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To cite this version:

Theo van Uem. GENERALIZED FIVE PERSON HAT GAME. Discrete Mathematics and Theoretical Computer Science, In press. hal-03859834

HAL Id: hal-03859834
https://hal.science/hal-03859834
Submitted on 18 Nov 2022

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GENERALIZED FIVE PERSON HAT GAME

THEO VAN UEM

ABSTRACT. The Hat Game (Ebert’s Hat Problem) got much attention in the beginning of this century; not in the last place by its connections to coding theory and computer science. All players guess simultaneously the color of their own hat observing only the hat colors of the other players. It is also allowed for each player to pass: no color is guessed. The team wins if at least one player guesses his or her hat color correct and none of the players has an incorrect guess. This paper studies Ebert’s hat problem with 5 players and two colors where the probabilities of the colors may be different (asymmetric case). Our goal is to maximize the probability of winning the game and to describe winning strategies. In this paper we use the notion of an adequate set. The construction of adequate sets is independent of underlying probabilities and we can use this fact in the analysis of the asymmetric case. Another point of interest is the fact that computational complexity using adequate sets is much less than using standard methods.

1. Introduction

Hat puzzles were formulated at least since Martin Gardner’s 1961 article [8]. They have got an impulse by Todd Ebert in his Ph.D. thesis in 1998 [6]. Buhler [2] stated: “It is remarkable that a purely recreational problem comes so close to the research frontier”. Also articles in The New York Times [16], Die Zeit [1] and abcNews [15] about this subject got broad attention. This paper studies generalized Ebert’s hat problem (symmetric and asymmetric): five distinguishable players are randomly fitted with a colored hat (two colors available), where the probabilities of getting a specific color may be different, but known to all the players. All players guess simultaneously the color of their own hat observing only the hat colors of the other four players. It is also allowed for each player to pass: no color is guessed. The team wins if at least one player guesses his or her hat color correctly and none of the players has an incorrect guess. Our goal is to maximize the probability of winning the game and to describe winning strategies. The symmetric two color hat problem (equal probabilities for each color) with \( N = 2^k - 1 \) players is solved in [7], using Hamming codes, and with \( N = 2^k \) players in [5] using extended Hamming codes. Burke et al. [3] try to solve the symmetric hat problem with \( N = 3, 4, 5, 7 \) players using genetic programming. Their conclusion: The \( N \)-prisoners puzzle (alternative names: Hat Problem, Hat Game) gives evolutionary computation and genetic programming a new challenge to overcome. Lenstra and
Seroussi [14] show that in the symmetric case of two hat colors, and for any value of $N$, playing strategies are equivalent to binary covering codes of radius one. Combining the result of Lenstra and Seroussi with Tables for Bounds on Covering Codes [11], we get:

\[
\begin{array}{cccccccc}
N & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
K(N,1) & 2 & 2 & 4 & 7 & 12 & 16 & 32 & 62 \\
\end{array}
\]

$K(N,1)$ is smallest size of a binary covering code of radius 1. Maximum probability for Ebert’s symmetric Hat Game is $1 - \frac{K(N,1)}{2^N}$. Lower bound on $K(9,1)$ was found in 2001 by Östergård-Blass, the upper bound in 2005 by Östergård. Krzywkowski [12] describes applications of the hat problem and its variations, and their connections to different areas of science. Krzywkowski [13], section 3, gives an optimal solution of the symmetric three person three color hat problem. Johnson [10] ends his presentation with an open problem: If the hat colors are not equally likely, how will the optimal strategy be affected? We will answer this question for five persons and two colors and our method gives also interesting results in the symmetric case.

In section 2 we define an adequate set. In section 3 we obtain results for the asymmetric five player, two color Hat Game. In section 4 we do the same for the symmetric Hat Game. In all situations all players know the underlying probabilities of each player.

