



HAL
open science

Resolvent and Spectral Asymptotics for Thin-Layer Transmission Problems in the Large Coupling Limit

Gilles Carbou, Victor P eron

► **To cite this version:**

Gilles Carbou, Victor P eron. Resolvent and Spectral Asymptotics for Thin-Layer Transmission Problems in the Large Coupling Limit. 2022. hal-03859342

HAL Id: hal-03859342

<https://hal.science/hal-03859342>

Preprint submitted on 18 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destin ee au d ep ot et  a la diffusion de documents scientifiques de niveau recherche, publi es ou non,  emanant des  tablissements d'enseignement et de recherche franais ou  trangers, des laboratoires publics ou priv es.

RESOLVENT AND SPECTRAL ASYMPTOTICS FOR THIN-LAYER TRANSMISSION PROBLEMS IN THE LARGE COUPLING LIMIT

VICTOR PÉRON¹, GILLES CARBOU¹

ABSTRACT. This work is concerned with a singular perturbation problem that involves a thin layer in the large coupling limit. Such a problem appears in the study of electromagnetic wave guides that contain supra-conductor bodies. We derive resolvent asymptotics for this problem by using boundary layer tools with uniform *a priori* estimates. This result will be useful to derive spectral asymptotics for this problem. This issue is a work in progress.

1. INTRODUCTION

This work is motivated by the study of the asymptotic behavior of eigenvalues related to a singular perturbation problem that involves a thin layer in the large coupling limit. Such a problem appears in the study of electromagnetic wave guides with supra-conductors. We refer the reader to several works that concern mathematical studies with electromagnetic wave guides, see for instance [1, 7, 2, 3].

An asymptotic expansion of the eigenvalues as the conductivity tends to infinity has been derived in [3]. However this result does not apply when the subdomain that represents a supra-conductor is a vanishing sheet. In this work we tackle this problem by considering vanishing supra-conductive layer of thickness ε where the conductivity is scaled like ε^{-2} . This is why there is a main difference between this work and the work in Ref. [3] all the more since we consider a transmission problem across a thin layer.

Let us describe now the mathematical model. Throughout the paper we denote by $\Omega \subset \mathbb{R}^3$ a simply connected domain. Let Ω_i be a simply connected subdomain of Ω and let ω_ε be a thin layer of constant thickness ε surrounding Ω_i such that Ω contains $\Omega_i \cup \overline{\omega_\varepsilon}$, see Figure 1. For the sake of simplicity we further assume that the boundaries $\partial\Omega$ and $\Gamma = \partial\Omega_i$ are connected. Now the exterior domain is defined as $\mathcal{U}_\varepsilon = \Omega \setminus (\Omega_i \cup \overline{\omega_\varepsilon})$ and the boundary of the subdomain $\overline{\Omega_i} \cup \omega_\varepsilon$ is denoted by Γ_ε . We denote by \mathbf{n} the outward unit normal at Γ .

In all that follows, unless specified, all the considered domains are smooth domains in \mathbb{R}^3 .

Notation 1.1. For any set $\mathcal{O} \subset \mathbb{R}^3$ we denote by $L^2(\mathcal{O})$ and $H^s(\mathcal{O})$ the classical Sobolev spaces.

Let $\varepsilon > 0$. We consider the self-adjoint operator \mathcal{H}^ε on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ and defined by

$$\mathcal{H}^\varepsilon = -\Delta + \frac{1}{\varepsilon^2} 1_{|\omega_\varepsilon}.$$

The limit operator is $\mathcal{H}^0 = -\Delta$ defined on $H^2(\Omega_i) \cap H_0^1(\Omega_i) \times H^2(\Omega_e) \cap H_0^1(\Omega_e)$ where Ω_e denotes the limit domain \mathcal{U}_ε when $\varepsilon \rightarrow 0$, i.e. $\Omega_e = \Omega \setminus \overline{\Omega_i}$ (see Figure 2).

Date: November 18, 2022, Version 1.

¹ Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, Laboratoire de Mathématiques et de leurs Applications de Pau, Avenue de l'Université - BP 1155, 64013 PAU CEDEX, France.

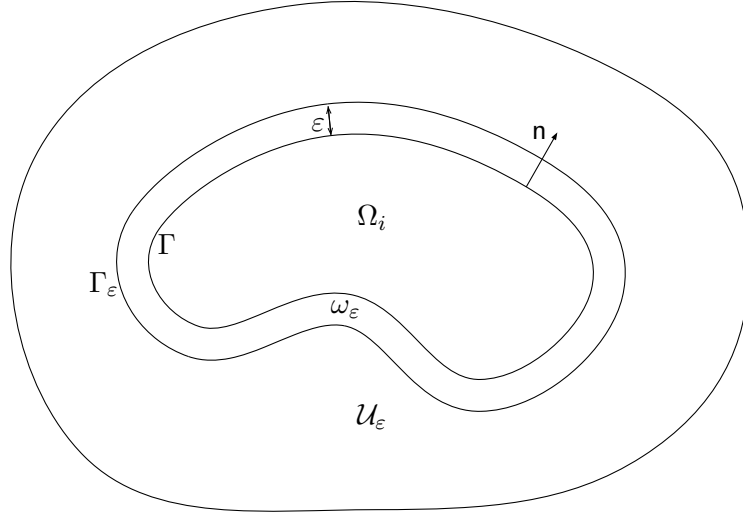


Figure 1 – A cross-section of the domain Ω and the subdomains $\Omega_i, \omega_\varepsilon, \mathcal{U}_\varepsilon$

In this work we derive an asymptotic expansion of the resolvent of \mathcal{H}^ε when ε tends to 0. We denote by $\mathcal{R}^\varepsilon(\xi) := (\mathcal{H}^\varepsilon - \xi\mathcal{I})^{-1}$ the resolvent of \mathcal{H}^ε , defined on $L^2(\Omega)$ with values in $H^2(\Omega) \cap H_0^1(\Omega)$ for $\xi \notin \text{spec } \mathcal{H}^\varepsilon$.

Let $f \in L^2(\Omega)$. The spectral problem writes

$$(1.1) \quad -\Delta u - \xi u + \frac{1}{\varepsilon^2} 1_{|\omega_\varepsilon} u = f \quad \text{in } \Omega.$$

This problem is well-posed and has a unique solution u_ε in $H^2(\Omega) \cap H_0^1(\Omega)$ when ξ is not an eigenvalue of the operator \mathcal{H}^ε .

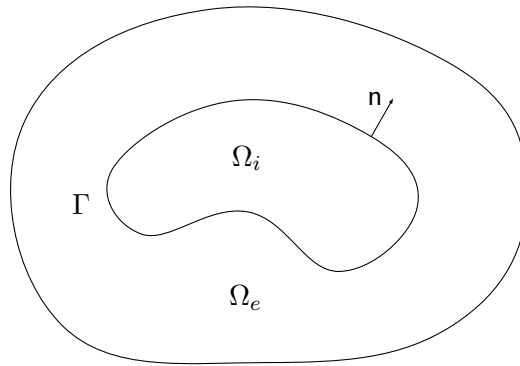


Figure 2 – A cross-section of the domain Ω and the subdomains Ω_i, Ω_e

Notation 1.2. We denote by f_i (resp. f_e) the restriction of any function f to Ω_i (resp. Ω_e).

