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RESOLVENT AND SPECTRAL ASYMPTOTICS FOR THIN-LAYER TRANSMISSION PROBLEMS IN THE LARGE COUPLING LIMIT

VICTOR PÉRON¹, GILLES CARBOU¹

ABSTRACT. This work is concerned with a singular perturbation problem that involves a thin layer in the large coupling limit. Such a problem appears in the study of electromagnetic wave guides that contain supra-conductor bodies. We derive resolvent asymptotics for this problem by using boundary layer tools with uniform *a priori* estimates. This result will be useful to derive spectral asymptotics for this problem. This issue is a work in progress.

1. INTRODUCTION

This work is motivated by the study of the asymptotic behavior of eigenvalues related to a singular perturbation problem that involves a thin layer in the large coupling limit. Such a problem appears in the study of electromagnetic wave guides with supra-conductors. We refer the reader to several works that concern mathematical studies with electromagnetic wave guides, see for instance [1, 7, 2, 3].

An asymptotic expansion of the eigenvalues as the conductivity tends to infinity has been derived in [3]. However this result does not apply when the subdomain that represents a supra-conductor is a vanishing sheet. In this work we tackle this problem by considering vanishing supra-conductive layer of thickness ε where the conductivity is scaled like ε^{-2} . This is why there is a main difference between this work and the work in Ref. [3] all the more since we consider a transmission problem across a thin layer.

Let us describe now the mathematical model. Throughout the paper we denote by $\Omega \subset \mathbb{R}^3$ a simply connected domain. Let Ω_i be a simply connected subdomain of Ω and let ω_ε be a thin layer of constant thickness ε surrounding Ω_i such that Ω contains $\Omega_i \cup \overline{\omega_\varepsilon}$, see Figure 1. For the sake of simplicity we further assume that the boundaries $\partial\Omega$ and $\Gamma = \partial\Omega_i$ are connected. Now the exterior domain is defined as $\mathcal{U}_\varepsilon = \Omega \setminus (\Omega_i \cup \overline{\omega_\varepsilon})$ and the boundary of the subdomain $\overline{\Omega_i} \cup \omega_\varepsilon$ is denoted by Γ_ε . We denote by \mathbf{n} the outward unit normal at Γ .

In all that follows, unless specified, all the considered domains are smooth domains in \mathbb{R}^3 .

Notation 1.1. For any set $\mathcal{O} \subset \mathbb{R}^3$ we denote by $L^2(\mathcal{O})$ and $H^s(\mathcal{O})$ the classical Sobolev spaces.

Let $\varepsilon > 0$. We consider the self-adjoint operator \mathcal{H}^ε on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ and defined by

$$\mathcal{H}^\varepsilon = -\Delta + \frac{1}{\varepsilon^2} 1_{|\omega_\varepsilon}.$$

The limit operator is $\mathcal{H}^0 = -\Delta$ defined on $H^2(\Omega_i) \cap H_0^1(\Omega_i) \times H^2(\Omega_e) \cap H_0^1(\Omega_e)$ where Ω_e denotes the limit domain \mathcal{U}_ε when $\varepsilon \rightarrow 0$, i.e. $\Omega_e = \Omega \setminus \overline{\Omega_i}$ (see Figure 2).

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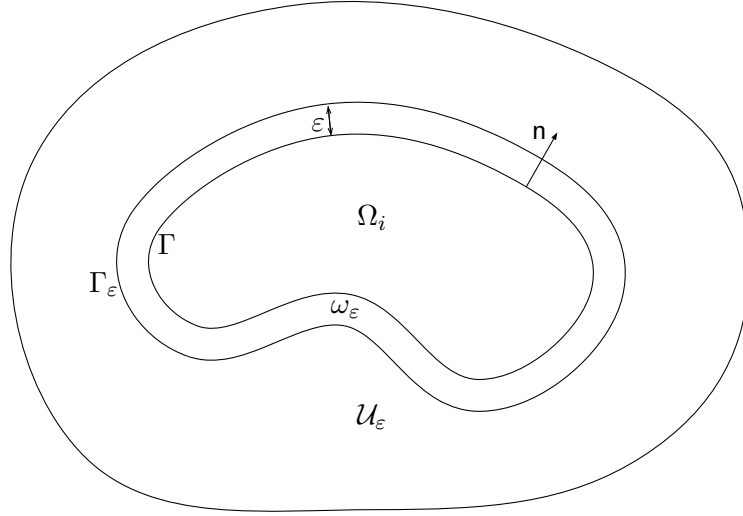


Figure 1 – A cross-section of the domain Ω and the subdomains $\Omega_i, \omega_\varepsilon, \mathcal{U}_\varepsilon$

In this work we derive an asymptotic expansion of the resolvent of \mathcal{H}^ε when ε tends to 0. We denote by $\mathcal{R}^\varepsilon(\xi) := (\mathcal{H}^\varepsilon - \xi\mathcal{I})^{-1}$ the resolvent of \mathcal{H}^ε , defined on $L^2(\Omega)$ with values in $H^2(\Omega) \cap H_0^1(\Omega)$ for $\xi \notin \text{spec } \mathcal{H}^\varepsilon$.

Let $f \in L^2(\Omega)$. The spectral problem writes

$$(1.1) \quad -\Delta u - \xi u + \frac{1}{\varepsilon^2} 1_{|\omega_\varepsilon} u = f \quad \text{in } \Omega.$$

This problem is well-posed and has a unique solution u_ε in $H^2(\Omega) \cap H_0^1(\Omega)$ when ξ is not an eigenvalue of the operator \mathcal{H}^ε .

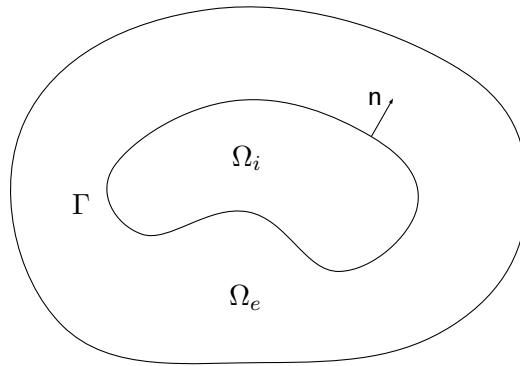


Figure 2 – A cross-section of the domain Ω and the subdomains Ω_i, Ω_e

Notation 1.2. We denote by f_i (resp. f_e) the restriction of any function f to Ω_i (resp. Ω_e).

2. ASYMPTOTIC EXPANSION FOR THE RESOLVENT

Let $\varepsilon > 0$. We consider the self-adjoint operator \mathcal{H}^ε on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ and defined by

$$\mathcal{H}^\varepsilon = -\Delta + \frac{1}{\varepsilon^2} 1_{|\omega_\varepsilon}.$$