2. Adequate sets

In this section we have $N$ players and $q$ colors. The $N$ persons in our game are distinguishable, so we can label them from 1 to $N$. We label the $q$ colors 0, 1, ..., $q - 1$. The probabilities of the colors are fixed and known to all players. The probability that color $i$ will be on a hat is $p_i$ ($i \in \{0, 1, ..., q - 1\}$, $\sum_{i=0}^{q-1} p_i = 1$).

Each possible configuration of the hats can be represented by an element of \( B = \{b_1b_2\ldots b_N|b_i \in \{0, 1, \ldots, q - 1\}, \ i = 1, 2, \ldots, N\} \). The Scode represents what the $N$ different players sees. Player $i$ sees q-ary code $b_1\ldots b_{i-1}b_{i+1}\ldots b_N$ with decimal value $s_i = \sum_{k=1}^{i-1} b_k \cdot q^{N-k-1} + \sum_{k=i+1}^{N} b_k \cdot q^{N-k}$, a value between 0 and $q^{N-1} - 1$.

Let $S$ be the set of all Scodes: \( S = \{s_1s_2\ldots s_N|s_i = \sum_{k=1}^{i-1} b_k \cdot q^{N-k-1} + \sum_{k=i+1}^{N} b_k \cdot q^{N-k}, b_i \in \{0, 1, \ldots, q - 1\}, i = 1, 2, \ldots, N\} \).

Each player has to make a choice out of $q + 1$ possibilities: 0=’guess color 0’, 1=’guess color 1’, ..., $q - 1$ =’guess color $q - 1$', $q$=’pass’.

We define a decision matrix $D = (a_{i,j})$ where $i \in \{1, 2, \ldots, N\}$ (players); $j \in \{0, 1, \ldots, q^{N-1} - 1\}$ (Scode of a player); $a_{i,j} \in \{0, 1, \ldots, q\}$. The meaning of $a_{i,j}$ is: player $i$ sees Scode $j$ and takes decision $a_{i,j}$ (guess a color or pass). We observe the total probability (sum) of our guesses. For each $b_1b_2\ldots b_N$ in B with $n_i$ times color $i$ ($i = 0, 1, \ldots, q - 1$, $\sum_{i=0}^{q-1} n_i = N$) we have:
CASE $b_1 b_2 \ldots b_N$
(Scode player $i$: $s_i = \sum_{k=1}^{i-1} b_k q^{N-k-1} + \sum_{k=i+1}^{N} b_k q^{N-k}$)

IF $a_{1,s_1} \in \{q, b_1\}$ AND $a_{2,s_2} \in \{q, b_2\}$ AND ... AND $a_{N,s_N} \in \{q, b_N\}$ AND NOT ($a_{1,s_1} = a_{2,s_2} = \cdots = a_{N,s_N} = q$) THEN $\sum = \sum + p_{0} \cdot p_{1} \pi \cdots p_{q-1}$.

Any choice of the $a_{i,j}$ in the decision matrix determines which CASES $b_1 b_2 \ldots b_N$ have a positive contribution to $\sum$ (we call it a GOOD CASE) and which CASES don’t contribute positive to $\sum$ (we call it a BAD CASE). Any GOOD CASE has at least one $a_{i,j}$ not equal to $q$. Let this specific $a_{i,j}$ have value $b_{i_0}$. Then our GOOD CASE generates $q - 1$ BAD CASES by only changing the value $b_{i_0}$ in any value of $0, 1, \ldots, q - 1$ except $b_{i_0}$. Let $A$ be the set of all BAD CASES (given a decision matrix $D$), then $B - A$ is the set of all GOOD CASES such that for each element $x$ in $B - A$ there are $(q - 1)$ elements in $A$ which are equal to $x$ up to one fixed position. We call $A$ an adequate set in relation to $B - A$ and the decision matrix $D$. The definition of adequate set is the same idea as the concept of strong covering, introduced by Lenstra and Seroussi [14]. The number of elements in an adequate set will be written as $das$ (dimension of adequate set). Adequate sets are generated by an adequate set generator (ASG) We have implemented an ASG in VBA/Excel (see Appendix A). Given an adequate set, we obtain a decision matrix $D = (a_{i,j})$ by the following procedure.