2. ASYMPTOTIC EXPANSION FOR THE RESOLVENT

Let $\varepsilon > 0$. We consider the self-adjoint operator \mathcal{H}^ε on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ and defined by

$$\mathcal{H}^\varepsilon = -\Delta + \frac{1}{\varepsilon^2} 1_{|\omega_\varepsilon}.$$

For $\xi \notin \text{spec } \mathcal{H}^\varepsilon$, we denote by $\mathcal{R}^\varepsilon(\xi) = (\mathcal{H}^\varepsilon - \xi \text{Id})^{-1}$ the resolvent operator. For $f \in L^2(\Omega)$, we denote by u^ε the restriction of $\mathcal{R}^\varepsilon(\xi)(f)$ to Ω_i , v^ε the restriction of $\mathcal{R}^\varepsilon(\xi)(f)$ to ω_ε , and w^ε the restriction of $\mathcal{R}^\varepsilon(\xi)(f)$ to \mathcal{U}_ε . By the resolvent equation, we have:

$$(2.1) \quad -\Delta u^\varepsilon - \xi u^\varepsilon = f|_{\Omega_i} \quad \text{in } \Omega_i$$

$$(2.2) \quad -\Delta v^\varepsilon - \xi v^\varepsilon + \frac{1}{\varepsilon^2} v^\varepsilon = f|_{\omega_\varepsilon} \quad \text{in } \omega_\varepsilon$$

$$(2.3) \quad -\Delta w^\varepsilon - \xi w^\varepsilon = f|_{\mathcal{U}_\varepsilon} \quad \text{in } \mathcal{U}_\varepsilon$$

$$(2.4) \quad u^\varepsilon = v^\varepsilon \quad \text{on } \Gamma$$

$$(2.5) \quad v^\varepsilon = w^\varepsilon \quad \text{on } \Gamma_\varepsilon$$

$$(2.6) \quad \partial_n u^\varepsilon = \partial_n v^\varepsilon \quad \text{on } \Gamma$$

$$(2.7) \quad \partial_n v^\varepsilon = \partial_n w^\varepsilon \quad \text{on } \Gamma_\varepsilon$$

$$(2.8) \quad w^\varepsilon = 0 \quad \text{on } \partial\Omega.$$

2.1. **Ansatz.** Outside the thin layer, we look for an expansion on the form:

$$(2.9) \quad u^\varepsilon(x) \approx \sum_{j \geq 0} \varepsilon^j U^j(x) \quad \text{in } \Omega_i,$$

$$(2.10) \quad w^\varepsilon(x) \approx \sum_{j \geq 0} \varepsilon^j W^j(x) \quad \text{in } \mathcal{U}_\varepsilon,$$

where the terms U^j and W^j are defined in ε -independent domains Ω_i and $\Omega_e := \Omega \setminus \Omega_i$. We emphasize that the terms W^j are defined in the thin layer ω_ε .

In the thin layer, we denote by $d : \omega_\varepsilon \rightarrow \mathbb{R}^+$ the euclidean distance to Γ , $d(x) = \text{dist}(x, \Gamma)$ and by $P(x)$ the orthogonal projection of x on Γ . We describe the solution v^ε on the following way:

$$(2.11) \quad v^\varepsilon(x) \approx \sum_{j \geq 0} \varepsilon^j \left(W^j(x) + V^j \left(\frac{d(x)}{\varepsilon}, P(x) \right) \right) \quad \text{in } \omega_\varepsilon,$$

where the profiles V^j are defined on $[0, 1] \times \Gamma$:

$$\begin{aligned} V^j : [0, 1] \times \Gamma &\rightarrow \mathbb{C} \\ (z, \sigma) &\mapsto V^j(z, \sigma). \end{aligned}$$

In ω_ε , we will work with the variables (z, σ) .

2.2. Geometrical and analytical tools for the thin layer. We use the following notations (see also [6]) : $x = (x_1, x_2, x_3)$ denotes the cartesian coordinates in \mathbb{R}^3 , and we set

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \partial_z = \frac{\partial}{\partial z}, \quad \partial_{zz} = \frac{\partial^2}{\partial z^2}$$

By simple calculations there holds (see also [5])

$$\begin{aligned} \Delta \left(x \mapsto V(P(x), \frac{d(x)}{\varepsilon}) \right) &= \varepsilon^{-2} \partial_{zz} V(P(x), z) + \varepsilon^{-1} G_{\varepsilon z}(P(x)) \partial_z V(P(x), z) \\ &\quad + \Delta_{\Gamma_{\varepsilon z}} V(P(x), z) \quad \text{with} \quad z = \frac{d(x)}{\varepsilon}, \end{aligned}$$

where G is defined as

$$G_s(\sigma) = \frac{1}{\gamma_s(\sigma)} \frac{\partial \gamma_s}{\partial s}(\sigma).$$

Here $\gamma_s(\sigma) = \det(\text{Id} + s \text{dn}(\sigma))$ and for $\sigma \in \Gamma$, the differential $\text{dn}(\sigma)$ is a linear map from the tangent plane $T_\sigma \Gamma$ of Γ at the point σ into $T_{n(\sigma)} S^2$ where S^2 is the unit sphere.

Proposition 2.1.

$$\frac{1}{2} \leq \gamma_s(\sigma) \leq \frac{3}{2}$$

In addition, since for $x \in \omega_\varepsilon$, we have: $x = P(x) + \varepsilon \frac{d(x)}{\varepsilon} \nu(P(x)) = \sigma + \varepsilon z \nu(\sigma)$, we perform the Taylor expansion of the functions $\varphi : \omega_\varepsilon \rightarrow \mathbb{C}$ writing:

$$\varphi(x) = \varphi(P(x)) + d(x) \frac{\partial \varphi}{\partial \nu}(P(x)) + \dots = \varphi(\sigma) + \varepsilon z \frac{\partial \varphi}{\partial \nu}(\sigma) + \dots$$

Then we insert the *Ansatz* (2.9)-(2.10)-(2.11) in equations (2.1)-(2.8) and we perform the identification of terms with the same power in ε .

2.3. Characterisation of the profiles at order ε^0 . Taking the order ε^0 in (2.1), we obtain that:

$$(2.12) \quad -\Delta U^0 - \xi U^0 = f \quad \text{in} \quad \Omega_i.$$

Taking the order ε^0 in (2.3), we have:

$$(2.13) \quad -\Delta W^0 - \xi W^0 = f \quad \text{in} \quad \Omega_e.$$

From (2.2) at order ε^{-2} , we obtain that:

$$(2.14) \quad -\partial_{zz} V^0 + V^0 + W^0(\sigma) = 0 \quad \text{in} \quad [0, 1] \times \Gamma.$$

By solving this ODE, we obtain that:

$$(2.15) \quad V^0(z, \sigma) = -W^0(\sigma) + \alpha_0(\sigma) \sinh z + \beta_0(\sigma) \cosh z.$$

From (2.6) at order ε^{-1} , we have

$$0 = \partial_z V^0(0, \sigma),$$

so we obtain that $\beta_0(\sigma) = 0$ for all $\sigma \in \Gamma$.

From (2.7) at order ε^{-1} , we obtain that

$$0 = \partial_z V^0(1, \sigma),$$

so, with (2.15), we obtain that

$$\beta_0(\sigma) = 0.$$

Thus, we have:

$$(2.16) \quad V^0(z, \sigma) = -W^0(\sigma).$$

Since W^0 satisfies (2.13), then W^0 and $\frac{\partial W^0}{\partial \nu}$ have no jump at Γ_ε . In particular, writing (2.5), we obtain that

$$V^0(1, \sigma) = 0.$$

Coupling this relation with (2.16), we obtain that

$$(2.17) \quad V^0(z, \sigma) = 0 \text{ for } z \in [0, 1] \text{ and } \sigma \in \Gamma.$$

Therefore, with (2.16),

$$(2.18) \quad W^0(\sigma) = 0 \text{ for all } \sigma \in \Gamma.$$

Writing (2.5) at order ε^0 , we have

$$U^0(\sigma) = V^0(0, \sigma) \text{ for } \sigma \in \Gamma.$$

So, with (2.17),

$$(2.19) \quad U^0(\sigma) = 0 \text{ for all } \sigma \in \Gamma.$$

Equation (2.8) at order ε^0 yields $W^0 = 0$ on $\partial\Omega$.

