda + \epsilon| < \lambda_1$ , the solution  $u_{\lambda+\epsilon}$  blows up in finite time,

and  $u_{0,\lambda+\epsilon} \rightarrow u_{0,\lambda}$  in  $L_0^2(\mathbb{S}^1)$  as  $\epsilon \rightarrow 0$ . Since finite time blowup is a closed condition with respect to the initial value by Theorem 1.2, it must be that  $u_{\lambda_0}$  blows up in finite time. Furthermore, it must be that  $\inf_{[0,2\pi]} \rho_{\lambda_0}(t) = 1$  is realized. If not, then  $\rho_{\lambda_0}(t) > 1$  for all  $t \in \mathbb{R}$ , and the solution would be global and  $C^\infty$ . By formula (2.7), this shows that whenever  $\lambda_0 < \lambda_1$ , it must be that  $\inf_{[0,2\pi]} \rho(t)$  is always realized at some value of  $t \in [0, 2\pi]$ .

If  $\lambda_0 = \lambda_1$ , then the infimum  $\lambda_0$  is realized at  $t = 0$  by definition. □

*Proof of Theorem 1.4.* Let  $u_0 \in L_{0,+}^2(\mathbb{S}^1)$ , of the form (1.10), be given by a trigonometric polynomial, as in (1.16), where  $a_k(0) \in \mathbb{C}$  for  $k = 1, 2, 3, \dots, m$ , for some integer  $m \geq 1$ . Let the functions  $\{a_k(t) : \mathbb{R} \rightarrow \mathbb{C}\}_{k=1,2,3,\dots}$  be the solutions of the system (1.18)-(1.19) with the initial values  $a_k(0) \in \mathbb{C}$  for  $k = 1, 2, 3, \dots, m$  and  $a_k(0) = 0$  for  $k > m$ . If  $\rho(t)$  is the radius of convergence as in Definition 2.2, then clearly  $\rho(0) = \infty$ . In the notation of Theorem 5.8, this means that  $\lambda_1 = \infty$ . Thus Theorem 1.4 follows from Theorem 5.8. □

**Remarks 5.9.** (i) In the context of Theorem 5.8, it is interesting to suppose that the initial value  $u_0$  given by the Fourier series (1.10) is in  $L_0^2(\mathbb{S}^1)$ , but not in  $C^\infty(\mathbb{S}^1)$ . Clearly, it must be the case that  $\rho(0) = 1$ , and by Proposition 2.5 that  $\rho(t) \equiv 1$  throughout the interval of existence of the maximal solution with initial value  $u_0$ . By Theorem 1.2, this solution is global if  $\|u_0\|_{L^2(\mathbb{S}^1)}$  is sufficiently small. In this case, for all  $\lambda \in \mathbb{C}$  nonzero,  $\rho_\lambda(t) \equiv |\lambda|^{-1}$  on  $\mathbb{R}$ .

If this particular choice of  $u_0$  is modified by changing (only) the value of  $a_1(0)$ , then the facts that  $u_0 \in L_0^2(\mathbb{S}^1)$ , but not  $C^\infty(\mathbb{S}^1)$ , and that  $\rho(0) = 1$  remain the same. Thus, as before  $\rho(t) \equiv 1$  throughout the interval of existence of the maximal solution with the new initial value  $u_0$ . However, if we choose  $|a_1(0)| \geq 6$ , it follows from Theorem 1.3 that the resulting solution is no longer global. In this case, there is not much immediate information about the behavior of  $\rho(t)$  outside the interval of existence of the maximal  $L^2$  solution. We do know that the formal Fourier expression for  $u(T_{\max}^+)$ , for example, can not converge in  $L_0^2(\mathbb{S}^1)$ , and so  $\rho(T_{\max}^+) \leq 1$ . In addition, it is not clear if  $\lambda_0 = \lambda_1 = \rho(0) = 1$  or if  $\lambda_0 < 1$ . It seems possible that in this case also,  $\rho(t) \equiv 1$  on  $\mathbb{R}$ .

(ii) Suppose next that  $u_0$  is a trigonometric polynomial as in Theorem 1.4. Here all one can say immediately is that  $\rho(0) = \infty$  and that  $\rho(t)$  must assume a finite value for some  $t \in (0, 2\pi)$ .

**Remark 5.10.** If instead of identifying the  $2\pi$ -periodic solutions of (1.1) of the form (1.11) with functions on  $\mathbb{S}^1$ , they are considered as functions on  $\mathbb{R}$ , it becomes natural to look for possible extensions to a portion of the complex plane  $\mathbb{C}$  which contains  $\mathbb{R}$ . To this end, consider solutions to

$$u_t + u_{zzz} + (u^2)_z = 0, \tag{5.24}$$

on  $\mathbb{R} \times \mathbb{C}$  of the form

$$u(t, z) = \sum_{k=1}^{\infty} a_k(t) e^{ikz}. \tag{5.25}$$

with initial value on  $\mathbb{C}$  of the form

$$u_0(z) = \sum_{k=1}^{\infty} a_k(0) e^{ikz}. \quad (5.26)$$

As functions of  $z \in \mathbb{C}$ ,  $u(t, z)$  is still periodic with period  $2\pi$  for each  $t \in \mathbb{R}$ . However, they are no longer considered as being defined on  $\mathbb{S}^1$ . For a function  $u(t, z)$  of the form (5.25) to be a solution of (5.24), the coefficients must constitute a solution of the system (1.18)-(1.19), or equivalently (1.20). Thus, the same coefficients give rise to a formal solution of (5.24) as well as a formal solution of (1.1).

Furthermore, for each  $t \in \mathbb{R}$ , a region in  $\mathbb{C}$  can be determined where the solution (5.25) is well-defined and analytic from information about  $\rho(t)$ . Indeed, if  $z = x + iy$ , then

$$u(t, z) = \sum_{k=1}^{\infty} a_k(t) e^{ikz} = \sum_{k=1}^{\infty} a_k(t) e^{-ky} e^{ikx} = \sum_{k=1}^{\infty} a_k(t) \lambda^k e^{ikx} = u_\lambda(t, x), \quad (5.27)$$

where  $\lambda = e^{-y} > 0$ . From the definition of  $\rho_\lambda(t)$  and the proof of Proposition 2.3, it is clear that  $u_\lambda(t, x)$  is well-defined and analytic in  $x \in \mathbb{R}$  precisely where  $\rho_\lambda(t) > 1$ , which is to say, for  $t \in (-T_{\max, \lambda}^-, T_{\max, \lambda}^+)$ . The condition  $\rho_\lambda(t) > 1$  translates as  $\rho(t) > e^{-y}$ , i.e.  $y > -\log \rho(t)$ . In other words, for each  $t \in \mathbb{R}$ , the function  $u(t, z)$  is analytic in the region

$$\Omega_t : \{z \in \mathbb{C} : \Im z > -\log \rho(t)\}.$$

For this region to include the real axis  $\Im(z) = y = 0$ , one needs that  $\rho(t) > 1$ .

Returning to the case of solutions on  $\mathbb{R} \times \mathbb{S}^1$  as in Theorem 5.8, assume for simplicity that  $\rho(0) > 1$ , i.e.  $\lambda_1 > 1$ . By part (i) of Theorem 5.8, the condition  $\rho(t) > 1$  for all  $t$  implies that the solution  $u$  is global. More interestingly, in the case of parts (ii) and (iii) of Theorem 5.8, Proposition 2.5 implies that blowup occurs at the precise time  $t$  that  $\rho(t)$  ceases to remain bigger than 1. In terms of the complex solutions  $u(t, z)$ , this is precisely the value of  $t$  at which the real axis is no longer included in the natural complex domain  $\Omega_t$  of analyticity of  $u(t, \cdot)$ . Thus, blowup happens as a singularity located away from the real axis moves onto the real axis. Of course it is not necessarily a pole, essential singularity or branch point, but it does correspond to a loss of analyticity somewhere on the boundary  $\partial\Omega_t$ .

This is reminiscent of the blowup behavior observed in [8, 10]. However, in the present context, we are not able to conclude that the singularity off the real axis moves *continuously* onto the real axis at blowup. That conclusion would require knowing that  $\rho(t) \rightarrow 1$  as  $t \rightarrow T_{\max}^+$ .

(ii) The above analysis could also be applied, at least formally, to solutions of (1.1) in  $L_0^2(\mathbb{S}^1)$ , where as noted in the introduction, the initial value problem is well-posed. In that case, consideration is given to solutions on  $\mathbb{R} \times \mathbb{C}$  of the form

$$u(t, z) = \sum_{k \neq 0} a_k(t) e^{ikz}, \quad (5.28)$$

with initial value

$$u_0(z) = \sum_{k \neq 0} a_k(0) e^{ikz}, \quad (5.29)$$



for which  $u_0(z)$  is periodic with period  $2\pi$  and analytic in some neighborhood of the real line, hence certainly lying in  $L_0^2(\mathbb{S}^1)$ . Let  $(-T_{\max}^-, T_{\max}^+)$  be the time interval of existence of the maximal solution to (1.1) in  $L_0^2(\mathbb{S}^1)$ , and suppose that for each  $t \in (-T_{\max}^-, T_{\max}^+)$  the solution  $u(t, z)$  remains analytic in some neighborhood of the real line. In particular, by periodicity, for each  $t \in (-T_{\max}^-, T_{\max}^+)$ , there is a maximal strip  $Y_n(t) < \Im z < Y_p(t)$ , with  $-\infty \leq Y_n(t) < 0 < Y_p(t) \leq \infty$ , such that the solution is well defined on the strip  $\{(t, z) : t \in (-T_{\max}^-, T_{\max}^+), Y_n(t) < \Im z < Y_p(t)\}$ .