For $\xi \notin \text{spec } \mathcal{H}^\varepsilon$, we denote by $\mathcal{R}^\varepsilon(\xi) = (\mathcal{H}^\varepsilon - \xi \text{Id})^{-1}$ the resolvent operator. For $f \in L^2(\Omega)$, we denote by u^ε the restriction of $\mathcal{R}^\varepsilon(\xi)(f)$ to Ω_i , v^ε the restriction of $\mathcal{R}^\varepsilon(\xi)(f)$ to ω_ε , and w^ε the restriction of $\mathcal{R}^\varepsilon(\xi)(f)$ to \mathcal{U}_ε . By the resolvent equation, we have:

$$(2.1) \quad -\Delta u^\varepsilon - \xi u^\varepsilon = f|_{\Omega_i} \quad \text{in } \Omega_i$$

$$(2.2) \quad -\Delta v^\varepsilon - \xi v^\varepsilon + \frac{1}{\varepsilon^2} v^\varepsilon = f|_{\omega_\varepsilon} \quad \text{in } \omega_\varepsilon$$

$$(2.3) \quad -\Delta w^\varepsilon - \xi w^\varepsilon = f|_{\mathcal{U}_\varepsilon} \quad \text{in } \mathcal{U}_\varepsilon$$

$$(2.4) \quad u^\varepsilon = v^\varepsilon \quad \text{on } \Gamma$$

$$(2.5) \quad v^\varepsilon = w^\varepsilon \quad \text{on } \Gamma_\varepsilon$$

$$(2.6) \quad \partial_n u^\varepsilon = \partial_n v^\varepsilon \quad \text{on } \Gamma$$

$$(2.7) \quad \partial_n v^\varepsilon = \partial_n w^\varepsilon \quad \text{on } \Gamma_\varepsilon$$

$$(2.8) \quad w^\varepsilon = 0 \quad \text{on } \partial\Omega.$$

2.1. **Ansatz.** Outside the thin layer, we look for an expansion on the form:

$$(2.9) \quad u^\varepsilon(x) \approx \sum_{j \geq 0} \varepsilon^j U^j(x) \quad \text{in } \Omega_i,$$

$$(2.10) \quad w^\varepsilon(x) \approx \sum_{j \geq 0} \varepsilon^j W^j(x) \quad \text{in } \mathcal{U}_\varepsilon,$$

where the terms U^j and W^j are defined in ε -independent domains Ω_i and $\Omega_e := \Omega \setminus \Omega_i$. We emphasize that the terms W^j are defined in the thin layer ω_ε .

In the thin layer, we denote by $d : \omega_\varepsilon \rightarrow \mathbb{R}^+$ the euclidean distance to Γ , $d(x) = \text{dist}(x, \Gamma)$ and by $P(x)$ the orthogonal projection of x on Γ . We describe the solution v^ε on the following way:

$$(2.11) \quad v^\varepsilon(x) \approx \sum_{j \geq 0} \varepsilon^j \left(W^j(x) + V^j \left(\frac{d(x)}{\varepsilon}, P(x) \right) \right) \quad \text{in } \omega_\varepsilon,$$

where the profiles V^j are defined on $[0, 1] \times \Gamma$:

$$\begin{aligned} V^j : [0, 1] \times \Gamma &\rightarrow \mathbb{C} \\ (z, \sigma) &\mapsto V^j(z, \sigma). \end{aligned}$$

In ω_ε , we will work with the variables (z, σ) .

2.2. Geometrical and analytical tools for the thin layer. We use the following notations (see also [6]): $x = (x_1, x_2, x_3)$ denotes the cartesian coordinates in \mathbb{R}^3 , and we set

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \partial_z = \frac{\partial}{\partial z}, \quad \partial_{zz} = \frac{\partial^2}{\partial z^2}$$

By simple calculations there holds (see also [5])

$$\begin{aligned} \Delta \left(x \mapsto V(P(x), \frac{d(x)}{\varepsilon}) \right) &= \varepsilon^{-2} \partial_{zz} V(P(x), z) + \varepsilon^{-1} G_{\varepsilon z}(P(x)) \partial_z V(P(x), z) \\ &\quad + \Delta_{\Gamma_{\varepsilon z}} V(P(x), z) \quad \text{with} \quad z = \frac{d(x)}{\varepsilon}, \end{aligned}$$

where G is defined as

$$G_s(\sigma) = \frac{1}{\gamma_s(\sigma)} \frac{\partial \gamma_s}{\partial s}(\sigma).$$

Here $\gamma_s(\sigma) = \det(\text{Id} + s \text{dn}(\sigma))$ and for $\sigma \in \Gamma$, the differential $\text{dn}(\sigma)$ is a linear map from the tangent plane $T_\sigma \Gamma$ of Γ at the point σ into $T_{n(\sigma)} S^2$ where S^2 is the unit sphere.

Proposition 2.1.

$$\frac{1}{2} \leq \gamma_s(\sigma) \leq \frac{3}{2}$$

In addition, since for $x \in \omega_\varepsilon$, we have: $x = P(x) + \varepsilon \frac{d(x)}{\varepsilon} \nu(P(x)) = \sigma + \varepsilon z \nu(\sigma)$, we perform the Taylor expansion of the functions $\varphi : \omega_\varepsilon \rightarrow \mathbb{C}$ writing:

$$\varphi(x) = \varphi(P(x)) + d(x) \frac{\partial \varphi}{\partial \nu}(P(x)) + \dots = \varphi(\sigma) + \varepsilon z \frac{\partial \varphi}{\partial \nu}(\sigma) + \dots$$

Then we insert the *Ansatz* (2.9)-(2.10)-(2.11) in equations (2.1)-(2.8) and we perform the identification of terms with the same power in ε .

2.3. Characterisation of the profiles at order ε^0 . Taking the order ε^0 in (2.1), we obtain that:

$$(2.12) \quad -\Delta U^0 - \xi U^0 = f \quad \text{in} \quad \Omega_i.$$

Taking the order ε^0 in (2.3), we have:

$$(2.13) \quad -\Delta W^0 - \xi W^0 = f \quad \text{in} \quad \Omega_e.$$

From (2.2) at order ε^{-2} , we obtain that:

$$(2.14) \quad -\partial_{zz} V^0 + V^0 + W^0(\sigma) = 0 \quad \text{in} \quad [0, 1] \times \Gamma.$$

By solving this ODE, we obtain that:

$$(2.15) \quad V^0(z, \sigma) = -W^0(\sigma) + \alpha_0(\sigma) \sinh z + \beta_0(\sigma) \cosh z.$$

From (2.6) at order ε^{-1} , we have

$$0 = \partial_z V^0(0, \sigma),$$

so we obtain that $\beta_0(\sigma) = 0$ for all $\sigma \in \Gamma$.

From (2.7) at order ε^{-1} , we obtain that

$$0 = \partial_z V^0(1, \sigma),$$

so, with (2.15), we obtain that

$$\beta_0(\sigma) = 0.$$

Thus, we have:

$$(2.16) \quad V^0(z, \sigma) = -W^0(\sigma).$$

Since W^0 satisfies (2.13), then W^0 and $\frac{\partial W^0}{\partial \nu}$ have no jump at Γ_ε . In particular, writing (2.5), we obtain that

$$V^0(1, \sigma) = 0.$$

Coupling this relation with (2.16), we obtain that

$$(2.17) \quad V^0(z, \sigma) = 0 \text{ for } z \in [0, 1] \text{ and } \sigma \in \Gamma.$$

Therefore, with (2.16),

$$(2.18) \quad W^0(\sigma) = 0 \text{ for all } \sigma \in \Gamma.$$

Writing (2.5) at order ε^0 , we have

$$U^0(\sigma) = V^0(0, \sigma) \text{ for } \sigma \in \Gamma.$$

So, with (2.17),

$$(2.19) \quad U^0(\sigma) = 0 \text{ for all } \sigma \in \Gamma.$$

Equation (2.8) at order ε^0 yields $W^0 = 0$ on $\partial\Omega$.