Therefore, coupling (2.12) and (2.19), U^0 is characterized by:

$$(2.20) \quad \begin{cases} -\Delta U^0 - \xi U^0 = f \text{ in } \Omega_i, \\ U^0 = 0 \text{ in } \partial\Omega_i. \end{cases}$$

In addition, coupling (2.13) and (2.18), W^0 is characterized by:

$$(2.21) \quad \begin{cases} -\Delta W^0 - \xi W^0 = f \text{ in } \Omega_e, \\ W^0 = 0 \text{ in } \partial\Omega_e. \end{cases}$$

2.4. Characterisation of the profiles at order ε^1 . Taking the order ε^1 in (2.1), we obtain that:

$$(2.22) \quad -\Delta U^1 - \xi U^1 = 0 \quad \text{in } \Omega_i.$$

Taking the order ε^1 in (2.3), we have:

$$(2.23) \quad -\Delta W^1 - \xi W^1 = 0 \quad \text{in } \Omega_e.$$

From (2.2) at order ε^{-1} , we obtain that:

$$(2.24) \quad -\partial_{zz} V^1 + V^1 + W^1(\sigma) + z \frac{\partial W^0}{\partial \nu}(\sigma) = 0 \quad \text{in } [0, 1] \times \Gamma.$$

By solving this ODE, we obtain that:

$$(2.25) \quad V^1(z, \sigma) = -W^1(\sigma) - z \frac{\partial W^0}{\partial \nu}(\sigma) + \alpha_1(\sigma) \sinh z + \beta_1(\sigma) \cosh z.$$

From (2.6) at order ε^0 , we have

$$\frac{\partial U^0}{\partial \nu}(\sigma) = \partial_z V^1(0, \sigma)(\sigma) + \frac{\partial W^0}{\partial \nu}(\sigma),$$

so we obtain that

$$(2.26) \quad \alpha_1(\sigma) = \frac{\partial U^0}{\partial \nu}(\sigma) \text{ for all } \sigma \in \Gamma.$$

From (2.7) at order ε^0 , we obtain that

$$0 = \partial_z V^1(1, \sigma) = -\frac{\partial W^0}{\partial \nu}(\sigma) + \alpha_1(\sigma) \cosh 1 + \beta_1(\sigma) \sinh 1,$$

so, with (2.25) and (2.26), we obtain that

$$\beta_1(\sigma) = \frac{1}{\sinh 1} \left(\frac{\partial W^0}{\partial \nu}(\sigma) - (\cosh 1) \frac{\partial U^0}{\partial \nu}(\sigma) \right).$$

Thus, we have:

$$(2.27) \quad V^1(z, \sigma) = -W^1(\sigma) + \frac{\partial W^0}{\partial \nu}(\sigma) \left(-z + \frac{\cosh z}{\sinh 1} \right) + \frac{\partial U^0}{\partial \nu}(\sigma) (\sinh z - \coth 1 \cosh z).$$

Using (2.5) at order ε^1 , we obtain that $V^1(1, \sigma) = 0$, so:

$$(2.28) \quad W^1(\sigma) = (-1 + \coth 1) \frac{\partial W^0}{\partial \nu}(\sigma) - \frac{1}{\sinh 1} \frac{\partial U^0}{\partial \nu}(\sigma)$$

Therefore, we obtain that

$$(2.29) \quad V^1(z, \sigma) = \frac{\partial W^0}{\partial \nu}(\sigma) \left(1 - z + \frac{\cosh z - \cosh 1}{\sinh 1} \right) + \frac{\partial U^0}{\partial \nu}(\sigma) \left(\sinh z - \coth 1 \cosh z + \frac{1}{\sinh 1} \right).$$

Using (2.5) at order ε^1 , we obtain that for $\sigma \in \Gamma$:

$$(2.30) \quad U^1(\sigma) = V^1(0, \sigma) + W^1(\sigma) = \frac{1}{\sinh 1} \frac{\partial W^0}{\partial \nu}(\sigma) - \frac{\cosh 1}{\sinh 1} \frac{\partial U^0}{\partial \nu}(\sigma).$$

Therefore, coupling (2.22) and (2.30), U^1 is characterized by:

$$(2.31) \quad \begin{cases} -\Delta U^1 - \xi U^1 = 0 \text{ in } \Omega_i, \\ U^1(\sigma) = \frac{1}{\sinh 1} \frac{\partial W^0}{\partial \nu}(\sigma) - \coth 1 \frac{\partial U^0}{\partial \nu}(\sigma) \text{ in } \partial\Omega_i. \end{cases}$$

In addition, coupling (2.23), (2.28), and (2.8) at order ε^1 , W^1 is characterized by:

$$(2.32) \quad \begin{cases} -\Delta W^1 - \xi W^1 = f \text{ in } \Omega_e, \\ W^1(\sigma) = (-1 + \coth 1) \frac{\partial W^0}{\partial \nu}(\sigma) - \frac{1}{\sinh 1} \frac{\partial U^0}{\partial \nu}(\sigma) \text{ in } \Gamma, \\ W^1 = 0 \text{ on } \partial\Omega. \end{cases}$$

We denote by \mathbf{H} the space of the functions $f \in L^2(\Omega)$ such that the restrictions of f respectively in Ω_i and Ω_e are respectively in $H^1(\Omega_i)$ and $H^1(\Omega_e)$, endowed with the Hilbert norm:

$$\|f\|_{\mathbf{H}} = \left(\|f|_{\Omega_i}\|_{H^1(\Omega_i)}^2 + \|f|_{\Omega_e}\|_{H^1(\Omega_e)}^2 \right)^{\frac{1}{2}}.$$

2.5. Equation for the remainder. We write:

$$\begin{cases} u^\varepsilon(x) = U^0(x) + \varepsilon U^1(x) + \varepsilon a^\varepsilon(x) & \text{for } x \in \Omega_i, \\ v^\varepsilon(x) = W^0(x) + \varepsilon W^1(x) + \varepsilon V^1(\mathbf{P}(x), \frac{d(x)}{\varepsilon}) + \varepsilon b^\varepsilon(x) & \text{for } x \in \omega_\varepsilon, \\ w^\varepsilon(x) = W^0(x) + \varepsilon W^1(x) + \varepsilon c^\varepsilon(x) & \text{for } x \in \mathcal{U}_\varepsilon. \end{cases}$$

The remainder term satisfies:

$$(2.33) \quad -\Delta a^\varepsilon - \xi a^\varepsilon = 0 \quad \text{in } \Omega_i$$

$$(2.34) \quad -\Delta b^\varepsilon - \xi b^\varepsilon + \frac{1}{\varepsilon^2} b^\varepsilon = d^\varepsilon \quad \text{in } \omega_\varepsilon$$

$$(2.35) \quad -\Delta c^\varepsilon - \xi c^\varepsilon = 0 \quad \text{in } \mathcal{U}_\varepsilon$$

$$(2.36) \quad a^\varepsilon = b^\varepsilon \quad \text{on } \Gamma$$

$$(2.37) \quad b^\varepsilon = c^\varepsilon \quad \text{on } \Gamma_\varepsilon$$

$$(2.38) \quad \partial_n a^\varepsilon = \partial_n b^\varepsilon + g^\varepsilon \quad \text{on } \Gamma$$

$$(2.39) \quad \partial_n b^\varepsilon = \partial_n c^\varepsilon \quad \text{on } \Gamma_\varepsilon$$

$$(2.40) \quad c^\varepsilon = 0 \quad \text{on } \partial\Omega,$$

where d^ε is defined on ω_ε by $d^\varepsilon = d_1^\varepsilon + d_2^\varepsilon + d_3^\varepsilon + d_4^\varepsilon$ with:

$$d_1^\varepsilon(x) = -\frac{1}{\varepsilon^3} \left(W^0(x) - d(x) \frac{\partial W^0}{\partial \nu}(\mathbf{P}(x)) \right),$$

$$d_2^\varepsilon(x) = -\frac{1}{\varepsilon^2} (W^1(x) - W^1(\mathbf{P}(x))),$$

$$d_3^\varepsilon(x) = \frac{1}{\varepsilon} G_{d(x)} \partial_z V^1\left(\frac{d(x)}{\varepsilon}, \mathbf{P}(x)\right) + \xi V^1\left(\frac{d(x)}{\varepsilon}, \mathbf{P}(x)\right),$$

$$d_4^\varepsilon(x) = \Delta_{\Gamma_{d(x)}} V^1\left(\frac{d(x)}{\varepsilon}, \mathbf{P}(x)\right),$$

and where g^ε is defined on Γ by:

$$g^\varepsilon = \frac{\partial W^1}{\partial \nu} - \frac{\partial U^1}{\partial \nu}.$$