One possible mechanism for blowup is that either or both of  $Y_n(t)$  and  $Y_p(t)$  tend to 0 as  $t \rightarrow T_{\max}^+$ , for example. This is indeed the blowup mechanism that appears in the example provided in [3].

## 6. SOLUTIONS WITH NONZERO MEAN VALUE

The results in the previous sections of this paper are based on the observation that the set of functions on  $\mathbb{S}^1$  whose Fourier series contains only positive modes is formally preserved by the equation (1.1). In addition, since (1.1) is locally well-posed in  $L_0^2(\mathbb{S}^1)$ , it follows that  $L_{0,+}^2(\mathbb{S}^1)$  is invariant under the flow in  $L_0^2(\mathbb{S}^1)$  determined by (1.1). In the same way, we may also consider solutions whose Fourier series contains only nonnegative modes, so having the form

$$\tilde{u}(t, x) = \sum_{k=0}^{\infty} \tilde{a}_k(t) e^{ikx}, \quad (6.1)$$

with initial values

$$\tilde{u}_0(x) = \sum_{k=0}^{\infty} \tilde{a}_k(0) e^{ikx}. \quad (6.2)$$

Functions of the form (6.2) are formally preserved by (1.1), or (1.7). However, we will see in this section that (1.1) is *not* locally well-posed in  $L^2(\mathbb{S}^1)$  if we understand  $L^2(\mathbb{S}^1)$  to include complex-valued functions. This is done by constructing solutions in  $L_+^2(\mathbb{S}^1)$  which reveal that continuous dependence at  $t = 0$  fails to hold. In addition, we will show explicitly how to produce initial values of the form (6.2) having no local solution in  $L_+^2(\mathbb{S}^1)$ . An important consequence of this ill-posedness is that we can not rule out the possibility that an initial value of the form (6.2) gives rise to a solution in  $L^2(\mathbb{S}^1)$  which is *not* of the form (6.1). See Remark 6.6 below for more details about this.

The basic approach continues to be the analysis of the coefficients in the Fourier development (6.1) of the solution. The analysis of the coefficients  $\tilde{a}_k(t)$  is not as straightforward as it was in the case of positive mode solutions since the nonlinear term in the equation for each  $\tilde{a}_k(t)$  does not depend only on the  $\tilde{a}_m(t)$  for  $0 \leq m < k$ . Nonetheless, we are able to provide a complete analysis of the  $\tilde{a}_k(t)$  as determined by the  $\tilde{a}_k(0)$ ,  $k = 0, 1, 2, \dots$ , by reducing it to the solutions  $a_k(t)$  of the system (1.18)–(1.19). If  $\tilde{u}$  is a solution of (1.1) of the form (6.1), then the coefficients  $\tilde{a}_k(t)$  must satisfy

$$\tilde{a}'_k(t) - ik^3 \tilde{a}_k(t) + ik \sum_{m=0}^k \tilde{a}_m(t) \tilde{a}_{k-m}(t) = 0, \quad k = 0, 1, 2, 3, \dots \quad (6.3)$$

Notice that  $\tilde{a}'_0(t) \equiv 0$ , so that  $\tilde{a}_0(t) = \tilde{a}_0(0)$ . Using this fact, equation (6.3) can be rewritten as

$$\tilde{a}'_k(t) + 2ik\tilde{a}_0(0)\tilde{a}_k(t) - ik^3\tilde{a}_k(t) + ik \sum_{m=1}^{k-1} \tilde{a}_m(t)\tilde{a}_{k-m}(t) = 0, \quad k = 1, 2, 3, \dots \quad (6.4)$$

If  $v = \tilde{u} - \tilde{a}_0(0)$ , then

$$v(t, x) = \sum_{k=1}^{\infty} \tilde{a}_k(t) e^{ikx}$$

and

$$v_t + 2\tilde{a}_0(0)v_x + v_{xxx} + (v^2)_x = 0. \quad (6.5)$$

Since  $v$  contains only positive Fourier modes, and each equation (6.4) for  $\tilde{a}_k(t)$  is a linear first order equation with constant coefficients with an inhomogeneous term depending only on the previous  $\tilde{a}_m(t)$ ,  $1 \leq m < k$ , these equations give rise to unique solutions  $\tilde{a}_k \in C^\infty(\mathbb{R}; \mathbb{C})$ , which can in principle be explicitly computed.

Indeed, the following proposition is straightforward to check, where for notational simplicity, we set  $a_0 = 2\tilde{a}_0(0)$ .

**Proposition 6.1.** *Let  $\{a_k\}_{k=1}^\infty$  be a sequence of smooth functions  $a_k : \mathbb{R} \rightarrow \mathbb{C}$  and let  $a_0 \in \mathbb{C}$ . Set*

$$\tilde{a}_k(t) = e^{-ika_0 t} a_k(t), \quad k \geq 1, \quad (6.6)$$

so that

$$\tilde{a}_k(0) = a_k(0), \quad k \geq 1. \quad (6.7)$$

Then the sequence  $\{a_k\}_{k=1}^\infty$  constitutes a solution of the system (1.18)–(1.19) if and only if the sequence of functions  $\{\tilde{a}_k\}_{k=1}^\infty$  constitutes a solution of the system

$$\tilde{a}'_k(t) + ik a_0 \tilde{a}_k(t) - ik^3 \tilde{a}_k(t) + ik \tilde{b}_k(t) = 0, \quad k = 1, 2, 3, \dots, \quad (6.8)$$

where  $\tilde{b}_1(t) \equiv 0$  and, for  $k \geq 2$ ,

$$\tilde{b}_k(t) = \sum_{m=1}^{k-1} \tilde{a}_m(t) \tilde{a}_{k-m}(t) = e^{-ika_0 t} b_k(t). \quad (6.9)$$

In other words, given  $\tilde{a}_k(0) \in \mathbb{C}$ ,  $k = 1, 2, 3, \dots$ , there exist unique functions  $\tilde{a}_k \in C^\infty(\mathbb{R}; \mathbb{C})$  which are solutions to the system (6.8)–(6.9) with these initial values  $\tilde{a}_k(0)$ . These solutions are given by (6.6), where the sequence  $\{a_k\}_{k=1}^\infty$  constitutes a solution of the system (1.18)–(1.19) with those same initial values, as in (6.7).

In particular, it follows that the Fourier series  $u$  given by (1.11) is a (formal) solution of (1.1) if and only if

$$\tilde{u}(t, x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} \tilde{a}_k(t) e^{ikx} = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k(t) e^{-ika_0 t} e^{ikx} \quad (6.10)$$

is a (formal) solution of (1.1). Note that

$$\tilde{u}(0, x) = \frac{1}{2} a_0 + u(0, x). \quad (6.11)$$

Finally, if  $\tilde{\rho}_{a_0}(t)$  denotes the radius of convergence of the power series

$$\sum_{k=1}^{\infty} \tilde{a}_k(t) z^k = \sum_{k=1}^{\infty} a_k(t) e^{-ika_0 t} z^k, \quad (6.12)$$

then

$$\tilde{\rho}_{a_0}(t) = e^{-(\Im a_0)t} \rho(t), \quad (6.13)$$

where  $\rho(t)$  is as in Definition 2.2. If  $\tilde{\rho}_{a_0}(t) > 1$  on some interval  $t_1 < t < t_2$ , then  $\tilde{u}$  given by (6.10) is a  $C^\infty((t_1, t_2) \times \mathbb{S}^1)$  solution of (1.1). If  $\tilde{\rho}_{a_0}(t) < 1$  for some fixed value of  $t \in \mathbb{R}$ , then the series in (6.10) does not converge even in  $\mathcal{D}'(\mathbb{S}^1)$ .

*Proof.* These assertions are straightforward consequences of Proposition 1.8 along with the arguments in the proof of Proposition 2.3.  $\square$

Note that the time-periodicity of the spatial Fourier coefficients is in general lost. Also, if  $a_0 \in \mathbb{R}$ , then

$$\tilde{u}(t, x) = \frac{1}{2} a_0 + u(t, x - a_0 t)$$

(see (1.8)) which is why it is easy to extend local well-posedness from  $L^2_0(\mathbb{S}^1)$  to  $L^2(\mathbb{S}^1)$  in the real-valued case.

The following corollary interprets the Proposition 6.1 in the context of  $L^2$  solutions.

**Corollary 6.2.** *Let  $\tilde{u}_0 \in L^2(\mathbb{S}^1)$  be of the form (6.2) and assume that*

$$\tilde{u} \in C((-T_-, T_+); L^2_+(\mathbb{S}^1)), \quad T_\pm > 0,$$

*is a solution of (1.1) with  $\tilde{u}(0) = \tilde{u}_0$ , so that  $\tilde{u}$  is of the form (6.1).*

*Let  $u_0 \in L^2_{0,+}(\mathbb{S}^1)$  be given in terms of  $\tilde{u}_0$  by (6.11), where  $a_0 = 2\tilde{a}_0(0)$ , and let*

$$u \in C((-T_{\max}^-, T_{\max}^+); L^2_{0,+}(\mathbb{S}^1))$$

*be the maximal  $L^2(\mathbb{S}^1)$  solution, necessarily of the form (1.11) with initial value  $u_0 \in L^2_{0,+}(\mathbb{S}^1)$ . It follows that  $\tilde{a}_0(t) \equiv a_0/2$  and the coefficients  $\tilde{a}_k(t), k = 1, 2, 3, \dots$  in (6.1) are related to the coefficients  $a_k(t), k = 1, 2, 3, \dots$  in (1.11) by (6.6).*

*The same conclusions hold if the solution  $\tilde{u}$  is only assumed to exist on a half-open interval, so that*

$$\tilde{u} \in C([0, T_+); L^2_+(\mathbb{S}^1)), \quad T_+ > 0,$$

*or*

$$\tilde{u} \in C((-T_-, 0]; L^2_+(\mathbb{S}^1)), \quad T_- > 0.$$

As already remarked, the theory does not guarantee that any possible solution  $\tilde{u} \in C((-T_-, T_+); L^2(\mathbb{S}^1))$  with initial value  $\tilde{u}_0 \in L^2_+(\mathbb{S}^1)$  will necessarily lie in  $L^2_+(\mathbb{S}^1)$  for  $t \neq 0$ . The point of the above corollary is that there is at most one such solution  $\tilde{u} \in C((-T_-, T_+); L^2_+(\mathbb{S}^1))$  and that it must be given explicitly in terms of a related solution  $u \in C((-T_{\max}^-, T_{\max}^+); L^2_0(\mathbb{S}^1))$  with a specific initial value.