Therefore, coupling (2.12) and (2.19), U^0 is characterized by:

$$(2.20) \quad \begin{cases} -\Delta U^0 - \xi U^0 = f \text{ in } \Omega_i, \\ U^0 = 0 \text{ in } \partial\Omega_i. \end{cases}$$

In addition, coupling (2.13) and (2.18), W^0 is characterized by:

$$(2.21) \quad \begin{cases} -\Delta W^0 - \xi W^0 = f \text{ in } \Omega_e, \\ W^0 = 0 \text{ in } \partial\Omega_e. \end{cases}$$

2.4. Characterisation of the profiles at order ε^1 . Taking the order ε^1 in (2.1), we obtain that:

$$(2.22) \quad -\Delta U^1 - \xi U^1 = 0 \quad \text{in } \Omega_i.$$

Taking the order ε^1 in (2.3), we have:

$$(2.23) \quad -\Delta W^1 - \xi W^1 = 0 \quad \text{in } \Omega_e.$$

From (2.2) at order ε^{-1} , we obtain that:

$$(2.24) \quad -\partial_{zz} V^1 + V^1 + W^1(\sigma) + z \frac{\partial W^0}{\partial \nu}(\sigma) = 0 \quad \text{in } [0, 1] \times \Gamma.$$

By solving this ODE, we obtain that:

$$(2.25) \quad V^1(z, \sigma) = -W^1(\sigma) - z \frac{\partial W^0}{\partial \nu}(\sigma) + \alpha_1(\sigma) \sinh z + \beta_1(\sigma) \cosh z.$$

From (2.6) at order ε^0 , we have

$$\frac{\partial U^0}{\partial \nu}(\sigma) = \partial_z V^1(0, \sigma)(\sigma) + \frac{\partial W^0}{\partial \nu}(\sigma),$$

so we obtain that

$$(2.26) \quad \alpha_1(\sigma) = \frac{\partial U^0}{\partial \nu}(\sigma) \text{ for all } \sigma \in \Gamma.$$

From (2.7) at order ε^0 , we obtain that

$$0 = \partial_z V^1(1, \sigma) = -\frac{\partial W^0}{\partial \nu}(\sigma) + \alpha_1(\sigma) \cosh 1 + \beta_1(\sigma) \sinh 1,$$

so, with (2.25) and (2.26), we obtain that

$$\beta_1(\sigma) = \frac{1}{\sinh 1} \left(\frac{\partial W^0}{\partial \nu}(\sigma) - (\cosh 1) \frac{\partial U^0}{\partial \nu}(\sigma) \right).$$

Thus, we have:

$$(2.27) \quad V^1(z, \sigma) = -W^1(\sigma) + \frac{\partial W^0}{\partial \nu}(\sigma) \left(-z + \frac{\cosh z}{\sinh 1} \right) + \frac{\partial U^0}{\partial \nu}(\sigma) (\sinh z - \coth 1 \cosh z).$$

Using (2.5) at order ε^1 , we obtain that $V^1(1, \sigma) = 0$, so:

$$(2.28) \quad W^1(\sigma) = (-1 + \coth 1) \frac{\partial W^0}{\partial \nu}(\sigma) - \frac{1}{\sinh 1} \frac{\partial U^0}{\partial \nu}(\sigma)$$

Therefore, we obtain that

$$(2.29) \quad V^1(z, \sigma) = \frac{\partial W^0}{\partial \nu}(\sigma) \left(1 - z + \frac{\cosh z - \cosh 1}{\sinh 1} \right) + \frac{\partial U^0}{\partial \nu}(\sigma) \left(\sinh z - \coth 1 \cosh z + \frac{1}{\sinh 1} \right).$$

Using (2.5) at order ε^1 , we obtain that for $\sigma \in \Gamma$:

$$(2.30) \quad U^1(\sigma) = V^1(0, \sigma) + W^1(\sigma) = \frac{1}{\sinh 1} \frac{\partial W^0}{\partial \nu}(\sigma) - \frac{\cosh 1}{\sinh 1} \frac{\partial U^0}{\partial \nu}(\sigma).$$

Therefore, coupling (2.22) and (2.30), U^1 is characterized by:

$$(2.31) \quad \begin{cases} -\Delta U^1 - \xi U^1 = 0 \text{ in } \Omega_i, \\ U^1(\sigma) = \frac{1}{\sinh 1} \frac{\partial W^0}{\partial \nu}(\sigma) - \coth 1 \frac{\partial U^0}{\partial \nu}(\sigma) \text{ in } \partial\Omega_i. \end{cases}$$

In addition, coupling (2.23), (2.28), and (2.8) at order ε^1 , W^1 is characterized by:

$$(2.32) \quad \begin{cases} -\Delta W^1 - \xi W^1 = f \text{ in } \Omega_e, \\ W^1(\sigma) = (-1 + \coth 1) \frac{\partial W^0}{\partial \nu}(\sigma) - \frac{1}{\sinh 1} \frac{\partial U^0}{\partial \nu}(\sigma) \text{ in } \Gamma, \\ W^1 = 0 \text{ on } \partial\Omega. \end{cases}$$

We denote by \mathbf{H} the space of the functions $f \in L^2(\Omega)$ such that the restrictions of f respectively in Ω_i and Ω_e are respectively in $H^1(\Omega_i)$ and $H^1(\Omega_e)$, endowed with the Hilbert norm:

$$\|f\|_{\mathbf{H}} = \left(\|f|_{\Omega_i}\|_{H^1(\Omega_i)}^2 + \|f|_{\Omega_e}\|_{H^1(\Omega_e)}^2 \right)^{\frac{1}{2}}.$$

2.5. Equation for the remainder. We write:

$$\begin{cases} u^\varepsilon(x) = U^0(x) + \varepsilon U^1(x) + \varepsilon a^\varepsilon(x) & \text{for } x \in \Omega_i, \\ v^\varepsilon(x) = W^0(x) + \varepsilon W^1(x) + \varepsilon V^1(\mathbf{P}(x), \frac{d(x)}{\varepsilon}) + \varepsilon b^\varepsilon(x) & \text{for } x \in \omega_\varepsilon, \\ w^\varepsilon(x) = W^0(x) + \varepsilon W^1(x) + \varepsilon c^\varepsilon(x) & \text{for } x \in \mathcal{U}_\varepsilon. \end{cases}$$

The remainder term satisfies:

$$(2.33) \quad -\Delta a^\varepsilon - \xi a^\varepsilon = 0 \quad \text{in } \Omega_i$$

$$(2.34) \quad -\Delta b^\varepsilon - \xi b^\varepsilon + \frac{1}{\varepsilon^2} b^\varepsilon = d^\varepsilon \quad \text{in } \omega_\varepsilon$$

$$(2.35) \quad -\Delta c^\varepsilon - \xi c^\varepsilon = 0 \quad \text{in } \mathcal{U}_\varepsilon$$

$$(2.36) \quad a^\varepsilon = b^\varepsilon \quad \text{on } \Gamma$$

$$(2.37) \quad b^\varepsilon = c^\varepsilon \quad \text{on } \Gamma_\varepsilon$$

$$(2.38) \quad \partial_n a^\varepsilon = \partial_n b^\varepsilon + g^\varepsilon \quad \text{on } \Gamma$$

$$(2.39) \quad \partial_n b^\varepsilon = \partial_n c^\varepsilon \quad \text{on } \Gamma_\varepsilon$$

$$(2.40) \quad c^\varepsilon = 0 \quad \text{on } \partial\Omega,$$

where d^ε is defined on ω_ε by $d^\varepsilon = d_1^\varepsilon + d_2^\varepsilon + d_3^\varepsilon + d_4^\varepsilon$ with:

$$d_1^\varepsilon(x) = -\frac{1}{\varepsilon^3} \left(W^0(x) - d(x) \frac{\partial W^0}{\partial \nu}(\mathbf{P}(x)) \right),$$

$$d_2^\varepsilon(x) = -\frac{1}{\varepsilon^2} (W^1(x) - W^1(\mathbf{P}(x))),$$

$$d_3^\varepsilon(x) = \frac{1}{\varepsilon} G_{d(x)} \partial_z V^1\left(\frac{d(x)}{\varepsilon}, \mathbf{P}(x)\right) + \xi V^1\left(\frac{d(x)}{\varepsilon}, \mathbf{P}(x)\right),$$

$$d_4^\varepsilon(x) = \Delta_{\Gamma_{d(x)}} V^1\left(\frac{d(x)}{\varepsilon}, \mathbf{P}(x)\right),$$

and where g^ε is defined on Γ by:

$$g^\varepsilon = \frac{\partial W^1}{\partial \nu} - \frac{\partial U^1}{\partial \nu}.$$