2.6. Estimates of the right-hand-side terms. Since ξ is in the circle of center λ^0 and of radius η_0 , there exists a constant C such that for all $f \in \mathbf{H}$,

$$(2.41) \quad \|U^0\|_{H^3(\Omega_i)} + \|W^0\|_{H^3(\Omega_e)} \leq C \|f\|_{\mathbf{H}},$$

and

$$(2.42) \quad \|U^1\|_{H^2(\Omega_i)} + \|W^1\|_{H^2(\Omega_e)} \leq C \|f\|_{\mathbf{H}}.$$

From (2.42), by trace theory, we obtain that:

$$\begin{aligned} \|g^\varepsilon\|_{L^2(\Gamma)} &\leq \left\| \frac{\partial W^1}{\partial \nu} \right\|_{H^{\frac{1}{2}}(\Gamma)} + \left\| \frac{\partial U^1}{\partial \nu} \right\|_{H^{\frac{1}{2}}(\Gamma)}, \\ &\leq K \left(\|\nabla W^1\|_{H^1(\Omega_e)} + \|\nabla U^1\|_{H^1(\Omega_i)} \right), \end{aligned}$$

so,

$$(2.43) \quad \|g^\varepsilon\|_{L^2(\Gamma)} \leq K \|f\|_{\mathbf{H}}.$$

In order to estimate d_1^ε and d_2^ε , we prove the following proposition:

Proposition 2.2. *We denote by N the dimension of the space. We define 2^* by $\frac{1}{2} = \frac{1}{N} + \frac{1}{2^*}$ (for $N = 2$, $2^* = +\infty$). Then for all $p \in [2, 2^*[$ ($[2, 2^*$] for $N \neq 2$), there exists a constant K_p such that for all $\varepsilon > 0$, we have:*

$$\forall v \in H^2(\Omega_e), \left\| \frac{1}{\varepsilon^2} (v(\cdot) - v(\mathbf{P}(\cdot))) \right\|_{L^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|v\|_{H^2(\Omega_e)}$$

and

$$\forall v \in H^3(\Omega_e), \left\| \frac{1}{\varepsilon^3} \left(v(\cdot) - v(\mathbf{P}(\cdot)) - \frac{\partial v}{\partial \nu}(\mathbf{P}(\cdot)) \right) \right\|_{L^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|v\|_{H^3(\Omega_e)}$$

Proof. We assume that v is smooth. For $x \in \omega_\varepsilon$, we have $x = \mathbf{P}(x) + d(x)\nu(\mathbf{P}(x))$, so, using the variables $s = d(x)$ and $\sigma = \mathbf{P}(x)$ and Taylor's formula:

$$\begin{aligned} \|v(\cdot) - v(\mathbf{P}(\cdot))\|_{L^2(\omega_\varepsilon)}^2 &= \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) |v(\sigma + s\nu(\sigma)) - v(\sigma)|^2 d\sigma ds, \\ &= \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) \left| \int_{\tau=0}^s \nabla v(\sigma + \tau\nu(\sigma)) \cdot \nu(\sigma) d\tau \right|^2 ds d\sigma, \\ &\leq \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) \left(\int_0^s |\nabla v(\sigma + \tau\nu(\sigma))|^2 d\tau \right) s ds d\sigma, \\ &\leq \varepsilon^2 3 \int_{\omega_\varepsilon} |\nabla v|^2 dx \quad (\text{using Proposition 2.1}). \end{aligned}$$

By Sobolev embedding, for all $p \in [2, 2^*[$ ($[2, 2^*$] for $N \neq 2$),

$$\|\nabla v\|_{L^p(\mathcal{U})} \leq C_p \|v\|_{H^2(\mathcal{U})}.$$

So, by Hölder inequality,

$$\begin{aligned} \|\nabla v\|_{L^2(\omega_\varepsilon)} &\leq \|\nabla v\|_{L^p(\omega_\varepsilon)} (\text{meas}(\omega_\varepsilon))^{\frac{1}{2} - \frac{1}{p}}, \\ &\leq (K\varepsilon)^{\frac{1}{2} - \frac{1}{p}} C_p \|v\|_{H^2(\mathcal{U})}. \end{aligned}$$

Therefore,

$$\left\| \frac{1}{\varepsilon^2} (v(\cdot) - v(\mathbf{P}(\cdot))) \right\|_{L^2(\omega_\varepsilon)} \leq \frac{\sqrt{3} K C_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|v\|_{H^2(\mathcal{U})}.$$

In addition, by Taylor expansion at order 2, for $\sigma \in \Gamma$ and $s \in [0, \varepsilon]$, we have:

$$\begin{aligned} \left| v(\sigma + s\nu(\sigma)) - v(\sigma) - \frac{\partial v}{\partial \nu}(\sigma)d(x) \right| &= \left| \int_0^s (s - \tau) D^2 v(\sigma + \tau s)(\nu(\sigma), \nu(\sigma)) d\tau \right|, \\ &\leq s \int_0^s |D^2 v(\sigma + \tau s)| d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| v(x) - v(\mathbf{P}(x)) - \frac{\partial v}{\partial \nu}(\mathbf{P}(x))d(x) \right\|_{L^2(\omega_\varepsilon)}^2 &= \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) \left| v(\sigma + s\nu(\sigma)) - v(\sigma) - \frac{\partial v}{\partial \nu}(\sigma)s \right|^2 ds d\sigma, \\ &\leq \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) \left| s \int_0^s |D^2 v(\sigma + \tau s)| d\tau \right|^2 ds d\sigma, \\ &\leq \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) s^3 \int_0^s |D^2 v(\sigma + \tau s)|^2 d\tau ds d\sigma, \\ &\leq 3\varepsilon^4 \int_{\omega_\varepsilon} |D^2 v(x)|^2 dx. \end{aligned}$$

By Sobolev embedding, for all $p \in [2, 2^*[$ ($[2, 2^*$ for $N \neq 2$),

$$\|D^2 v\|_{L^p(\mathcal{U})} \leq C_p \|v\|_{H^3(\mathcal{U})},$$

so

$$\|D^2 v\|_{L^2(\omega_\varepsilon)} \leq \|D^2 v\|_{L^p(\mathcal{U})} (\text{meas}(\omega_\varepsilon))^{\frac{1}{2} - \frac{1}{p}} \leq C_p \|v\|_{H^3(\mathcal{U})} (K\varepsilon)^{\frac{1}{2} - \frac{1}{p}},$$

thus,

$$\left\| \frac{1}{\varepsilon^3} \left(v(x) - v(\mathbf{P}(x)) - \frac{\partial v}{\partial \nu}(\mathbf{P}(x))d(x) \right) \right\|_{L^2(\omega_\varepsilon)} \leq \frac{\sqrt{3}KC_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|v\|_{H^3(\mathcal{U})}.$$

We take $K_p = \sqrt{3}KC_p$, and we conclude the proof of Proposition 2.2 by a density argument. \square

