Resolvent and Spectral Asymptotics for Thin-Layer Transmission Problems in the Large Coupling Limit

Gilles Carbou, Victor Péron

RESOLVENT AND SPECTRAL ASYMPTOTICS FOR THIN-LAYER TRANSMISSION PROBLEMS IN THE LARGE COUPLING LIMIT

VICTOR PÉRON[1], GILLES CARBOU[1]

ABSTRACT. This work is concerned with a singular perturbation problem that involves a thin layer in the large coupling limit. Such a problem appears in the study of electromagnetic wave guides that contain supra-conductor bodies. We derive resolvent asymptotics for this problem by using boundary layer tools with uniform *a priori* estimates. This result will be useful to derive spectral asymptotics for this problem. This issue is a work in progress.

1. INTRODUCTION

This work is motivated by the study of the asymptotic behavior of eigenvalues related to a singular perturbation problem that involves a thin layer in the large coupling limit. Such a problem appears in the study of electromagnetic wave guides with supra-conductors. We refer the reader to several works that concern mathematical studies with electromagnetic wave guides, see for instance [1, 7, 2, 3].

An asymptotic expansion of the eigenvalues as the conductivity tends to infinity has been derived in [3]. However this result does not apply when the subdomain that represents a supra-conductor is a vanishing sheet. In this work we tackle this problem by considering vanishing supra-conductive layer of thickness ε where the conductivity is scaled like ε^{-2}. This is why there is a main difference between this work and the work in Ref. [3] all the more since we consider a transmission problem across a thin layer.

Let us describe now the mathematical model. Throughout the paper we denote by $\Omega \subset \mathbb{R}^3$ a simply connected domain. Let Ω_i be a simply connected subdomain of Ω and let ω_ε be a thin layer of constant thickness ε surrounding Ω_i such that Ω contains $\Omega_i \cup \overline{\omega_\varepsilon}$, see Figure 1. For the sake of simplicity we further assume that the boundaries $\partial\Omega$ and $\Gamma = \partial\Omega_i$ are connected. Now the exterior domain is defined as $\mathcal{U}_\varepsilon = \Omega \setminus (\Omega_i \cup \overline{\omega_\varepsilon})$ and the boundary of the subdomain $\overline{\Omega_i} \cup \omega_\varepsilon$ is denoted by Γ_ε. We denote by n the outward unit normal at Γ.

In all that follows, unless specified, all the considered domains are smooth domains in $\mathbb{R}^3$.

Notation 1.1. *For any set $\mathcal{O} \subset \mathbb{R}^3$ we denote by $\mathrm{L}^2(\mathcal{O})$ and $\mathrm{H}^s(\mathcal{O})$ the classical Sobolev spaces.*

Let $\varepsilon > 0$. We consider the self-adjoint operator $\mathcal{H}^\varepsilon$ on $\mathrm{L}^2(\Omega)$ with domain $\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ and defined by

$$\mathcal{H}^\varepsilon = -\Delta + \frac{1}{\varepsilon^2} 1_{|\omega_\varepsilon} .$$

The limit operator is $\mathcal{H}^0 = -\Delta$ defined on $\mathrm{H}^2(\Omega_i) \cap \mathrm{H}^1_0(\Omega_i) \times \mathrm{H}^2(\Omega_e) \cap \mathrm{H}^1_0(\Omega_e)$ where Ω_e denotes the limit domain $\mathcal{U}_\varepsilon$ when $\varepsilon \to 0$, *i.e.* $\Omega_e = \Omega \setminus \overline{\Omega_i}$ (see Figure 2).

Date: November 18, 2022, Version 1.

[1] Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, Laboratoire de Mathématiques et de leurs Applications de Pau, Avenue de l'Université - BP 1155, 64013 PAU CEDEX, France.

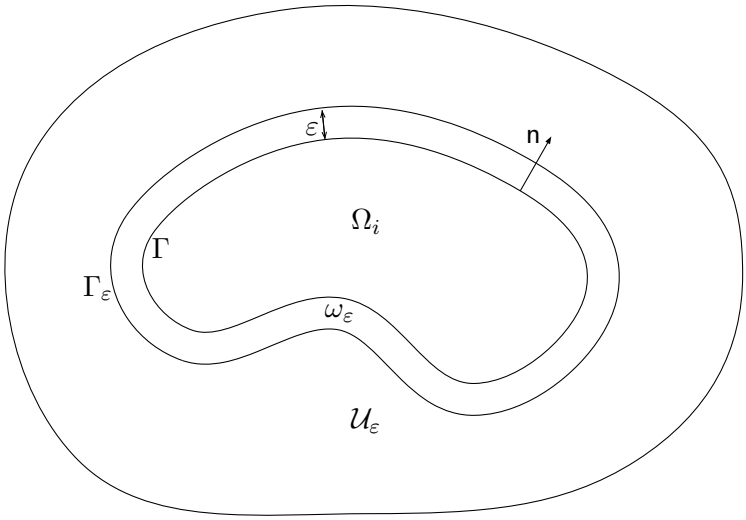

Figure 1 – A cross-section of the domain Ω and the subdomains Ω_i, ω_ε, $\mathcal{U}_\varepsilon$

In this work we derive an asymptotic expansion of the resolvent of $\mathcal{H}^\varepsilon$ when ε tends to 0. We denote by $\mathcal{R}^\varepsilon(\xi) := (\mathcal{H}^\varepsilon - \xi\mathcal{I})^{-1}$ the resolvent of $\mathcal{H}^\varepsilon$, defined on $\mathrm{L}^2(\Omega)$ with values in $\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ for $\xi \notin \mathrm{spec}\ \mathcal{H}^\varepsilon$.

Let $f \in \mathrm{L}^2(\Omega)$. The spectral problem writes

$$-\Delta u - \xi u + \frac{1}{\varepsilon^2} 1_{|\omega_\varepsilon} u = f \quad \text{in} \quad \Omega\,. \tag{1.1}$$

This problem is well-posed and has a unique solution u_ε in $\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ when ξ is not an eigenvalue of the operator $\mathcal{H}^\varepsilon$.

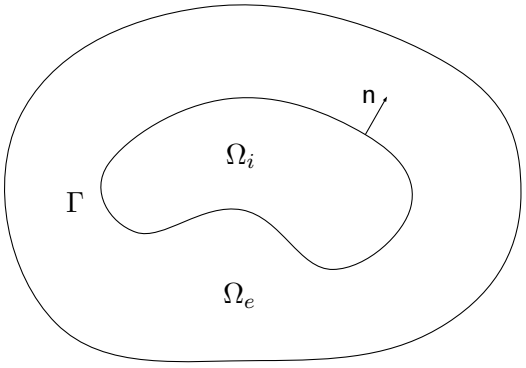

Figure 2 – A cross-section of the domain Ω and the subdomains Ω_i, Ω_e

Notation 1.2. *We denote by f_i (resp. f_e) the restriction of any function f to Ω_i (resp. Ω_e).*

2. Asymptotic expansion for the resolvent

Let $\varepsilon > 0$. We consider the self-adjoint operator $\mathcal{H}^\varepsilon$ on $\mathrm{L}^2(\Omega)$ with domain $\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ and defined by

$$\mathcal{H}^\varepsilon = -\Delta + \frac{1}{\varepsilon^2} 1_{|\omega_\varepsilon} .$$