We now turn to the task of constructing solutions  $\tilde{u} \in C((-T_-, T_+); L^2_+(\mathbb{S}^1))$  with specified properties, and showing that in certain cases no such solution exists.

**Proposition 6.3.** Fix  $\lambda > 0$  and  $\sigma > 1$ . Suppose that the sequence  $\{\tilde{a}_k(0)\}_{k=1,2,3,\dots}$  belongs to  $\mathcal{W}_{\lambda,\sigma}$ , defined in Definition 3.2, and let  $M > 0$  be such that the inequality (3.14) holds. Let  $a_0 \in \mathbb{C}$ , and let  $\tilde{a}_k(t) \in C^\infty(\mathbb{R}; \mathbb{C})$ ,  $k = 1, 2, 3, \dots$ , be the resulting solutions of the system (6.8)-(6.9). It follows that

$$|\tilde{a}_k(t)| \leq Mk^{-\sigma} \lambda^k e^{MLk|t|} e^{(\Im a_0)kt}, \quad (6.14)$$

for all  $k = 1, 2, 3, \dots$  and all  $t \in \mathbb{R}$ , where  $L = L(\sigma)$  is as specified in (3.10). In other words,  $\{\tilde{a}_k(t)\}_{k=1,2,3,\dots} \in \mathcal{W}_{\lambda e^{ML|t|} e^{(\Im a_0)kt}, \sigma}$  for all  $t \in \mathbb{R}$ . In particular, if  $\tilde{\rho}(t)$  is the radius of convergence of the power series  $\sum_{k=1}^{\infty} \tilde{a}_k(t) z^k$ , then

$$\tilde{\rho}(t) \geq \lambda^{-1} e^{-ML|t|} e^{-(\Im a_0)t}. \quad (6.15)$$

Suppose in addition that  $0 < \lambda < 1$ , so that  $\tilde{\rho}(0) > 1$ , and let  $\tilde{u}_0 \in L^2_+(\mathbb{S}^1)$  be given by

$$\tilde{u}_0(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} \tilde{a}_k(0) e^{ikx}. \quad (6.16)$$

There is a unique solution  $\tilde{u} \in C((-T_-, T_+); H^3_+(\mathbb{S}^1)) \cap C^1((-T_-, T_+); L^2_+(\mathbb{S}^1))$  of (1.1) such that  $\tilde{u}(0, \cdot) = \tilde{u}_0$ , where

$$T_+ = \begin{cases} \frac{-\log \lambda}{ML(\sigma) + \Im a_0}, & \Im a_0 > -ML(\sigma), \\ \infty, & \Im a_0 \leq -ML(\sigma); \end{cases} \quad (6.17)$$

$$T_- = \begin{cases} \frac{-\log \lambda}{ML(\sigma) - \Im a_0}, & \Im a_0 < ML(\sigma), \\ \infty, & \Im a_0 \geq ML(\sigma). \end{cases} \quad (6.18)$$

This solution is given by the Fourier series

$$\tilde{u}(t, x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} \tilde{a}_k(t) e^{ikx}, \quad (6.19)$$

for all  $t \in (-T_-, T_+)$ . Furthermore,  $\tilde{u} \in C^1((-T_-, T_+); H^m(\mathbb{S}^1))$  for all  $m = 1, 2, 3, \dots$ , so certainly  $\tilde{u} \in C([-T_-, T_+]; C(\mathbb{S}^1))$ .

In the special case where  $\tilde{a}_k(0) = 0$  for all  $k \geq 2$ , the interval  $(-T_-, T_+)$  can be specified as the largest interval containing 0 such that

$$2L(\sigma) |a_1(0)| |t| e^{(\Im a_0)t} < 1, \quad (6.20)$$

for any choice of  $\sigma > 1$ , and the coefficients  $\tilde{a}_k(t)$  obey the inequality

$$|\tilde{a}_k(t)| \leq (2L)^{k-1} |a_1(0)|^k k^{-\sigma} |t|^{k-1} e^{(\Im a_0)kt}, \quad (6.21)$$

for all  $k = 1, 2, 3, \dots$ , where  $L = L(\sigma)$  is as before.

Finally, if  $\Im a_0 < 0$  and  $T_+ > 2\pi$ , then in fact  $\tilde{u}$  given by (6.19) is a positively global solution, i.e.  $\tilde{u} \in C^1((-T_-, \infty); H^m(\mathbb{S}^1))$  for all  $m = 1, 2, 3, \dots$ . Likewise, if  $\Im a_0 > 0$  and  $T_- > 2\pi$ , then in fact  $\tilde{u}$  given by (6.19) is a negatively global solution, i.e.  $\tilde{u} \in C^1((-\infty, T_+); H^m(\mathbb{S}^1))$  for all  $m = 1, 2, 3, \dots$ .

*Proof.* With the exception of the last part, this proposition is a straightforward consequence of Proposition 6.1 and Corollary 6.2 along with Theorem 3.10. In particular, the choice of  $T_+$  and  $T_-$  is such that  $\tilde{\rho}(t) > 1$  on the interval  $(-T_-, T_+)$ .

The last statement uses the periodicity of the coefficients  $a_k(t)$ . In more detail, by Proposition 4.1 we know that  $\rho(t)$  is  $2\pi$ -periodic. Since  $\tilde{\rho}(t) > 1$  on the interval  $(-T_-, T_+)$ , it follows that  $\tilde{\rho}(t) > 1$  on  $[0, 2\pi]$ . But  $\tilde{\rho}(t) = e^{-(\Im a_0)t} \rho(t)$ , so that if  $t \in [0, 2\pi]$ ,

$$\tilde{\rho}(t + 2\pi) = e^{-(\Im a_0)(t+2\pi)} \rho(t + 2\pi) = e^{-(\Im a_0)(t+2\pi)} \rho(t) \geq e^{-(\Im a_0)t} \rho(t) = \tilde{\rho}(t) > 1. \quad (6.22)$$

In other words,  $\tilde{\rho}(t) > 1$  on  $[0, 4\pi]$ . Iterating this argument proves the claim about positively global solutions. As similar argument establishes the claim about negatively global solutions.  $\square$

**Remark 6.4.** Proposition 6.3 can be used to show that there are solutions of (1.1) which are global for positive time, but blowup in finite negative time (and vice versa). This example is inspired by and similar to the example in [16, Remark 2.6].

Let  $u_0 \in L^2_{0,+}(\mathbb{S}^1)$  be a trigonometric polynomial of the form displayed in (1.16), and suppose that  $|a_1(0)| > 6$ , so that the solution with initial value  $u_0$  blows up in finite time, in both directions by Theorem 1.3. Next, for  $a_0 \in \mathbb{C}$ , let  $\tilde{u}$  denote the solution given by (6.10) as in Proposition 6.1. Since  $u_0$  is a trigonometric polynomial, Proposition 6.3 applies for any given value of  $\lambda$  in  $(0, 1)$ . If  $\Im a_0 < 0$  is such that  $T_+ = \infty$ , or, what is the same,  $T_+ > 2\pi$ , then the solution is global in positive time. On the other hand, for  $t < 0$ ,

$$|\tilde{a}_k(t)| = e^{(\Im a_0)kt} |a_k(t)| \geq |a_k(t)|$$

so that  $\|\tilde{u}(t)\|_{L^2(\mathbb{S}^1)} \geq \|u(t)\|_{L^2(\mathbb{S}^1)}$ . This shows that  $\tilde{u}$  must blowup in negative finite time.

Along these lines, note more generally that if the maximal solution with initial value (1.10) blows up in finite positive time, and if  $\Im a_0 \geq 0$ , the same is true for the solution with initial value (6.2) constructed by Proposition 6.1, and the blowup time could be earlier. Likewise, if the maximal solution with initial value (1.10) blows up in finite negative time, and if  $\Im a_0 \leq 0$ , the same is true for the solution with initial value (6.2) constructed by Proposition 6.1, and the blowup time could be earlier (in the negative direction). These assertions are simple consequences of the fact that  $|\tilde{a}_k(t)| = e^{(\Im a_0)kt} |a_k(t)|$ .

*Proof of Theorem 1.5.* Fix some positive integer  $\mu > 1$  and consider initial values of the form

$$u_0^\mu(x) = a_\mu(0)e^{i\mu x}, \quad (6.23)$$

and

$$\tilde{u}_0^\mu(x) = \frac{1}{2}a_0 + u_0^\mu(x) = \frac{1}{2}a_0 + a_\mu(0)e^{i\mu x}. \quad (6.24)$$

In particular, for all  $s \geq 0$ ,

$$\|\tilde{u}_0^\mu\|_{H^s(\mathbb{S}^1)}^2 = \frac{1}{4}|a_0|^2 + (1 + \mu^2)^s |a_\mu(0)|^2. \quad (6.25)$$

Let  $u_\mu(t, x) = \sum_{k=1}^{\infty} a_k(t)e^{ikx}$  be the (formal) solution of (1.1) with initial value  $u_0^\mu$ , and let

$$\tilde{u}_\mu(t, x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \tilde{a}_k(t)e^{ikx} \quad (6.26)$$

be the corresponding (formal) solution of (1.1) with initial value  $\tilde{u}_0^\mu$  as given by Proposition 6.1, where the coefficients are related by formula (6.6). By Corollary 6.2, if there is an  $L^2_+(\mathbb{S}^1)$  solution of (1.1) of the form (6.1) with initial value  $\tilde{u}_0^\mu$  it must be the  $\tilde{u}_\mu$  displayed above, since the coefficients are uniquely determined by the initial value.