Corollary 2.3. *For all $p \in [2, 2^*[$ ($[2, +\infty[$ in the case $N = 2$), there exists K_p such that for all $\varepsilon > 0$, for all $f \in \mathbf{H}$,*

$$\|d_1^\varepsilon\|_{L^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|f\|_{\mathbf{H}},$$

and

$$\|d_2^\varepsilon\|_{L^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|f\|_{\mathbf{H}}.$$

From (2.29), we have:

$$\begin{aligned} \|d_3^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 &= \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) \left| \frac{1}{\varepsilon} G_s(\sigma) \partial_z V^1\left(\frac{s}{\varepsilon}, \sigma\right) + \xi V^1\left(\frac{s}{\varepsilon}, \sigma\right) \right|^2 ds d\sigma \\ &\leq \frac{K}{\varepsilon^2} \int_0^\varepsilon \int_\Gamma \left(\left| \frac{\partial W^0}{\partial \nu}(\sigma) \right|^2 + \left| \frac{\partial U^0}{\partial \nu}(\sigma) \right|^2 \right) d\sigma ds, \\ &\leq \frac{K}{\varepsilon} \left(\|W^0\|_{H^3(\Omega_e)}^2 + \|U^0\|_{H^3(\Omega_i)}^2 \right). \end{aligned}$$

Therefore

$$(2.44) \quad \|d_3^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon^{\frac{1}{2}}} \|f\|_{\mathbf{H}}.$$

2.7. Variational estimates for the remainder term. Multiplying (2.33) by the conjugate of a^ε and integrating on Ω_i , multiplying (2.34) by the conjugate of b^ε and integrating on ω_ε , multiplying (2.35) by the conjugate of c^ε and integrating on \mathcal{U}_ε , after integrations by part and after summation, we obtain:

$$\begin{aligned} \|\nabla a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|\nabla c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 + \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 = \\ \int_\Gamma g^\varepsilon \bar{b}^\varepsilon + \int_{\omega_\varepsilon} d^\varepsilon \bar{b}^\varepsilon - \xi \left(\|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \right). \end{aligned}$$

So,

$$\begin{aligned} \|\nabla a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|\nabla c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 + \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 \leq \\ |\xi| \left(\|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \right) \\ (2.45) \quad + \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|d_1^\varepsilon + d_2^\varepsilon + d_3^\varepsilon\|_{L^2(\omega_\varepsilon)} + \|g^\varepsilon\|_{L^2(\Gamma)} \|b^\varepsilon\|_{L^2(\Gamma)} \\ + \left| \int_{\omega_\varepsilon} d_4^\varepsilon \bar{b}^\varepsilon \right|. \end{aligned}$$

We claim the following lemma proved in Section 3.1.

Lemma 2.4. *There exists a constant $C > 0$ independent of ε such that for any $\xi \in C(\lambda^0, \eta)$, for any $u \in H^1(\Omega_i)$ such that $-\Delta u - \xi u \in L^2(\Omega_e)$, we have :*

$$(2.46) \quad \|u\|_{L^2(\Omega_i)} \leq C \left(\|u\|_{L^2(\Gamma)} + \|(-\Delta u - \xi u)\|_{L^2(\Omega_i)} \right),$$

and for any $v \in H^1(\mathcal{U}_\varepsilon)$ such that $v = 0$ on $\partial\Omega$ and such that $-\Delta v - \xi v \in L^2(\mathcal{U}_\varepsilon)$, we have :

$$(2.47) \quad \|v\|_{L^2(\mathcal{U}_\varepsilon)} \leq C \left(\|v\|_{L^2(\Gamma_\varepsilon)} + \|(-\Delta v - \xi v)\|_{L^2(\mathcal{U}_\varepsilon)} \right).$$

We claim the following Lemma, proved in [4] (see Inequality (2.1) in Proposition 2.1., page 822).

Lemma 2.5. *There exists constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, for all $\mathbf{q} \in \mathbf{H}^1(\omega_\varepsilon)$,*

$$\|\mathbf{q}\|_{L^2(\partial\omega_\varepsilon)}^2 \leq C \left(\frac{1}{\varepsilon} \|\mathbf{q}\|_{L^2(\omega_\varepsilon)}^2 + \|\mathbf{q}\|_{L^2(\omega_\varepsilon)} \|\nabla \mathbf{q}\|_{L^2(\omega_\varepsilon)} \right)$$

Since a^ε (resp. c^ε) satisfies (2.33) in Ω_i (resp. (2.35) in \mathcal{U}_ε), we obtain with Lemma 2.4 and the transmission condition (2.36) (resp. (2.37)) that there exists C independent of $\varepsilon > 0$ and $\xi \in \mathcal{C}(\lambda^0, \eta)$ such that:

$$\|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \leq C \|b^\varepsilon\|_{L^2(\partial\omega_\varepsilon)}^2.$$

Using now Lemma 2.5, we obtain that:

$$(2.48) \quad \|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \leq \frac{C}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + C \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}.$$

Using Corollary 2.3, Estimate (2.44), we obtain that there exists a constant C such that for all $\varepsilon > 0$, for all $\xi \in \mathcal{C}(\lambda^0, \eta)$ and for all $f \in \mathbf{H}$,

$$(2.49) \quad \|d_1^\varepsilon + d_2^\varepsilon + d_3^\varepsilon\|_{L^2(\omega_\varepsilon)} \leq \frac{C}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|f\|_{\mathbf{H}}.$$

In addition, coupling Lemma 2.5 and Estimate (2.43), we obtain that:

$$(2.50) \quad \|g^\varepsilon\|_{L^2(\Gamma)} \|b^\varepsilon\|_{L^2(\Gamma)} \leq C \sqrt{\varepsilon} \|f\|_{\mathbf{H}} \frac{1}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} + C \varepsilon^{\frac{1}{2}} \|f\|_{\mathbf{H}} \left(\frac{1}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \right)^{\frac{1}{2}} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^{\frac{1}{2}}.$$

Let us estimate the last right-hand-side term in (2.45). We denote by $B^\varepsilon = b^\varepsilon \circ \Psi$, where:

$$\begin{aligned} \Psi : [0, \eta_0] \times \Gamma &\rightarrow \omega_{\eta_0} \\ (s, \sigma) &\mapsto \sigma + s\nu(\sigma). \end{aligned}$$

$$\begin{aligned} \int_{\omega_\varepsilon} d_4^\varepsilon \overline{b^\varepsilon} &= \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) (\Delta_{\Gamma_s} V^1(\frac{s}{\varepsilon}, \sigma)) \overline{B^\varepsilon(s, \sigma)} ds d\sigma, \\ &= - \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) \nabla_{\Gamma_s} V^1(\frac{s}{\varepsilon}, \sigma) \overline{\nabla_{\Gamma_s} B^\varepsilon(s, \sigma)} ds, \end{aligned}$$

So,

$$\begin{aligned} \left| \int_{\omega_\varepsilon} d_4^\varepsilon \overline{b^\varepsilon} \right| &\leq \int_{s=0}^\varepsilon \int_{\sigma \in \Gamma} \gamma_s(\sigma) \left| \nabla_{\Gamma_s} V^1(\frac{s}{\varepsilon}, \sigma) \right| \left| \nabla_{\Gamma_s} B^\varepsilon(s, \sigma) \right| ds d\sigma \\ &\leq \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|\nabla_{\Gamma_s} V^1(\frac{s}{\varepsilon}, \sigma)\|_{L^2(\omega_\varepsilon)}, \\ &\leq \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} \left(\left\| \frac{\partial U^0}{\partial \nu} \right\|_{\mathbf{H}^1(\Gamma)} + \left\| \frac{\partial W^0}{\partial \nu} \right\|_{\mathbf{H}^1(\Gamma)} \right) \sqrt{\varepsilon}, \end{aligned}$$

thus, using (2.41), we obtain:

$$(2.51) \quad \left| \int_{\omega_\varepsilon} d_4^\varepsilon \overline{b^\varepsilon} \right| \leq C \sqrt{\varepsilon} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|f\|_{\mathbf{H}}.$$