2.6. Estimates of the right-hand-side terms. Since ξ is in the circle of center λ^0 and of radius η_0 , there exists a constant C such that for all $f \in \mathbf{H}$,

$$(2.41) \quad \|U^0\|_{H^3(\Omega_i)} + \|W^0\|_{H^3(\Omega_e)} \leq C \|f\|_{\mathbf{H}},$$

and

$$(2.42) \quad \|U^1\|_{H^2(\Omega_i)} + \|W^1\|_{H^2(\Omega_e)} \leq C \|f\|_{\mathbf{H}}.$$

From (2.42), by trace theory, we obtain that:

$$\begin{aligned} \|g^\varepsilon\|_{L^2(\Gamma)} &\leq \left\| \frac{\partial W^1}{\partial \nu} \right\|_{H^{\frac{1}{2}}(\Gamma)} + \left\| \frac{\partial U^1}{\partial \nu} \right\|_{H^{\frac{1}{2}}(\Gamma)}, \\ &\leq K \left(\|\nabla W^1\|_{H^1(\Omega_e)} + \|\nabla U^1\|_{H^1(\Omega_i)} \right), \end{aligned}$$

so,

$$(2.43) \quad \|g^\varepsilon\|_{L^2(\Gamma)} \leq K \|f\|_{\mathbf{H}}.$$

In order to estimate d_1^ε and d_2^ε , we prove the following proposition:

Proposition 2.2. *We denote by N the dimension of the space. We define 2^* by $\frac{1}{2} = \frac{1}{N} + \frac{1}{2^*}$ (for $N = 2$, $2^* = +\infty$). Then for all $p \in [2, 2^*[$ ($[2, 2^*$] for $N \neq 2$), there exists a constant K_p such that for all $\varepsilon > 0$, we have:*

$$\forall v \in H^2(\Omega_e), \left\| \frac{1}{\varepsilon^2} (v(\cdot) - v(\mathbf{P}(\cdot))) \right\|_{L^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|v\|_{H^2(\Omega_e)}$$

and

$$\forall v \in H^3(\Omega_e), \left\| \frac{1}{\varepsilon^3} \left(v(\cdot) - v(\mathbf{P}(\cdot)) - \frac{\partial v}{\partial \nu}(\mathbf{P}(\cdot)) \right) \right\|_{L^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|v\|_{H^3(\Omega_e)}$$

Proof. We assume that v is smooth. For $x \in \omega_\varepsilon$, we have $x = \mathbf{P}(x) + d(x)\nu(\mathbf{P}(x))$, so, using the variables $s = d(x)$ and $\sigma = \mathbf{P}(x)$ and Taylor's formula:

$$\begin{aligned} \|v(\cdot) - v(\mathbf{P}(\cdot))\|_{L^2(\omega_\varepsilon)}^2 &= \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) |v(\sigma + s\nu(\sigma)) - v(\sigma)|^2 d\sigma ds, \\ &= \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) \left| \int_{\tau=0}^s \nabla v(\sigma + \tau\nu(\sigma)) \cdot \nu(\sigma) d\tau \right|^2 ds d\sigma, \\ &\leq \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) \left(\int_0^s |\nabla v(\sigma + \tau\nu(\sigma))|^2 d\tau \right) s ds d\sigma, \\ &\leq \varepsilon^2 3 \int_{\omega_\varepsilon} |\nabla v|^2 dx \quad (\text{using Proposition 2.1}). \end{aligned}$$

By Sobolev embedding, for all $p \in [2, 2^*[$ ($[2, 2^*$] for $N \neq 2$),

$$\|\nabla v\|_{L^p(\mathcal{U})} \leq C_p \|v\|_{H^2(\mathcal{U})}.$$

So, by Hölder inequality,

$$\begin{aligned} \|\nabla v\|_{L^2(\omega_\varepsilon)} &\leq \|\nabla v\|_{L^p(\omega_\varepsilon)} (\text{meas}(\omega_\varepsilon))^{\frac{1}{2} - \frac{1}{p}}, \\ &\leq (K\varepsilon)^{\frac{1}{2} - \frac{1}{p}} C_p \|v\|_{H^2(\mathcal{U})}. \end{aligned}$$

Therefore,

$$\left\| \frac{1}{\varepsilon^2} (v(\cdot) - v(\mathbf{P}(\cdot))) \right\|_{L^2(\omega_\varepsilon)} \leq \frac{\sqrt{3} K C_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|v\|_{H^2(\mathcal{U})}.$$

In addition, by Taylor expansion at order 2, for $\sigma \in \Gamma$ and $s \in [0, \varepsilon]$, we have:

$$\begin{aligned} \left| v(\sigma + s\nu(\sigma)) - v(\sigma) - \frac{\partial v}{\partial \nu}(\sigma)d(x) \right| &= \left| \int_0^s (s - \tau) D^2 v(\sigma + \tau s)(\nu(\sigma), \nu(\sigma)) d\tau \right|, \\ &\leq s \int_0^s |D^2 v(\sigma + \tau s)| d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| v(x) - v(\mathbf{P}(x)) - \frac{\partial v}{\partial \nu}(\mathbf{P}(x))d(x) \right\|_{L^2(\omega_\varepsilon)}^2 &= \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) \left| v(\sigma + s\nu(\sigma)) - v(\sigma) - \frac{\partial v}{\partial \nu}(\sigma)s \right|^2 ds d\sigma, \\ &\leq \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) \left| s \int_0^s |D^2 v(\sigma + \tau s)| d\tau \right|^2 ds d\sigma, \\ &\leq \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) s^3 \int_0^s |D^2 v(\sigma + \tau s)|^2 d\tau ds d\sigma, \\ &\leq 3\varepsilon^4 \int_{\omega_\varepsilon} |D^2 v(x)|^2 dx. \end{aligned}$$

By Sobolev embedding, for all $p \in [2, 2^*[$ ($[2, 2^*]$ for $N \neq 2$),

$$\|D^2 v\|_{L^p(\mathcal{U})} \leq C_p \|v\|_{H^3(\mathcal{U})},$$

so

$$\|D^2 v\|_{L^2(\omega_\varepsilon)} \leq \|D^2 v\|_{L^p(\mathcal{U})} (\text{meas}(\omega_\varepsilon))^{\frac{1}{2} - \frac{1}{p}} \leq C_p \|v\|_{H^3(\mathcal{U})} (K\varepsilon)^{\frac{1}{2} - \frac{1}{p}},$$

thus,

$$\left\| \frac{1}{\varepsilon^3} \left(v(x) - v(\mathbf{P}(x)) - \frac{\partial v}{\partial \nu}(\mathbf{P}(x))d(x) \right) \right\|_{L^2(\omega_\varepsilon)} \leq \frac{\sqrt{3} K C_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|v\|_{H^3(\mathcal{U})}.$$

We take $K_p = \sqrt{3} K C_p$, and we conclude the proof of Proposition 2.2 by a density argument. \square

Corollary 2.3. *For all $p \in [2, 2^*]$ ($[2, +\infty[$ in the case $N = 2$), there exists K_p such that for all $\varepsilon > 0$, for all $f \in \mathbf{H}$,*

$$\|d_1^\varepsilon\|_{L^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|f\|_{\mathbf{H}},$$

and

$$\|d_2^\varepsilon\|_{L^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|f\|_{\mathbf{H}}.$$