It follows from Proposition 2.1 that

$$\|\tilde{u}_\mu(t)\|_{H^s(\mathbb{S}^1)}^2 \geq \frac{1}{4}|a_0|^2 + (1 + \mu^2)^s |a_\mu(0)|^2 e^{2\mu\Im a_0 t}, \quad (6.27)$$

throughout its interval of existence in  $H^s(\mathbb{S}^1)$ .

Let  $s \geq 0$ ,  $\epsilon > 0$ ,  $M > 0$ , and  $T > 0$  be given. Choose the following specific values of  $a_0$  and  $a_\mu(0)$ , as functions of  $\mu$ :

- $a_0 = i\mu^{-\frac{1}{2}}$  ;
- $a_\mu(0) = \mu^{-\frac{1}{2}}(1 + \mu^2)^{-\frac{s}{2}}$ ,

so that

- $\|\tilde{u}_0^\mu\|_{H^s(\mathbb{S}^1)}^2 = \frac{1}{4}|a_0|^2 + (1 + \mu^2)^s |a_\mu(0)|^2 \leq \frac{2}{\mu}$ ,
- $\|\tilde{u}_\mu(t)\|_{H^s(\mathbb{S}^1)}^2 \geq \frac{1}{\mu} e^{2t\sqrt{\mu}}$ .

To complete the proof, it suffices to choose  $\mu$  sufficiently large that

- $\frac{2}{\mu} \leq \epsilon^2$
- $\frac{1}{\mu} e^{2T\sqrt{\mu}} \geq M^2$ .

Indeed, with such choices, either the function  $\tilde{u}_\mu(t)$  does not remain regular enough to be a solution on  $[0, T]$ , or if it does, then  $\sup_{[0, T]} \|\tilde{u}_\mu(t)\|_{H^s(\mathbb{S}^1)} \geq M$ .

Since, by Corollary 6.2,  $\tilde{u}_\mu(t)$  is the only possible solution in  $H^s_+(\mathbb{S}^1)$  with initial value  $\tilde{u}_0^\mu$ , this concludes the proof.  $\square$

*Proof of Theorem 1.6.* The proof of this result follows the same lines as the previous proof, but we need to insure that the formal solution constructed is a smooth solution on the interval  $[0, T]$ . To accomplish this,  $T$  will be allowed to depend on  $\mu$ , and as a consequence, it seems that  $s > 0$  is needed.

First, repeat the previous proof up through formula (6.27), and then continue from there. The solution given by the Fourier series (6.26) is a  $C^\infty$  solution of (1.1) on any time interval over which  $\tilde{\rho}_{a_0}(t) > 1$ , where  $\tilde{\rho}_{a_0}(t)$  is the radius of convergence of the power series (6.12) for this specific solution. From the estimate (5.22) and the relation (6.13), it follows that

$$\tilde{\rho}_{a_0}(t) \geq \frac{e^{-(\Im a_0)t}}{[2L|a_\mu(0)|\mu|t|]^{1/\mu}}, \quad (6.28)$$

where  $L = L(\sigma)$  is as before and  $\sigma > 1$ .

Let  $s \geq 0$ ,  $\epsilon > 0$  and  $M > 0$  be given. The aim is to specify  $T > 0$ ,  $a_0$ ,  $\mu$  and  $a_\mu(0)$  so that

- $\|\tilde{u}_0^\mu\|_{H^s(\mathbb{S}^1)} < \epsilon$  ;
- $\tilde{\rho}_{a_0}(t) > 1$  on  $[0, T]$  ;
- $\|\tilde{u}_\mu(T)\|_{H^s(\mathbb{S}^1)} \geq M$ .

Assuming  $\Im a_0 > 0$ , so that the right-hand side of (6.28) is decreasing on  $[0, T]$ , the condition that  $\tilde{\rho}_{a_0}(t) > 1$  on  $[0, T]$  is implied by

$$\frac{e^{-(\Im a_0)\mu T}}{2L|a_\mu(0)|\mu T} > 1, \quad (6.29)$$

or, equivalently,

$$2L|a_\mu(0)|\mu T e^{(\Im a_0)\mu T} < 1. \quad (6.30)$$

In addition to (6.30), the following two additional conditions are needed:

$$\frac{1}{4}|a_0|^2 + (1 + \mu^2)^s |a_\mu(0)|^2 < \epsilon^2, \quad (6.31)$$

and

$$\frac{1}{4}|a_0|^2 + (1 + \mu^2)^s |a_\mu(0)|^2 e^{2(\Im a_0)\mu T} \geq M^2. \quad (6.32)$$

For convenience, square condition (6.30), *viz.*

$$|a_\mu(0)|^2 \mu^2 T^2 e^{2(\Im a_0)\mu T} < \frac{1}{4L^2}. \quad (6.33)$$

Conditions (6.32) and (6.33) can be gathered together in the two sided inequality,

$$M^2 - \frac{|a_0|^2}{4} < (1 + \mu^2)^s |a_\mu(0)|^2 e^{2(\Im a_0)\mu T} < \frac{(1 + \mu^2)^s}{4L^2 \mu^2 T^2}. \quad (6.34)$$

This time, let

- $a_0 = ip\mu^{-1+r} \log \mu$ ,
- $a_\mu(0) = \mu^{-q}(1 + \mu^2)^{-\frac{s}{2}}$ ,
- $T = \mu^{-r}$ ,

where  $p > 0$ ,  $q > 0$  and  $r > 0$  are to be chosen depending on the value of  $s > 0$ . With these specifications, condition (6.31) becomes

$$\frac{p^2(\log \mu)^2}{4\mu^{2(1-r)}} + \frac{1}{\mu^{2q}} < \epsilon^2. \quad (6.35)$$

As long as  $r < 1$  and  $q > 0$ , this is satisfied for the given value of  $\epsilon$  and all large values of  $\mu$ . The inequality (6.34) with these choices is

$$M^2 - \frac{p^2(\log \mu)^2}{4\mu^{2(1-r)}} < \frac{1}{\mu^{2q}} e^{(2p \log \mu)} = \frac{\mu^{2p}}{\mu^{2q}} < \frac{(1 + \mu^2)^s}{4L^2 \mu^{2(1-r)}}. \quad (6.36)$$

For the given  $s > 0$ , specify  $0 < r < 1$  so that  $1 - r < s$  and then choose  $p, q > 0$  such that  $p - q < s + r - 1$ . It then follows that for the given  $s > 0$ , these choices of  $p, q$  and  $r$  and any  $\epsilon \in (0, 1/2)$  and  $M > 2$ , say, conditions (6.35) and (6.36) are satisfied for sufficiently large  $\mu > 0$ .

To recover the precise statement of the theorem, let  $M = M_n = n$ , for large integers  $n$ , and let  $\epsilon = \epsilon_n = \frac{1}{n}$ , for example. For the fixed value of  $s > 0$  and associated choices of  $p, q, r$  and these values of  $M$  and  $\epsilon$ , choose  $\mu = \mu_n$  so that conditions (6.35) and (6.36) are satisfied. Without loss of generality, we may assume that the  $\mu = \mu_n$  are increasing and it is clear that as the  $M_n \rightarrow \infty$  the  $\mu_n$  must do the same. For the value  $\mu = \mu_n$ , let  $t_n = T = \mu^{-r}$ . Also, let  $\varphi_n = \tilde{u}_0^\mu$  as in (6.24), with  $a_0 = ip\mu^{-1+r} \log \mu$  and

$a_\mu(0) = \mu^{-q}(1 + \mu^2)^{-\frac{s}{2}}$  as before. The claim now is that the conditions (i) – (v) of the theorem are verified. Indeed, (iv) and (v) are immediate. Condition (iii) is true because of (6.25) and (6.35). Condition (i) follows since the solution is determined by the Fourier coefficients in (6.26), which themselves are uniquely determined by  $a_0$  and  $a_\mu(0)$ . For small  $t \geq 0$  the solution is regular, and since (6.30) is true, the solution is regular at least on the time interval  $[0, T] = [0, t_n]$ . Finally, condition (ii) is true by formula (6.32), which follows from (6.36). This completes the proof.  $\square$

**Theorem 6.5.** *Let  $u_0(x) = \sum_{k=1}^{\infty} a_k(0)e^{ikx} \in L^2_{0,+}(\mathbb{S}^1)$  be such that the radius of convergence of the power series  $\sum_{k=1}^{\infty} a_k(0)z^k$  is equal to 1. Let  $\tilde{u}_0 \in L^2_+(\mathbb{S}^1)$  be given by (6.11) for some  $a_0 \in \mathbb{C}$  such that  $\Im a_0 \neq 0$ . Then there does not exist a solution  $\tilde{u} \in C((-T, T); L^2_+(\mathbb{S}^1))$  of (1.1), with initial value  $\tilde{u}_0$ , for any  $T > 0$ .*