Procedure DMG (Decision Matrix Generator):

Begin Procedure
For each element in the adequate set:

- Determine the $q$-ary representation $b_1 b_2 \ldots b_N$
- Calculate Scodes $s_i = \sum_{k=1}^{i-1} b_k q^{N-k-1} + \sum_{k=i+1}^{N} b_k q^{N-k}$ ($i=1,..,N$)
- For each player $i$: fill decision matrix with $a_{i,s_i} = b_i$ ($i=1,..,N$), where each cell may contain several values.

Matrix $D$ is filled with BAD COLORS. We can extract the GOOD COLORS by considering all $a_{i,j}$ with $q - 1$ BAD COLORS and then choose the only missing color. In all situations with less than $q - 1$ BAD COLORS we pass. When there is an $a_{i,j}$ with $q$ BAD COLORS all colors are bad, so the first option is to pass. But when we choose any color, we get a situation with $q - 1$ colors. So in case of $q$ BAD COLORS we are free to choose any color or pass. The code for pass is $q$, but in our decision matrices we prefer a blank, which supports readability. The code for ‘any color or pass will do’ is defined $q + 1$, but in our decision matrices we prefer a $\ast$.

End Procedure.

Decision matrices are generated by an Decision Matrix Generator (DMG). We have implemented a DMG in VBA/Excel (see Appendix B).
3. Asymmetric five person Hat Game

Theorem 1. In asymmetric five person (two color) hat game we have maximal probability $\Psi (5, p)$ of winning the game:

\[
\Psi (5, p) = \begin{cases} 
1 - p + 2p^2 - 2p^3 + p^5 & (0 \leq p \leq \sqrt{2} - 1) \\
5p - 10p^2 + 6p^3 + p^4 - p^5 & (\sqrt{2} - 1 \leq p \leq 0.5) \\
1 - 2p + 4p^2 - 4p^4 + p^5 & (0.5 \leq p \leq 2 - \sqrt{2}) \\
1 - 2p + 6p^2 - 8p^3 + 5p^4 - p^5 & (2 - \sqrt{2} \leq p \leq 1)
\end{cases}
\]

with optimal decision matrices:

**CASE 0.5 < p < 2 - \sqrt{2}:**

\[
\begin{array}{cccccccccccccccc}
0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

**CASE 2 - \sqrt{2} \leq p < 1:**

\[
\begin{array}{cccccccccccccccc}
0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 & 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

(* means: any color or pass; stars are independent)

Proof. We run the application ASG with parameters $N=5$, $p=0.55$ and $das=7$. This yields 320 adequate sets. When we sort these sets by probability, we get 12 different classes (see Appendix C where we show the first element of each class). The adequate set and the number of zero’s in each element in an adequate set are independent of $p$. We note the following structure in Appendix C:

<table>
<thead>
<tr>
<th>012345 probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>024001 $2pq^4 + 4p^2q^3 + p^5$</td>
</tr>
<tr>
<td>022210 $2pq^4 + 2p^2q^3 + 2p^3q^2 + p^4q$</td>
</tr>
<tr>
<td>111310 $q^5 + pq^4 + p^2q^3 + 3p^3q^2 + p^4q$</td>
</tr>
<tr>
<td>013210 $pq^4 + 3p^2q^3 + 2p^3q^2 + p^4q$</td>
</tr>
<tr>
<td>102310 $q^5 + 2pq^4 + 3p^3q^2 + p^4q$</td>
</tr>
<tr>
<td>012310 $pq^4 + 2p^2q^3 + 3p^3q^2 + p^4q$</td>
</tr>
<tr>
<td>120130 $q^5 + 2pq^4 + p^3q^2 + 3p^4q$</td>
</tr>
<tr>
<td>013201 $pq^4 + 3p^2q^3 + 2p^3q^2 + p^5$</td>
</tr>
<tr>
<td>012220 $pq^4 + 2p^2q^3 + 2p^3q^2 + 2p^4q$</td>
</tr>
<tr>
<td>031021 $3pq^4 + p^2q^3 + 2p^3q + p^5$</td>
</tr>
<tr>
<td>013111 $pq^4 + 3p^2q^3 + 3pq^2 + p^4q + p^5$</td>
</tr>
<tr>
<td>100420 $q^5 + 4p^3q^2 + 2p^4q$</td>
</tr>
</tbody>
</table>