From (2.29), we have:

$$\begin{aligned} \|d_3^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 &= \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) \left| \frac{1}{\varepsilon} G_s(\sigma) \partial_z V^1\left(\frac{s}{\varepsilon}, \sigma\right) + \xi V^1\left(\frac{s}{\varepsilon}, \sigma\right) \right|^2 ds d\sigma \\ &\leq \frac{K}{\varepsilon^2} \int_0^\varepsilon \int_\Gamma \left(\left| \frac{\partial W^0}{\partial \nu}(\sigma) \right|^2 + \left| \frac{\partial U^0}{\partial \nu}(\sigma) \right|^2 \right) d\sigma ds, \\ &\leq \frac{K}{\varepsilon} \left(\|W^0\|_{H^3(\Omega_e)}^2 + \|U^0\|_{H^3(\Omega_i)}^2 \right). \end{aligned}$$

Therefore

$$(2.44) \quad \|d_3^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon^{\frac{1}{2}}} \|f\|_{\mathbf{H}}.$$

2.7. Variational estimates for the remainder term. Multiplying (2.33) by the conjugate of a^ε and integrating on Ω_i , multiplying (2.34) by the conjugate of b^ε and integrating on ω_ε , multiplying (2.35) by the conjugate of c^ε and integrating on \mathcal{U}_ε , after integrations by part and after summation, we obtain:

$$\begin{aligned} \|\nabla a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|\nabla c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 + \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 = \\ \int_\Gamma g^\varepsilon \bar{b}^\varepsilon + \int_{\omega_\varepsilon} d^\varepsilon \bar{b}^\varepsilon - \xi \left(\|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \right). \end{aligned}$$

So,

$$\begin{aligned} \|\nabla a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|\nabla c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 + \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 \leq \\ |\xi| \left(\|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \right) \\ (2.45) \quad + \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|d_1^\varepsilon + d_2^\varepsilon + d_3^\varepsilon\|_{L^2(\omega_\varepsilon)} + \|g^\varepsilon\|_{L^2(\Gamma)} \|b^\varepsilon\|_{L^2(\Gamma)} \\ + \left| \int_{\omega_\varepsilon} d_4^\varepsilon \bar{b}^\varepsilon \right|. \end{aligned}$$

We claim the following lemma proved in Section 3.1.

Lemma 2.4. *There exists a constant $C > 0$ independent of ε such that for any $\xi \in C(\lambda^0, \eta)$, for any $u \in H^1(\Omega_i)$ such that $-\Delta u - \xi u \in L^2(\Omega_e)$, we have :*

$$(2.46) \quad \|u\|_{L^2(\Omega_i)} \leq C \left(\|u\|_{L^2(\Gamma)} + \|(-\Delta u - \xi u)\|_{L^2(\Omega_i)} \right),$$

and for any $v \in H^1(\mathcal{U}_\varepsilon)$ such that $v = 0$ on $\partial\Omega$ and such that $-\Delta v - \xi v \in L^2(\mathcal{U}_\varepsilon)$, we have :

$$(2.47) \quad \|v\|_{L^2(\mathcal{U}_\varepsilon)} \leq C \left(\|v\|_{L^2(\Gamma_\varepsilon)} + \|(-\Delta v - \xi v)\|_{L^2(\mathcal{U}_\varepsilon)} \right).$$

We claim the following Lemma, proved in [4] (see Inequality (2.1) in Proposition 2.1., page 822).

Lemma 2.5. *There exists constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, for all $\mathbf{q} \in \mathbf{H}^1(\omega_\varepsilon)$,*

$$\|\mathbf{q}\|_{L^2(\partial\omega_\varepsilon)}^2 \leq C \left(\frac{1}{\varepsilon} \|\mathbf{q}\|_{L^2(\omega_\varepsilon)}^2 + \|\mathbf{q}\|_{L^2(\omega_\varepsilon)} \|\nabla \mathbf{q}\|_{L^2(\omega_\varepsilon)} \right)$$

Since a^ε (resp. c^ε) satisfies (2.33) in Ω_i (resp. (2.35) in \mathcal{U}_ε), we obtain with Lemma 2.4 and the transmission condition (2.36) (resp. (2.37)) that there exists C independent of $\varepsilon > 0$ and $\xi \in \mathcal{C}(\lambda^0, \eta)$ such that:

$$\|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \leq C \|b^\varepsilon\|_{L^2(\partial\omega_\varepsilon)}^2.$$

Using now Lemma 2.5, we obtain that:

$$(2.48) \quad \|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \leq \frac{C}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + C \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}.$$

Using Corollary 2.3, Estimate (2.44), we obtain that there exists a constant C such that for all $\varepsilon > 0$, for all $\xi \in \mathcal{C}(\lambda^0, \eta)$ and for all $f \in \mathbf{H}$,

$$(2.49) \quad \|d_1^\varepsilon + d_2^\varepsilon + d_3^\varepsilon\|_{L^2(\omega_\varepsilon)} \leq \frac{C}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|f\|_{\mathbf{H}}.$$

In addition, coupling Lemma 2.5 and Estimate (2.43), we obtain that:

$$(2.50) \quad \|g^\varepsilon\|_{L^2(\Gamma)} \|b^\varepsilon\|_{L^2(\Gamma)} \leq C \sqrt{\varepsilon} \|f\|_{\mathbf{H}} \frac{1}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} + C \varepsilon^{\frac{1}{2}} \|f\|_{\mathbf{H}} \left(\frac{1}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \right)^{\frac{1}{2}} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^{\frac{1}{2}}.$$

Let us estimate the last right-hand-side term in (2.45). We denote by $B^\varepsilon = b^\varepsilon \circ \Psi$, where:

$$\begin{aligned} \Psi : [0, \eta_0] \times \Gamma &\rightarrow \omega_{\eta_0} \\ (s, \sigma) &\mapsto \sigma + s\nu(\sigma). \end{aligned}$$

$$\begin{aligned} \int_{\omega_\varepsilon} d_4^\varepsilon \overline{b^\varepsilon} &= \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) (\Delta_{\Gamma_s} V^1(\frac{s}{\varepsilon}, \sigma)) \overline{B^\varepsilon(s, \sigma)} ds d\sigma, \\ &= - \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) \nabla_{\Gamma_s} V^1(\frac{s}{\varepsilon}, \sigma) \overline{\nabla_{\Gamma_s} B^\varepsilon(s, \sigma)} ds, \end{aligned}$$

So,

$$\begin{aligned} \left| \int_{\omega_\varepsilon} d_4^\varepsilon \overline{b^\varepsilon} \right| &\leq \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) \left| \nabla_{\Gamma_s} V^1(\frac{s}{\varepsilon}, \sigma) \right| \left| \nabla_{\Gamma_s} B^\varepsilon(s, \sigma) \right| ds d\sigma \\ &\leq \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|\nabla_{\Gamma_s} V^1(\frac{s}{\varepsilon}, \sigma)\|_{L^2(\omega_\varepsilon)}, \\ &\leq \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} \left(\left\| \frac{\partial U^0}{\partial \nu} \right\|_{\mathbf{H}^1(\Gamma)} + \left\| \frac{\partial W^0}{\partial \nu} \right\|_{\mathbf{H}^1(\Gamma)} \right) \sqrt{\varepsilon}, \end{aligned}$$

thus, using (2.41), we obtain:

$$(2.51) \quad \left| \int_{\omega_\varepsilon} d_4^\varepsilon \overline{b^\varepsilon} \right| \leq C \sqrt{\varepsilon} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|f\|_{\mathbf{H}}.$$

