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# RESOLVENT AND SPECTRAL ASYMPTOTICS FOR THIN-LAYER TRANSMISSION PROBLEMS IN THE LARGE COUPLING LIMIT 

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#### Abstract

This work is concerned with a singular perturbation problem that involves a thin layer in the large coupling limit. Such a problem appears in the study of electromagnetic wave guides that contain supra-conductor bodies. We derive resolvent asymptotics for this problem by using boundary layer tools with uniform a priori estimates. This result will be useful to derive spectral asymptotics for this problem. This issue is a work in progress.


## 1. Introduction

This work is motivated by the study of the asymptotic behavior of eigenvalues related to a singular perturbation problem that involves a thin layer in the large coupling limit. Such a problem appears in the study of electromagnetic wave guides with supra-conductors. We refer the reader to several works that concern mathematical studies with electromagnetic wave guides, see for instance [1, 7, 2, 3].

An asymptotic expansion of the eigenvalues as the conductivity tends to infinity has been derived in [3]. However this result does not apply when the subdomain that represents a supraconductor is a vanishing sheet. In this work we tackle this problem by considering vanishing supra-conductive layer of thickness $\varepsilon$ where the conductivity is scaled like $\varepsilon^{-2}$. This is why there is a main difference between this work and the work in Ref. [3] all the more since we consider a transmission problem across a thin layer.

Let us describe now the mathematical model. Throughout the paper we denote by $\Omega \subset \mathbb{R}^{3}$ a simply connected domain. Let $\Omega_{i}$ be a simply connected subdomain of $\Omega$ and let $\omega_{\varepsilon}$ be a thin layer of constant thickness $\varepsilon$ surrounding $\Omega_{i}$ such that $\Omega$ contains $\Omega_{i} \cup \overline{\omega_{\varepsilon}}$, see Figure 1. For the sake of simplicity we further assume that the boundaries $\partial \Omega$ and $\Gamma=\partial \Omega_{i}$ are connected. Now the exterior domain is defined as $\mathcal{U}_{\varepsilon}=\Omega \backslash\left(\Omega_{i} \cup \overline{\omega_{\varepsilon}}\right)$ and the boundary of the subdomain $\overline{\Omega_{i}} \cup \omega_{\varepsilon}$ is denoted by $\Gamma_{\varepsilon}$. We denote by $n$ the outward unit normal at $\Gamma$.

In all that follows, unless specified, all the considered domains are smooth domains in $\mathbb{R}^{3}$.
Notation 1.1. For any set $\mathcal{O} \subset \mathbb{R}^{3}$ we denote by $\mathrm{L}^{2}(\mathcal{O})$ and $\mathrm{H}^{s}(\mathcal{O})$ the classical Sobolev spaces.
Let $\varepsilon>0$. We consider the self-adjoint operator $\mathcal{H}^{\varepsilon}$ on $\mathrm{L}^{2}(\Omega)$ with domain $\mathrm{H}^{2}(\Omega) \cap \mathrm{H}_{0}^{1}(\Omega)$ and defined by

$$
\mathcal{H}^{\varepsilon}=-\Delta+\frac{1}{\varepsilon^{2}} 1_{\left.\right|_{\varepsilon}} .
$$

The limit operator is $\mathcal{H}^{0}=-\Delta$ defined on $\mathrm{H}^{2}\left(\Omega_{i}\right) \cap \mathrm{H}_{0}^{1}\left(\Omega_{i}\right) \times \mathrm{H}^{2}\left(\Omega_{e}\right) \cap \mathrm{H}_{0}^{1}\left(\Omega_{e}\right)$ where $\Omega_{e}$ denotes the limit domain $\mathcal{U}_{\varepsilon}$ when $\varepsilon \rightarrow 0$, i.e. $\Omega_{e}=\Omega \backslash \bar{\Omega}_{i}$ (see Figure 2).

[^0]

Figure 1 - A cross-section of the domain $\Omega$ and the subdomains $\Omega_{i}, \omega_{\varepsilon}, \mathcal{U}_{\varepsilon}$

In this work we derive an asymptotic expansion of the resolvent of $\mathcal{H}^{\varepsilon}$ when $\varepsilon$ tends to 0 . We denote by $\mathcal{R}^{\varepsilon}(\xi):=\left(\mathcal{H}^{\varepsilon}-\xi \mathcal{I}\right)^{-1}$ the resolvent of $\mathcal{H}^{\varepsilon}$, defined on $\mathrm{L}^{2}(\Omega)$ with values in $\mathrm{H}^{2}(\Omega) \cap \mathrm{H}_{0}^{1}(\Omega)$ for $\xi \notin \operatorname{spec} \mathcal{H}^{\varepsilon}$.

Let $f \in \mathrm{~L}^{2}(\Omega)$. The spectral problem writes

$$
\begin{equation*}
-\Delta u-\xi u+\frac{1}{\varepsilon^{2}} 1_{\mid \omega_{\varepsilon}} u=f \quad \text { in } \quad \Omega . \tag{1.1}
\end{equation*}
$$

This problem is well-posed and has a unique solution $u_{\varepsilon}$ in $\mathrm{H}^{2}(\Omega) \cap \mathrm{H}_{0}^{1}(\Omega)$ when $\xi$ is not an eigenvalue of the operator $\mathcal{H}^{\varepsilon}$.


Figure 2 - A cross-section of the domain $\Omega$ and the subdomains $\Omega_{i}, \Omega_{e}$

Notation 1.2. We denote by $f_{i}$ (resp. $f_{e}$ ) the restriction of any function $f$ to $\Omega_{i}$ (resp. $\Omega_{e}$ ).

## 2. ASymptotic expansion for the resolvent

Let $\varepsilon>0$. We consider the self-adjoint operator $\mathcal{H}^{\varepsilon}$ on $\mathrm{L}^{2}(\Omega)$ with domain $\mathrm{H}^{2}(\Omega) \cap \mathrm{H}_{0}^{1}(\Omega)$ and defined by

$$
\mathcal{H}^{\varepsilon}=-\Delta+\frac{1}{\varepsilon^{2}} 1_{\left.\right|_{\varepsilon}} .
$$

For $\xi \notin \operatorname{spec} \mathcal{H}^{\varepsilon}$, we denote by $\mathcal{R}^{\varepsilon}(\xi)=\left(\mathcal{H}^{\varepsilon}-\xi \mathrm{Id}\right)^{-1}$ the resolvent operator. For $f \in \mathrm{~L}^{2}(\Omega)$, we denote by $u^{\varepsilon}$ the restriction of $\mathcal{R}^{\varepsilon}(\xi)(f)$ to $\Omega_{i}, v^{\varepsilon}$ the restriction of $\mathcal{R}^{\varepsilon}(\xi)(f)$ to $\omega_{\varepsilon}$, and $w^{\varepsilon}$ the restriction of $\mathcal{R}^{\varepsilon}(\xi)(f)$ to $\mathcal{U}_{\varepsilon}$, By the resolvent equation, we have:

$$
\begin{array}{ll}
-\Delta u^{\varepsilon}-\xi u^{\varepsilon}=f_{\mid \Omega_{i}} & \text { in } \Omega_{i} \\
-\Delta v^{\varepsilon}-\xi v^{\varepsilon}+\frac{1}{\varepsilon^{2}} v^{\varepsilon}=f_{\mid \omega_{\varepsilon}} & \text { in } \omega_{\varepsilon} \\
-\Delta w^{\varepsilon}-\xi w^{\varepsilon}=f_{\mid \mathcal{U}^{\varepsilon}} & \text { in } \mathcal{U}_{\varepsilon} \\
u^{\varepsilon}=v^{\varepsilon} & \text { on } \Gamma \\
v^{\varepsilon}=w^{\varepsilon} & \text { on } \Gamma_{\varepsilon} \\
\partial_{\mathrm{n}} u^{\varepsilon}=\partial_{\mathrm{n}} v^{\varepsilon} & \text { on } \Gamma \\
\partial_{\mathrm{n}} v^{\varepsilon}=\partial_{\mathrm{n}} w^{\varepsilon} & \text { on } \Gamma_{\varepsilon} \\
w^{\varepsilon}=0 & \text { on } \partial \Omega . \tag{2.8}
\end{array}
$$

2.1. Ansatz. Outside the thin layer, we look for an expansion on the form:

$$
\begin{align*}
u^{\varepsilon}(x) & \approx \sum_{j \geqslant 0} \varepsilon^{j} U^{j}(x) \quad \text { in } \quad \Omega_{i}  \tag{2.9}\\
w^{\varepsilon}(x) & \approx \sum_{j \geqslant 0} \varepsilon^{j} W^{j}(x) \quad \text { in } \quad \mathcal{U}_{\varepsilon} \tag{2.10}
\end{align*}
$$

where the terms $U^{j}$ and $W^{j}$ are defined in $\varepsilon$-independent domains $\Omega_{i}$ and $\Omega_{e}:=\Omega \backslash \Omega_{i}$. We emphasize that the terms $W^{j}$ are defined in the thin layer $\omega_{\varepsilon}$.