Figure I shows the dominance relations between this 12 different classes.
A class is dominant over another class if the probability of that class is less than the probability of the other class.

An arrow with a S means that one or more shifts of bits to the left (of the dominated one) results to the pattern of the dominant one (shifts to the left are 'cheaper', because of $p > q$). More formally, for fixed $\text{das}$ we have: the pattern $b(0)\ b(1)\ b(2)\ b(3)\ b(4)\ b(5)$ is dominant over pattern $a(0)\ a(1)\ a(2)\ a(3)\ a(4)\ a(5)$ when in each position there is enough compensation:

$b(0) \geq a(0)$

$b(0)+b(1) \geq a(0)+a(1)$

...  

$b(0)+b(1)+b(2)+b(3)+b(4)+b(5) \geq a(0)+a(1)+a(2)+a(3)+a(4)+a(5)$

The last inequality may be omitted (we have $\text{das} \geq \text{das}$).

The dominance relation is transitive.

The arrows A,B,C and D needs some attention.

We use $\succ$ as symbol for dominance.

A: $022210 \succ 111310$

If $p > q > 0$ then

$-q^5 + pq^4 + p^2q^3 - p^3q^2 = -q^5 + pq^4 + p^2q^3 - p^3q^2 = -q^2(p - q)^2 < 0$

B and C: $102310 \succ 120310$ and $013201 \succ 031021$
if \( p > q > 0 \) then \(-2pq^4 + 2p^2q^3 + 2p^3q^2 - 2p^4q = -2pq(p - q)^3 < 0\)

D: If \( 2 - \sqrt{2} < p < 1 \) then \( 0.22210 \succ 0.24001 \) and if \( 0.5 < p < 2 - \sqrt{2} \)
then \( 0.24001 \succ 0.22210 \)

\[-2pq^3 + 2p^3q^2 + p^4q - p^5 = p^2(2p^3 - 9p^2 + 8p - 2) =
\]
\[p^2(p - 1)(p^2 - 4p + 2) = p^2(p - 1)(p - 2 - \sqrt{2})(p - 2 + \sqrt{2})\]

Let \( \Psi (N, p) \) be the maximum probability of correct guessing.

If \( 2 - \sqrt{2} < p < 1 \) then \( 0.22210 \succ 0.24001 \) and

\[\Psi (5, p) = 1 - (2pq^4 + 2p^2q^3 + 2p^3q^2 + p^4q) = 1 - 2p + 6p^2 - 8p^3 + 5p^4 - p^5\]

If \( 0.5 < p < 2 - \sqrt{2} \) then \( 0.24001 \succ 0.22210 \) and

\[\Psi (5, p) = 1 - (2pq^4 + 4p^2q^3 + p^5) = 1 - 2p + 4p^2 - 4p^4 + p^5\]

By changing the roles of \( p \) and \( q \) we get: \([1]\)

Graph of \( \Psi (5, p) \):

We remark: minimum is at \( \left(\frac{1}{2}, \frac{25}{32}\right) \) and \( \Psi (5, p) \) is not differentiable at \( \frac{1}{2} \), \( \sqrt{2} - 1 \) and \( 2 - \sqrt{2} \). When \( N=5 \) we have 320 adequate sets. Using the program \( \text{ASG } p=0.55 \text{ das}=7 \) we get after sorting on sum: when \( 0.5 < p < 2 - \sqrt{2} \) we have 10 optimal adequate sets. Using the program \( \text{ASG } p=0.9 \text{ das}=7 \) we get after sorting on sum: when \( 2 - \sqrt{2} < p < 1 \) we have 30 optimal adequate sets.