Therefore, from (2.45), (2.48), (2.49), (2.50), and (2.51), we obtain that there exists a constant C independent of $\varepsilon > 0$, $\xi \in \mathcal{C}(\lambda^0, \eta)$ and $f \in \mathbf{H}$ such that

$$\begin{aligned} & \|\nabla a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|\nabla c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 + \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 \leq \\ & \frac{C}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + C \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} + \frac{C}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|f\|_{\mathbf{H}} \\ & + C\sqrt{\varepsilon} \|f\|_{\mathbf{H}} \frac{1}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)} + C\varepsilon^{\frac{1}{2}} \|f\|_{\mathbf{H}} \left(\frac{1}{\varepsilon} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}\right)^{\frac{1}{2}} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^{\frac{1}{2}} + C\varepsilon^{\frac{1}{2}} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)} \|f\|_{\mathbf{H}} \\ & \leq \frac{1}{2} \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \frac{1}{2} \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + C\varepsilon^{1-\frac{2}{p}} \|f\|_{\mathbf{H}}^2 \quad (\text{by Young inequality}). \end{aligned}$$

So, after absorption, we obtain that there exists C such that:

$$(2.52) \quad \|\nabla a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|\nabla b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 + \|\nabla c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 + \frac{1}{\varepsilon^2} \|b^\varepsilon\|_{L^2(\omega_\varepsilon)}^2 \leq C\varepsilon^{1-\frac{2}{p}} \|f\|_{\mathbf{H}}^2.$$

Reusing (2.48), we obtain also that:

$$(2.53) \quad \|a^\varepsilon\|_{L^2(\Omega_i)}^2 + \|c^\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)}^2 \leq C\varepsilon^{2-\frac{2}{p}} \|f\|_{\mathbf{H}}^2.$$

3. TECHNICAL LEMMA : UNIFORM A PRIORI ESTIMATES

3.1. Geometrical tools in the thin layer. In this part, we refer to [4]. For $\sigma \in \Gamma$, we denote by $\nu(\sigma)$ the outward unitary normal. We remark that $\nu : \Gamma \rightarrow S^{N-1}$ is smooth since Γ is regular. We define Ψ by:

$$\begin{aligned} \Psi :]0, \eta_0[\times \Gamma & \rightarrow \omega_{\eta_0} \\ (s, \sigma) & \mapsto \sigma + s\nu(\sigma). \end{aligned}$$

Since Γ is a smooth compact surface without boundary, we can fix $\eta_0 > 0$ such that Ψ is a C^∞ -diffeomorphism and so that for all $\varepsilon \in]0, \eta_0[$,

$$\Psi(]0, \varepsilon[\times \Gamma) = \omega_\varepsilon.$$

We define $\gamma : [0, \eta_0] \times \Gamma \rightarrow \mathbb{R}$ by:

$$\gamma_s(\sigma) = \det(\text{Id} + s d\nu(\sigma)),$$

and we have:

$$\int_{\omega_{\eta_0}} F(x) dx = \int_{s=0}^{\eta_0} \int_{\sigma \in \Gamma} F(\Psi(s, \sigma)) \gamma_s(\sigma) d\sigma ds.$$

Since $\gamma_0(\sigma) = 1$ for all $\sigma \in \Gamma$, even if it means reducing η_0 , we assume that

$$(3.1) \quad \forall s \in [0, \eta_0[, \forall \sigma \in \Gamma, \frac{1}{2} \leq \gamma_s(\sigma) \leq \frac{3}{2}.$$

We denote ∇_Γ , div_Γ and Δ_Γ the gradient, divergence and Laplacian operators on the sub-manifold Γ .

For $v : \Gamma \rightarrow \mathbb{C}$ and $s \in [0, \eta_0[$, we define $\nabla_{\Gamma_s} v$ by:

$$\nabla_{\Gamma_s} v(\sigma) = (\text{Id} + s d\nu(\sigma))^{-1} \nabla_\Gamma v(\sigma),$$

so that: for $v : \omega_{\eta_0} \rightarrow \mathbb{C}$, denoting $V = v \circ \Psi$, we have:

$$\nabla v(x) = \frac{\partial V}{\partial s}(d(x), P(x))\nu(P(x)) + \left(\nabla_{\Gamma_{d(x)}} V \right) (d(x), P(x)).$$

We define the operator div_{Γ_s} as the adjoint of ∇_{Γ_s} for the $L^2(\Gamma)$ -inner product: for $Y : \Gamma \rightarrow T_\Gamma$ a tangent vector field defined on Γ , we set:

$$(\text{div}_{\Gamma_s} Y)(\sigma) = \frac{1}{\gamma_s(\sigma)} \text{div}_\Gamma (\gamma_s(\text{Id} + s d\nu)^{-1} Y)(\sigma),$$

so that if $z : \omega_{\eta_0} \rightarrow \mathbb{R}^N$ is a vector field, denoting $Z = z \circ \Psi$, $Z_N(s, \sigma) = (Z(s, \sigma) \cdot \nu(\sigma))$ and $Z_T(s, \sigma) = Z(s, \sigma) - Z_N(s, \sigma)\nu(\sigma)$, we have:

$$\text{div} z(x) = \frac{\partial Z_N}{\partial s}(d(x), P(x)) + G_{d(x)}(P(x))Z_N(d(x), P(x)) + \left(\text{div}_{\Gamma_{d(x)}} Z_T \right) (d(x), P(x)),$$

where

$$G_s(\sigma) = \frac{1}{\gamma_s(\sigma)} \frac{\partial \gamma_s}{\partial s}(\sigma).$$

We define the operator Δ_{Γ_s} by:

$$\Delta_{\Gamma_s} = \text{div}_{\Gamma_s} \circ \nabla_{\Gamma_s},$$

so that for $v : \omega_{\eta_0} \rightarrow \mathbb{C}$, denoting $V = v \circ \Psi$, we have:

$$\Delta v(x) = \frac{\partial^2 V}{\partial s^2}(d(x), P(x)) + G_{d(x)}(P(x)) \frac{\partial V}{\partial s}(d(x), P(x)) + (\Delta_{\Gamma_{d(x)}} V)(d(x), P(x)).$$

3.2. Geometrical tools for \mathcal{U}_ε . Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth non-decreasing function such that:

$$\forall x \leq 0, \chi(x) = -1,$$

$$\forall x \geq \eta_0, \chi(x) = 0.$$

For $\varepsilon > 0$ such that $\varepsilon \leq \frac{\eta_0}{4}$, we define ϕ_ε by:

$$\begin{cases} \phi_\varepsilon : \mathcal{U}_\varepsilon \rightarrow \Omega_\varepsilon \\ x \mapsto x + \varepsilon \chi(d(x) - \varepsilon) \nu(P(x)) \end{cases}$$

We remark that if $x \in \mathcal{U}_\varepsilon$ with $d(x) \geq \varepsilon + \eta_0$, then $\phi_\varepsilon(x) = x$. In addition, ϕ_ε is a \mathcal{C}^∞ -diffeomorphism from \mathcal{U}_ε onto Ω_ε . We denote by $\phi_{\varepsilon,k}$ the k -th coordinate of ϕ_ε .