In the thin layer, we denote by $d: \omega_{\varepsilon} \rightarrow \mathbb{R}^{+}$the euclidean distance to $\Gamma, \mathrm{d}(x)=\operatorname{dist}(x, \Gamma)$ and by $\mathrm{P}(x)$ the orthogonal projection of $x$ on $\Gamma$. We describe the solution $v^{\varepsilon}$ on the following way:

$$
\begin{equation*}
v^{\varepsilon}(x) \approx \sum_{j \geqslant 0} \varepsilon^{j}\left(W^{j}(x)+V^{j}\left(\frac{d(x)}{\varepsilon}, \mathrm{P}(x)\right)\right) \quad \text { in } \quad \omega_{\varepsilon} \tag{2.11}
\end{equation*}
$$

where the profiles $V^{j}$ are defined on $[0,1] \times \Gamma$ :

$$
\begin{aligned}
V^{j}:[0,1] \times \Gamma & \rightarrow \mathbb{C} \\
(z, \sigma) & \mapsto V^{j}(z, \sigma) .
\end{aligned}
$$

In $\omega_{\varepsilon}$, we will work with the variables $(z, \sigma)$.
2.2. Geometrical and analytical tools for the thin layer. We use the following notations (see also [6]) : $x=\left(x_{1}, x_{2}, x_{3}\right)$ denotes the cartesian coordinates in $\mathbb{R}^{3}$, and we set

$$
\partial_{i}=\frac{\partial}{\partial x_{i}}, \quad \partial_{z}=\frac{\partial}{\partial z}, \quad \partial_{z z}=\frac{\partial^{2}}{\partial z^{2}}
$$

By simple calculations there holds (see also [5])

$$
\begin{aligned}
\Delta\left(x \mapsto V\left(\mathrm{P}(x), \frac{d(x)}{\varepsilon}\right)\right)= & \varepsilon^{-2} \partial_{\mathrm{zz}} V(\mathrm{P}(x), \mathrm{z})+\varepsilon^{-1} G_{\varepsilon \mathrm{z}}(\mathrm{P}(x)) \partial_{\mathrm{z}} V(\mathrm{P}(x), \mathrm{z}) \\
& +\Delta_{\Gamma_{\varepsilon z}} V(\mathrm{P}(x), \mathrm{z}) \quad \text { with } \quad \mathrm{z}=\frac{d(x)}{\varepsilon},
\end{aligned}
$$

where $G$ is defined as

$$
G_{s}(\sigma)=\frac{1}{\gamma_{s}(\sigma)} \frac{\partial \gamma_{s}}{\partial s}(\sigma)
$$

Here $\gamma_{s}(\sigma)=\operatorname{det}(\operatorname{Id}+s \operatorname{dn}(\sigma))$ and for $\sigma \in \Gamma$, the differential $\operatorname{dn}(\sigma)$ is a linear map from the tangent plane $\mathrm{T}_{\sigma} \Gamma$ of $\Gamma$ at the point $\sigma$ into $\mathrm{T}_{\mathrm{n}(\sigma)} \mathrm{S}^{2}$ where $\mathrm{S}^{2}$ is the unit sphere.

## Proposition 2.1.

$$
\frac{1}{2} \leq \gamma_{s}(\sigma) \leq \frac{3}{2}
$$

In addition, since for $x \in \omega_{\varepsilon}$, we have: $x=\mathrm{P}(x)+\varepsilon \frac{\mathrm{d}(x)}{\varepsilon} \nu(\mathrm{P}(x))=\sigma+\varepsilon z \nu(\sigma)$, we perform the Taylor expansion of the functions $\varphi: \omega_{\varepsilon} \longrightarrow \mathbb{C}$ writing:

$$
\varphi(x)=\varphi(\mathrm{P}(x))+\mathrm{d}(x) \frac{\partial \varphi}{\partial \nu}(\mathrm{P}(x))+\ldots,=\varphi(\sigma)+\varepsilon z \frac{\partial \varphi}{\partial \nu}(\sigma)+\ldots
$$

Then we insert the Ansatz (2.9)-(2.10)-(2.11) in equations (2.1)-(2.8) and we perform the identification of terms with the same power in $\varepsilon$.
2.3. Characterisation of the profiles at order $\varepsilon^{0}$. Taking the order $\varepsilon^{0}$ in (2.1), we obtain that:

$$
\begin{equation*}
-\Delta U^{0}-\xi U^{0}=f \quad \text { in } \quad \Omega_{i} . \tag{2.12}
\end{equation*}
$$

Taking the order $\varepsilon^{0}$ in (2.3), we have:

$$
\begin{equation*}
-\Delta W^{0}-\xi W^{0}=f \quad \text { in } \quad \Omega_{e} . \tag{2.13}
\end{equation*}
$$

From (2.2) at order $\varepsilon^{-2}$, we obtain that:

$$
\begin{equation*}
-\partial_{z z} V^{0}+V^{0}+W^{0}(\sigma)=0 \quad \text { in } \quad[0,1] \times \Gamma . \tag{2.14}
\end{equation*}
$$

By solving this ODE, we obtain that:

$$
\begin{equation*}
V^{0}(z, \sigma)=-W^{0}(\sigma)+\alpha_{0}(\sigma) \sinh z+\beta_{0}(\sigma) \cosh z . \tag{2.15}
\end{equation*}
$$

From (2.6) at order $\varepsilon^{-1}$, we have

$$
0=\partial_{\mathbf{z}} V^{0}(0, \sigma),
$$

so we obtain that $\beta_{0}(\sigma)=0$ for all $\sigma \in \Gamma$.
From (2.7) at order $\varepsilon^{-1}$, we obtain that

$$
0=\partial_{z} V^{0}(1, \sigma),
$$

so, with (2.15), we obtain that

$$
\beta_{0}(\sigma)=0
$$

Thus, we have:

$$
\begin{equation*}
V^{0}(z, \sigma)=-W^{0}(\sigma) \tag{2.16}
\end{equation*}
$$

Since $W^{0}$ satisfies (2.13), then $W^{0}$ and $\frac{\partial W^{0}}{\partial \nu}$ have no jump at $\Gamma_{\varepsilon}$. In particular, writing (2.5), we obtain that

$$
V^{0}(1, \sigma)=0
$$

Coupling this relation with (2.16), we obtain that

$$
\begin{equation*}
V^{0}(z, \sigma)=0 \text { for } z \in[0,1] \text { and } \sigma \in \Gamma . \tag{2.17}
\end{equation*}
$$

Therefore, with (2.16),

$$
\begin{equation*}
W^{0}(\sigma)=0 \text { for all } \sigma \in \Gamma \tag{2.18}
\end{equation*}
$$

Writing (2.5) at order $\varepsilon^{0}$, we have

$$
U^{0}(\sigma)=V^{0}(0, \sigma) \text { for } \sigma \in \Gamma .
$$

So, with (2.17),

$$
\begin{equation*}
U^{0}(\sigma)=0 \text { for all } \sigma \in \Gamma \tag{2.19}
\end{equation*}
$$