For $\xi \notin \operatorname{spec} \mathcal{H}^\varepsilon$, we denote by $\mathcal{R}^\varepsilon(\xi) = (\mathcal{H}^\varepsilon - \xi \mathrm{Id})^{-1}$ the resolvent operator. For $f \in \mathrm{L}^2(\Omega)$, we denote by u^ε the restriction of $\mathcal{R}^\varepsilon(\xi)(f)$ to Ω_i, v^ε the restriction of $\mathcal{R}^\varepsilon(\xi)(f)$ to ω_ε, and w^ε the restriction of $\mathcal{R}^\varepsilon(\xi)(f)$ to $\mathcal{U}_\varepsilon$, By the resolvent equation, we have:

$$-\Delta u^\varepsilon - \xi u^\varepsilon = f_{|\Omega_i} \quad \text{in} \quad \Omega_i \tag{2.1}$$

$$-\Delta v^\varepsilon - \xi v^\varepsilon + \frac{1}{\varepsilon^2} v^\varepsilon = f_{|\omega_\varepsilon} \quad \text{in} \quad \omega_\varepsilon \tag{2.2}$$

$$-\Delta w^\varepsilon - \xi w^\varepsilon = f_{|\mathcal{U}^\varepsilon} \quad \text{in} \quad \mathcal{U}_\varepsilon \tag{2.3}$$

$$u^\varepsilon = v^\varepsilon \quad \text{on} \quad \Gamma \tag{2.4}$$

$$v^\varepsilon = w^\varepsilon \quad \text{on} \quad \Gamma_\varepsilon \tag{2.5}$$

$$\partial_{\mathsf{n}} u^\varepsilon = \partial_{\mathsf{n}} v^\varepsilon \quad \text{on} \quad \Gamma \tag{2.6}$$

$$\partial_{\mathsf{n}} v^\varepsilon = \partial_{\mathsf{n}} w^\varepsilon \quad \text{on} \quad \Gamma_\varepsilon \tag{2.7}$$

$$w^\varepsilon = 0 \quad \text{on} \quad \partial\Omega. \tag{2.8}$$

2.1. **Ansatz.**

Outside the thin layer, we look for an expansion on the form:

$$u^\varepsilon(x) \approx \sum_{j \geqslant 0} \varepsilon^j U^j(x) \quad \text{in} \quad \Omega_i \,, \tag{2.9}$$

$$w^\varepsilon(x) \approx \sum_{j \geqslant 0} \varepsilon^j W^j(x) \quad \text{in} \quad \mathcal{U}_\varepsilon \,, \tag{2.10}$$

where the terms U^j and W^j are defined in ε-independent domains Ω_i and $\Omega_e := \Omega \setminus \Omega_i$. We emphasize that the terms W^j are defined in the thin layer ω_ε.

In the thin layer, we denote by $d : \omega_\varepsilon \to \mathbb{R}^+$ the euclidean distance to Γ, $\mathrm{d}(x) = \operatorname{dist}(x, \Gamma)$ and by $\mathsf{P}(x)$ the orthogonal projection of x on Γ. We describe the solution v^ε on the following way:

$$v^\varepsilon(x) \approx \sum_{j \geqslant 0} \varepsilon^j \left(W^j(x) + V^j \left(\frac{d(x)}{\varepsilon}, \mathsf{P}(x) \right) \right) \quad \text{in} \quad \omega_\varepsilon \,, \tag{2.11}$$

where the profiles V^j are defined on $[0,1] \times \Gamma$:

$$\begin{array}{rccc} V^j : & [0,1] \times \Gamma & \to & \mathbb{C} \\ & (z, \sigma) & \mapsto & V^j(z, \sigma). \end{array}$$

In ω_ε, we will work with the variables (z, σ).

2.2. **Geometrical and analytical tools for the thin layer.** We use the following notations (see also [6]) : $x = (x_1, x_2, x_3)$ denotes the cartesian coordinates in $\mathbb{R}^3$, and we set

$$\partial_i = \frac{\partial}{\partial x_i}\,, \quad \partial_{\mathsf{z}} = \frac{\partial}{\partial \mathsf{z}}\,, \quad \partial_{\mathsf{zz}} = \frac{\partial^2}{\partial \mathsf{z}^2}$$

By simple calculations there holds (see also [5])

$$\begin{aligned}\Delta\left(x \mapsto V(\mathsf{P}(x), \tfrac{d(x)}{\varepsilon})\right) = \quad & \varepsilon^{-2}\partial_{\mathsf{zz}}V(\mathsf{P}(x),\mathsf{z}) + \varepsilon^{-1}G_{\varepsilon\mathsf{z}}(\mathsf{P}(x))\partial_{\mathsf{z}}V(\mathsf{P}(x),\mathsf{z}) \\ & +\Delta_{\Gamma_{\varepsilon\mathsf{z}}}V(\mathsf{P}(x),\mathsf{z}) \quad \text{with} \quad \mathsf{z} = \tfrac{d(x)}{\varepsilon}\,,\end{aligned}$$

where G is defined as

$$G_s(\sigma) = \frac{1}{\gamma_s(\sigma)}\frac{\partial \gamma_s}{\partial s}(\sigma)\,.$$

Here $\gamma_s(\sigma) = \det\left(\mathrm{Id} + s\mathrm{d}\mathsf{n}(\sigma)\right)$ and for $\sigma \in \Gamma$, the differential $\mathrm{d}\mathsf{n}(\sigma)$ is a linear map from the tangent plane $\mathrm{T}_\sigma\Gamma$ of Γ at the point σ into $\mathrm{T}_{\mathsf{n}(\sigma)}\mathrm{S}^2$ where S^2 is the unit sphere.

Proposition 2.1.

$$\frac{1}{2} \leq \gamma_s(\sigma) \leq \frac{3}{2}$$

In addition, since for $x \in \omega_\varepsilon$, we have: $x = \mathsf{P}(x) + \varepsilon\frac{\mathrm{d}(x)}{\varepsilon}\nu(\mathsf{P}(x)) = \sigma + \varepsilon z\nu(\sigma)$, we perform the Taylor expansion of the functions $\varphi : \omega_\varepsilon \longrightarrow \mathbb{C}$ writing:

$$\varphi(x) = \varphi(\mathsf{P}(x)) + \mathrm{d}(x)\frac{\partial\varphi}{\partial\nu}(\mathsf{P}(x)) + \ldots, = \varphi(\sigma) + \varepsilon z\frac{\partial\varphi}{\partial\nu}(\sigma) + \ldots$$

Then we insert the *Ansatz* (2.9)-(2.10)-(2.11) in equations (2.1)-(2.8) and we perform the identification of terms with the same power in ε.

2.3. **Characterisation of the profiles at order ε^0.** Taking the order ε^0 in (2.1), we obtain that:

$$-\Delta U^0 - \xi U^0 = f \quad \text{in} \quad \Omega_i\,. \tag{2.12}$$

Taking the order ε^0 in (2.3), we have:

$$-\Delta W^0 - \xi W^0 = f \quad \text{in} \quad \Omega_e\,. \tag{2.13}$$

From (2.2) at order ε^{-2}, we obtain that:

$$-\partial_{\mathsf{zz}}V^0 + V^0 + W^0(\sigma) = 0 \quad \text{in} \quad [0,1]\times\Gamma\,. \tag{2.14}$$

By solving this ODE, we obtain that:

$$V^0(z,\sigma) = -W^0(\sigma) + \alpha_0(\sigma)\sinh z + \beta_0(\sigma)\cosh z. \tag{2.15}$$

From (2.6) at order ε^{-1}, we have

$$0 = \partial_{\mathsf{z}}V^0(0,\sigma),$$

so we obtain that $\beta_0(\sigma) = 0$ for all $\sigma \in \Gamma$.

From (2.7) at order ε^{-1}, we obtain that

$$0 = \partial_{\mathsf{z}}V^0(1,\sigma),$$

so, with (2.15), we obtain that

$$\beta_0(\sigma) = 0.$$

Thus, we have:

$$V^0(z,\sigma) = -W^0(\sigma). \tag{2.16}$$

Since W^0 satisfies (2.13), then W^0 and $\frac{\partial W^0}{\partial \nu}$ have no jump at Γ_ε. In particular, writing (2.5), we obtain that

$$V^0(1,\sigma) = 0.$$

Coupling this relation with (2.16), we obtain that

$$V^0(z,\sigma) = 0 \text{ for } z \in [0,1] \text{ and } \sigma \in \Gamma. \tag{2.17}$$

Therefore, with (2.16),

$$W^0(\sigma) = 0 \text{ for all } \sigma \in \Gamma. \tag{2.18}$$

Writing (2.5) at order ε^0, we have

$$U^0(\sigma) = V^0(0,\sigma) \text{ for } \sigma \in \Gamma.$$

So, with (2.17),

$$U^0(\sigma) = 0 \text{ for all } \sigma \in \Gamma. \tag{2.19}$$

Equation (2.8) at order ε^0 yields $W^0 = 0$ on $\partial\Omega$.