When \( p = 2 - \sqrt{2} \): 40 optimal adequate sets (the union of the two foregoing adequate sets).

Using procedure \( \text{DMG} \), we give the first element in each case:

CASE \( 0.5 < p < 2 - \sqrt{2} \) : adequate set \( \{0,7,11,19,28,29,30\} \);

Use: \( \text{DMG } p=0.55 \) to obtain the decision matrices. Note: players 1, 2, 3 have same strategy; players 4 and 5 also.

CASE \( 2 - \sqrt{2} < p < 1 \) : adequate set \( \{1,6,14,22,24,27,29\} \);
Use DMG to obtain the decision matrices.

Note: players 1,2 have same strategy, players 3,4 also; player 5 has her own strategy.

CASE $p = 2 - \sqrt{2}$: we get the union of 10 optimal sets in case of $0.5 < p < 2 - \sqrt{2}$ and 30 optimal sets in case of $2 - \sqrt{2} < p < 1$.

We call two solutions isomorphic when we can transform one solution to the other by renumbering the players. Observing all 40 decision matrices and making use of the positions (players) of the stars in these matrices, we get the result in Appendix D where the column STARS gives the positions of the two stars and the column CYCLES gives the renumbering to obtain the same solution. This can be verified by writing each adequate set in the binary representation and then applying the cycles. Appendix D shows that there are only two non-isomorphic solutions (these two solutions can’t be isomorphic for isomorphic sets have always the same probability).

The last point here is to convince ourselves that any adequate set with $das > 7$ doesn’t yield better solutions.

Let $b = das(b()$, $a = das(a())$. When $e = a - b \geq 0$ then we delete the 'cheapest' $a - b$ elements from $a()$, which gives a 'cheaper' situation. We get:

$b(0) \geq a(0)-e$

$b(0)+b(1) \geq a(0)+a(1)-e$

$\ldots$

$b(0)+b(1)+b(2)+b(3)+b(4)+b(5) \geq a(0)+a(1)+a(2)+a(3)+a(4)+a(5)-e$

The last inequality may be omitted (we have $b \geq a - (a - b)$.)

These dominance relations are implemented as procedure dom() in ASG.

Run the program ASG with $das=8,9, \ldots, 32$. where we have 5 calls to dom() (see figure 1; we omitted the S patterns, for they are detected by the procedure). No (dominant) adequate sets are found.

Conclusion: $das = 7$ is optimal and $\{(022210), (024001)\}$ dominates all adequate sets.

3.1. Computational complexity. We consider the number of strategies to be examined to solve the hat problem with $N$ players and two colors. Each of the $N$ players has $2^{N-1}$ possible situations to observe and in each situation there are three possible guesses: white, black or pass. So we have $(3^{2^{N-1}})^{N}$ possible strategies. Krzywkowski [14] shows that is suffices to examine $(3^{2^{N-1}-2})^{N}$ strategies.

The adequate set method has to deal where $\{i_1, i_2, \ldots, i_{das}\}$ with $0 \leq i_1 < i_2 < \ldots < i_{das} \leq 2^N - 1$. 
The number of strategies for fixed $das$ is the number of subsets of dimension $das$ of $\{0,1,\ldots,2^N-1\}$: \( \binom{2^N}{das} \). But we have to test all possible values of $das$. So the correct expression is: \( \sum_{das} \binom{2^N}{das} = 2^{(2^N)} \). To get an idea of the power of the adequate set method, we compare the number of strategies (brute force, Krzywkowski and adequate set method):

<table>
<thead>
<tr>
<th>$N$</th>
<th>$das$</th>
<th>$(3^{2^N-1})^N$</th>
<th>$(3^{2^N-1}-2)^N$</th>
<th>$2^{(2^N)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>7</td>
<td>1.50E+38</td>
<td>2.50E+33</td>
<td>4.29E+09</td>
</tr>
</tbody>
</table>

4. Symmetric two color five person Hat Game

In this section with five persons the two colors have the same probability: $p = q = \frac{1}{2}$.