Equation (2.8) at order $\varepsilon^{0}$ yields $W^{0}=0$ on $\partial \Omega$.
Therefore, coupling (2.12) and (2.19), $U^{0}$ is characterized by:

$$
\left\{\begin{array}{l}
-\Delta U^{0}-\xi U^{0}=f \text { in } \Omega_{i},  \tag{2.20}\\
U^{0}=0 \text { in } \partial \Omega_{i} .
\end{array}\right.
$$

In addition, coupling (2.13) and (2.18), $W^{0}$ is characterized by:

$$
\left\{\begin{array}{l}
-\Delta W^{0}-\xi W^{0}=f \text { in } \Omega_{e}  \tag{2.21}\\
W^{0}=0 \text { in } \partial \Omega_{e}
\end{array}\right.
$$

2.4. Characterisation of the profiles at order $\varepsilon^{1}$. Taking the order $\varepsilon^{1}$ in (2.1), we obtain that:

$$
\begin{equation*}
-\Delta U^{1}-\xi U^{1}=0 \quad \text { in } \quad \Omega_{i} \tag{2.22}
\end{equation*}
$$

Taking the order $\varepsilon^{1}$ in (2.3), we have:

$$
\begin{equation*}
-\Delta W^{1}-\xi W^{1}=0 \quad \text { in } \quad \Omega_{e} \tag{2.23}
\end{equation*}
$$

From (2.2) at order $\varepsilon^{-1}$, we obtain that:

$$
\begin{equation*}
-\partial_{\mathrm{zz}} V^{1}+V^{1}+W^{1}(\sigma)+\mathrm{z} \frac{\partial W^{0}}{\partial \nu}(\sigma)=0 \quad \text { in } \quad[0,1] \times \Gamma \tag{2.24}
\end{equation*}
$$

By solving this ODE, we obtain that:

$$
\begin{equation*}
V^{1}(\mathbf{z}, \sigma)=-W^{1}(\sigma)-\mathbf{z} \frac{\partial W^{0}}{\partial \nu}(\sigma)+\alpha_{1}(\sigma) \sinh \mathbf{z}+\beta_{1}(\sigma) \cosh \mathbf{z} \tag{2.25}
\end{equation*}
$$

From (2.6) at order $\varepsilon^{0}$, we have

$$
\frac{\partial U^{0}}{\partial \nu}(\sigma)=\partial_{\mathbf{z}} V^{1}(0, \sigma)(\sigma)+\frac{\partial W^{0}}{\partial \nu}(\sigma)
$$

so we obtain that

$$
\begin{equation*}
\alpha_{1}(\sigma)=\frac{\partial U^{0}}{\partial \nu}(\sigma) \text { for all } \sigma \in \Gamma \tag{2.26}
\end{equation*}
$$

From (2.7) at order $\varepsilon^{0}$, we obtain that

$$
0=\partial_{\mathbf{z}} V^{1}(1, \sigma)=-\frac{\partial W^{0}}{\partial \nu}(\sigma)+\alpha_{1}(\sigma) \cosh 1+\beta_{1}(\sigma) \sinh 1
$$

so, with (2.25) and (2.26), we obtain that

$$
\beta_{1}(\sigma)=\frac{1}{\sinh 1}\left(\frac{\partial W^{0}}{\partial \nu}(\sigma)-(\cosh 1) \frac{\partial U^{0}}{\partial \nu}(\sigma)\right) .
$$

Thus, we have:
(2.27) $V^{1}(z, \sigma)=-W^{1}(\sigma)+\frac{\partial W^{0}}{\partial \nu}(\sigma)\left(-\mathbf{z}+\frac{\cosh \mathbf{z}}{\sinh 1}\right)+\frac{\partial U^{0}}{\partial \nu}(\sigma)(\sinh \mathbf{z}-\operatorname{coth} 1 \cosh \mathbf{z})$.

Using (2.5) at order $\varepsilon^{1}$, we obtain that $V^{1}(1, \sigma)=0$, so:

$$
\begin{equation*}
W^{1}(\sigma)=(-1+\operatorname{coth} 1) \frac{\partial W^{0}}{\partial \nu}(\sigma)-\frac{1}{\sinh 1} \frac{\partial U^{0}}{\partial \nu}(\sigma) \tag{2.28}
\end{equation*}
$$

Therefore, we obtain that
$V^{1}(\mathrm{z}, \sigma)=\frac{\partial W^{0}}{\partial \nu}(\sigma)\left(1-\mathrm{z}+\frac{\cosh \mathrm{z}-\cosh 1}{\sinh 1}\right)+\frac{\partial U^{0}}{\partial \nu}(\sigma)\left(\sinh \mathrm{z}-\operatorname{coth} 1 \cosh \mathrm{z}+\frac{1}{\sinh 1}\right)$.
Using (2.5) at order $\varepsilon^{1}$, we obtain that for $\sigma \in \Gamma$ :

$$
\begin{equation*}
U^{1}(\sigma)=V^{1}(0, \sigma)+W^{1}(\sigma)=\frac{1}{\sinh 1} \frac{\partial W^{0}}{\partial \nu}(\sigma)-\frac{\cosh 1}{\sinh 1} \frac{\partial U^{0}}{\partial \nu}(\sigma) \tag{2.30}
\end{equation*}
$$

Therefore, coupling (2.22) and (2.30), $U^{1}$ is characterized by:

$$
\left\{\begin{array}{l}
-\Delta U^{1}-\xi U^{1}=0 \text { in } \Omega_{i},  \tag{2.31}\\
U^{1}(\sigma)=\frac{1}{\sinh 1} \frac{\partial W^{0}}{\partial \nu}(\sigma)-\operatorname{coth} 1 \frac{\partial U^{0}}{\partial \nu}(\sigma) \text { in } \partial \Omega_{i}
\end{array}\right.
$$

In addition, coupling (2.23), (2.28), and (2.8) at order $\varepsilon^{1}, W^{1}$ is characterized by:

$$
\left\{\begin{array}{l}
-\Delta W^{1}-\xi W^{1}=f \text { in } \Omega_{e}  \tag{2.32}\\
W^{1}(\sigma)=(-1+\operatorname{coth} 1) \frac{\partial W^{0}}{\partial \nu}(\sigma)-\frac{1}{\sinh 1} \frac{\partial U^{0}}{\partial \nu}(\sigma) \text { in } \Gamma \\
W^{1}=0 \text { on } \partial \Omega
\end{array}\right.
$$

We denote by $\mathbf{H}$ the space of the functions $f \in L^{2}(\Omega)$ such that the restrictions of $f$ respectively in $\Omega_{i}$ and $\Omega_{e}$ are respectively in $\mathrm{H}^{1}\left(\Omega_{i}\right)$ and $\mathrm{H}^{1}\left(\Omega_{e}\right)$, endowed with the Hilbert norm:

$$
\|f\|_{\mathbf{H}}=\left(\left\|f_{\mid \Omega_{i}}\right\|_{\mathrm{H}^{1}\left(\Omega_{i}\right)}^{2}+\left\|f_{\mid \Omega_{e}}\right\|_{\mathrm{H}^{1}\left(\Omega_{e}\right)}^{2}\right)^{\frac{1}{2}} .
$$

2.5. Equation for the remainder. We write:

$$
\begin{cases}u^{\varepsilon}(x)=U^{0}(x)+\varepsilon U^{1}(x)+\varepsilon a^{\varepsilon}(x) & \text { for } x \in \Omega_{i}, \\ v^{\varepsilon}(x)=W^{0}(x)+\varepsilon W^{1}(x)+\varepsilon V^{1}\left(\mathrm{P}(x), \frac{\mathrm{d}(x)}{\varepsilon}\right)+\varepsilon b^{\varepsilon}(x) & \text { for } x \in \omega_{\varepsilon}, \\ w^{\varepsilon}(x)=W^{0}(x)+\varepsilon W^{1}(x)+\varepsilon c^{\varepsilon}(x) & \text { for } x \in \mathcal{U}_{\varepsilon} .\end{cases}
$$