*Proof.* If there exists such a solution, it must be of the form (6.10) as in Proposition 6.1 and Corollary 6.2. Furthermore, again in the notation of Proposition 6.1, we have  $\tilde{\rho}_{a_0}(t) = e^{-(\Im a_0)t}\rho(t)$  for all  $t \in \mathbb{R}$  by (6.13). On the other hand, since by assumption,  $\rho(0) = 1$ , it must be that  $\rho(t) \equiv 1$  for all  $t \in (-T, T)$  for sufficiently small  $T > 0$ , by Proposition 2.5. Since  $\Im a_0 \neq 0$ , this implies that  $\tilde{\rho}_{a_0}(t) < 1$  on all intervals either of the form  $(0, T)$  or  $(-T, 0)$  for sufficiently small  $T > 0$ . Finally, since if  $\tilde{\rho}_{a_0}(t) < 1$  on some interval, the corresponding formal solution of (1.1) is not an  $L^2$  solution, by the last part of Proposition 6.1, and therefore the putative solution  $\tilde{u} \in C((-T, T); L^2_+(\mathbb{S}^1))$  of (1.1) cannot exist.  $\square$

*Proof of Theorem 1.7.* Let  $\tilde{u}_0 \in L^2_0(\mathbb{S}^1)$  be as in the statement of the theorem. Then  $\tilde{u}_0$  is given by (6.11) for some appropriate  $u_0 \in L^2_{0,+}(\mathbb{S}^1)$  and  $a_0 \in \mathbb{C}$ . The assumption that the mean value of  $\tilde{u}_0$  has non-zero imaginary part means precisely that  $\Im a_0 \neq 0$ . Furthermore, since  $u_0 = \sum_{k=1}^{\infty} a_k(0)e^{ikx} \in L^2(\mathbb{S}^1)$ , the radius of convergence of the series  $\sum_{k=1}^{\infty} a_k(0)z^k$  must be at least one. Since  $u_0$  is not  $C^\infty$ , the radius of convergence can not be bigger than one. Thus, Theorem 6.5 implies the non-existence of an  $L^2_+(\mathbb{S}^1)$  solution with initial value  $\tilde{u}_0$ .  $\square$

**Remark 6.6.** As has been mentioned earlier, Theorems 1.7 and 6.5 do not exclude the possibility that an initial value  $\tilde{u}_0$  of the form (6.2) in  $L^2_+(\mathbb{S}^1)$  could give rise to a local solution  $\tilde{u} \in C((-T, T); L^2(\mathbb{S}^1))$  which does not remain in  $L^2_+(\mathbb{S}^1)$ . This would be very surprising and counter-intuitive since the subspace  $L^2_+(\mathbb{S}^1) \subset L^2(\mathbb{S}^1)$  is invariant under the natural iterative procedure which would normally be used to construct solutions.

The technical difficulty in proving that such a solution could not exist, using the methods in this paper, is related to the infinitely coupled system of ODEs. More precisely, if

$$\tilde{u}(t, x) = \frac{1}{2}a_0 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \tilde{a}_k(t)e^{ikx}, \tag{6.37}$$

is a solution of (1.1) then the coefficients must satisfy

$$\tilde{a}'_k(t) + ik a_0 \tilde{a}_k(t) - ik^3 \tilde{a}_k(t) + ik \tilde{b}_k(t) = 0, \quad k \in \mathbb{Z}, k \neq 0, \tag{6.38}$$



where

$$\tilde{b}_k(t) = \sum_{\substack{m=-\infty \\ m \neq 0, m \neq k}}^{\infty} \tilde{a}_m(t) \tilde{a}_{k-m}(t). \quad (6.39)$$

If the initial value  $\tilde{u}(0, x)$  is such that  $\tilde{a}_k(0) = 0$  for all  $k < 0$ , then we know there is a solution to the system as described in Proposition 6.1. However, we do not know that this solution is unique. Since the nonlinear terms are infinitely coupled, there is no immediate way to reduce the system (6.38) - (6.39) to something explicitly solvable, or even to prove a uniqueness theorem in this general context. Hence, there could be a solution of the form (6.37) which has nontrivial negative Fourier coefficients, even if the initial value does not. Of course if  $a_0 = 0$ , or more generally if  $a_0 \in \mathbb{R}$ , this anomaly is excluded by the well-posedness result [22, Theorem 1.6]. If  $\Im a_0 \neq 0$ , there is no well-posedness result.

A similar phenomenon occurs with the nonlinear heat equation  $u_t = \Delta u + |u|^\alpha u$ , on  $\mathbb{R}^N$ , for example. By the maximum principle, as well as by the equivalent integral equation, the set of nonnegative functions is preserved under the resulting flow. Indeed, in every case where the initial value problem is locally well-posed, nonnegativity of the initial value is preserved by the solution. However, this equation is not locally well-posed in  $L^p(\mathbb{R}^N)$  if  $1 \leq p < \frac{N\alpha}{2}$ . In this case, it was first observed in [27] that there exist nonnegative functions in  $L^p(\mathbb{R}^N)$  for which no nonnegative solution to the evolution equation exists on any nontrivial time interval. It was therefore somewhat surprising to learn 40 years later that in this case, there could indeed exist sign-changing solutions [14, 15]. In other words, the natural invariance of the equation was not respected by solutions in spaces where the equation was not locally well-posed. We refer the reader to the introduction to [15] for a careful discussion of this phenomenon, including historical references.

The moral is that, as unlikely as it might seem, in the current context of complex-valued solutions to KdV, there may indeed exist solutions which do not respect the natural invariance studied in this paper.

## 7. COMPLEX KdV AS A REAL SYSTEM

The purpose of this section is to recast the results obtained above about periodic complex-valued solutions to the Korteweg-de Vries equation (1.1) in the context of real-valued solutions of dispersive systems. Indeed, slightly changing the notation used previously, if  $w$  is a complex-valued solution of (1.1), then the real and imaginary parts  $u = \Re w$  and  $v = \Im w$  constitute a solution of the coupled system

$$\begin{cases} u_t + u_{xxx} + (u^2 - v^2)_x = 0, \\ v_t + v_{xxx} + (2uv)_x = 0. \end{cases} \quad (7.1)$$

Likewise, if the real-valued pair  $(u, v)$  is a solution of the system (7.1), then  $w = u + iv$  is a complex-valued solution of (1.1). This suggests consideration of more general systems of the form

$$\begin{cases} u_t + u_{xxx} + P(u, v)_x = 0, \\ v_t + v_{xxx} + Q(u, v)_x = 0, \end{cases} \quad (7.2)$$

where  $u$  and  $v$  are real-valued functions of the two real variables  $x$  and  $t$ . Such equations and their near relatives arise as models of wave propagation in several geophysical situations (see [4] and [24] for example).

If  $P$  and  $Q$  are homogeneous, quadratic polynomials in  $(u, v)$ , there is theory for local, and in some circumstances, global well posedness for initial-value problems for such models (see [5]). This theory includes the case of (7.1). The well-posedness holds in the  $L^2(\mathbb{R})$ -based Sobolev spaces  $H^s(\mathbb{R})$  for suitable values of  $s$ . For (7.1) itself, it is also known that there are smooth solutions that blow up in finite time. The blowup solutions can even have smooth initial data that is small in  $H^1(\mathbb{R})$ -norm (see [5], [10]).

The focus here is on the case of periodic solutions where the underlying spatial domain is  $\mathbb{S}^1$  rather than  $\mathbb{R}$ . A natural question is the local well-posedness in  $L^2(\mathbb{S}^1) \times L^2(\mathbb{S}^1)$ , or more generally in the Sobolev space  $H^s(\mathbb{S}^1) \times H^s(\mathbb{S}^1)$ . In the special case of (7.1), the results of Theorems 1.5, 1.6, and 1.7 make it clear that (7.1) can not be locally well-posed in  $L^2(\mathbb{S}^1) \times L^2(\mathbb{S}^1)$ . On the other hand, [22, Theorem 1.6] implies that (7.1) is indeed locally well-posed in  $L_0^2(\mathbb{S}^1) \times L_0^2(\mathbb{S}^1)$ . In other words, a mean-value zero condition is needed for local well-posedness. In [25], the special case

$$\begin{cases} u_t + u_{xxx} + vv_x = 0, \\ v_t + v_{xxx} + (uv)_x = 0, \end{cases} \quad (7.3)$$

of the class of systems (7.2) is studied. This system, derived by Majda and Biello [24], is different from the system (7.1). In fact, the second equation of the system studied in [25] includes an additional coefficient,  $v_t + \alpha v_{xxx} + (uv)_x = 0$ , where  $0 < \alpha \leq 1$ . The main thrust of that paper is to study the case  $0 < \alpha < 1$ . Interest here is precisely in the case  $\alpha = 1$ . The result [25, Theorem 5] states that the system (7.3), i.e. the case  $\alpha = 1$ , is locally well-posed in  $H^s(\mathbb{S}^1) \times H^s(\mathbb{S}^1)$  for any  $s \geq -\frac{1}{2}$ . The result emphasizes that a mean-value zero condition is not needed for local well-posedness. In light of the results in the last section pertaining to (7.1), this immediately raises the question of understanding which systems of the form (7.2) are locally well-posed in the space  $L^2(\mathbb{S}^1) \times L^2(\mathbb{S}^1)$  without a mean-value zero condition, and which require such a condition for well-posedness. We return to this question at the end of this section.

It would also be very interesting to study whether or not the finite time blowup results of Theorems 1.3 and 1.4 hold in the more general context of the systems (7.2). It turns out, unfortunately, that the main tool used in the proof of these theorems is inoperative in the more general context. More precisely, the starting point for the analysis in the main body of the paper is the property that complex-valued functions on the circle only having nontrivial Fourier modes for positive indices are preserved by the KdV flow. The first task in extending the result to the systems (7.2) would be to translate this property in terms of the pair of real-valued function  $(u, v)$  on the circle, and to determine if it holds for the system (7.2). It transpires that in general, this property is not preserved.