For $w : \mathcal{U}_\varepsilon \rightarrow \mathbb{C}$, we define $w : \Omega_\varepsilon \rightarrow \mathbb{C}$ by:

$$w = \mathbf{w} \circ \phi_\varepsilon.$$

We have, for all $j \in \{1, \dots, N\}$,

$$\frac{\partial w}{\partial x_j} = \sum_{k=1}^N \frac{\partial \mathbf{w}}{\partial x_k}(\phi_\varepsilon(x)) \frac{\partial \phi_{\varepsilon,k}}{\partial x_j}(x)$$

so

$$\frac{\partial^2 w}{\partial x_j^2} = \sum_{k=1}^N \sum_{l=1}^N \frac{\partial^2 \mathbf{w}}{\partial x_k \partial x_l}(\phi_\varepsilon(x)) \frac{\partial \phi_{\varepsilon,k}}{\partial x_j}(x) \frac{\partial \phi_{\varepsilon,l}}{\partial x_j}(x) + \sum_{k=1}^N \frac{\partial \mathbf{w}}{\partial x_k}(\phi_\varepsilon(x)) \frac{\partial^2 \phi_{\varepsilon,k}}{\partial x_j^2}(x)$$

Therefore,

$$\Delta w(x) = \sum_{k,l} a_{k,l}^\varepsilon(\phi_\varepsilon(x)) \frac{\partial^2 w}{\partial x_k \partial x_l}(\phi_\varepsilon(x)) + \sum_{k=1}^N b_k^\varepsilon(\phi_\varepsilon(x)) \frac{\partial w}{\partial x_k}(\phi_\varepsilon(x)),$$

where $a_{k,l}^\varepsilon : \Omega_e \rightarrow \mathbb{R}$ and $b_k^\varepsilon : \Omega_e \rightarrow \mathbb{R}$ are defined by:

$$\forall x \in \mathcal{U}_\varepsilon, \quad a_{kl}^\varepsilon(\phi_\varepsilon(x)) = \nabla \phi_{\varepsilon,k}(x) \cdot \nabla \phi_{\varepsilon,l}(x) \quad \text{and} \quad b_k^\varepsilon(\phi_\varepsilon(x)) = \Delta \phi_{\varepsilon,k}(x).$$

For $x \in \Omega_e$ such that $d(x) \geq \varepsilon + \eta_0$, $\phi_\varepsilon(x) = x$ so that $a_{kl}^\varepsilon(x) = \delta_{kl}$ and $b_k^\varepsilon(x) = 0$.

For $x \in \Omega_e$ such that $d(x) \leq \varepsilon + \eta_0$, we have: $x = \phi_\varepsilon^{-1}(x) \in \omega_{\eta_0}$ and:

$$\frac{\partial \phi_{\varepsilon,k}}{\partial x_j}(x) = \delta_{jk} + \varepsilon \left(\chi'(d(x) - \varepsilon) \nu_j(\mathbf{P}(x)) \nu_k(\mathbf{P}(x)) + \chi(d(x) - \varepsilon) \nabla \nu_k(\mathbf{P}(x)) \cdot \frac{\partial \mathbf{P}(x)}{\partial x_j} \right).$$

Thus, $a_{k,l}^\varepsilon(x) = \delta_{kl} + \widetilde{a_{k,l}^\varepsilon}(x)$, where $\widetilde{a_{k,l}^\varepsilon}(x)$ is of order ε and tends uniformly to zero when ε tends to zero. In the same way, $b_k^\varepsilon(x)$ is of order ε and tends uniformly to zero when ε tends to zero. We denote, for $w \in H^2(\Omega_e)$:

$$\mathcal{K}^\varepsilon w = \sum_{(k,l), k \neq l} a_{kl}^\varepsilon \frac{\partial^2 w}{\partial x_k \partial x_l} + \sum_{k=1}^N b_k^\varepsilon \frac{\partial w}{\partial x_k},$$

so that

$$\Delta w(x) = (\Delta w + \mathcal{K}^\varepsilon w)(\phi_\varepsilon(x)),$$

and there exists a constant C such that for all $\varepsilon > 0$, for all $w \in H^2(\Omega_e)$, we have:

$$(3.2) \quad \|\mathcal{K}^\varepsilon w\|_{L^2(\Omega_e)} \leq C\varepsilon \|w\|_{H^2(\Omega_e)}.$$

For $\varepsilon \geq 0$, we denote by \mathcal{L}^ε the operator $-\Delta$ defined on $H^2(\mathcal{U}_\varepsilon) \cap H_0^1(\mathcal{U}_\varepsilon)$. This operator is positive and self-adjoint for the $L^2(\mathcal{U}_\varepsilon)$ -scalar product. Its resolvent is compact, so that its spectrum only contains a sequence of real eigenvalues tending to $+\infty$.

Let us prove the following lemma:

Lemma 3.1. *Let $\lambda_0 \in \text{Spec}(\mathcal{H}^0)$ and $\eta > 0$ such that $\text{Spec}(\mathcal{H}^0) \cap \overline{B(\lambda_0, \eta)} = \{\lambda_0\}$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$,*

$$\text{Spec}(\mathcal{L}^\varepsilon) \cap \mathcal{C}(\lambda_0, \eta) = \emptyset.$$

Proof. If it is not the case, there exists a sequence of positive numbers $(\varepsilon_n)_n$ tending to zero when n tends to $+\infty$, there exists a sequence $(\xi_n)_n$ of points of $\mathcal{C}(\lambda_0, \eta)$, and there exists a sequence $(u_n)_n$ such that for all n , $u_n \in H^2(\mathcal{U}_{\varepsilon_n}) \cap H_0^1(\mathcal{U}_{\varepsilon_n})$ is an eigenvector of \mathcal{L}^ε associated to the eigenvalue ξ_n .

We denote $u_n = u_n \circ (\phi_{\varepsilon_n})^{-1}$, so that we have:

$$(3.3) \quad u_n \in H^2(\Omega_e) \cap H_0^1(\Omega_e) \quad \text{and} \quad -\Delta u_n - \mathcal{K}^{\varepsilon_n} u_n = \xi_n u_n.$$

Since $u_n \neq 0$, we can assume that $\|u_n\|_{H^2(\Omega_e)} = 1$. Thus we can extract subsequences still denoted (ε_n) , (ξ_n) and (u_n) such that ξ_n tends to $\xi_\infty \in \mathcal{C}(\lambda_0, \eta)$ and u_n tends to u_∞ weakly in $H^2(\Omega_e)$ and strongly in $H^1(\Omega_e)$ by compactness of the embedding $H^2(\Omega_e) \subset H^1(\Omega_e)$.

By taking the weak limit in (3.3) and using (3.2), we obtain that

$$(3.4) \quad u_\infty \in H^2(\Omega_e) \cap H_0^1(\Omega_e) \quad \text{and} \quad -\Delta u_\infty = \xi_\infty u_\infty.$$

In addition, writing that $-\Delta u_n = \xi_n u_n + \mathcal{K}^{\varepsilon_n} u_n$, using again (3.2), we obtain that $-\Delta u_n$ tends strongly in $L^2(\Omega_e)$ to $-\Delta u_\infty$, so that u_n tends strongly to u_∞ in $H^2(\Omega_e)$. So, $\|u_\infty\|_{H^2(\Omega_e)} = 1$, thus $u_\infty \neq 0$ and by (3.4), we obtain that $\xi_\infty \in \text{Spec}(\mathcal{L}^0) \cap \mathcal{C}(\lambda_0, \eta)$, which is false. \square

With the previous lemma, for $\varepsilon > 0$ small enough, for all $\xi \in \mathcal{C}(\lambda_0, \eta)$, $\mathcal{L}^\varepsilon - \xi\mathbb{I}$ is invertible. We give a uniform estimate of the resolvent operator in the following lemma:

Lemma 3.2. *There exists $\varepsilon_0 > 0$ and C such that for all $\xi \in \mathcal{C}(\lambda_0, \eta)$, for all $\varepsilon \in]0, \varepsilon_0[$, for all $f \in L^2(\mathcal{U}_\varepsilon)$, there exists a unique $u \in H^2(\mathcal{U}_\varepsilon) \cap H_0^1(\mathcal{U}_\varepsilon)$ such that $-\Delta u - \xi u = f$ in \mathcal{U}_ε . Moreover,*

$$\|u\|_{H^2(\mathcal{U}_\varepsilon)} \leq C \|f\|_{L^2(\mathcal{U}_\varepsilon)}.$$