The remainder term satisfies:

$$
\begin{array}{ll}
-\Delta a^{\varepsilon}-\xi a^{\varepsilon}=0 & \text { in } \Omega_{i} \\
-\Delta b^{\varepsilon}-\xi b^{\varepsilon}+\frac{1}{\varepsilon^{2}} b^{\varepsilon}=d^{\varepsilon} & \text { in } \omega_{\varepsilon} \\
-\Delta c^{\varepsilon}-\xi c^{\varepsilon}=0 & \text { in } \mathcal{U}_{\varepsilon} \\
a^{\varepsilon}=b^{\varepsilon} & \text { on } \Gamma \\
b^{\varepsilon}=c^{\varepsilon} & \text { on } \Gamma_{\varepsilon} \\
\partial_{\mathrm{n}} a^{\varepsilon}=\partial_{\mathrm{n}} b^{\varepsilon}+g^{\varepsilon} & \text { on } \Gamma \\
\partial_{\mathrm{n}} b^{\varepsilon}=\partial_{\mathrm{n}} c^{\varepsilon} & \text { on } \Gamma_{\varepsilon} \\
c^{\varepsilon}=0 & \text { on } \partial \Omega, \tag{2.40}
\end{array}
$$

where $d^{\varepsilon}$ is defined on $\omega_{\varepsilon}$ by $d^{\varepsilon}=d_{1}^{\varepsilon}+d_{2}^{\varepsilon}+d_{3}^{\varepsilon}+d_{4}^{\varepsilon}$ with:

$$
\begin{aligned}
& d_{1}^{\varepsilon}(x)=-\frac{1}{\varepsilon^{3}}\left(W^{0}(x)-\mathrm{d}(x) \frac{\partial W^{0}}{\partial \nu}(\mathrm{P}(x))\right), \\
& d_{2}^{\varepsilon}(x)=-\frac{1}{\varepsilon^{2}}\left(W^{1}(x)-W^{1}(\mathrm{P}(x))\right), \\
& d_{3}^{\varepsilon}(x)=\frac{1}{\varepsilon} G_{\mathrm{d}(x)} \partial_{z} V^{1}\left(\frac{\mathrm{~d}(x)}{\varepsilon}, \mathrm{P}(x)\right)+\xi V^{1}\left(\frac{\mathrm{~d}(x)}{\varepsilon}, \mathrm{P}(x)\right), \\
& d_{4}^{\varepsilon}(x)=\Delta_{\Gamma_{\mathrm{d}(x)}} V^{1}\left(\frac{\mathrm{~d}(x)}{\varepsilon}, \mathrm{P}(x)\right),
\end{aligned}
$$

and where $g^{\varepsilon}$ is defined on $\Gamma$ by:

$$
g^{\varepsilon}=\frac{\partial W^{1}}{\partial \nu}-\frac{\partial U^{1}}{\partial \nu}
$$

2.6. Estimates of the right-hand-side terms. Since $\xi$ is in the circle of center $\lambda^{0}$ and of radius $\eta_{0}$, there exists a constant $C$ such that for all $f \in \mathbf{H}$,

$$
\begin{equation*}
\left\|U^{0}\right\|_{H^{3}\left(\Omega_{i}\right)}+\left\|W^{0}\right\|_{H^{3}\left(\Omega_{e}\right)} \leq C\|f\|_{\mathbf{H}}, \tag{2.41}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|U^{1}\right\|_{H^{2}\left(\Omega_{i}\right)}+\left\|W^{1}\right\|_{H^{2}\left(\Omega_{e}\right)} \leq C\|f\|_{\mathbf{H}} \tag{2.42}
\end{equation*}
$$

From (2.42), by trace theory, we obtain that:

$$
\begin{aligned}
\left\|g^{\varepsilon}\right\|_{\mathrm{L}^{2}(\Gamma)} & \leq\left\|\frac{\partial W^{1}}{\partial \nu}\right\|_{H^{\frac{1}{2}}(\Gamma)}+\left\|\frac{\partial U^{1}}{\partial \nu}\right\|_{H^{\frac{1}{2}}(\Gamma)}, \\
& \leq K\left(\left\|\nabla W^{1}\right\|_{\mathrm{H}^{1}\left(\Omega_{e}\right)}+\left\|\nabla U^{1}\right\|_{\mathrm{H}^{1}\left(\Omega_{i}\right)}\right)
\end{aligned}
$$

so,

$$
\begin{equation*}
\left\|g^{\varepsilon}\right\|_{\mathrm{L}^{2}(\Gamma)} \leq K\|f\|_{\mathbf{H}} \tag{2.43}
\end{equation*}
$$

In order to estimate $d_{1}^{\varepsilon}$ and $d_{2}^{\varepsilon}$, we prove the following proposition:
Proposition 2.2. We denote by $N$ the dimension of the space. We define $2^{\star}$ by $\frac{1}{2}=\frac{1}{N}+\frac{1}{2^{\star}}$ (for $\left.N=2,2^{\star}=+\infty\right)$. Then for all $p \in\left[2,2^{\star}\left[\left(\left[2,2^{\star}\right]\right.\right.\right.$ for $\left.N \neq 2\right)$, there exists a constant $K_{p}$ such that for all $\varepsilon>0$, we have:

$$
\forall v \in H^{2}\left(\Omega_{e}\right),\left\|\frac{1}{\varepsilon^{2}}(v(\cdot)-v(\mathrm{P}(\cdot)))\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)} \leq \frac{K_{p}}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\|v\|_{H^{2}\left(\Omega_{e}\right)}
$$

and

$$
\forall v \in H^{3}\left(\Omega_{e}\right),\left\|\frac{1}{\varepsilon^{3}}\left(v(\cdot)-v(\mathrm{P}(\cdot))-\frac{\partial v}{\partial \nu}(\mathrm{P}(\cdot))\right)\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)} \leq \frac{K_{p}}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\|v\|_{H^{3}\left(\Omega_{e}\right)}
$$

Proof. We assume that $v$ is smooth. For $x \in \omega_{\mathcal{E}}$, we have $x=\mathrm{P}(x)+\mathrm{d}(x) \nu(\mathrm{P}(x))$, so, using the variables $s=\mathrm{d}(x)$ and $\sigma=\mathrm{P}(x)$ and Taylor's formula:

$$
\begin{aligned}
\|v(\cdot)-v(\mathrm{P}(\cdot))\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2} & =\int_{s=0}^{\varepsilon} \int_{\sigma \in \Gamma} \gamma_{s}(\sigma)|v(\sigma+s \nu(\sigma))-v(\sigma)|^{2} d \sigma \mathrm{~d} s \\
& =\int_{s=0}^{\varepsilon} \int_{\sigma \in \Gamma} \gamma_{s}(\sigma)\left|\int_{\tau=0}^{s} \nabla v(\sigma+\tau \nu(\sigma)) \cdot \nu(\sigma) d \tau\right|^{2} d s d \sigma \\
& \leq \int_{s=0}^{\varepsilon} \int_{\sigma \in \Gamma} \gamma_{s}(\sigma)\left(\int_{0}^{s}|\nabla v(\sigma+\tau \nu(\sigma))|^{2} d \tau\right) s d s d \sigma \\
& \leq \varepsilon^{2} 3 \int_{\omega_{\varepsilon}}|\nabla v|^{2} d x \quad \text { (using Proposition 2.1). }
\end{aligned}
$$