Therefore, coupling (2.12) and (2.19), U^0 is characterized by:

$$\begin{cases} -\Delta U^0 - \xi U^0 = f \text{ in } \Omega_i, \\ \\ U^0 = 0 \text{ in } \partial\Omega_i. \end{cases} \tag{2.20}$$

In addition, coupling (2.13) and (2.18), W^0 is characterized by:

$$\begin{cases} -\Delta W^0 - \xi W^0 = f \text{ in } \Omega_e, \\ \\ W^0 = 0 \text{ in } \partial\Omega_e. \end{cases} \tag{2.21}$$

2.4. **Characterisation of the profiles at order ε^1.** Taking the order ε^1 in (2.1), we obtain that:

$$-\Delta U^1 - \xi U^1 = 0 \quad \text{in} \quad \Omega_i\ . \tag{2.22}$$

Taking the order ε^1 in (2.3), we have:

$$-\Delta W^1 - \xi W^1 = 0 \quad \text{in} \quad \Omega_e\ . \tag{2.23}$$

From (2.2) at order ε^{-1}, we obtain that:

$$-\partial_{\mathsf{zz}} V^1 + V^1 + W^1(\sigma) + \mathsf{z}\frac{\partial W^0}{\partial \nu}(\sigma) = 0 \quad \text{in} \quad [0,1]\times\Gamma\ . \tag{2.24}$$

By solving this ODE, we obtain that:

$$V^1(\mathsf{z},\sigma) = -W^1(\sigma) - \mathsf{z}\frac{\partial W^0}{\partial \nu}(\sigma) + \alpha_1(\sigma)\sinh\mathsf{z} + \beta_1(\sigma)\cosh\mathsf{z}. \tag{2.25}$$

From (2.6) at order ε^0, we have

$$\frac{\partial U^0}{\partial \nu}(\sigma) = \partial_{\mathsf{z}} V^1(0,\sigma)(\sigma) + \frac{\partial W^0}{\partial \nu}(\sigma),$$

so we obtain that

$$\alpha_1(\sigma) = \frac{\partial U^0}{\partial \nu}(\sigma) \text{ for all } \sigma \in \Gamma. \tag{2.26}$$

From (2.7) at order ε^0, we obtain that

$$0 = \partial_{\mathsf{z}} V^1(1,\sigma) = -\frac{\partial W^0}{\partial \nu}(\sigma) + \alpha_1(\sigma)\cosh 1 + \beta_1(\sigma)\sinh 1,$$

so, with (2.25) and (2.26), we obtain that

$$\beta_1(\sigma) = \frac{1}{\sinh 1}\left(\frac{\partial W^0}{\partial \nu}(\sigma) - (\cosh 1)\frac{\partial U^0}{\partial \nu}(\sigma)\right).$$

Thus, we have:

$$V^1(z,\sigma) = -W^1(\sigma) + \frac{\partial W^0}{\partial \nu}(\sigma)\left(-\mathsf{z} + \frac{\cosh \mathsf{z}}{\sinh 1}\right) + \frac{\partial U^0}{\partial \nu}(\sigma)\left(\sinh \mathsf{z} - \coth 1 \cosh \mathsf{z}\right). \tag{2.27}$$

Using (2.5) at order ε^1, we obtain that $V^1(1,\sigma) = 0$, so:

$$W^1(\sigma) = (-1 + \coth 1)\frac{\partial W^0}{\partial \nu}(\sigma) - \frac{1}{\sinh 1}\frac{\partial U^0}{\partial \nu}(\sigma) \tag{2.28}$$

Therefore, we obtain that

$$V^1(\mathsf{z},\sigma) = \frac{\partial W^0}{\partial \nu}(\sigma)\left(1 - \mathsf{z} + \frac{\cosh \mathsf{z} - \cosh 1}{\sinh 1}\right) + \frac{\partial U^0}{\partial \nu}(\sigma)\left(\sinh \mathsf{z} - \coth 1 \cosh \mathsf{z} + \frac{1}{\sinh 1}\right). \tag{2.29}$$

Using (2.5) at order ε^1, we obtain that for $\sigma \in \Gamma$:

$$U^1(\sigma) = V^1(0,\sigma) + W^1(\sigma) = \frac{1}{\sinh 1}\frac{\partial W^0}{\partial \nu}(\sigma) - \frac{\cosh 1}{\sinh 1}\frac{\partial U^0}{\partial \nu}(\sigma). \tag{2.30}$$

Therefore, coupling (2.22) and (2.30), U^1 is characterized by:

$$\begin{cases} -\Delta U^1 - \xi U^1 = 0 \text{ in } \Omega_i, \\ \\ U^1(\sigma) = \dfrac{1}{\sinh 1}\dfrac{\partial W^0}{\partial \nu}(\sigma) - \coth 1 \dfrac{\partial U^0}{\partial \nu}(\sigma) \text{ in } \partial\Omega_i. \end{cases} \tag{2.31}$$

In addition, coupling (2.23), (2.28), and (2.8) at order ε^1, W^1 is characterized by:

$$\begin{cases} -\Delta W^1 - \xi W^1 = f \text{ in } \Omega_e, \\ \\ W^1(\sigma) = (-1 + \coth 1)\dfrac{\partial W^0}{\partial \nu}(\sigma) - \dfrac{1}{\sinh 1}\dfrac{\partial U^0}{\partial \nu}(\sigma) \text{ in } \Gamma, \\ \\ W^1 = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.32}$$

We denote by $\mathbf{H}$ the space of the functions $f \in L^2(\Omega)$ such that the restrictions of f respectively in Ω_i and Ω_e are respectively in $\mathrm{H}^1(\Omega_i)$ and $\mathrm{H}^1(\Omega_e)$, endowed with the Hilbert norm:

$$\|f\|_{\mathbf{H}} = \left(\|f_{|\Omega_i}\|^2_{\mathrm{H}^1(\Omega_i)} + \|f_{|\Omega_e}\|^2_{\mathrm{H}^1(\Omega_e)}\right)^{\frac{1}{2}}.$$

2.5. **Equation for the remainder.** We write:

$$\begin{cases} u^\varepsilon(x) = U^0(x) + \varepsilon U^1(x) + \varepsilon a^\varepsilon(x) & \text{for } x \in \Omega_i, \\ v^\varepsilon(x) = W^0(x) + \varepsilon W^1(x) + \varepsilon V^1(\mathsf{P}(x), \frac{\mathrm{d}(x)}{\varepsilon}) + \varepsilon b^\varepsilon(x) & \text{for } x \in \omega_\varepsilon, \\ w^\varepsilon(x) = W^0(x) + \varepsilon W^1(x) + \varepsilon c^\varepsilon(x) & \text{for } x \in \mathcal{U}_\varepsilon. \end{cases}$$

The remainder term satisfies:

$$-\Delta a^\varepsilon - \xi a^\varepsilon = 0 \quad \text{in} \quad \Omega_i \tag{2.33}$$

$$-\Delta b^\varepsilon - \xi b^\varepsilon + \frac{1}{\varepsilon^2} b^\varepsilon = d^\varepsilon \quad \text{in} \quad \omega_\varepsilon \tag{2.34}$$

$$-\Delta c^\varepsilon - \xi c^\varepsilon = 0 \quad \text{in} \quad \mathcal{U}_\varepsilon \tag{2.35}$$

$$a^\varepsilon = b^\varepsilon \quad \text{on} \quad \Gamma \tag{2.36}$$

$$b^\varepsilon = c^\varepsilon \quad \text{on} \quad \Gamma_\varepsilon \tag{2.37}$$

$$\partial_{\mathsf{n}} a^\varepsilon = \partial_{\mathsf{n}} b^\varepsilon + g^\varepsilon \quad \text{on} \quad \Gamma \tag{2.38}$$

$$\partial_{\mathsf{n}} b^\varepsilon = \partial_{\mathsf{n}} c^\varepsilon \quad \text{on} \quad \Gamma_\varepsilon \tag{2.39}$$

$$c^\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \tag{2.40}$$

where d^ε is defined on ω_ε by $d^\varepsilon = d_1^\varepsilon + d_2^\varepsilon + d_3^\varepsilon + d_4^\varepsilon$ with:

$$d_1^\varepsilon(x) = -\frac{1}{\varepsilon^3}\left(W^0(x) - \mathrm{d}(x)\frac{\partial W^0}{\partial \nu}(\mathsf{P}(x))\right),$$

$$d_2^\varepsilon(x) = -\frac{1}{\varepsilon^2}\left(W^1(x) - W^1(\mathsf{P}(x))\right),$$

$$d_3^\varepsilon(x) = \frac{1}{\varepsilon} G_{\mathrm{d}(x)} \partial_z V^1(\frac{\mathrm{d}(x)}{\varepsilon}, \mathsf{P}(x)) + \xi V^1(\frac{\mathrm{d}(x)}{\varepsilon}, \mathsf{P}(x)),$$

$$d_4^\varepsilon(x) = \Delta_{\Gamma_{\mathrm{d}(x)}} V^1(\frac{\mathrm{d}(x)}{\varepsilon}, \mathsf{P}(x)),$$

and where g^ε is defined on Γ by:

$$g^\varepsilon = \frac{\partial W^1}{\partial \nu} - \frac{\partial U^1}{\partial \nu}.$$