**Theorem 2.** The maximal probability for symmetric five person two color Hat Game is $\frac{25}{32}$. All 12 non-isomorphic optimal decision matrices are given in Appendices $E$ and $F$.

**Proof.** Running ASG with any value of $p$ gives 320 adequate sets. When $p = 0.5$ all these sets are optimal (with probability $\frac{25}{32}$). We split the 320 sets in 12 probability classes (see section 3). In Appendices $E$ and $F$ we give the first element of each class, the decision matrix and the number of isomorphic elements in each class, where isomorphic sets can be detected in the same way (STARS, CYCLES) as in section 3. All 320 decision matrices are generated by procedure DMG.

The 12 probability classes generates 12 non-isomorphic decision matrices. \( \square \)
Dim k1, k2, k3, k4, k5, k6 As Integer
Public dominant As Integer
Private Declare Function SetThreadExecutionState Lib "Kernel32.dll" (ByVal esFlags As Long) As Long
Sub dominant_adequate_set_generator() Dim d() As Integer Dim i() As Integer Dim check() As Integer Dim a() As Integer Dim c() As Integer Dim j() As Integer p = 0.55 q = 0.45 Count = 0 'number of already found adequate sets n = 5 'five players H = 2 ^ n - 1 '2^n elements in B das = 7 'dimension of adequate set ReDim a(0 To 5) ReDim c(0 To H, 1 To n) As Integer ReDim i(0 To H, 1 To n) As Integer ReDim j(1 To n) As Integer ReDim check(0 To H) As Integer KeepPowerOn "for each number from 0 to H: " "first calculate binary digits " 00000 .... 11111 and put it in matrix c: For k = 0 To H g = k
For $Z = 1$ To $n$
  $c(k, Z) = g \mod 2$
  $g = g$
Next $Z$
Next $k$

$x = 0$ ' $x$: row in Excel where result is displayed

' adequate set: $\{i_1, i_2, \ldots, i_{das}\}$
' brute force for all potential adequate sets:
For $i_1 = 0$ To $H + 1$
  $i(1) = i_1$
  For $i_2 = i_1 + 1$ To $H + 2$
    $i(2) = i_2$
    For $i_3 = i_2 + 1$ To $H + 3$
      $i(3) = i_3$
      For $i_4 = i_3 + 1$ To $H + 4$
        $i(4) = i_4$
        For $i_5 = i_4 + 1$ To $H + 5$
          $i(5) = i_5$
          For $i_6 = i_5 + 1$ To $H + 6$
            $i(6) = i_6$
            For $i_7 = i_6 + 1$ To $H + 7$
              ' binary digits for adequate set:
              For $k = 1$ To $das$
                $g = i(k)$
                For $Z = 1$ To $n$
                  $j(k, Z) = g \mod 2$
                  $g = (g - d(Z)) / 2$
              Next $Z$
              Next $k$

'' check on adequate set property;
For $Z = 1$ To $n$
  $c(k, Z) = g \mod 2$
  $g = (g - d(Z)) / 2$
Next $Z$
Next $k$

' each element of $B$ has distance 0 or 1 to $A$
For $k = 0$ To $H$
  $check(k) = 0$
  For $m = 1$ To $das$
    ' distance<2
    If $\text{Abs}(c(k, 1) - j(m, 1)) + \text{Abs}(c(k, 2) - j(m, 2)) + \text{Abs}(c(k, 3) - j(m, 3)) + \text{Abs}(c(k, 4) - j(m, 4)) + \text{Abs}(c(k, 5) - j(m, 5)) < 2$ Then $check(k) = 1$
  Next $m$
Next $k$