By Sobolev embedding, for all $p \in\left[2,2^{\star}\left[\left(\left[2,2^{\star}\right]\right.\right.\right.$ for $\left.N \neq 2\right)$,

$$
\|\nabla v\|_{L^{p}(\mathcal{U})} \leq C_{p}\|v\|_{H^{2}(\mathcal{U})} .
$$

So, by Hölder inequality,

$$
\begin{aligned}
\|\nabla v\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)} & \leq\|\nabla v\|_{\mathrm{L}^{p}\left(\omega_{\varepsilon}\right)}\left(\operatorname{meas}\left(\omega_{\varepsilon}\right)\right)^{\frac{1}{2}-\frac{1}{p}} \\
& \leq(K \varepsilon)^{\frac{1}{2}-\frac{1}{p}} C_{p}\|v\|_{H^{2}(\mathcal{U})} .
\end{aligned}
$$

Therefore,

$$
\left\|\frac{1}{\varepsilon^{2}}(v(\cdot)-v(\mathrm{P}(\cdot)))\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)} \leq \frac{\sqrt{3} K C_{p}}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\|v\|_{H^{2}(\mathcal{U})} .
$$

In addition, by Taylor expansion at order 2 , for $\sigma \in \Gamma$ and $s \in[0, \varepsilon]$, we have:

$$
\begin{aligned}
\left|v(\sigma+s \nu(\sigma))-v(\sigma)-\frac{\partial v}{\partial \nu}(\sigma) \mathrm{d}(x)\right| & =\left|\int_{0}^{s}(s-\tau) D^{2} v(\sigma+\tau s)(\nu(\sigma), \nu(\sigma)) d \tau\right| \\
& \leq s \int_{0}^{s}\left|D^{2} v(\sigma+\tau s)\right| d \tau
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\|v(x)-v(\mathrm{P}(x))-\frac{\partial v}{\partial \nu}(\mathrm{P}(x)) \mathrm{d}(x)\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2} & =\int_{0}^{\varepsilon} \int_{\Gamma} \gamma_{s}(\sigma)\left|v(\sigma+s \nu(\sigma))-v(\sigma)-\frac{\partial v}{\partial \nu}(\sigma) s\right|^{2} d s d \sigma \\
& \leq \int_{0}^{\varepsilon} \int_{\Gamma} \gamma_{s}(\sigma)\left|s \int_{0}^{s}\right| D^{2} v(\sigma+\tau s)|d \tau|^{2} d s d \sigma \\
& \leq \int_{0}^{\varepsilon} \int_{\Gamma} \gamma_{s}(\sigma) s^{3} \int_{0}^{s}\left|D^{2} v(\sigma+\tau s)\right|^{2} d \tau d s d \sigma \\
& \leq 3 \varepsilon^{4} \int_{\omega_{\varepsilon}}\left|D^{2} v(x)\right|^{2} d x
\end{aligned}
$$

By Sobolev embedding, for all $p \in\left[2,2^{\star}\left[\left(\left[2,2^{\star}\right]\right.\right.\right.$ for $\left.N \neq 2\right)$,

$$
\left\|D^{2} v\right\|_{\mathrm{L}^{p}(\mathcal{U})} \leq C_{p}\|v\|_{H^{3}(\mathcal{U})}
$$

so

$$
\left\|D^{2} v\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)} \leq\left\|D^{2} v\right\|_{\mathrm{L}^{p}(\mathcal{U})}\left(\operatorname{meas}\left(\omega_{\varepsilon}\right)\right)^{\frac{1}{2}-\frac{1}{p}} \leq C_{p}\|v\|_{H^{3}(\mathcal{U})}(K \varepsilon)^{\frac{1}{2}-\frac{1}{p}}
$$

thus,

$$
\left\|\frac{1}{\varepsilon^{3}}\left(v(x)-v(\mathrm{P}(x))-\frac{\partial v}{\partial \nu}(\mathrm{P}(x)) \mathrm{d}(x)\right)\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)} \leq \frac{\sqrt{3} K C_{p}}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\|v\|_{H^{3}(\mathcal{U})} .
$$

We take $K_{p}=\sqrt{3} K C_{p}$, and we conclude the proof of Proposition 2.2 by a density argument.

Corollary 2.3. For all $p \in\left[2,2^{\star}\right]\left(\left[2,+\infty[\right.\right.$ in the case $N=2)$, there exists $K_{p}$ such that for all $\varepsilon>0$, for all $f \in \mathbf{H}$,

$$
\left\|d_{1}^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)} \leq \frac{K_{p}}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\|f\|_{\mathbf{H}}
$$

and

$$
\left\|d_{2}^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)} \leq \frac{K_{p}}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\|f\|_{\mathbf{H}}
$$

From (2.29), we have:

$$
\begin{aligned}
\left\|d_{3}^{\varepsilon}\right\|_{L^{2}\left(\omega_{\varepsilon}\right)}^{2} & =\int_{0}^{\varepsilon} \int_{\Gamma} \gamma_{s}(\sigma)\left|\frac{1}{\varepsilon} G_{s}(\sigma) \partial_{z} V^{1}\left(\frac{s}{\varepsilon}, \sigma\right)+\xi V^{1}\left(\frac{s}{\varepsilon}, \sigma\right)\right|^{2} d s d \sigma \\
& \leq \frac{K}{\varepsilon^{2}} \int_{0}^{\varepsilon} \int_{\Gamma}\left(\left|\frac{\partial W^{0}}{\partial \nu}(\sigma)\right|^{2}+\left|\frac{\partial U^{0}}{\partial \nu}(\sigma)\right|^{2}\right) d \sigma d s, \\
& \leq \frac{K}{\varepsilon}\left(\left\|W^{0}\right\|_{H^{3}\left(\Omega_{e}\right)}^{2}+\left\|U^{0}\right\|_{H^{3}\left(\Omega_{i}\right)}^{2}\right) .
\end{aligned}
$$

Therefore

$$
\begin{equation*}
\left\|d_{3}^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{\varepsilon}\right)} \leq \frac{C}{\varepsilon^{\frac{1}{2}}}\|f\|_{\mathbf{H}} . \tag{2.44}
\end{equation*}
$$

2.7. Variational estimates for the remainder term. Multiplying (2.33) by the conjugate of $a^{\varepsilon}$ and integrating on $\Omega_{i}$, multiplying (2.34) by the conjugate of $b^{\varepsilon}$ and integrating on $\omega_{\varepsilon}$, multiplying (2.35) by the conjugate of $c^{\varepsilon}$ and integrating on $\mathcal{U}_{\mathcal{E}}$, after integrations by part and after summation, we obtain:

$$
\begin{aligned}
& \left\|\nabla a^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+\left\|\nabla c^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}^{2}+\frac{1}{\varepsilon^{2}}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}= \\
& \quad \int_{\Gamma} g^{\varepsilon} \overline{b^{\varepsilon}}+\int_{\omega_{\varepsilon}} d^{\varepsilon} \overline{b^{\varepsilon}}-\xi\left(\left\|a^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+\left\|c^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}^{2}\right) .
\end{aligned}
$$

So,

$$
\begin{array}{r}
\left\|\nabla a^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+\left\|\nabla c^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}^{2}+\frac{1}{\varepsilon^{2}}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2} \leq \\
|\xi|\left(\left\|a^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+\left\|c^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}^{2}\right)  \tag{2.45}\\
+\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}\left\|d_{1}^{\varepsilon}+d_{2}^{\varepsilon}+d_{3}^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}+\left\|g^{\varepsilon}\right\|_{\mathrm{L}^{2}(\Gamma)}\| \|^{\varepsilon} \|_{\mathrm{L}^{2}(\Gamma)} \\
+\left|\int_{\omega_{\varepsilon}} d_{4}^{\varepsilon} \overline{\bar{\varepsilon}}\right| .
\end{array}
$$

We claim the following lemma proved in Section 3.1.
Lemma 2.4. There exists a constant $C>0$ independent of $\varepsilon$ such that for any $\xi \in \mathcal{C}\left(\lambda^{0}, \eta\right)$, for any $u \in \mathrm{H}^{1}\left(\Omega_{i}\right)$ such that $-\Delta u-\xi u \in \mathrm{~L}^{2}\left(\Omega_{e}\right)$, we have :

$$
\begin{equation*}
\|u\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)} \leqslant C\left(\|u\|_{\mathrm{L}^{2}(\Gamma)}+\|(-\Delta u-\xi u)\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}\right) \tag{2.46}
\end{equation*}
$$

and for any $v \in \mathrm{H}^{1}\left(\mathcal{U}_{\varepsilon}\right)$ such that $v=0$ on $\partial \Omega$ and such that $-\Delta v-\xi v \in \mathrm{~L}^{2}\left(\mathcal{U}_{e}\right)$, we have :

$$
\begin{equation*}
\|v\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)} \leqslant C\left(\|v\|_{\mathrm{L}^{2}\left(\Gamma_{\varepsilon}\right)}+\|(-\Delta v-\xi v)\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}\right) . \tag{2.47}
\end{equation*}
$$

We claim the following Lemma, proved in [4] (see Inequality (2.1) in Proposition 2.1., page 822).