2.6. **Estimates of the right-hand-side terms.** Since ξ is in the circle of center λ^0 and of radius η_0, there exists a constant C such that for all $f \in \mathbf{H}$,

$$\|U^0\|_{H^3(\Omega_i)} + \|W^0\|_{H^3(\Omega_e)} \leq C\|f\|_{\mathbf{H}}, \tag{2.41}$$

and

$$\|U^1\|_{H^2(\Omega_i)} + \|W^1\|_{H^2(\Omega_e)} \leq C\|f\|_{\mathbf{H}}. \tag{2.42}$$

From (2.42), by trace theory, we obtain that:

$$\begin{aligned}\|g^\varepsilon\|_{\mathrm{L}^2(\Gamma)} \leq & \ \|\tfrac{\partial W^1}{\partial \nu}\|_{H^{\frac12}(\Gamma)} + \|\tfrac{\partial U^1}{\partial \nu}\|_{H^{\frac12}(\Gamma)},\\ \leq & \ K\left(\|\nabla W^1\|_{\mathrm{H}^1(\Omega_e)} + \|\nabla U^1\|_{\mathrm{H}^1(\Omega_i)}\right),\end{aligned}$$

so,

$$\|g^\varepsilon\|_{\mathrm{L}^2(\Gamma)} \leq K\|f\|_{\mathbf{H}}. \tag{2.43}$$

In order to estimate d_1^ε and d_2^ε, we prove the following proposition:

Proposition 2.2. *We denote by N the dimension of the space. We define $2^\star$ by $\frac12 = \frac1N + \frac{1}{2^\star}$ (for $N=2$, $2^\star = +\infty$). Then for all $p \in [2, 2^\star[$ ($[2,2^\star]$ for $N \neq 2$), there exists a constant K_p such that for all $\varepsilon > 0$, we have:*

$$\forall v \in H^2(\Omega_e),\ \left\|\frac{1}{\varepsilon^2}\left(v(\cdot) - v(\mathsf{P}(\cdot))\right)\right\|_{\mathrm{L}^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac12+\frac1p}}\|v\|_{H^2(\Omega_e)}$$

and

$$\forall v \in H^3(\Omega_e),\ \left\|\frac{1}{\varepsilon^3}\left(v(\cdot) - v(\mathsf{P}(\cdot)) - \frac{\partial v}{\partial \nu}(\mathsf{P}(\cdot))\right)\right\|_{\mathrm{L}^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac12+\frac1p}}\|v\|_{H^3(\Omega_e)}$$

Proof. We assume that v is smooth. For $x \in \omega_\varepsilon$, we have $x = \mathsf{P}(x) + \mathrm{d}(x)\nu(\mathsf{P}(x))$, so, using the variables $s = \mathrm{d}(x)$ and $\sigma = \mathsf{P}(x)$ and Taylor's formula:

$$\begin{aligned}\|v(\cdot) - v(\mathsf{P}(\cdot))\|^2_{\mathrm{L}^2(\omega_\varepsilon)} = & \int_{s=0}^{\varepsilon}\int_{\sigma\in\Gamma}\gamma_s(\sigma)|v(\sigma + s\nu(\sigma)) - v(\sigma)|^2 d\sigma \mathrm{d}s,\\ = & \int_{s=0}^{\varepsilon}\int_{\sigma\in\Gamma}\gamma_s(\sigma)\left|\int_{\tau=0}^{s}\nabla v(\sigma+\tau\nu(\sigma))\cdot\nu(\sigma)d\tau\right|^2 ds\, d\sigma,\\ \leq & \int_{s=0}^{\varepsilon}\int_{\sigma\in\Gamma}\gamma_s(\sigma)\left(\int_0^s|\nabla v(\sigma+\tau\nu(\sigma))|^2 d\tau\right) s\, ds\, d\sigma,\\ \leq & \ \varepsilon^2 3\int_{\omega_\varepsilon}|\nabla v|^2 dx \quad \text{(using Proposition 2.1).}\end{aligned}$$

By Sobolev embedding, for all $p \in [2, 2^\star[$ ($[2,2^\star]$ for $N \neq 2$),

$$\|\nabla v\|_{\mathrm{L}^p(\mathcal{U})} \leq C_p\|v\|_{H^2(\mathcal{U})}.$$

So, by Hölder inequality,

$$\begin{aligned}\|\nabla v\|_{\mathrm{L}^2(\omega_\varepsilon)} \leq & \ \|\nabla v\|_{\mathrm{L}^p(\omega_\varepsilon)}(\mathrm{meas}(\omega_\varepsilon))^{\frac12-\frac1p},\\ \leq & \ (K\varepsilon)^{\frac12-\frac1p}C_p\|v\|_{H^2(\mathcal{U})}.\end{aligned}$$

Therefore,

$$\left\|\frac{1}{\varepsilon^2}\left(v(\cdot) - v(\mathsf{P}(\cdot))\right)\right\|_{\mathrm{L}^2(\omega_\varepsilon)} \leq \frac{\sqrt{3}KC_p}{\varepsilon^{\frac12+\frac1p}}\|v\|_{H^2(\mathcal{U})}.$$

In addition, by Taylor expansion at order 2, for $\sigma \in \Gamma$ and $s \in [0, \varepsilon]$, we have:

$$\begin{aligned}\left| v(\sigma + s\nu(\sigma)) - v(\sigma) - \frac{\partial v}{\partial \nu}(\sigma)\mathrm{d}(x) \right| = & \left| \int_0^s (s-\tau) D^2 v(\sigma + \tau s)(\nu(\sigma), \nu(\sigma)) d\tau \right|, \\ \leq & \ s \int_0^s \left| D^2 v(\sigma + \tau s) \right| d\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\| v(x) - v(\mathsf{P}(x)) - \frac{\partial v}{\partial \nu}(\mathsf{P}(x))\mathrm{d}(x) \right\|^2_{\mathrm{L}^2(\omega_\varepsilon)} = & \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) \left| v(\sigma + s\nu(\sigma)) - v(\sigma) - \frac{\partial v}{\partial \nu}(\sigma) s \right|^2 ds\, d\sigma, \\ \leq & \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) \left| s \int_0^s |D^2 v(\sigma + \tau s)| d\tau \right|^2 ds d\sigma, \\ \leq & \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) s^3 \int_0^s |D^2 v(\sigma + \tau s)|^2 d\tau ds d\sigma, \\ \leq & \ 3\varepsilon^4 \int_{\omega_\varepsilon} \left| D^2 v(x) \right|^2 dx.\end{aligned}$$

By Sobolev embedding, for all $p \in [2, 2^\star[$ ($[2, 2^\star]$ for $N \neq 2$),

$$\|D^2 v\|_{\mathrm{L}^p(\mathcal{U})} \leq C_p \|v\|_{H^3(\mathcal{U})},$$

so

$$\|D^2 v\|_{\mathrm{L}^2(\omega_\varepsilon)} \leq \|D^2 v\|_{\mathrm{L}^p(\mathcal{U})} (\mathrm{meas}(\omega_\varepsilon))^{\frac{1}{2} - \frac{1}{p}} \leq C_p \|v\|_{H^3(\mathcal{U})} (K\varepsilon)^{\frac{1}{2} - \frac{1}{p}},$$

thus,

$$\left\| \frac{1}{\varepsilon^3} \left(v(x) - v(\mathsf{P}(x)) - \frac{\partial v}{\partial \nu}(\mathsf{P}(x))\mathrm{d}(x) \right) \right\|_{\mathrm{L}^2(\omega_\varepsilon)} \leq \frac{\sqrt{3} K C_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|v\|_{H^3(\mathcal{U})}.$$

We take $K_p = \sqrt{3} K C_p$, and we conclude the proof of Proposition 2.2 by a density argument. □

Corollary 2.3. *For all $p \in [2, 2^\star]$ ($[2, +\infty[$ in the case $N = 2$), there exists K_p such that for all $\varepsilon > 0$, for all $f \in \mathbf{H}$,*

$$\|d_1^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|f\|_{\mathbf{H}},$$

and

$$\|d_2^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)} \leq \frac{K_p}{\varepsilon^{\frac{1}{2} + \frac{1}{p}}} \|f\|_{\mathbf{H}}.$$