To abbreviate the discussion somewhat, a more discursive style than the usual formal theorem-proof presentation will be taken. It will be clear how to fill in the few technical details that are omitted. In what follows, the letter  $w$  will denote complex-valued functions, while  $u$  and  $v$  will be used for real-valued functions. Since functions are defined on the circle, the real-variable  $x$  is replaced by the angle variable  $\theta$ . Moreover, we will be

considering extensions of functions defined on the circle to the interior of the unit disc, where  $z = re^{i\theta}$  becomes the variable. See, in this regard, Remark 2.6.

Suppose that the complex-valued function  $w$  on  $\mathbb{S}^1$ , expressed as a Fourier series, only has nontrivial positive modes, and that  $w = u + iv$ , where

$$w(\theta) = \sum_{k=1}^{\infty} a_k e^{ik\theta}, \quad u(\theta) = \Re w(\theta) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} b_k e^{ik\theta}, \quad v(\theta) = \Im w(\theta) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} c_k e^{ik\theta}. \quad (7.4)$$

The condition of only having positive index Fourier coefficients entails that

$$b_k + ic_k = 0, \quad k < 0. \quad (7.5)$$

Furthermore, since  $u$  and  $v$  are real-valued, it is required that

$$b_k = \overline{b_{-k}}, \quad c_k = \overline{c_{-k}}, \quad (7.6)$$

for all non-zero integers  $k$ . These two conditions yield

$$c_k = \begin{cases} -ib_k, & k > 0, \\ ib_k, & k < 0. \end{cases} \quad (7.7)$$

In other words,  $v = \mathcal{H}u$  is the Hilbert transform of  $u$ , a conclusion that is not at all surprising. The fact that the negative Fourier coefficients of  $w$  all vanish means that  $w$  defined on  $\mathbb{S}^1$  is the boundary value of a function  $W$  defined and holomorphic in the disc, a fact that has been used implicitly many times, starting in Section 2. The real and imaginary parts of  $W$  are conjugate harmonic functions, whose boundary values are related by the Hilbert transform.

Consequently, the question to resolve is under what circumstances the system (7.2) preserves the set of pairs  $(u, v)$  of real-valued functions on  $\mathbb{S}^1$  such that  $v = \mathcal{H}u$ . This appears on the face of it to be a rather formidable calculation.

To cast light on this issue, the question is reformulated on the unit disc  $\mathcal{U} \subset \mathbb{C}$ , rather than on its boundary  $\mathbb{S}^1$  where it is originally posed. Let  $w(t, \theta) = \sum_{k=1}^{\infty} a_k(t) e^{ik\theta}$  be a solution of the KdV equation, written in the form

$$w_t + w_{\theta\theta\theta} + (w^2)_{\theta} = 0, \quad (7.8)$$

and, for each  $t$ , let  $W(t, z) = W(t, re^{i\theta})$  be its extension to a holomorphic function in the disc, so that

$$W(t, z) = W(t, re^{i\theta}) = \sum_{k=1}^{\infty} a_k(t) z^k = \sum_{k=1}^{\infty} a_k(t) r^k e^{ik\theta}. \quad (7.9)$$

As has been observed in the main body of the paper (Remark 2.4), for each fixed  $r > 0$ , the function  $(t, \theta) \rightarrow W(t, re^{i\theta})$  is likewise a (perhaps formal) solution of (7.8), with

$$W_t + W_{\theta\theta\theta} + (W^2)_{\theta} = 0, \quad (7.10)$$

where  $r > 0$  appears as a parameter, having the same role as does  $\lambda$  in Remark 2.4. The condition that the constant term in the Fourier expansion of  $w$  is zero, means, in terms of  $W$ , simply that  $W(t, 0) = 0$ .

One way to understand the fact that the KdV flow preserves the set of functions whose only non-zero Fourier coefficients have positive indices is to observe that this set is preserved under the map

$$w \rightarrow (w^2)_\theta \quad (7.11)$$

and, by extension, that the map

$$W \rightarrow (W^2)_\theta \quad (7.12)$$

preserves the set of holomorphic functions on the disc that vanish at the origin. (The linear terms in the KdV equation obviously preserve this class of functions, so it is the nonlinear term in the equation that becomes the central point.) Focusing on this property enables us to put aside the variable  $t \in \mathbb{R}$ .

The task before us is now clear. Let  $W(re^{i\theta}) = U(re^{i\theta}) + iV(re^{i\theta})$  be a holomorphic function in the unit disc  $\mathcal{U} = \{re^{i\theta} : 0 \leq r < 1\} \subset \mathbb{C}$ , where  $U = \Re W$  and  $V = \Im W$ . Suppose that  $W(0) = 0$ , so that  $U(0) = 0$  and  $V(0) = 0$ . In particular,  $W(z) = W(re^{i\theta}) = \sum_{k=1}^{\infty} a_k z^k = \sum_{k=1}^{\infty} a_k r^k e^{ik\theta}$  so that

$$U(re^{i\theta}) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} b_k r^{|k|} e^{ik\theta}, \quad V(re^{i\theta}) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} c_k r^{|k|} e^{ik\theta},$$

where

$$b_k = \begin{cases} \frac{1}{2}a_k, & k > 0 \\ \frac{1}{2}\bar{a}_{-k}, & k < 0 \end{cases} \quad \text{and} \quad c_k = \begin{cases} -\frac{i}{2}a_k, & k > 0 \\ \frac{i}{2}\bar{a}_{-k}, & k < 0 \end{cases},$$

so the condition (7.7) is respected. As mentioned already, it is the nonlinear terms in (7.10) that might disturb the condition of being holomorphic. So, let  $\widetilde{W} = \widetilde{U} + i\widetilde{V}$ , where

$$\begin{cases} \widetilde{U} = P(U, V)_\theta, \\ \widetilde{V} = Q(U, V)_\theta. \end{cases} \quad (7.13)$$

We need to determine what conditions on the functions  $P$  and  $Q$  imply that  $\widetilde{W}$  is holomorphic on the disc  $\mathcal{U}$  and vanishes at the origin. The Cauchy-Riemann equations come to our rescue. In polar coordinates, they assert that

$$r\widetilde{U}_r = \widetilde{V}_\theta, \quad r\widetilde{V}_r = -\widetilde{U}_\theta, \quad (7.14)$$

which, in terms of  $P$  and  $Q$  becomes

$$rP(U, V)_{\theta r} = Q(U, V)_{\theta\theta}, \quad rQ(U, V)_{\theta r} = -P(U, V)_{\theta\theta}. \quad (7.15)$$

Taking the primitives in the variable  $\theta$  of the equations in (7.15) yields

$$rP(U, V)_r - Q(U, V)_\theta = f(r), \quad rQ(U, V)_r + P(U, V)_\theta = g(r), \quad (7.16)$$

where  $f$  and  $g$  are functions only of  $r$ . In verifying whether or not (7.15) holds, use must be made of the fact that  $U$  and  $V$  themselves verify the Cauchy-Riemann equations (7.14).

As an illustration, consider the system (7.1), where  $P(U, V) = U^2 - V^2$  and  $Q(U, V) = 2UV$ . The left-hand side  $rP(U, V)_r - Q(U, V)_\theta$  of the first equation in (7.16) becomes

$$r(U^2 - V^2)_r - 2(UV)_\theta = 2rUU_r - 2rVV_r - 2UV_\theta - 2VU_\theta = 2U[rU_r - V_\theta] - 2V[rV_r + U_\theta]$$

which equals zero since  $U$  and  $V$  satisfy the Cauchy Riemann equations. A similar calculation reveals that the left-hand side  $rQ(U, V)_r + P(U, V)_\theta$  of the second equation in (7.16) likewise vanishes identically.

Next, consider the Majda-Biello system (7.3) where  $P(U, V) = \frac{1}{2}V^2$  and  $Q(U, V) = UV$ . Here, the left-hand side  $rP(U, V)_r - Q(U, V)_\theta$  of the first equation in (7.16) becomes

$$rVV_r - UV_\theta - VU_\theta = -VU_\theta - UV_\theta - VU_\theta = -2VU_\theta - UV_\theta$$

and this will not in general be independent of  $\theta$ . The conclusion, formal at this point, but pretty clear nonetheless, is that the Majda-Biello system (7.3), as a system governing the evolution of real-valued functions on the circle  $\mathbb{S}^1$ , does not have the property of preserving the set of functions whose non-positive Fourier coefficients are all zero.

Consider now the general case of homogeneous quadratic polynomials, *viz.*

$$\begin{cases} P(U, V) = \alpha_1 U^2 + 2\beta_1 UV + \gamma_1 V^2, \\ Q(U, V) = \alpha_2 U^2 + 2\beta_2 UV + \gamma_2 V^2. \end{cases} \quad (7.17)$$

Calculate as follows:

$$\begin{aligned} rP(U, V)_r &= 2\alpha_1 rUU_r + 2\beta_1 rUV_r + 2\beta_1 rU_rV + 2\gamma_1 rVV_r \\ &= 2\alpha_1 UV_\theta - 2\beta_1 UU_\theta + 2\beta_1 V_\theta V - 2\gamma_1 VU_\theta \end{aligned}$$

and

$$Q(U, V)_\theta = 2\alpha_2 UU_\theta + 2\beta_2 UV_\theta + 2\beta_2 U_\theta V + 2\gamma_2 VV_\theta$$

so that  $-rP(U, V)_r + Q(U, V)_\theta$  is, in detail,

$$2(\alpha_2 + \beta_1)UU_\theta + 2(\beta_2 - \alpha_1)UV_\theta + 2(\beta_2 + \gamma_1)U_\theta V + 2(\gamma_2 - \beta_1)VV_\theta \quad (7.18)$$

$$= 2(U_\theta, V_\theta) \begin{pmatrix} \alpha_2 + \beta_1 & \beta_2 + \gamma_1 \\ \beta_2 - \alpha_1 & \gamma_2 - \beta_1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = 2(U_\theta, V_\theta) \mathcal{M} \begin{pmatrix} U \\ V \end{pmatrix}. \quad (7.19)$$