Lemma 2.5. There exists constants $C, \varepsilon_{0}>0$ such that for all $\varepsilon \in\left(0, \varepsilon_{0}\right)$, for all $\boldsymbol{q} \in H^{1}\left(\omega_{\varepsilon}\right)$,

$$
\|\mathrm{q}\|_{\mathrm{L}^{2}\left(\partial \omega_{\varepsilon}\right)}^{2} \leqslant C\left(\frac{1}{\varepsilon}\|\mathrm{q}\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+\|\mathrm{q}\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}\|\nabla \mathrm{q}\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}\right)
$$

Since $a^{\varepsilon}$ (resp. $c^{\varepsilon}$ ) satisfies (2.33) in $\Omega_{i}$ (resp. (2.35) in $\mathcal{U}_{\varepsilon}$ ), we obtain with Lemma 2.4 and the transmission condition (2.36) (resp. (2.37)) that there exists $C$ independent of $\varepsilon>0$ and $\xi \in \mathcal{C}\left(\lambda^{0}, \eta\right)$ such that:

$$
\left\|a^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|c^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}^{2} \leq C\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\partial \omega_{\varepsilon}\right)}^{2} .
$$

Using now Lemma 2.5 , we obtain that:

$$
\begin{equation*}
\left\|a^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|c^{\varepsilon}\right\|_{\left.\mathrm{L}^{2} \mathcal{U}_{\varepsilon}\right)}^{2} \leq \frac{C}{\varepsilon}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+C\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)} . \tag{2.48}
\end{equation*}
$$

Using Corollary 2.3, Estimate (2.44), we obtain that there exists a constant $C$ such that for all $\varepsilon>0$, for all $\xi \in \mathcal{C}\left(\lambda^{0}, \eta\right)$ and for all $f \in \mathbf{H}$,

$$
\begin{equation*}
\left\|d_{1}^{\varepsilon}+d_{2}^{\varepsilon}+d_{3}^{\varepsilon}\right\|_{L^{2}\left(\omega_{\varepsilon}\right)} \leq \frac{C}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\|f\|_{\mathbf{H}} \tag{2.49}
\end{equation*}
$$

In addition, coupling Lemma 2.5 and Estimate (2.43), we obtain that:

$$
\begin{equation*}
\left\|g^{\varepsilon}\right\|_{\mathrm{L}^{2}(\Gamma)}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}(\Gamma)} \leq C \sqrt{\varepsilon}\|f\|_{\mathbf{H}} \frac{1}{\varepsilon}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}+C \varepsilon^{\frac{1}{2}}\|f\|_{\mathbf{H}}\left(\frac{1}{\varepsilon}\left\|b_{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}\right)^{\frac{1}{2}}\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{\frac{1}{2}} . \tag{2.50}
\end{equation*}
$$

Let us estimate the last right-hand-side term in (2.45). We denote by $B^{\varepsilon}=b^{\varepsilon} \circ \Psi$, where:

$$
\begin{aligned}
& \Psi:\left[0, \eta_{0}\right] \times \Gamma \rightarrow \omega_{\eta_{0}} \\
&(s, \sigma) \mapsto \quad \sigma+s \nu(\sigma) \\
& \int_{\omega_{\varepsilon}} d_{4}^{\varepsilon} \overline{b^{\varepsilon}}=\int_{s=0}^{\varepsilon} \int_{\sigma \in \Gamma} \gamma_{s}(\sigma)\left(\Delta_{\Gamma_{s}} V^{1}\left(\frac{s}{\varepsilon}, \sigma\right)\right) \overline{B^{\varepsilon}(s, \sigma)} d s d \sigma \\
&=-\int_{s=0}^{\varepsilon} \int_{\sigma \in \Gamma} \gamma_{s}(\sigma) \nabla_{\Gamma_{s}} V^{1}\left(\frac{s}{\varepsilon}, \sigma\right) \overline{\nabla_{\Gamma_{s}} B^{\varepsilon}(s, \sigma)} d s
\end{aligned}
$$

So,

$$
\begin{aligned}
\left|\int_{\omega_{\varepsilon}} d_{4}^{\bar{\varepsilon}} \overline{b^{\varepsilon}}\right| & \leq \int_{s=0}^{\varepsilon} \int_{\sigma \in \Gamma} \gamma_{s}(\sigma)\left|\nabla_{\Gamma_{s}} V^{1}\left(\frac{s}{\varepsilon}, \sigma\right)\right|\left|\nabla_{\Gamma_{s}} B^{\varepsilon}(s, \sigma)\right| d s d \sigma \\
& \leq\left\|\nabla b^{\varepsilon}\right\|_{L^{2}\left(\omega_{\varepsilon}\right)}\left\|\nabla_{\Gamma_{s}} V^{1}\left(\frac{s}{\varepsilon}, \sigma\right)\right\|_{L^{2}\left(\omega_{\varepsilon}\right)}, \\
& \leq\left\|\nabla b^{\varepsilon}\right\|_{L^{2}\left(\omega_{\varepsilon}\right)}\left(\left\|\frac{\partial U^{0}}{\partial \nu}\right\|_{\mathrm{H}^{1}(\Gamma)}+\left\|\frac{\partial W^{0}}{\partial \nu}\right\|_{\mathrm{H}^{1}(\Gamma)}\right) \sqrt{\varepsilon},
\end{aligned}
$$

thus, using (2.41), we obtain:

$$
\begin{equation*}
\left|\int_{\omega_{\varepsilon}} d_{4}^{\varepsilon} \overline{\bar{b}}\right| \leq C \sqrt{\varepsilon}\left\|\nabla b^{\varepsilon}\right\|_{L^{2}\left(\omega_{\varepsilon}\right)}\|f\|_{\mathbf{H}} . \tag{2.51}
\end{equation*}
$$

Therefore, from (2.45), (2.48), (2.49), (2.50), and (2.51), we obtain that there exists a constant $C$ independent of $\varepsilon>0, \xi \in \mathcal{C}\left(\lambda^{0}, \eta\right)$ and $f \in \mathbf{H}$ such that

$$
\begin{array}{r}
\left\|\nabla a^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+\left\|\nabla c^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}^{2}+\frac{1}{\varepsilon^{2}}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2} \leq \\
\frac{C}{\varepsilon}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+C\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}+\frac{C}{\varepsilon^{\frac{1}{2}+\frac{1}{p}}}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}\|f\|_{\mathbf{H}} \\
+C \sqrt{\varepsilon}\|f\|_{\mathbf{H}} \frac{1}{\varepsilon}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}+C \varepsilon^{\frac{1}{2}}\|f\|_{\mathbf{H}}\left(\frac{1}{\varepsilon}\left\|b_{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}\right)^{\frac{1}{2}}\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{\frac{1}{2}}+C \varepsilon^{\frac{1}{2}}\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}\|f\|_{\mathbf{H}} \\
\leq \frac{1}{2} \frac{1}{\varepsilon^{2}}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+\frac{1}{2}\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+C \varepsilon^{1-\frac{2}{p}}\|f\|_{\mathbf{H}}^{2} \quad \text { (by Young inequality). }
\end{array}
$$