From (2.29), we have:

$$\begin{aligned}\|d_3^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} &= \int_0^\varepsilon \int_\Gamma \gamma_s(\sigma) \left| \frac{1}{\varepsilon} G_s(\sigma) \partial_z V^1(\frac{s}{\varepsilon}, \sigma) + \xi V^1(\frac{s}{\varepsilon}, \sigma) \right|^2 ds\, d\sigma \\ &\leq \frac{K}{\varepsilon^2} \int_0^\varepsilon \int_\Gamma \left(|\frac{\partial W^0}{\partial \nu}(\sigma)|^2 + |\frac{\partial U^0}{\partial \nu}(\sigma)|^2 \right) d\sigma ds, \\ &\leq \frac{K}{\varepsilon} \left(\|W^0\|^2_{H^3(\Omega_e)} + \|U^0\|^2_{H^3(\Omega_i)} \right).\end{aligned}$$

Therefore

$$\|d_3^\varepsilon\|_{\mathrm{L}^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon^{\frac{1}{2}}} \|f\|_{\mathbf{H}}. \tag{2.44}$$

2.7. **Variational estimates for the remainder term.** Multiplying (2.33) by the conjugate of a^ε and integrating on Ω_i, multiplying (2.34) by the conjugate of b^ε and integrating on ω_ε, multiplying (2.35) by the conjugate of c^ε and integrating on $\mathcal{U}_\varepsilon$, after integrations by part and after summation, we obtain:

$$\|\nabla a^\varepsilon\|^2_{\mathrm{L}^2(\Omega_i)} + \|\nabla b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + \|\nabla c^\varepsilon\|^2_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} + \frac{1}{\varepsilon^2}\|b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} =$$

$$\int_\Gamma g^\varepsilon \overline{b^\varepsilon} + \int_{\omega_\varepsilon} d^\varepsilon \overline{b^\varepsilon} - \xi \left(\|a^\varepsilon\|^2_{\mathrm{L}^2(\Omega_i)} + \|b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + \|c^\varepsilon\|^2_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} \right).$$

So,

$$\begin{aligned}\|\nabla a^\varepsilon\|^2_{\mathrm{L}^2(\Omega_i)} + \|\nabla b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + \|\nabla c^\varepsilon\|^2_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} + \frac{1}{\varepsilon^2}\|b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} \leq \\ |\xi| \left(\|a^\varepsilon\|^2_{\mathrm{L}^2(\Omega_i)} + \|b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + \|c^\varepsilon\|^2_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} \right) \\ + \|b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)} \|d_1^\varepsilon + d_2^\varepsilon + d_3^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)} + \|g^\varepsilon\|_{\mathrm{L}^2(\Gamma)} \|b^\varepsilon\|_{\mathrm{L}^2(\Gamma)} \\ + \left| \int_{\omega_\varepsilon} d_4^\varepsilon \overline{b^\varepsilon} \right|.\end{aligned} \tag{2.45}$$

We claim the following lemma proved in Section 3.1.

Lemma 2.4. *There exists a constant $C > 0$ independent of ε such that for any $\xi \in \mathcal{C}(\lambda^0, \eta)$, for any $u \in \mathrm{H}^1(\Omega_i)$ such that $-\Delta u - \xi u \in \mathrm{L}^2(\Omega_e)$, we have :*

$$\|u\|_{\mathrm{L}^2(\Omega_i)} \leqslant C \left(\|u\|_{\mathrm{L}^2(\Gamma)} + \|(-\Delta u - \xi u)\|_{\mathrm{L}^2(\Omega_i)} \right), \tag{2.46}$$

and for any $v \in \mathrm{H}^1(\mathcal{U}_\varepsilon)$ such that $v = 0$ on $\partial\Omega$ and such that $-\Delta v - \xi v \in \mathrm{L}^2(\mathcal{U}_e)$,, we have :

$$\|v\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} \leqslant C \left(\|v\|_{\mathrm{L}^2(\Gamma_\varepsilon)} + \|(-\Delta v - \xi v)\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} \right). \tag{2.47}$$

We claim the following Lemma, proved in [4] (see Inequality (2.1) in Proposition 2.1., page 822).

Lemma 2.5. *There exists constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, for all $\mathsf{q} \in \mathrm{H}^1(\omega_\varepsilon)$,*

$$\|\mathsf{q}\|^2_{\mathrm{L}^2(\partial\omega_\varepsilon)} \leqslant C\left(\frac{1}{\varepsilon}\|\mathsf{q}\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + \|\mathsf{q}\|_{\mathrm{L}^2(\omega_\varepsilon)}\|\nabla\mathsf{q}\|_{\mathrm{L}^2(\omega_\varepsilon)}\right)$$

Since a^ε (resp. c^ε) satisfies (2.33) in Ω_i (resp. (2.35) in $\mathcal{U}_\varepsilon$), we obtain with Lemma 2.4 and the transmission condition (2.36) (resp. (2.37)) that there exists C independent of $\varepsilon > 0$ and $\xi \in \mathcal{C}(\lambda^0, \eta)$ such that:

$$\|a^\varepsilon\|^2_{\mathrm{L}^2(\Omega_i)} + \|c^\varepsilon\|^2_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} \leq C\|b^\varepsilon\|^2_{\mathrm{L}^2(\partial\omega_\varepsilon)}.$$

Using now Lemma 2.5, we obtain that:

$$\|a^\varepsilon\|^2_{\mathrm{L}^2(\Omega_i)} + \|c^\varepsilon\|^2_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} \leq \frac{C}{\varepsilon}\|b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + C\|b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}\|\nabla b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}. \tag{2.48}$$

Using Corollary 2.3, Estimate (2.44), we obtain that there exists a constant C such that for all $\varepsilon > 0$, for all $\xi \in \mathcal{C}(\lambda^0, \eta)$ and for all $f \in \mathbf{H}$,

$$\|d_1^\varepsilon + d_2^\varepsilon + d_3^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)} \leq \frac{C}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\|f\|_{\mathbf{H}}. \tag{2.49}$$

In addition, coupling Lemma 2.5 and Estimate (2.43), we obtain that:

$$\|g^\varepsilon\|_{\mathrm{L}^2(\Gamma)}\|b^\varepsilon\|_{\mathrm{L}^2(\Gamma)} \leq C\sqrt{\varepsilon}\|f\|_{\mathbf{H}}\frac{1}{\varepsilon}\|b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)} + C\varepsilon^{\frac{1}{2}}\|f\|_{\mathbf{H}}\left(\frac{1}{\varepsilon}\|b_\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}\right)^{\frac{1}{2}}\|\nabla b^\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2(\omega_\varepsilon)}. \tag{2.50}$$

Let us estimate the last right-hand-side term in (2.45). We denote by $B^\varepsilon = b^\varepsilon \circ \Psi$, where:

$$\begin{array}{rccc} \Psi: & [0,\eta_0]\times\Gamma & \to & \omega_{\eta_0} \\ & (s,\sigma) & \mapsto & \sigma + s\nu(\sigma). \end{array}$$

$$\begin{aligned} \int_{\omega_\varepsilon} d_4^\varepsilon\overline{b^\varepsilon} &= \int_{s=0}^{\varepsilon}\int_{\sigma\in\Gamma}\gamma_s(\sigma)(\Delta_{\Gamma_s}V^1(\frac{s}{\varepsilon},\sigma))\overline{B^\varepsilon(s,\sigma)}ds\,d\sigma, \\ &= -\int_{s=0}^{\varepsilon}\int_{\sigma\in\Gamma}\gamma_s(\sigma)\nabla_{\Gamma_s}V^1(\frac{s}{\varepsilon},\sigma)\overline{\nabla_{\Gamma_s}B^\varepsilon(s,\sigma)}ds, \end{aligned}$$

So,

$$\begin{aligned} \left|\int_{\omega_\varepsilon} d_4^\varepsilon\overline{b^\varepsilon}\right| &\leq \int_{s=0}^{\varepsilon}\int_{\sigma\in\Gamma}\gamma_s(\sigma)\left|\nabla_{\Gamma_s}V^1(\frac{s}{\varepsilon},\sigma)\right|\left|\nabla_{\Gamma_s}B^\varepsilon(s,\sigma)\right|ds\,d\sigma \\ &\leq \|\nabla b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}\|\nabla_{\Gamma_s}V^1(\tfrac{s}{\varepsilon},\sigma)\|_{\mathrm{L}^2(\omega_\varepsilon)}, \\ &\leq \|\nabla b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}\left(\|\frac{\partial U^0}{\partial\nu}\|_{\mathrm{H}^1(\Gamma)} + \|\frac{\partial W^0}{\partial\nu}\|_{\mathrm{H}^1(\Gamma)}\right)\sqrt{\varepsilon}, \end{aligned}$$

thus, using (2.41), we obtain:

$$\left|\int_{\omega_\varepsilon} d_4^\varepsilon\overline{b^\varepsilon}\right| \leq C\sqrt{\varepsilon}\|\nabla b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}\|f\|_{\mathbf{H}}. \tag{2.51}$$