Therefore, from (2.45), (2.48), (2.49), (2.50), and (2.51), we obtain that there exists a constant C independent of $\varepsilon > 0$, $\xi \in \mathcal{C}(\lambda^0, \eta)$ and $f \in \mathbf{H}$ such that

$$\begin{aligned} & \|\nabla a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|\nabla c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 + \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 \leq \\ & \frac{C}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + C \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} + \frac{C}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|f\|_{\mathbf{H}} \\ & + C\sqrt{\varepsilon} \|f\|_{\mathbf{H}} \frac{1}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} + C\varepsilon^{\frac{1}{2}} \|f\|_{\mathbf{H}} \left(\frac{1}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}\right)^{\frac{1}{2}} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^{\frac{1}{2}} + C\varepsilon^{\frac{1}{2}} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|f\|_{\mathbf{H}} \\ & \leq \frac{1}{2} \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \frac{1}{2} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + C\varepsilon^{1-\frac{2}{p}} \|f\|_{\mathbf{H}}^2 \quad (\text{by Young inequality}). \end{aligned}$$

So, after absorption, we obtain that there exists C such that:

$$(2.52) \quad \|\nabla a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|\nabla c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 + \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 \leq C\varepsilon^{1-\frac{2}{p}} \|f\|_{\mathbf{H}}^2.$$

Reusing (2.48), we obtain also that:

$$(2.53) \quad \|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \leq C\varepsilon^{2-\frac{2}{p}} \|f\|_{\mathbf{H}}^2.$$

3. TECHNICAL LEMMA : UNIFORM A PRIORI ESTIMATES

3.1. Geometrical tools in the thin layer. In this part, we refer to [4]. For $\sigma \in \Gamma$, we denote by $\nu(\sigma)$ the outward unitary normal. We remark that $\nu : \Gamma \rightarrow S^{N-1}$ is smooth since Γ is regular. We define Ψ by:

$$\begin{aligned} \Psi :]0, \eta_0[\times \Gamma & \rightarrow \omega_{\eta_0} \\ (s, \sigma) & \mapsto \sigma + s\nu(\sigma). \end{aligned}$$

Since Γ is a smooth compact surface without boundary, we can fix $\eta_0 > 0$ such that Ψ is a C^∞ -diffeomorphism and so that for all $\varepsilon \in]0, \eta_0[$,

$$\Psi(]0, \varepsilon[\times \Gamma) = \omega_\varepsilon.$$

We define $\gamma : [0, \eta_0] \times \Gamma \rightarrow \mathbb{R}$ by:

$$\gamma_s(\sigma) = \det(\text{Id} + s d\nu(\sigma)),$$

and we have:

$$\int_{\omega_{\eta_0}} F(x) dx = \int_{s=0}^{\eta_0} \int_{\sigma \in \Gamma} F(\Psi(s, \sigma)) \gamma_s(\sigma) d\sigma ds.$$

Since $\gamma_0(\sigma) = 1$ for all $\sigma \in \Gamma$, even if it means reducing η_0 , we assume that

$$(3.1) \quad \forall s \in [0, \eta_0[, \forall \sigma \in \Gamma, \frac{1}{2} \leq \gamma_s(\sigma) \leq \frac{3}{2}.$$

We denote ∇_Γ , div_Γ and Δ_Γ the gradient, divergence and Laplacian operators on the sub-manifold Γ .

For $v : \Gamma \rightarrow \mathbb{C}$ and $s \in [0, \eta_0[$, we define $\nabla_{\Gamma_s} v$ by:

$$\nabla_{\Gamma_s} v(\sigma) = (\text{Id} + s d\nu(\sigma))^{-1} \nabla_\Gamma v(\sigma),$$

so that: for $v : \omega_{\eta_0} \rightarrow \mathbb{C}$, denoting $V = v \circ \Psi$, we have:

$$\nabla v(x) = \frac{\partial V}{\partial s}(d(x), P(x))\nu(P(x)) + \left(\nabla_{\Gamma_{d(x)}} V \right) (d(x), P(x)).$$

We define the operator div_{Γ_s} as the adjoint of ∇_{Γ_s} for the $L^2(\Gamma)$ -inner product: for $Y : \Gamma \rightarrow T_\Gamma$ a tangent vector field defined on Γ , we set:

$$(\text{div}_{\Gamma_s} Y)(\sigma) = \frac{1}{\gamma_s(\sigma)} \text{div}_\Gamma (\gamma_s(\text{Id} + s d\nu)^{-1} Y)(\sigma),$$

so that if $z : \omega_{\eta_0} \rightarrow \mathbb{R}^N$ is a vector field, denoting $Z = z \circ \Psi$, $Z_N(s, \sigma) = (Z(s, \sigma) \cdot \nu(\sigma))$ and $Z_T(s, \sigma) = Z(s, \sigma) - Z_N(s, \sigma)\nu(\sigma)$, we have:

$$\text{div} z(x) = \frac{\partial Z_N}{\partial s}(d(x), P(x)) + G_{d(x)}(P(x))Z_N(d(x), P(x)) + \left(\text{div}_{\Gamma_{d(x)}} Z_T \right) (d(x), P(x)),$$

where

$$G_s(\sigma) = \frac{1}{\gamma_s(\sigma)} \frac{\partial \gamma_s}{\partial s}(\sigma).$$

We define the operator Δ_{Γ_s} by:

$$\Delta_{\Gamma_s} = \text{div}_{\Gamma_s} \circ \nabla_{\Gamma_s},$$

so that for $v : \omega_{\eta_0} \rightarrow \mathbb{C}$, denoting $V = v \circ \Psi$, we have:

$$\Delta v(x) = \frac{\partial^2 V}{\partial s^2}(d(x), P(x)) + G_{d(x)}(P(x)) \frac{\partial V}{\partial s}(d(x), P(x)) + (\Delta_{\Gamma_{d(x)}} V)(d(x), P(x)).$$

3.2. Geometrical tools for \mathcal{U}_ε . Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth non-decreasing function such that:

$$\forall x \leq 0, \chi(x) = -1,$$

$$\forall x \geq \eta_0, \chi(x) = 0.$$

For $\varepsilon > 0$ such that $\varepsilon \leq \frac{\eta_0}{4}$, we define ϕ_ε by:

$$\begin{cases} \phi_\varepsilon : \mathcal{U}_\varepsilon \rightarrow \Omega_\varepsilon \\ x \mapsto x + \varepsilon \chi(d(x) - \varepsilon) \nu(P(x)) \end{cases}$$

We remark that if $x \in \mathcal{U}_\varepsilon$ with $d(x) \geq \varepsilon + \eta_0$, then $\phi_\varepsilon(x) = x$. In addition, ϕ_ε is a \mathcal{C}^∞ -diffeomorphism from \mathcal{U}_ε onto Ω_ε . We denote by $\phi_{\varepsilon,k}$ the k -th coordinate of ϕ_ε .