So, after absorption, we obtain that there exists $C$ such that:

$$
\begin{equation*}
\left\|\nabla a^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|\nabla b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2}+\left\|\nabla c^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}^{2}+\frac{1}{\varepsilon^{2}}\left\|b^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\omega_{\varepsilon}\right)}^{2} \leq C \varepsilon^{1-\frac{2}{p}}\|f\|_{\mathbf{H}}^{2} . \tag{2.52}
\end{equation*}
$$

Reusing (2.48), we obtain also that:

$$
\begin{equation*}
\left\|a^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\Omega_{i}\right)}^{2}+\left\|c^{\varepsilon}\right\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}^{2} \leq C \varepsilon^{2-\frac{2}{p}}\|f\|_{\mathbf{H}}^{2} \tag{2.53}
\end{equation*}
$$

## 3. Technical Lemma : Uniform a priori estimates

3.1. Geometrical tools in the thin layer. In this part, we refer to [4]. For $\sigma \in \Gamma$, we denote by $\nu(\sigma)$ the outward unitary normal. We remark that $\nu: \Gamma \longrightarrow S^{N-1}$ is smooth since $\Gamma$ is regular. We define $\Psi$ by:

$$
\begin{aligned}
\Psi: \quad] 0, \eta_{0}[\times \Gamma & \rightarrow \omega_{\eta_{0}} \\
(s, \sigma) & \mapsto \sigma+s \nu(\sigma) .
\end{aligned}
$$

Since $\Gamma$ is a smooth compact surface without boundary, we can fix $\eta_{0}>0$ such that $\Psi$ is a $\mathcal{C}^{\infty}$-diffeomorphism and so that for all $\left.\varepsilon \in\right] 0, \eta_{0}[$,

$$
\Psi(] 0, \varepsilon[\times \Gamma)=\omega_{\varepsilon}
$$

We define $\gamma:\left[0, \eta_{0}\right] \times \Gamma \rightarrow \mathbb{R}$ by:

$$
\gamma_{s}(\sigma)=\operatorname{det}(I d+s d \nu(\sigma))
$$

and we have:

$$
\int_{\omega_{\eta_{0}}} F(x) d x=\int_{s=0}^{\eta_{0}} \int_{\sigma \in \Gamma} F(\Psi(s, \sigma)) \gamma_{s}(\sigma) d \sigma d s
$$

Since $\gamma_{0}(\sigma)=1$ for all $\sigma \in \Gamma$, even if it means reducing $\eta_{0}$, we assume that

$$
\begin{equation*}
\forall s \in\left[0, \eta_{0}\left[, \forall \sigma \in \Gamma, \frac{1}{2} \leq \gamma_{s}(\sigma) \leq \frac{3}{2}\right.\right. \tag{3.1}
\end{equation*}
$$

We denote $\nabla_{\Gamma}$, $\operatorname{div}_{\Gamma}$ and $\Delta_{\Gamma}$ the gradient, divergence and Laplacian operators on the submanifold $\Gamma$.

For $\mathrm{v}: \Gamma \longrightarrow \mathbb{C}$ and $s \in\left[0, \eta_{0}\left[\right.\right.$, we define $\nabla_{\Gamma_{s}} \mathrm{v}$ by:

$$
\nabla_{\Gamma_{s}} \mathrm{v}(\sigma)=(I d+s d \nu(\sigma))^{-1} \nabla_{\Gamma} \mathrm{v}(\sigma)
$$

so that: for $v: \omega_{\eta_{0}} \rightarrow \mathbb{C}$, denoting $V=v \circ \Psi$, we have:

$$
\nabla v(x)=\frac{\partial V}{\partial s}(\mathrm{~d}(x), \mathrm{P}(x)) \nu(\mathrm{P}(x))+\left(\nabla_{\Gamma_{\mathrm{d}(x)}} V\right)(\mathrm{d}(x), \mathrm{P}(x))
$$

We define the operator div $\Gamma_{s}$ as the adjoint of $\nabla_{\Gamma_{s}}$ for the $\mathrm{L}^{2}(\Gamma)$-inner product: for $Y: \Gamma \longrightarrow$ $T_{\Gamma}$ a tangent vector field defined on $\Gamma$, we set:

$$
\left(\operatorname{div}_{\Gamma_{s}} Y\right)(\sigma)=\frac{1}{\gamma_{s}(\sigma)} \operatorname{div}_{\Gamma}\left(\gamma_{s}(I d+s d \nu)^{-1} Y\right)(\sigma)
$$

so that if $z: \omega_{\eta_{0}} \longrightarrow \mathbb{R}^{N}$ is a vector field, denoting $Z=z \circ \Psi, Z_{N}(s, \sigma)=(Z(s, \sigma) \cdot \nu(\sigma))$ and $Z_{T}(s, \sigma)=Z(s, \sigma)-Z_{N}(s, \sigma) \nu(\sigma)$, we have:

$$
\operatorname{div} z(x)=\frac{\partial Z_{N}}{\partial s}(\mathrm{~d}(x), \mathrm{P}(x))+G_{\mathrm{d}(x)}(\mathrm{P}(x)) Z_{N}(\mathrm{~d}(x), \mathrm{P}(x))+\left(\operatorname{div}_{\Gamma_{\mathrm{d}(x)}} Z_{T}\right)(\mathrm{d}(x), \mathrm{P}(x))
$$

where

$$
G_{s}(\sigma)=\frac{1}{\gamma_{s}(\sigma)} \frac{\partial \gamma_{s}}{\partial s}(\sigma)
$$

We define the operator $\Delta_{\Gamma_{s}}$ by:

$$
\Delta_{\Gamma_{s}}=\operatorname{div}_{\Gamma_{s}} \circ \nabla_{\Gamma_{s}},
$$

so that for $v: \omega_{\eta_{0}} \rightarrow \mathbb{C}$, denoting $V=v \circ \Psi$, we have:

$$
\Delta v(x)=\frac{\partial^{2} V}{\partial s^{2}}(\mathrm{~d}(x), \mathrm{P}(x))+G_{\mathrm{d}(x)}(\mathrm{P}(x)) \frac{\partial V}{\partial s}(\mathrm{~d}(x), \mathrm{P}(x))+\left(\Delta_{\Gamma_{\mathrm{d}(x)}} V\right)(\mathrm{d}(x), \mathrm{P}(x))
$$

3.2. Geometrical tools for $\mathcal{U}_{\varepsilon}$. Let $\chi: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth non-decreasing function such that:

$$
\begin{aligned}
& \forall x \leq 0, \chi(x)=-1 \\
& \forall x \geq \eta_{0}, \chi(x)=0
\end{aligned}
$$

For $\varepsilon>0$ such that $\varepsilon \leq \frac{\eta_{0}}{4}$, we define $\phi_{\varepsilon}$ by:

$$
\left\{\begin{array}{rl}
\phi_{\varepsilon}: \mathcal{U}_{\varepsilon} & \rightarrow \Omega_{e} \\
x & \mapsto
\end{array} x+\varepsilon \chi(\mathrm{d}(x)-\varepsilon) \nu(\mathrm{P}(x)) .\right.
$$

We remark that if $x \in \mathcal{U}_{\varepsilon}$ with $\mathrm{d}(x) \geq \varepsilon+\eta_{0}$, then $\phi_{\varepsilon}(x)=x$. In addition, $\phi_{\varepsilon}$ is a $\mathcal{C}^{\infty}$ diffeomorphism from $\mathcal{U}_{\varepsilon}$ onto $\Omega_{e}$. We denote by $\phi_{\varepsilon, k}$ the $k$-th coordinate of $\phi_{\varepsilon}$.