Likewise,

$$\begin{aligned} rQ(U, V)_r &= 2\alpha_2 rUU_r + 2\beta_2 rUV_r + 2\beta_2 rU_rV + 2\gamma_2 rVV_r \\ &= 2\alpha_2 UV_\theta - 2\beta_2 UU_\theta + 2\beta_2 V_\theta V - 2\gamma_2 VU_\theta \end{aligned}$$

and

$$P(U, V)_\theta = 2\alpha_1 UU_\theta + 2\beta_1 UV_\theta + 2\beta_1 U_\theta V + 2\gamma_1 VV_\theta$$

so that  $rQ(U, V)_r + P(U, V)_\theta$  is given by

$$2(\alpha_1 - \beta_2)UU_\theta + 2(\beta_1 + \alpha_2)UV_\theta + 2(\beta_1 - \gamma_2)U_\theta V + 2(\gamma_1 + \beta_2)VV_\theta. \quad (7.20)$$

$$= 2(V_\theta, -U_\theta) \begin{pmatrix} \alpha_2 + \beta_1 & \beta_2 + \gamma_1 \\ \beta_2 - \alpha_1 & \gamma_2 - \beta_1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = 2(V_\theta, -U_\theta) \mathcal{M} \begin{pmatrix} U \\ V \end{pmatrix}, \quad (7.21)$$

where

$$\mathcal{M} = \begin{pmatrix} \alpha_2 + \beta_1 & \beta_2 + \gamma_1 \\ \beta_2 - \alpha_1 & \gamma_2 - \beta_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (7.22)$$

Search for those matrices  $\mathcal{M}$  for which the expressions (7.18)-(7.19) and (7.20)-(7.21) must be radial functions in the disc, for all pairs  $(U, V)$  of conjugate harmonic functions in the disc which vanish at the origin. Consider in particular the choice

$$U = \Re(z + z^2) = r \cos \theta + r^2 \cos 2\theta \quad \text{and} \quad V = \Im(z + z^2) = r \sin \theta + r^2 \sin 2\theta,$$

so that

$$U_\theta = -r \sin \theta - 2r^2 \sin 2\theta, \quad \text{while} \quad V_\theta = r \cos \theta + 2r^2 \cos 2\theta.$$

Calculating the quantity in (7.21) for this selection of  $U$  and  $V$ , there obtains

$$\begin{aligned} & (V_\theta, -U_\theta) \mathcal{M} \begin{pmatrix} U \\ V \end{pmatrix} \\ &= (r \cos \theta + 2r^2 \cos 2\theta, r \sin \theta + 2r^2 \sin 2\theta) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r \cos \theta + r^2 \cos 2\theta \\ r \sin \theta + r^2 \sin 2\theta \end{pmatrix} \\ &= A(r^2 \cos^2 \theta + 3r^3 \cos \theta \cos 2\theta + 2r^4 \cos^2 2\theta) \\ &\quad + B(r^2 \cos \theta \sin \theta + r^3 \cos \theta \sin 2\theta + 2r^3 \cos 2\theta \sin \theta + 2r^4 \cos 2\theta \sin 2\theta) \\ &\quad + C(r^2 \cos \theta \sin \theta + r^3 \cos 2\theta \sin \theta + 2r^3 \cos \theta \sin 2\theta + 2r^4 \cos 2\theta \sin 2\theta) \\ &\quad + D(r^2 \sin^2 \theta + 3r^3 \sin \theta \sin 2\theta + 2r^4 \sin^2 2\theta) \\ &= r^2(A \cos^2 \theta + D \sin^2 \theta + (B + C) \cos \theta \sin \theta) \\ &\quad + r^3(3A \cos \theta \cos 2\theta + (B + 2C) \cos \theta \sin 2\theta + (2B + C) \cos 2\theta \sin \theta + 3D \sin \theta \sin 2\theta) \\ &\quad + 2r^4(A \cos^2 2\theta + D \sin^2 2\theta + (B + C) \cos 2\theta \sin 2\theta). \end{aligned}$$

For this to be a radially symmetric function in the disc, the coefficients of the powers  $r^2$ ,  $r^3$ , and  $r^4$  must have no dependence on  $\theta$ . In particular, the coefficient of  $r^2$  is

$$\begin{aligned} & A \cos^2 \theta + D \sin^2 \theta + (B + C) \cos \theta \sin \theta \\ &= \frac{1}{2} [A(1 + \cos 2\theta) + D(1 - \cos 2\theta) + (B + C) \sin 2\theta] \\ &= \frac{1}{2} [(A + D) + (A - D) \cos 2\theta + (B + C) \sin 2\theta]. \end{aligned}$$

Since  $\cos 2\theta$  and  $\sin 2\theta$  are linearly independent, it must be the case that  $A = D$  and  $B = -C$ . Using this information along with elementary trigonometric identities, one deduces from the fact that the coefficient of  $r^3$  must vanish that

$$3A \cos \theta = B \sin \theta$$

for all  $\theta$ , from whence the conclusion that  $A = B = 0$  follows. As  $D = A$  and  $C = -B$ , all four of  $A, B, C$  and  $D$  must therefore vanish.

Reverting to the definitions of  $A, B, C$  and  $D$ , this implies that

$$\alpha_1 = \beta_2 = -\gamma_1, \quad \alpha_2 = -\beta_1 = -\gamma_2.$$

If we set  $\alpha = \alpha_1$  and  $\beta = -\beta_1$ , then the system has the form

$$\begin{cases} P(U, V) = \alpha(U^2 - V^2) - 2\beta UV, \\ Q(U, V) = 2\alpha UV + \beta(U^2 - V^2). \end{cases}$$

In other words, the function  $w = u + iv$ , where the pair  $(u, v)$  of real-valued functions satisfies the system (7.2), must be a solution of the equation

$$w_t + w_{\theta\theta\theta} + (\alpha + i\beta)(w^2)_\theta = 0. \quad (7.23)$$

The simple change of variables  $h = (\alpha + i\beta)w$  reduces (7.23) to

$$h_t + h_{\theta\theta\theta} + (h^2)_\theta = 0,$$

the complex KdV equation itself.

Just to be clear about what has been shown, let us return to the context of functions on the circle. Let  $w : \mathbb{S}^1 \rightarrow \mathbb{C}$  be given by  $w(\theta) = e^{i\theta} + e^{2i\theta}$ , which is of course a very simple trigonometric polynomial, as in the hypotheses of Theorem 1.4. Note that its real and imaginary parts  $u = \Re w$  and  $v = \Im w$  are given by  $u(\theta) = \cos \theta + \cos 2\theta$  and  $v(\theta) = \sin \theta + \sin 2\theta$ . Next consider the nonlinear operator on  $C^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1)$  given by  $(u, v) \rightarrow (\tilde{u}, \tilde{v})$  where

$$\begin{cases} \tilde{u} = [\alpha_1 u^2 + 2\beta_1 uv + \gamma_1 v^2]_\theta, \\ \tilde{v} = [\alpha_2 u^2 + 2\beta_2 uv + \gamma_2 v^2]_\theta. \end{cases} \quad (7.24)$$

for some real-valued functions  $u$  and  $v$  on the circle satisfying (7.6) and (7.7). Let  $\tilde{w} = \tilde{u} + i\tilde{v}$ . For the Fourier series of  $\tilde{w}$  on  $\mathbb{S}^1$  to contain only nontrivial modes for positive values of the index  $k$ , it is necessary that  $\tilde{w} = (\alpha + i\beta)(w^2)_\theta$  for some  $\alpha, \beta \in \mathbb{R}$ . Thus, up to the trivial change of variables

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

the only KdV-type system of the form (7.2) where  $P$  and  $Q$  are homogeneous, quadratic polynomials as in (7.17) whose flows preserve having only positive Fourier modes is the complex KdV equation.

**Remark 7.1.** Returning to the question of local non-wellposedness in  $L^2(\mathbb{S}^1) \times L^2(\mathbb{S}^1)$ , the proofs of Theorems 1.5, 1.6, and 1.7 all use the same property of invariance of the set of functions on the circle with only nontrivial positive (or nonnegative) modes. Hence, those constructions can not be adapted to the more general systems discussed in this section. This leads us to conjecture that the non-wellposedness in  $L^2(\mathbb{S}^1) \times L^2(\mathbb{S}^1)$  is limited to systems that arise from the real and imaginary parts of systems of the form (7.23).

## 8. APPENDIX

As mentioned earlier, this Appendix is a set of remarks attached to the theory developed in Section 3. These remarks are comprised of some observations, an indication that the analysis made in Section 3 has prospects beyond the KdV equation, an interesting example and some conjectures.

- (1) It is instructive to compare the two estimates (3.18) and (3.19), which give lower bounds for the existence time of a solution. In the case of an initial value of the form  $u_0(x) = a_1(0)e^{ix}$ , both estimates can be used, for any choice of  $\sigma > 1$ . In the case of (3.18), one can choose any values of  $M > 0$ ,  $\sigma > 1$  and  $0 < \lambda < 1$  such that  $|a_1(0)| \leq M\lambda$ . In particular  $\sigma > 1$  is arbitrary. To maximize the value of  $T$  given

by (3.18), one needs to choose  $M > 0$  and  $0 < \lambda < 1$  both as small as possible. Taking  $M = |a_1(0)|/\lambda$  gives

$$T = \frac{-\lambda \log \lambda}{|a_1(0)|L(\sigma)}$$

for any choice of  $0 < \lambda < 1$  and  $\sigma > 1$ . The right-hand side of the last expression attains its maximum for  $\lambda \in (0, 1)$  at  $\lambda = 1/e$ , where it has the value

$$T = \frac{1}{e|a_1(0)|L(\sigma)}. \tag{8.1}$$

Since  $2 < e$ , the estimate (3.19) provides a longer existence time for the solution with initial value  $u_0(x) = a_1(0)e^{ix}$  than does (3.18).