Proof. If it is not the case, there exists a sequence (ε_n) tending to 0, a sequence $(\xi_n)_n$ of points of $\mathcal{C}(\lambda_0, \eta)$, a sequence (f_n) with $f_n \in L^2(\mathcal{U}_{\varepsilon_n})$ and $u_n \in H^2(\mathcal{U}_{\varepsilon_n}) \cap H_0^1(\mathcal{U}_{\varepsilon_n})$ such that $-\Delta u_n - \xi_n u_n = f_n$ in $\mathcal{U}_{\varepsilon_n}$ and:

$$\|u_n\|_{H^2(\mathcal{U}_{\varepsilon_n})} > n \|f_n\|_{L^2(\mathcal{U}_{\varepsilon_n})}.$$

We denote $\mathbf{u}_n = u_n \circ (\phi_{\varepsilon_n})^{-1}$ and $\mathbf{f}_n = f_n \circ (\phi_{\varepsilon_n})^{-1}$. We have:

$$-\Delta \mathbf{u}_n - \mathcal{K}^{\varepsilon_n} \mathbf{u}_n - \xi_n \mathbf{u}_n = \mathbf{f}_n \text{ in } \Omega_e,$$

and:

$$\|\mathbf{u}_n\|_{H^2(\Omega_e)} > C n \|\mathbf{f}_n\|_{L^2(\Omega_e)}.$$

Since $\mathbf{u}_n \neq 0$, we can divide by its $H^2(\Omega_e)$ -norm. We denote

$$\tilde{\mathbf{u}}_n = \frac{\mathbf{u}_n}{\|\mathbf{u}_n\|_{H^2(\Omega_e)}} \quad \text{and} \quad \tilde{\mathbf{f}}_n = \frac{\mathbf{f}_n}{\|\mathbf{u}_n\|_{H^2(\Omega_e)}}.$$

We have, for all n :

$$\|\tilde{\mathbf{u}}_n\|_{H^2(\Omega_e)} = 1 > n \|\tilde{\mathbf{f}}_n\|_{L^2(\Omega_e)}$$

and

$$(3.5) \quad -\Delta \tilde{\mathbf{u}}_n - \mathcal{K}^{\varepsilon_n} \tilde{\mathbf{u}}_n - \xi_n \tilde{\mathbf{u}}_n = \tilde{\mathbf{f}}_n \text{ in } \Omega_e.$$

By extracting a sub sequence, we can assume that ξ_n tends to $\xi_\infty \in \mathcal{C}(\lambda_0, \eta)$, and $\tilde{\mathbf{u}}_n$ tends to a limit $\tilde{\mathbf{u}}_\infty$ weakly in $H^2(\Omega_e)$ and strongly in $H^1(\Omega_e)$. In addition, we remark that $\tilde{\mathbf{f}}_n$ tends to zero strongly in $L^2(\Omega_e)$. Taking the weak limit in (3.5), we obtain that:

$$(3.6) \quad \tilde{\mathbf{u}}_\infty \in H^2(\Omega_e) \cap H_0^1(\Omega_e) \quad \text{and} \quad -\Delta \tilde{\mathbf{u}}_\infty - \xi_\infty \tilde{\mathbf{u}}_\infty = 0 \text{ in } \Omega_e.$$

In addition, using (3.2) and (3.5), we obtain that $\Delta \tilde{\mathbf{u}}_n$ converges strongly in $L^2(\Omega_e)$. Since its limit is the weak limit $\Delta \tilde{\mathbf{u}}_\infty$, then $\tilde{\mathbf{u}}_n$ converges strongly in $H^2(\Omega_e)$ to $\tilde{\mathbf{u}}_\infty$. So $\|\tilde{\mathbf{u}}_\infty\|_{H^2(\Omega_e)} = 1$, and $\tilde{\mathbf{u}}_\infty \neq 0$. Therefore, with (3.6), ξ_∞ is in $\text{Spec}(\mathcal{L}^0)$, which leads to a contradiction. \square

Proof of Lemma 2.4

Inequality (2.46) is proved by duality arguments in [3] (see Lemma 2.1, on page 95). We use the same method and uniform estimates proved in Lemma 3.2 to establish (2.47). Let ε_0 given by Lemma 3.2. Let $\varepsilon \in (0, \varepsilon_0)$ and $\xi \in \mathcal{C}(\lambda_0, \eta)$. We fix $v \in H^1(\mathcal{U}_\varepsilon)$ such that $-\Delta v - \xi v$ belongs to $L^2(\mathcal{U}_\varepsilon)$. Let $f \in L^2(\mathcal{U}_\varepsilon)$. By Lemma 3.2, there exists $w \in H^2(\mathcal{U}_\varepsilon) \cap H_0^1(\mathcal{U}_\varepsilon)$ such that $-\Delta w - \xi w = f$ in \mathcal{U}_ε . Moreover,

$$\|w\|_{H^2(\mathcal{U}_\varepsilon)} \leq C \|f\|_{L^2(\mathcal{U}_\varepsilon)},$$

where C does not depend on ε , f or ξ . We have:

$$\int_{\mathcal{U}_\varepsilon} v f = \int_{\mathcal{U}_\varepsilon} v(-\Delta w - \xi w) = \int_{\mathcal{U}_\varepsilon} (-\Delta v - \xi v)w + \int_{\Gamma_\varepsilon} v \frac{\partial w}{\partial \nu}.$$

Therefore:

$$\begin{aligned} \left| \int_{\mathcal{U}_\varepsilon} v f \right| &\leq \| -\Delta v - \xi v \|_{L^2(\mathcal{U}_\varepsilon)} \|w\|_{L^2(\mathcal{U}_\varepsilon)} + \|v\|_{L^2(\Gamma_\varepsilon)} \left\| \frac{\partial w}{\partial \nu} \right\|_{L^2(\Gamma_\varepsilon)} \\ &\leq C (\| -\Delta v - \xi v \|_{L^2(\mathcal{U}_\varepsilon)} + \|v\|_{L^2(\Gamma_\varepsilon)}) \|f\|_{L^2(\mathcal{U}_\varepsilon)} \end{aligned}$$

This is true for all $f \in L^2(\mathcal{U}_\varepsilon)$ so:

$$\|v\|_{L^2(\mathcal{U}_\varepsilon)} \leq C (\| -\Delta v - \xi v \|_{L^2(\mathcal{U}_\varepsilon)} + \|v\|_{L^2(\Gamma_\varepsilon)}).$$

□

REFERENCES

- [1] A. Bamberger and A. S. Bonnet-Ben Dhia. Mathematical analysis of the guided modes of an optical fiber. *SIAM Journal on Mathematical Analysis*, 21(6):1487–1510, 1990.
- [2] A. S. Bonnet-Ben Dhia and Joly P. Théorie spectrale des opérateurs autoadjoints et application à l'étude des ondes guidées. 1999.
- [3] V. Bruneau and G. Carbou. Spectral asymptotic in the large coupling limit. *Asymptotic Analysis*, 29(2):91–113, 2002.
- [4] G. Carbou. Penalization method for viscous incompressible flow around a porous thin layer. *Nonlinear Anal. Real World Appl.*, 5(5):815–855, 2004.
- [5] G. Carbou. Brinkmann model and double penalization method for the flow around a porous thin layer. *J. Math. Fluid Mech.*, 10(1):126–158, 2008.
- [6] G. Carbou and P. Fabrie. Boundary layer for a penalization method for viscous incompressible flow. *Adv. Differential Equations*, 8(12):1453–1480, 2003.
- [7] P. Joly and C. Poirier. Mathematical analysis of electromagnetic open waveguides. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, 29(5):505–575, 1995.