State = 1    ' potential adequate set
For $k = 0$ To $H$
  State = State * $check(k)$
Next $k$

Sum = 0
Cell(1, das + 2) = Count

If State = 1 And dominant = 1 Then
  x = x + 1
  Count = Count + 1
' next row in Excel sheet:
  For $k = 1$ To $das$
    Cells(x, $k$) = $i(k)$
  Next $k$
  Cells(x, 3 + das) = Sum
  For $b = 0$ To 5
    Cells(x, das + 4 + $b$) = $a(b)$
  Next $b$
End If
Next $i_7$
Next $i_6$
Next $i_5$
Next $i_4$
Next $i_3$
Next $i_2$
Next $i_1$
End Sub

Sub dom(h1, h2, h3, h4, h5, h6)
  $c = 0$
  $e = 0$
  If $h_1 \geq k_1 - e$ Then $c = c + 1$
  If $h_1 + h_2 \geq k_1 + k_2 - e$ Then $c = c + 1$
  If $h_1 + h_2 + h_3 \geq k_1 + k_2 + k_3 - e$ Then $c = c + 1$
  If $h_1 + h_2 + h_3 + h_4 \geq k_1 + k_2 + k_3 + k_4 - e$ Then $c = c + 1$
  If $h_1 + h_2 + h_3 + h_4 + h_5 \geq k_1 + k_2 + k_3 + k_4 + k_5 - e$ Then $c = c + 1$
  If $c = 5$ Then $dominant = 0$
  If $h_1 = k_1$ And $h_2 = k_2$ And $h_3 = k_3$ And $h_4 = k_4$ And $h_5 = k_5$ And $h_6 = k_6$ Then $dominant = 1$
End Sub

Sub KeepPowerOn()
  Const ES_SYSTEM_REQUIRED As Long = &H1
  Const ES_DISPLAY_REQUIRED As Long = &H2
  Const ES_CONTINUOUS As Long = &H80000000
  Dim PrevState As Long
  PrevState = SetThreadExecutionState(ES_CONTINUOUS Or ES_DISPLAY_REQUIRED Or ES_SYSTEM_REQUIRED)
  Application.OnTime Now() + TimeValue("00:01:00"), "KeepPowerOn", , True
End Sub

Sub decision_matrix_generator_n5_q2()
Dim d() As Integer
Dim c() As Integer
Dim l() As Integer
Dim m() As Integer
Dim check() As Integer
Dim a() As Integer
Dim tot() As Integer
p = 0.55
q = 1 - p
n = 5
H = 2 ^ n - 1
v = 2 ^ (n - 1) - 1
das = 7
ReDim d(1 To n) As Integer
ReDim c(0 To H, 1 To n) As Integer
ReDim l(1 To das) As Integer
ReDim m(1 To das, 1 To n) As Integer
ReDim check(0 To H) As Integer
ReDim a(1 To n, 0 To v) As Integer
ReDim tot(1 To n, 0 To v) As Integer
" for each number from 0 to H:
" first calculate binary digits
" 00000 ..... 11111 and put it in matrix c:
For k = 0 To H
    g = k
    For Z = 1 To n
        c(k, Z) = g Mod 2
        g = g \ 2
    Next Z
Next k
For i1 = 0 To H - das + 1
   i(1) = i1
   " VBA
   EXCEL can't handle with an array in for to next
For i2 = i1 + 1 To H - das + 2
   i(2) = i2
For i3 = i2 + 1 To H - das + 3
   i(3) = i3
For i4 = i3 + 1 To H - das + 4
   i(4) = i4
For i5 = i4 + 1 To H - das + 5
   i(5) = i5
For i6 = i5 + 1 To H - das + 6
   i(6) = i6
For i7 = i6 + 1 To H - das + 7
   i(7) = i7