We will see in part (4) of this Appendix what is likely to be the best possible estimate obtainable by this method for the existence of the solution of (1.1) with initial value  $u_0(x) = a_1(0)e^{ix}$ .

- (2) In addition to providing a means for estimating the coefficients  $a_k(t)$  in the system (1.18)–(1.19), the system (3.2)–(3.4) can be used directly to prove the existence of complex valued solutions to certain partial differential equations. For example, if we set  $f(t, z) = \sum_{k=1}^{\infty} c_k(t)z^k$ , then the system (3.2)–(3.4) corresponds to the partial differential equation

$$\frac{\partial f}{\partial t} = z \frac{\partial}{\partial z} f^2 = 2zf \frac{\partial f}{\partial z} \tag{8.2}$$

in the region of  $\mathbb{C}$  where the series converges. Note that this region may vary in time. In more detail, Proposition 3.7 proves the existence of a solution to (8.2) with initial data  $f(0, z) = \sum_{k=1}^{\infty} c_k(0)z^k$ , when the coefficients  $c_k(0)$  form a sequence in the space  $\mathcal{W}_{\lambda, \sigma}$ . For each time  $t \in \mathbb{R}$ , this solution exists at least in the open ball of radius  $\lambda^{-1}e^{-ML(\sigma)|t|}$  about 0. This solution is unique (in a neighborhood of 0, and so by analytic continuation, in its maximal domain in  $\mathbb{C}$ ), since any solution  $f(t, z)$  would have a power series expansion in  $z$  and the coefficients would have to satisfy the system (3.2)–(3.4). Note that equation (8.2) implies that  $\frac{\partial f}{\partial t}(t, 0) \equiv 0$ , so if  $f(0, 0) = 0$ , then  $f(t, 0) = 0$  for all  $t > 0$ .

- (3) If we set  $g(t, x) = \sum_{k=1}^{\infty} c_k(t)e^{ikx}$ , where  $x \in \mathbb{R}$ , then the system (3.2)–(3.4) corresponds to the partial differential equation

$$\frac{\partial g}{\partial t} + i \frac{\partial}{\partial x} g^2 = 0 \tag{8.3}$$

as long as the Fourier series converges. Thus, Proposition 3.7 proves the existence of a spatially periodic solution to (8.3) with initial data  $g(0, x) = \sum_{k=1}^{\infty} c_k(0)e^{ikx}$ , where the coefficients  $c_k(0)$  are a sequence in the space  $\mathcal{W}_{\lambda, \sigma}$ . It is required that  $0 < \lambda < 1$  to guarantee convergence of the Fourier series giving the initial value. The solution exists as long as the Fourier series converges. Proposition 3.7, respectively Proposition 3.9 in the case where it applies, tell us that this convergence holds at least as long as  $\lambda e^{ML(\sigma)|t|} < 1$ , respectively (3.19). In particular, it follows that  $g : (-T, T) \rightarrow C^\infty(\mathbb{S}^1)$  where  $T$  is given by (3.18), respectively (3.19). This solution



is unique among solutions of the form  $\sum_{k=1}^{\infty} c_k(t)e^{ikx}$ , since the coefficients would have to satisfy the system (3.2)–(3.4).

- (4) As previously remarked, the system (3.2)–(3.4) can in principle be solved explicitly, by recursive calculation. In the special case where  $c_1(0) = c \in \mathbb{C}$  and  $c_k(0) = 0$  for all  $k \geq 2$ , calculations with Maple<sup>2</sup> suggest the following formula:

$$c_k(t) = \frac{(2k)^{k-1}}{k!} c^k t^{k-1}, \quad (8.4)$$

for all  $k \geq 1$  and  $t \in \mathbb{R}$ . While this formula has not yet been analytically verified, assume it to be correct for the purposes of this remark. Consider the power series

$$\sum_{k=1}^{\infty} \frac{(2k)^{k-1}}{k!} z^k. \quad (8.5)$$

Its radius of convergence  $R$  is given by

$$\begin{aligned} -\log R &= \lim_{k \rightarrow \infty} \frac{1}{k} \log \frac{(2k)^{k-1}}{k!} \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} \left( \log(2k)^{k-1} - \log k! \right) \\ &= \lim_{k \rightarrow \infty} \left( \frac{k-1}{k} \log(2k) - \frac{1}{k} \log k! \right) \\ &= \lim_{k \rightarrow \infty} \left( \frac{k-1}{k} \log(2k) - \frac{1}{k} [k \log k - k + O(\log k)] \right) \\ &= \lim_{k \rightarrow \infty} \left( \frac{k-1}{k} \log 2 - \frac{1}{k} \log k + 1 + \frac{1}{k} [O(\log k)] \right) \\ &= 1 + \log 2, \end{aligned}$$

where Stirling's approximation of  $\log k!$  has been used. It is concluded that

$$R = \frac{1}{2e}. \quad (8.6)$$

Therefore, the function

$$g(t, x) = \sum_{k=1}^{\infty} c_k(t)e^{ikx} = \sum_{k=1}^{\infty} \frac{(2k)^{k-1}}{k!} c^k t^{k-1} e^{ikx}$$

is the solution of (8.3) with initial value  $g(0, x) = ce^{ix}$  with only positive Fourier modes, and is defined (and smooth) on the time interval  $t \in (-\frac{1}{2e|c|}, \frac{1}{2e|c|})$ . This is the same time interval as given by the estimate (3.19), except that  $L(\sigma)$  is replaced by  $e$ . Standard approximation of  $k!$  as in Stirling's formula has it that

$$\sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} \leq k! \leq e k^{k+\frac{1}{2}} e^{-k},$$

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<sup>2</sup>The authors thank Gilles Scarella for these calculations.

valid for  $k = 1, 2, \dots$ . Hence,  $k! \geq \frac{\sqrt{2\pi}}{e} k^{k+1/2} e^{-k+1}$  for all  $k = 1, 2, 3, \dots$ , so it follows that

$$\left\{ \begin{aligned} \sum_{k=1}^{\infty} \frac{(2k)^{k-1}}{k!} \frac{1}{(2e)^{k-1}} &\leq \frac{\sqrt{2\pi}}{e} \sum_{k=1}^{\infty} \frac{(2k)^{k-1}}{k^{k+1/2} e^{-k+1}} \frac{1}{(2e)^{k-1}} \\ &= \frac{\sqrt{2\pi}}{e} \sum_{k=1}^{\infty} k^{-3/2} < \infty. \end{aligned} \right. \quad (8.7)$$

This implies that  $g(\pm \frac{1}{2e|c|}, \cdot)$  is a continuous function on  $\mathbb{S}^1$ . On the other hand,

$$\frac{\partial g}{\partial x}(t, x) = i \sum_{k=1}^{\infty} k c_k(t) e^{ikx} = i \sum_{k=1}^{\infty} \frac{2^{k-1} k^k}{k!} c^k t^{k-1} e^{ikx}.$$

Since  $k! \leq k^{k+1/2} e^{-k+1}$  for  $k = 1, 2, 3, \dots$ , it transpires that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{2^{k-1} k^k}{k!} \frac{1}{(2e)^{k-1}} &\geq \sum_{k=1}^{\infty} \frac{2^{k-1} k^k}{k^{k+1/2} e^{-k+1}} \frac{1}{(2e)^{k-1}} \\ &= \sum_{k=1}^{\infty} k^{-1/2} = \infty. \end{aligned}$$

Hence, if  $c > 0$ , then  $\frac{\partial g}{\partial x}(t, 0) \rightarrow \infty$  as  $t \rightarrow \frac{1}{2ec}$ . Thus, this solution cannot be continued smoothly past  $T = \frac{1}{2ec}$ .

In particular, it must be that  $L(\sigma) \geq e$  for all  $\sigma > 1$ , since otherwise the function  $g$  in (8.3) could be continued as a smooth solution beyond  $T = \frac{1}{2ec}$  by Proposition 3.9. We remind the reader that this is all predicated on the validity of formula (8.4).

- (5) In the special case where  $c_k(0) = 1$ , for all  $k = 1, 2, 3, \dots$ , calculations by hand of  $c_1(t)$  through  $c_5(t)$  suggest the following formula for the solution of the system (3.2)–(3.4):

$$c_k(t) = \sum_{m=0}^{k-1} \binom{k-1}{m} \frac{(2kt)^m}{(m+1)!}, \quad (8.8)$$

where

$$\binom{k-1}{m} = \frac{(k-1)!}{m!(k-1-m)!}$$

is the usual binomial coefficient.

Indeed, we conjecture more generally that the solution to the system (3.2)–(3.4) is given by

$$c_k(t) = \sum_{m=0}^{k-1} \left( \sum_{i_1+i_2+\dots+i_{m+1}=k} \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_{m+1}} \right) \frac{(2kt)^m}{(m+1)!} \quad (8.9)$$

where  $c_k(0) = \gamma_k$ .

- (6) The system (3.2)–(3.4) exhibits some simple invariances. If the functions  $c_k(t)$  constitute a solution to the system, then so do the functions

$$\gamma\lambda^k c_k(\gamma t), \tag{8.10}$$

for all  $\gamma > 0$  and  $\lambda > 0$ . Indeed, it was the explicit calculations described above and these invariances which led to consideration of the spaces  $\mathcal{W}_{\lambda,\sigma}$  in Propositions 3.7 and 3.9. In particular, it was the appearance of the terms  $k^{-3/2}$  in formula (8.7) which led to the introduction of the factors  $k^{-\sigma}$  in the definition of  $\mathcal{W}_{\lambda,\sigma}$ .

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