For $w : \mathcal{U}_\varepsilon \rightarrow \mathbb{C}$, we define $w : \Omega_\varepsilon \rightarrow \mathbb{C}$ by:

$$w = \mathbf{w} \circ \phi_\varepsilon.$$

We have, for all $j \in \{1, \dots, N\}$,

$$\frac{\partial w}{\partial x_j} = \sum_{k=1}^N \frac{\partial \mathbf{w}}{\partial x_k}(\phi_\varepsilon(x)) \frac{\partial \phi_{\varepsilon,k}}{\partial x_j}(x)$$

so

$$\frac{\partial^2 w}{\partial x_j^2} = \sum_{k=1}^N \sum_{l=1}^N \frac{\partial^2 \mathbf{w}}{\partial x_k \partial x_l}(\phi_\varepsilon(x)) \frac{\partial \phi_{\varepsilon,k}}{\partial x_j}(x) \frac{\partial \phi_{\varepsilon,l}}{\partial x_j}(x) + \sum_{k=1}^N \frac{\partial \mathbf{w}}{\partial x_k}(\phi_\varepsilon(x)) \frac{\partial^2 \phi_{\varepsilon,k}}{\partial x_j^2}(x)$$

Therefore,

$$\Delta w(x) = \sum_{k,l} a_{k,l}^\varepsilon(\phi_\varepsilon(x)) \frac{\partial^2 w}{\partial x_k \partial x_l}(\phi_\varepsilon(x)) + \sum_{k=1}^N b_k^\varepsilon(\phi_\varepsilon(x)) \frac{\partial w}{\partial x_k}(\phi_\varepsilon(x)),$$

where $a_{k,l}^\varepsilon : \Omega_e \rightarrow \mathbb{R}$ and $b_k^\varepsilon : \Omega_e \rightarrow \mathbb{R}$ are defined by:

$$\forall x \in \mathcal{U}_\varepsilon, \quad a_{kl}^\varepsilon(\phi_\varepsilon(x)) = \nabla \phi_{\varepsilon,k}(x) \cdot \nabla \phi_{\varepsilon,l}(x) \quad \text{and} \quad b_k^\varepsilon(\phi_\varepsilon(x)) = \Delta \phi_{\varepsilon,k}(x).$$

For $x \in \Omega_e$ such that $d(x) \geq \varepsilon + \eta_0$, $\phi_\varepsilon(x) = x$ so that $a_{kl}^\varepsilon(x) = \delta_{kl}$ and $b_k^\varepsilon(x) = 0$.

For $x \in \Omega_e$ such that $d(x) \leq \varepsilon + \eta_0$, we have: $x = \phi_\varepsilon^{-1}(x) \in \omega_{\eta_0}$ and:

$$\frac{\partial \phi_{\varepsilon,k}}{\partial x_j}(x) = \delta_{jk} + \varepsilon \left(\chi'(d(x) - \varepsilon) \nu_j(\mathbf{P}(x)) \nu_k(\mathbf{P}(x)) + \chi(d(x) - \varepsilon) \nabla \nu_k(\mathbf{P}(x)) \cdot \frac{\partial \mathbf{P}(x)}{\partial x_j} \right).$$

Thus, $a_{k,l}^\varepsilon(x) = \delta_{kl} + \widetilde{a_{k,l}^\varepsilon}(x)$, where $\widetilde{a_{k,l}^\varepsilon}(x)$ is of order ε and tends uniformly to zero when ε tends to zero. In the same way, $b_k^\varepsilon(x)$ is of order ε and tends uniformly to zero when ε tends to zero. We denote, for $w \in H^2(\Omega_e)$:

$$\mathcal{K}^\varepsilon w = \sum_{(k,l), k \neq l} a_{kl}^\varepsilon \frac{\partial^2 w}{\partial x_k \partial x_l} + \sum_{k=1}^N b_k^\varepsilon \frac{\partial w}{\partial x_k},$$

so that

$$\Delta w(x) = (\Delta w + \mathcal{K}^\varepsilon w)(\phi_\varepsilon(x)),$$

and there exists a constant C such that for all $\varepsilon > 0$, for all $w \in H^2(\Omega_e)$, we have:

$$(3.2) \quad \|\mathcal{K}^\varepsilon w\|_{L^2(\Omega_e)} \leq C\varepsilon \|w\|_{H^2(\Omega_e)}.$$

For $\varepsilon \geq 0$, we denote by \mathcal{L}^ε the operator $-\Delta$ defined on $H^2(\mathcal{U}_\varepsilon) \cap H_0^1(\mathcal{U}_\varepsilon)$. This operator is positive and self-adjoint for the $L^2(\mathcal{U}_\varepsilon)$ -scalar product. Its resolvent is compact, so that its spectrum only contains a sequence of real eigenvalues tending to $+\infty$.

Let us prove the following lemma:

Lemma 3.1. *Let $\lambda_0 \in \text{Spec}(\mathcal{H}^0)$ and $\eta > 0$ such that $\text{Spec}(\mathcal{H}^0) \cap \overline{B(\lambda_0, \eta)} = \{\lambda_0\}$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$,*

$$\text{Spec}(\mathcal{L}^\varepsilon) \cap \mathcal{C}(\lambda_0, \eta) = \emptyset.$$

Proof. If it is not the case, there exists a sequence of positive numbers $(\varepsilon_n)_n$ tending to zero when n tends to $+\infty$, there exists a sequence $(\xi_n)_n$ of points of $\mathcal{C}(\lambda_0, \eta)$, and there exists a sequence $(u_n)_n$ such that for all n , $u_n \in H^2(\mathcal{U}_{\varepsilon_n}) \cap H_0^1(\mathcal{U}_{\varepsilon_n})$ is an eigenvector of \mathcal{L}^ε associated to the eigenvalue ξ_n .

We denote $u_n = u_n \circ (\phi_{\varepsilon_n})^{-1}$, so that we have:

$$(3.3) \quad u_n \in H^2(\Omega_e) \cap H_0^1(\Omega_e) \quad \text{and} \quad -\Delta u_n - \mathcal{K}^{\varepsilon_n} u_n = \xi_n u_n.$$

Since $u_n \neq 0$, we can assume that $\|u_n\|_{H^2(\Omega_e)} = 1$. Thus we can extract subsequences still denoted (ε_n) , (ξ_n) and (u_n) such that ξ_n tends to $\xi_\infty \in \mathcal{C}(\lambda_0, \eta)$ and u_n tends to u_∞ weakly in $H^2(\Omega_e)$ and strongly in $H^1(\Omega_e)$ by compactness of the embedding $H^2(\Omega_e) \subset H^1(\Omega_e)$.

By taking the weak limit in (3.3) and using (3.2), we obtain that

$$(3.4) \quad u_\infty \in H^2(\Omega_e) \cap H_0^1(\Omega_e) \quad \text{and} \quad -\Delta u_\infty = \xi_\infty u_\infty.$$

In addition, writing that $-\Delta u_n = \xi_n u_n + \mathcal{K}^{\varepsilon_n} u_n$, using again (3.2), we obtain that $-\Delta u_n$ tends strongly in $L^2(\Omega_e)$ to $-\Delta u_\infty$, so that u_n tends strongly to u_∞ in $H^2(\Omega_e)$. So, $\|u_\infty\|_{H^2(\Omega_e)} = 1$, thus $u_\infty \neq 0$ and by (3.4), we obtain that $\xi_\infty \in \text{Spec}(\mathcal{L}^0) \cap \mathcal{C}(\lambda_0, \eta)$, which is false. \square

With the previous lemma, for $\varepsilon > 0$ small enough, for all $\xi \in \mathcal{C}(\lambda_0, \eta)$, $\mathcal{L}^\varepsilon - \xi \mathbb{I}$ is invertible. We give a uniform estimate of the resolvent operator in the following lemma:

Lemma 3.2. *There exists $\varepsilon_0 > 0$ and C such that for all $\xi \in \mathcal{C}(\lambda_0, \eta)$, for all $\varepsilon \in]0, \varepsilon_0[$, for all $f \in L^2(\mathcal{U}_\varepsilon)$, there exists a unique $u \in H^2(\mathcal{U}_\varepsilon) \cap H_0^1(\mathcal{U}_\varepsilon)$ such that $-\Delta u - \xi u = f$ in \mathcal{U}_ε . Moreover,*

$$\|u\|_{H^2(\mathcal{U}_\varepsilon)} \leq C \|f\|_{L^2(\mathcal{U}_\varepsilon)}.$$