Therefore, from (2.45), (2.48), (2.49), (2.50), and (2.51), we obtain that there exists a constant C independent of $\varepsilon > 0$, $\xi \in \mathcal{C}(\lambda^0, \eta)$ and $f \in \mathbf{H}$ such that

$$\|\nabla a^\varepsilon\|^2_{\mathrm{L}^2(\Omega_i)} + \|\nabla b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + \|\nabla c^\varepsilon\|^2_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} + \frac{1}{\varepsilon^2}\|b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} \leq$$

$$\frac{C}{\varepsilon}\|b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + C\|b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}\|\nabla b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)} + \frac{C}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\|b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}\|f\|_{\mathbf{H}}$$

$$+C\sqrt{\varepsilon}\|f\|_{\mathbf{H}}\tfrac{1}{\varepsilon}\|b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)} + C\varepsilon^{\frac{1}{2}}\|f\|_{\mathbf{H}}\left(\tfrac{1}{\varepsilon}\|b_\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}\right)^{\frac{1}{2}}\|\nabla b^\varepsilon\|^{\frac{1}{2}}_{\mathrm{L}^2(\omega_\varepsilon)} + C\varepsilon^{\frac{1}{2}}\|\nabla b^\varepsilon\|_{\mathrm{L}^2(\omega_\varepsilon)}\|f\|_{\mathbf{H}}$$

$$\leq \frac{1}{2}\frac{1}{\varepsilon^2}\|b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + \frac{1}{2}\|\nabla b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + C\varepsilon^{1-\frac{2}{p}}\|f\|^2_{\mathbf{H}} \quad \text{(by Young inequality).}$$

So, after absorption, we obtain that there exists C such that:

$$\|\nabla a^\varepsilon\|^2_{\mathrm{L}^2(\Omega_i)} + \|\nabla b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} + \|\nabla c^\varepsilon\|^2_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} + \frac{1}{\varepsilon^2}\|b^\varepsilon\|^2_{\mathrm{L}^2(\omega_\varepsilon)} \leq C\varepsilon^{1-\frac{2}{p}}\|f\|^2_{\mathbf{H}}. \tag{2.52}$$

Reusing (2.48), we obtain also that:

$$\|a^\varepsilon\|^2_{\mathrm{L}^2(\Omega_i)} + \|c^\varepsilon\|^2_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} \leq C\varepsilon^{2-\frac{2}{p}}\|f\|^2_{\mathbf{H}}. \tag{2.53}$$

3. Technical Lemma : Uniform a priori estimates

3.1. **Geometrical tools in the thin layer.** In this part, we refer to [4]. For $\sigma \in \Gamma$, we denote by $\nu(\sigma)$ the outward unitary normal. We remark that $\nu : \Gamma \longrightarrow S^{N-1}$ is smooth since Γ is regular. We define Ψ by:

$$\begin{array}{rccc} \Psi : &]0,\eta_0[\times\Gamma & \to & \omega_{\eta_0} \\ & (s,\sigma) & \mapsto & \sigma + s\nu(\sigma). \end{array}$$

Since Γ is a smooth compact surface without boundary, we can fix $\eta_0 > 0$ such that Ψ is a $\mathcal{C}^\infty$-diffeomorphism and so that for all $\varepsilon \in]0,\eta_0[$,

$$\Psi(]0,\varepsilon[\times\Gamma) = \omega_\varepsilon.$$

We define $\gamma : [0,\eta_0]\times\Gamma \to \mathbb{R}$ by:

$$\gamma_s(\sigma) = \det(Id + s\, d\nu(\sigma)),$$

and we have:

$$\int_{\omega_{\eta_0}} F(x)\,dx = \int_{s=0}^{\eta_0}\int_{\sigma\in\Gamma} F(\Psi(s,\sigma))\gamma_s(\sigma)d\sigma\, ds.$$

Since $\gamma_0(\sigma) = 1$ for all $\sigma \in \Gamma$, even if it means reducing η_0, we assume that

$$\forall\, s \in [0,\eta_0[,\ \forall\,\sigma\in\Gamma,\ \frac{1}{2} \leq \gamma_s(\sigma) \leq \frac{3}{2}. \tag{3.1}$$

We denote ∇_Γ, div_Γ and Δ_Γ the gradient, divergence and Laplacian operators on the submanifold Γ.

For $\mathrm{v} : \Gamma \longrightarrow \mathbb{C}$ and $s \in [0,\eta_0[$, we define $\nabla_{\Gamma_s}\mathrm{v}$ by:

$$\nabla_{\Gamma_s}\mathrm{v}(\sigma) = (Id + s\, d\nu(\sigma))^{-1}\nabla_\Gamma \mathrm{v}(\sigma),$$

so that: for $v : \omega_{\eta_0} \to \mathbb{C}$, denoting $V = v \circ \Psi$, we have:

$$\nabla v(x) = \frac{\partial V}{\partial s}(\mathrm{d}(x), \mathsf{P}(x))\nu(\mathsf{P}(x)) + \left(\nabla_{\Gamma_{\mathrm{d}(x)}} V\right)(\mathrm{d}(x), \mathsf{P}(x)).$$

We define the operator div_{Γ_s} as the adjoint of ∇_{Γ_s} for the $\mathrm{L}^2(\Gamma)$-inner product: for $Y : \Gamma \longrightarrow T_\Gamma$ a tangent vector field defined on Γ, we set:

$$\left(\mathrm{div}_{\Gamma_s} Y\right)(\sigma) = \frac{1}{\gamma_s(\sigma)} \mathrm{div}_\Gamma \left(\gamma_s (Id + s\, d\nu)^{-1} Y\right)(\sigma),$$

so that if $z : \omega_{\eta_0} \longrightarrow \mathbb{R}^N$ is a vector field, denoting $Z = z \circ \Psi$, $Z_N(s,\sigma) = (Z(s,\sigma) \cdot \nu(\sigma))$ and $Z_T(s,\sigma) = Z(s,\sigma) - Z_N(s,\sigma)\nu(\sigma)$, we have:

$$\mathrm{div} z(x) = \frac{\partial Z_N}{\partial s}(\mathrm{d}(x), \mathsf{P}(x)) + G_{\mathrm{d}(x)}(\mathsf{P}(x)) Z_N(\mathrm{d}(x), \mathsf{P}(x)) + \left(\mathrm{div}_{\Gamma_{\mathrm{d}(x)}} Z_T\right)(\mathrm{d}(x), \mathsf{P}(x)),$$

where

$$G_s(\sigma) = \frac{1}{\gamma_s(\sigma)} \frac{\partial \gamma_s}{\partial s}(\sigma).$$

We define the operator Δ_{Γ_s} by:

$$\Delta_{\Gamma_s} = \mathrm{div}_{\Gamma_s} \circ \nabla_{\Gamma_s},$$

so that for $v : \omega_{\eta_0} \to \mathbb{C}$, denoting $V = v \circ \Psi$, we have:

$$\Delta v(x) = \frac{\partial^2 V}{\partial s^2}(\mathrm{d}(x), \mathsf{P}(x)) + G_{\mathrm{d}(x)}(\mathsf{P}(x)) \frac{\partial V}{\partial s}(\mathrm{d}(x), \mathsf{P}(x)) + (\Delta_{\Gamma_{\mathrm{d}(x)}} V)(\mathrm{d}(x), \mathsf{P}(x)).$$

3.2. **Geometrical tools for $\mathcal{U}_\varepsilon$.** Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth non-decreasing function such that:

$$\forall\, x \leq 0,\ \chi(x) = -1,$$

$$\forall\, x \geq \eta_0,\ \chi(x) = 0.$$

For $\varepsilon > 0$ such that $\varepsilon \leq \frac{\eta_0}{4}$, we define ϕ_ε by:

$$\left\{ \begin{array}{rccl} \phi_\varepsilon : & \mathcal{U}_\varepsilon & \to & \Omega_e \\ & x & \mapsto & x + \varepsilon \chi(\mathrm{d}(x) - \varepsilon)\nu(\mathsf{P}(x)) \end{array} \right.$$

We remark that if $x \in \mathcal{U}_\varepsilon$ with $\mathrm{d}(x) \geq \varepsilon + \eta_0$, then $\phi_\varepsilon(x) = x$. In addition, ϕ_ε is a $\mathcal{C}^\infty$-diffeomorphism from $\mathcal{U}_\varepsilon$ onto Ω_e. We denote by $\phi_{\varepsilon,k}$ the k-th coordinate of ϕ_ε.