" binary digits for adequate set:
For k = 1 To das
   g = i(k)
   For Z = 1 To n
      j(k, Z) = g Mod 2
   Next Z
Next k

" check on adequate set property;
" each element of B has distance 0 or 1 to A
For k = 0 To H
   check(k) = 0
   For m = 1 To das
      'Distance<2
      If Abs(c(k, 1) - j(m, 1)) + Abs(c(k, 2) - j(m, 2)) + Abs(c(k, 3) - j(m, 3)) + Abs(c(k, 4) - j(m, 4)) + Abs(c(k, 5) - j(m, 5)) < 2 Then check(k) = 1
   Next m
Next k
State = 1
For k = 0 To H
   State = State * check(k)
Next k
Sum = 0
If State = 1 Then
   For k = 1 To das
      g = i(k)
      For Z = 1 To n
         d(Z) = g Mod 2
      Next Z
      Count = 0 ' count of zero's in element of adequate set
      For t = 1 To n
         If d(t) = 0 Then Count = Count + 1
      Next t
      term = p ^ Count * q ^ (n - Count)
      Sum = Sum + term
   Next k
End If
If State = 1 Then
   x = x + n + 2
   Cells(x, 1) = 1 + x 
   For k = 1 To das 
      Cells(x, k + 2) = i(k) 
   Next k 
   For s = 1 To n 
      If Abs(c(k, 1) - j(m, 1)) + Abs(c(k, 2) - j(m, 2)) + Abs(c(k, 3) - j(m, 3)) + Abs(c(k, 4) - j(m, 4)) + Abs(c(k, 5) - j(m, 5)) < 2 Then 
         x = x + n + 2 
         Cells(x, 1) = 1 + x 
         For k = 1 To das 
            Cells(x, k + 2) = i(k) 
         Next k 
         For s = 1 To n 
      Next s 
   Next s 
End If 
Next k 
Next i7 
Next i6 
Next i5 
Next i4 
Next i3 
Next i2 
Next i1 
End Sub
### Appendix C. Q=2, N=5, 12 classes

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Appendix D. $Q=2$, $N=5$, Two isomorphic sets

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### Appendix E. \( Q=2, N=5 \), decision matrices 12 classes, PART I

**Matrix 1**

\[
\begin{array}{ccccccc}
1 & 0 & 7 & 11 & 19 & 28 & 29 & 30 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\( \binom{5}{3} = 10 \)

**Matrix 2**

\[
\begin{array}{ccccccc}
2 & 1 & 6 & 14 & 22 & 24 & 27 & 29 \\
1 & * & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & * & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

\( \binom{5}{2,1} = 30 \)

**Matrix 3**

\[
\begin{array}{ccccccc}
3 & 1 & 6 & 10 & 18 & 28 & 29 & 31 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\( \binom{5}{3,1,1} = 20 \)

**Matrix 4**

\[
\begin{array}{ccccccc}
4 & 1 & 6 & 11 & 13 & 22 & 23 & 24 \\
1 & * & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

\( \binom{5}{2,1,1} = 60 \)

**Matrix 5**

\[
\begin{array}{ccccccc}
5 & 1 & 3 & 5 & 14 & 22 & 24 & 31 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
* & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
* & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{array}
\]

\( \binom{5}{2,2,1} = 30 \)

**Matrix 6**

\[
\begin{array}{ccccccc}
6 & 1 & 3 & 7 & 12 & 20 & 26 & 29 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & * & 0 & 0 & 1 & 0 & 0 & 0 \\
* & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{array}
\]

\( \binom{5}{2,1,1} = 60 \)
### Appendix F. Q=2, N=5, Decision Matrices 12 Classes, Part II

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- $^5\text{(10)}$
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- $^5\text{(10)}$
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References


Email address: tjvanuem@gmail.com