Proof. If it is not the case, there exists a sequence (ε_n) tending to 0, a sequence $(\xi_n)_n$ of points of $\mathcal{C}(\lambda_0, \eta)$, a sequence (f_n) with $f_n \in L^2(\mathcal{U}_{\varepsilon_n})$ and $u_n \in H^2(\mathcal{U}_{\varepsilon_n}) \cap H_0^1(\mathcal{U}_{\varepsilon_n})$ such that $-\Delta u_n - \xi_n u_n = f_n$ in $\mathcal{U}_{\varepsilon_n}$ and:

$$\|u_n\|_{H^2(\mathcal{U}_{\varepsilon_n})} > n \|f_n\|_{L^2(\mathcal{U}_{\varepsilon_n})}.$$

We denote $\mathbf{u}_n = u_n \circ (\phi_{\varepsilon_n})^{-1}$ and $\mathbf{f}_n = f_n \circ (\phi_{\varepsilon_n})^{-1}$. We have:

$$-\Delta \mathbf{u}_n - \mathcal{K}^{\varepsilon_n} \mathbf{u}_n - \xi_n \mathbf{u}_n = \mathbf{f}_n \text{ in } \Omega_e,$$

and:

$$\|\mathbf{u}_n\|_{H^2(\Omega_e)} > C n \|\mathbf{f}_n\|_{L^2(\Omega_e)}.$$

Since $\mathbf{u}_n \neq 0$, we can divide by its $H^2(\Omega_e)$ -norm. We denote

$$\tilde{\mathbf{u}}_n = \frac{\mathbf{u}_n}{\|\mathbf{u}_n\|_{H^2(\Omega_e)}} \quad \text{and} \quad \tilde{\mathbf{f}}_n = \frac{\mathbf{f}_n}{\|\mathbf{u}_n\|_{H^2(\Omega_e)}}.$$

We have, for all n :

$$\|\tilde{\mathbf{u}}_n\|_{H^2(\Omega_e)} = 1 > n \|\tilde{\mathbf{f}}_n\|_{L^2(\Omega_e)}$$

and

$$(3.5) \quad -\Delta \tilde{\mathbf{u}}_n - \mathcal{K}^{\varepsilon_n} \tilde{\mathbf{u}}_n - \xi_n \tilde{\mathbf{u}}_n = \tilde{\mathbf{f}}_n \text{ in } \Omega_e.$$

By extracting a sub sequence, we can assume that ξ_n tends to $\xi_\infty \in \mathcal{C}(\lambda_0, \eta)$, and $\tilde{\mathbf{u}}_n$ tends to a limit $\tilde{\mathbf{u}}_\infty$ weakly in $H^2(\Omega_e)$ and strongly in $H^1(\Omega_e)$. In addition, we remark that $\tilde{\mathbf{f}}_n$ tends to zero strongly in $L^2(\Omega_e)$. Taking the weak limit in (3.5), we obtain that:

$$(3.6) \quad \tilde{\mathbf{u}}_\infty \in H^2(\Omega_e) \cap H_0^1(\Omega_e) \quad \text{and} \quad -\Delta \tilde{\mathbf{u}}_\infty - \xi_\infty \tilde{\mathbf{u}}_\infty = 0 \text{ in } \Omega_e.$$

In addition, using (3.2) and (3.5), we obtain that $\Delta \tilde{\mathbf{u}}_n$ converges strongly in $L^2(\Omega_e)$. Since its limit is the weak limit $\Delta \tilde{\mathbf{u}}_\infty$, then $\tilde{\mathbf{u}}_n$ converges strongly in $H^2(\Omega_e)$ to $\tilde{\mathbf{u}}_\infty$. So $\|\tilde{\mathbf{u}}_\infty\|_{H^2(\Omega_e)} = 1$, and $\tilde{\mathbf{u}}_\infty \neq 0$. Therefore, with (3.6), ξ_∞ is in $\text{Spec}(\mathcal{L}^0)$, which leads to a contradiction. \square

Proof of Lemma 2.4

Inequality (2.46) is proved by duality arguments in [3] (see Lemma 2.1, on page 95). We use the same method and uniform estimates proved in Lemma 3.2 to establish (2.47). Let ε_0 given by Lemma 3.2. Let $\varepsilon \in (0, \varepsilon_0)$ and $\xi \in \mathcal{C}(\lambda_0, \eta)$. We fix $v \in H^1(\mathcal{U}_\varepsilon)$ such that $-\Delta v - \xi v$ belongs to $L^2(\mathcal{U}_\varepsilon)$. Let $f \in L^2(\mathcal{U}_\varepsilon)$. By Lemma 3.2, there exists $w \in H^2(\mathcal{U}_\varepsilon) \cap H_0^1(\mathcal{U}_\varepsilon)$ such that $-\Delta w - \xi w = f$ in \mathcal{U}_ε . Moreover,

$$\|w\|_{H^2(\mathcal{U}_\varepsilon)} \leq C \|f\|_{L^2(\mathcal{U}_\varepsilon)},$$

where C does not depend on ε , f or ξ . We have:

$$\int_{\mathcal{U}_\varepsilon} v f = \int_{\mathcal{U}_\varepsilon} v(-\Delta w - \xi w) = \int_{\mathcal{U}_\varepsilon} (-\Delta v - \xi v)w + \int_{\Gamma_\varepsilon} v \frac{\partial w}{\partial \nu}.$$

Therefore:

$$\begin{aligned} \left| \int_{\mathcal{U}_\varepsilon} v f \right| &\leq \| -\Delta v - \xi v \|_{L^2(\mathcal{U}_\varepsilon)} \|w\|_{L^2(\mathcal{U}_\varepsilon)} + \|v\|_{L^2(\Gamma_\varepsilon)} \left\| \frac{\partial w}{\partial \nu} \right\|_{L^2(\Gamma_\varepsilon)} \\ &\leq C (\| -\Delta v - \xi v \|_{L^2(\mathcal{U}_\varepsilon)} + \|v\|_{L^2(\Gamma_\varepsilon)}) \|f\|_{L^2(\mathcal{U}_\varepsilon)} \end{aligned}$$

This is true for all $f \in L^2(\mathcal{U}_\varepsilon)$ so:

$$\|v\|_{L^2(\mathcal{U}_\varepsilon)} \leq C (\| -\Delta v - \xi v \|_{L^2(\mathcal{U}_\varepsilon)} + \|v\|_{L^2(\Gamma_\varepsilon)}).$$

□

REFERENCES

- [1] A. Bamberger and A. S. Bonnet-Ben Dhia. Mathematical analysis of the guided modes of an optical fiber. *SIAM Journal on Mathematical Analysis*, 21(6):1487–1510, 1990.
- [2] A. S. Bonnet-Ben Dhia and Joly P. Théorie spectrale des opérateurs autoadjoints et application à l'étude des ondes guidées. 1999.
- [3] V. Bruneau and G. Carbou. Spectral asymptotic in the large coupling limit. *Asymptotic Analysis*, 29(2):91–113, 2002.
- [4] G. Carbou. Penalization method for viscous incompressible flow around a porous thin layer. *Nonlinear Anal. Real World Appl.*, 5(5):815–855, 2004.
- [5] G. Carbou. Brinkmann model and double penalization method for the flow around a porous thin layer. *J. Math. Fluid Mech.*, 10(1):126–158, 2008.
- [6] G. Carbou and P. Fabrie. Boundary layer for a penalization method for viscous incompressible flow. *Adv. Differential Equations*, 8(12):1453–1480, 2003.
- [7] P. Joly and C. Poirier. Mathematical analysis of electromagnetic open waveguides. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, 29(5):505–575, 1995.