For $w : \mathcal{U}_\varepsilon \to \mathbb{C}$, we define $\mathsf{w} : \Omega_e \to \mathbb{C}$ by:

$$w = \mathsf{w} \circ \phi_\varepsilon.$$

We have, for all $j \in \{1, \dots N\}$,

$$\frac{\partial w}{\partial x_j} = \sum_{k=1}^{N} \frac{\partial \mathsf{w}}{\partial \mathrm{x}_k}(\phi_\varepsilon(x)) \frac{\partial \phi_{\varepsilon,k}}{\partial x_j}(x)$$

so

$$\frac{\partial^2 w}{\partial x_j^2} = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^2 \mathsf{w}}{\partial \mathrm{x}_k \partial \mathrm{x}_l}(\phi_\varepsilon(x)) \frac{\partial \phi_{\varepsilon,k}}{\partial x_j}(x) \frac{\partial \phi_{\varepsilon,l}}{\partial x_j}(x) + \sum_{k=1}^{N} \frac{\partial \mathsf{w}}{\partial \mathrm{x}_k}(\phi_\varepsilon(x)) \frac{\partial^2 \phi_{\varepsilon,k}}{\partial x_j^2}(x)$$

Therefore,

$$\Delta w(x) = \sum_{k,l} a^\varepsilon_{k,l}(\phi_\varepsilon(x)) \frac{\partial^2 \mathsf{w}}{\partial \mathrm{x}_k \partial \mathrm{x}_l}(\phi_\varepsilon(x)) + \sum_{k=1}^N b^\varepsilon_k(\phi_\varepsilon(x)) \frac{\partial \mathsf{w}}{\partial \mathrm{x}_k}(\phi_\varepsilon(x)),$$

where $a^\varepsilon_{k,l} : \Omega_e \longrightarrow \mathbb{R}$ and $b^\varepsilon_k : \Omega_e \longrightarrow \mathbb{R}$ are defined by:

$$\forall x \in \mathcal{U}_\varepsilon, \quad a^\varepsilon_{kl}(\phi_\varepsilon(x)) = \nabla \phi_{\varepsilon,k}(x) \cdot \nabla \phi_{\varepsilon,l}(x) \quad \text{and} \quad b^\varepsilon_k(\phi_\varepsilon(x)) = \Delta \phi_{\varepsilon,k}(x).$$

For $\mathrm{x} \in \Omega_e$ such that $\mathrm{d}(\mathrm{x}) \geq \varepsilon + \eta_0$, $\phi_\varepsilon(\mathrm{x}) = \mathrm{x}$ so that $a^\varepsilon_{kl}(\mathrm{x}) = \delta_{kl}$ and $b^\varepsilon_k(\mathrm{x}) = 0$.
For $\mathrm{x} \in \Omega_e$ such that $\mathrm{d}(\mathrm{x}) \leq \varepsilon + \eta_0$, we have: $x = \phi_\varepsilon^{-1}(\mathrm{x}) \in \omega_{\eta_0}$ and:

$$\frac{\partial \phi_{\varepsilon,k}}{\partial x_j}(x) = \delta_{jk} + \varepsilon \left(\chi'(\mathrm{d}(x) - \varepsilon) \nu_j(\mathsf{P}(x)) \nu_k(\mathsf{P}(x)) + \chi(\mathrm{d}(x) - \varepsilon) \nabla \nu_k(\mathsf{P}(x)) \cdot \frac{\partial \mathsf{P}(x)}{\partial x_j} \right).$$

Thus, $a^\varepsilon_{k,l}(\mathrm{x}) = \delta_{kj} + \widetilde{a_{k,l}}^\varepsilon(\mathrm{x})$, where $\widetilde{a_{k,l}}^\varepsilon(\mathrm{x})$ is of order ε and tends uniformly to zero when ε tends to zero. In the same way, $b^\varepsilon_k(\mathrm{x})$ is of order ε and tends uniformly to zero when ε tends to zero. We denote, for $\mathsf{w} \in \mathrm{H}^2(\Omega_e)$:

$$\mathcal{K}^\varepsilon \mathsf{w} = \sum_{(k,l),\, k \neq l} a^\varepsilon_{kl} \frac{\partial^2 \mathsf{w}}{\partial \mathrm{x}_k \partial \mathrm{x}_l} + \sum_{k=1}^N b^\varepsilon_k \frac{\partial \mathsf{w}}{\partial \mathrm{x}_k},$$

so that

$$\Delta w(x) = \left(\Delta \mathsf{w} + \mathcal{K}^\varepsilon \mathsf{w} \right)(\phi_\varepsilon(x)),$$

and there exists a constant C such that for all $\varepsilon > 0$, for all $\mathsf{w} \in \mathrm{H}^2(\Omega_e)$, we have:

$$\|\mathcal{K}^\varepsilon \mathsf{w}\|_{\mathrm{L}^2(\Omega_e)} \leq C\varepsilon \|\mathsf{w}\|_{\mathrm{H}^2(\Omega_e)}. \tag{3.2}$$

For $\varepsilon \geq 0$, we denote by $\mathcal{L}^\varepsilon$ the operator $-\Delta$ defined on $\mathrm{H}^2(\mathcal{U}_\varepsilon) \cap \mathrm{H}^1_0(\mathcal{U}_\varepsilon)$. This operator is positive and self-adjoint for the $\mathrm{L}^2(\mathcal{U}_\varepsilon)$-scalar product. Its resolvent is compact, so that its spectrum only contains a sequence of real eigenvalues tending to $+\infty$.

Let us prove the following lemma:

Lemma 3.1. *Let $\lambda_0 \in \mathsf{Spec}(\mathcal{H}^0)$ and $\eta > 0$ such that $\mathsf{Spec}(\mathcal{H}^0) \cap \overline{B(\lambda_0, \eta)} = \{\lambda_0\}$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$,*

$$\mathsf{Spec}(\mathcal{L}^\varepsilon) \cap \mathcal{C}(\lambda_0, \eta) = \emptyset.$$

Proof. If it is not the case, there exists a sequence of positive numbers $(\varepsilon_n)_n$ tending to zero when n tends to $+\infty$, there exists a sequence $(\xi_n)_n$ of points of $\mathcal{C}(\lambda_0, \eta)$, and there exists a sequence $(u_n)_n$ such that for all n, $u_n \in \mathrm{H}^2(\mathcal{U}_{\varepsilon_n}) \cap \mathrm{H}^1_0(\mathcal{U}_{\varepsilon_n})$ is an eigenvector of $\mathcal{L}^\varepsilon$ associated to the eigenvalue ξ_n.

We denote $\mathsf{u}_n = u_n \circ (\phi_{\varepsilon_n})^{-1}$, so that we have:

$$\mathsf{u}_n \in \mathrm{H}^2(\Omega_e) \cap \mathrm{H}^1_0(\Omega_e) \text{ and } -\Delta \mathsf{u}_n - \mathcal{K}^{\varepsilon_n} \mathsf{u}_n = \xi_n \mathsf{u}_n. \tag{3.3}$$

Since $\mathsf{u}_n \neq 0$, we can assume that $\|\mathsf{u}_n\|_{\mathbf{H}^2(\Omega_e)} = 1$. Thus we can extract subsequences still denoted (ε_n), (ξ_n) and (u_n) such that ξ_n tends to $\xi_\infty \in \mathcal{C}(\lambda_0, \eta)$ and u_n tends to u_∞ weakly in $\mathrm{H}^2(\Omega_e)$ and strongly in $\mathrm{H}^1(\Omega_e)$ by compactness of the embedding $\mathrm{H}^2(\Omega_e) \subset \mathrm{H}^1(\Omega_e)$.

By taking the weak limit in (3.3) and using (3.2), we obtain that

$$\mathsf{u}_\infty \in \mathrm{H}^2(\Omega_e) \cap \mathrm{H}^1_0(\Omega_e) \text{ and } -\Delta \mathsf{u}_\infty = \xi_\infty \mathsf{u}_\infty. \tag{3.4}$$

In addition, writing that $-\Delta \mathsf{u}_n = \xi_n \mathsf{u}_n + \mathcal{K}^{\varepsilon_n}\mathsf{u}_n$, using again (3.2), we obtain that $-\Delta \mathsf{u}_n$ tends strongly in $\mathrm{L}^2(\Omega_e)$ to $-\Delta \mathsf{u}_\infty$, so that u_n tends strongly to u_∞ in $\mathrm{H}^2(\Omega_e)$. So, $\|\mathsf{u}_\infty\|_{\mathrm{H}^2(\Omega_e)} = 1$, thus $\mathsf{u}_\infty \neq 0$ and by (3.4), we obtain that $\xi_\infty \in \mathsf{Spec}(\mathcal{L}^0) \cap \mathcal{C}(\lambda_0, \eta)$, which is false.

□

With the previous lemma, for $\varepsilon > 0$ small enough, for all $\xi \in \mathcal{C}(\lambda_0, \eta)$, $\mathcal{L}^\varepsilon - \xi\mathbb{I}$ is inversible. We give a uniform estimate of the resolvent operator in the following lemma:

Lemma 3.2. *There exists $\varepsilon_0 > 0$ and C such that for all $\xi \in \mathcal{C}(\lambda_0, \eta)$, for all $\varepsilon \in]0, \varepsilon_0[$, for all $f \in \mathrm{L}^2(\mathcal{U}_\varepsilon)$, there exists a unique $u \in \mathrm{H}^2(\mathcal{U}_\varepsilon) \cap \mathrm{H}^1_0(\mathcal{U}_\varepsilon)$ such that $-\Delta u - \xi u = f$ in $\mathcal{U}_\varepsilon$. Moreover,*

$$\|u\|_{\mathrm{H}^2(\mathcal{U}_\varepsilon)} \leq C \|f\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)}.$$

Proof. If it is not the case, there exists a sequence (ε_n) tending to 0, a sequence $(\xi_n)_n$ of points of $\mathcal{C}(\lambda_0, \eta)$, a sequence (f_n) with $f_n \in \mathrm{L}^2(\mathcal{U}_{\varepsilon_n})$ and $u_n \in \mathrm{H}^2(\mathcal{U}_{\varepsilon_n}) \cap \mathrm{H}^1_0(\mathcal{U}_{\varepsilon_n})$ such that $-\Delta u_n - \xi_n u_n = f_n$ in $\mathcal{U}_{\varepsilon_n}$ and:

$$\|u_n\|_{\mathrm{H}^2(\mathcal{U}_{\varepsilon_n})} > n \|f_n\|_{\mathrm{L}^2(\mathcal{U}_{\varepsilon_n})}.$$

We denote $\mathsf{u}_n = u_n \circ (\phi_{\varepsilon_n})^{-1}$ and $\mathsf{f}_n = f_n \circ (\phi_{\varepsilon_n})^{-1}$. We have:

$$-\Delta \mathsf{u}_n - \mathcal{K}^{\varepsilon_n}\mathsf{u}_n - \xi_n \mathsf{u}_n = \mathsf{f}_n \text{ in } \Omega_e,$$

and:

$$\|\mathsf{u}_n\|_{\mathrm{H}^2(\Omega_e)} > Cn \|\mathsf{f}_n\|_{\mathrm{L}^2(\Omega_e)}.$$

Since $\mathsf{u}_n \neq 0$, we can divide by its $\mathrm{H}^2(\Omega_e)$-norm. We denote

$$\tilde{\mathsf{u}}_n = \frac{\mathsf{u}_n}{\|\mathsf{u}_n\|_{\mathrm{H}^2(\Omega_e)}} \quad \text{and} \quad \tilde{\mathsf{f}}_n = \frac{\mathsf{f}_n}{\|\mathsf{u}_n\|_{\mathrm{H}^2(\Omega_e)}}.$$

We have, for all n:

$$\|\tilde{\mathsf{u}}_n\|_{\mathrm{H}^2(\Omega_e)} = 1 > n \|\tilde{\mathsf{f}}_n\|_{\mathrm{L}^2(\Omega_e)}$$

and

$$-\Delta \tilde{\mathsf{u}}_n - \mathcal{K}^{\varepsilon_n}\tilde{\mathsf{u}}_n - \xi_n \tilde{\mathsf{u}}_n = \tilde{\mathsf{f}}_n \text{ in } \Omega_e. \tag{3.5}$$

By extracting a sub sequence, we can assume that ξ_n tends to $\xi_\infty \in \mathcal{C}(\lambda_0, \eta)$, and $\tilde{\mathsf{u}}_n$ tends to a limit $\tilde{\mathsf{u}}_\infty$ weakly in $\mathrm{H}^2(\Omega_e)$ and strongly in $\mathrm{H}^1(\Omega_e)$. In addition, we remark that $\tilde{\mathsf{f}}_n$ tends to zero strongly in $\mathrm{L}^2(\Omega_e)$. Taking the weak limit in (3.5), we obtain that:

$$\tilde{\mathsf{u}}_\infty \in \mathrm{H}^2(\Omega_e) \cap \mathrm{H}^1_0(\Omega_e) \quad \text{and} \quad -\Delta \tilde{\mathsf{u}}_\infty - \xi_\infty \tilde{\mathsf{u}}_\infty = 0 \text{ in } \Omega_e. \tag{3.6}$$

In addition, using (3.2) and (3.5), we obtain that $\Delta \tilde{\mathsf{u}}_n$ converges strongly in $\mathrm{L}^2(\Omega_e)$. Since its limit is the weak limit $\Delta \tilde{\mathsf{u}}_\infty$, then $\tilde{\mathsf{u}}_n$ converges strongly in $\mathrm{H}^2(\Omega_e)$ to $\tilde{\mathsf{u}}_\infty$. So $\|\tilde{\mathsf{u}}_\infty\|_{\mathrm{H}^2(\Omega_e)} = 1$, and $\tilde{\mathsf{u}}_\infty \neq 0$. Therefore, with (3.6), ξ_∞ is in $\mathsf{Spec}(\mathcal{L}^0)$, which leads to a contradiction.

□

Proof of Lemma 2.4

Inequality (2.46) is proved by duality arguments in [3] (see Lemma 2.1, on page 95). We use the same method and uniform estimates proved in Lemma 3.2 to establish (2.47). Let ε_0 given by Lemma 3.2. Let $\varepsilon \in (0, \varepsilon_0)$ and $\xi \in \mathcal{C}(\lambda_0, \eta)$. We fix $v \in \mathrm{H}^1(\mathcal{U}_\varepsilon)$ such that $-\Delta v - \xi v$ belongs to $\mathrm{L}^2(\mathcal{U}_\varepsilon)$. Let $f \in \mathrm{L}^2(\mathcal{U}_\varepsilon)$. By Lemma 3.2, there exists $w \in \mathrm{H}^2(\mathcal{U}_\varepsilon) \cap \mathrm{H}^1_0(\mathcal{U}_\varepsilon)$ such that $-\Delta w - \xi w = f$ in $\mathcal{U}_\varepsilon$. Moreover,

$$\|w\|_{\mathrm{H}^2(\mathcal{U}_\varepsilon)} \leq C \|f\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)},$$

where C does not depend on ε, f or ξ. We have:

$$\int_{\mathcal{U}_\varepsilon} vf = \int_{\mathcal{U}_\varepsilon} v(-\Delta w - \xi w) = \int_{\mathcal{U}_\varepsilon} (-\Delta v - \xi v)w + \int_{\Gamma_\varepsilon} v\frac{\partial w}{\partial \nu}.$$

Therefore:

$$\begin{aligned}\left|\int_{\mathcal{U}_\varepsilon} vf\right| \leq & \ \| - \Delta v - \xi v\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)}\|w\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} + \|v\|_{\mathrm{L}^2(\Gamma_\varepsilon)}\|\frac{\partial w}{\partial \nu}\|_{\mathrm{L}^2(\Gamma_\varepsilon)} \\ \leq & \ C\left(\| - \Delta v - \xi v\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} + \|v\|_{\mathrm{L}^2(\Gamma_\varepsilon)}\right)\|f\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)}\end{aligned}$$

This is true for all $f \in \mathrm{L}^2(\mathcal{U}_\varepsilon)$ so:

$$\|v\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} \leq C\left(\| - \Delta v - \xi v\|_{\mathrm{L}^2(\mathcal{U}_\varepsilon)} + \|v\|_{\mathrm{L}^2(\Gamma_\varepsilon)}\right).$$

□

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