For $w: \mathcal{U}_{\varepsilon} \rightarrow \mathbb{C}$, we define $\mathrm{w}: \Omega_{e} \rightarrow \mathbb{C}$ by:

$$
w=w \circ \phi_{\varepsilon} .
$$

We have, for all $j \in\{1, \ldots N\}$,

$$
\frac{\partial w}{\partial x_{j}}=\sum_{k=1}^{N} \frac{\partial \mathrm{w}}{\partial \mathrm{x}_{k}}\left(\phi_{\varepsilon}(x)\right) \frac{\partial \phi_{\varepsilon, k}}{\partial x_{j}}(x)
$$

so

$$
\frac{\partial^{2} w}{\partial x_{j}^{2}}=\sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^{2} \mathrm{w}}{\partial \mathrm{x}_{k} \partial \mathrm{x}_{l}}\left(\phi_{\varepsilon}(x)\right) \frac{\partial \phi_{\varepsilon, k}}{\partial x_{j}}(x) \frac{\partial \phi_{\varepsilon, l}}{\partial x_{j}}(x)+\sum_{k=1}^{N} \frac{\partial \mathrm{w}}{\partial \mathrm{x}_{k}}\left(\phi_{\varepsilon}(x)\right) \frac{\partial^{2} \phi_{\varepsilon, k}}{\partial x_{j}^{2}}(x)
$$

Therefore,

$$
\Delta w(x)=\sum_{k, l} a_{k, l}^{\varepsilon}\left(\phi_{\varepsilon}(x)\right) \frac{\partial^{2} \mathrm{w}}{\partial \mathrm{x}_{k} \partial \mathrm{x}_{l}}\left(\phi_{\varepsilon}(x)\right)+\sum_{k=1}^{N} b_{k}^{\varepsilon}\left(\phi_{\varepsilon}(x)\right) \frac{\partial \mathrm{w}}{\partial \mathrm{x}_{k}}\left(\phi_{\varepsilon}(x)\right),
$$

where $a_{k, l}^{\varepsilon}: \Omega_{e} \longrightarrow \mathbb{R}$ and $b_{k}^{\varepsilon}: \Omega_{e} \longrightarrow \mathbb{R}$ are defined by:

$$
\forall x \in \mathcal{U}_{\varepsilon}, \quad a_{k l}^{\varepsilon}\left(\phi_{\varepsilon}(x)\right)=\nabla \phi_{\varepsilon, k}(x) \cdot \nabla \phi_{\varepsilon, l}(x) \quad \text { and } \quad b_{k}^{\varepsilon}\left(\phi_{\varepsilon}(x)\right)=\Delta \phi_{\varepsilon, k}(x) .
$$

For $\mathrm{x} \in \Omega_{e}$ such that $\mathrm{d}(\mathrm{x}) \geq \varepsilon+\eta_{0}, \phi_{\varepsilon}(\mathrm{x})=\mathrm{x}$ so that $a_{k l}^{\varepsilon}(\mathrm{x})=\delta_{k l}$ and $b_{k}^{\varepsilon}(\mathrm{x})=0$.
For $\mathrm{x} \in \Omega_{e}$ such that $\mathrm{d}(\mathrm{x}) \leq \varepsilon+\eta_{0}$, we have: $x=\phi_{\varepsilon}^{-1}(\mathrm{x}) \in \omega_{\eta_{0}}$ and:

$$
\frac{\partial \phi_{\varepsilon, k}}{\partial x_{j}}(x)=\delta_{j k}+\varepsilon\left(\chi^{\prime}(\mathrm{d}(x)-\varepsilon) \nu_{j}(\mathrm{P}(x)) \nu_{k}(\mathrm{P}(x))+\chi(\mathrm{d}(x)-\varepsilon) \nabla \nu_{k}(\mathrm{P}(x)) \cdot \frac{\partial \mathrm{P}(x)}{\partial x_{j}}\right) .
$$

Thus, $a_{k, l}^{\varepsilon}(\mathrm{x})=\delta_{k j}+{\widetilde{a_{k, l}}}^{\varepsilon}(\mathrm{x})$, where ${\widetilde{a_{k, l}}}^{\varepsilon}(\mathrm{x})$ is of order $\varepsilon$ and tends uniformly to zero when $\varepsilon$ tends to zero. In the same way, $b_{k}^{\varepsilon}(\mathrm{x})$ is of order $\varepsilon$ and tends uniformly to zero when $\varepsilon$ tends to zero. We denote, for $\mathrm{w} \in \mathrm{H}^{2}\left(\Omega_{e}\right)$ :

$$
\mathcal{K}^{\varepsilon} \mathrm{w}=\sum_{(k, l), k \neq l} a_{k l}^{\varepsilon} \frac{\partial^{2} \mathrm{w}}{\partial \mathrm{x}_{k} \partial \mathrm{x}_{l}}+\sum_{k=1}^{N} b_{k}^{\varepsilon} \frac{\partial \mathrm{w}}{\partial \mathrm{x}_{k}},
$$

so that

$$
\Delta w(x)=\left(\Delta \mathbf{w}+\mathcal{K}^{\varepsilon} \mathbf{w}\right)\left(\phi_{\varepsilon}(x)\right)
$$

and there exists a constant $C$ such that for all $\varepsilon>0$, for all $\mathrm{w} \in \mathrm{H}^{2}\left(\Omega_{e}\right)$, we have:

$$
\begin{equation*}
\left\|\mathcal{K}^{\varepsilon} \mathrm{w}\right\|_{\mathrm{L}^{2}\left(\Omega_{e}\right)} \leq C \varepsilon\|\mathrm{w}\|_{\mathrm{H}^{2}\left(\Omega_{e}\right)} \tag{3.2}
\end{equation*}
$$

For $\varepsilon \geq 0$, we denote by $\mathcal{L}^{\varepsilon}$ the operator $-\Delta$ defined on $\mathrm{H}^{2}\left(\mathcal{U}_{\varepsilon}\right) \cap \mathrm{H}_{0}^{1}\left(\mathcal{U}_{\varepsilon}\right)$. This operator is positive and self-adjoint for the $\mathrm{L}^{2}\left(\mathcal{U}_{\mathcal{E}}\right)$-scalar product. Its resolvent is compact, so that its spectrum only contains a sequence of real eigenvalues tending to $+\infty$.

Let us prove the following lemma:
Lemma 3.1. Let $\lambda_{0} \in \operatorname{Spec}\left(\mathcal{H}^{0}\right)$ and $\eta>0$ such that $\operatorname{Spec}\left(\mathcal{H}^{0}\right) \cap \overline{B\left(\lambda_{0}, \eta\right)}=\left\{\lambda_{0}\right\}$. Then there exists $\varepsilon_{0}>0$ such that for all $\left.\varepsilon \in\right] 0, \varepsilon_{0}[$,

$$
\operatorname{Spec}\left(\mathcal{L}^{\varepsilon}\right) \cap \mathcal{C}\left(\lambda_{0}, \eta\right)=\emptyset
$$

Proof. If it is not the case, there exists a sequence of positive numbers $\left(\varepsilon_{n}\right)_{n}$ tending to zero when $n$ tends to $+\infty$, there exists a sequence $\left(\xi_{n}\right)_{n}$ of points of $\mathcal{C}\left(\lambda_{0}, \eta\right)$, and there exists a sequence $\left(u_{n}\right)_{n}$ such that for all $n, u_{n} \in \mathrm{H}^{2}\left(\mathcal{U}_{\varepsilon_{n}}\right) \cap \mathrm{H}_{0}^{1}\left(\mathcal{U}_{\varepsilon_{n}}\right)$ is an eigenvector of $\mathcal{L}^{\varepsilon}$ associated to the eigenvalue $\xi_{n}$.

We denote $\mathrm{u}_{n}=u_{n} \circ\left(\phi_{\varepsilon_{n}}\right)^{-1}$, so that we have:

$$
\begin{equation*}
\mathbf{u}_{n} \in \mathrm{H}^{2}\left(\Omega_{e}\right) \cap \mathrm{H}_{0}^{1}\left(\Omega_{e}\right) \text { and }-\Delta \mathrm{u}_{n}-\mathcal{K}^{\varepsilon_{n}} \mathbf{u}_{n}=\xi_{n} \mathbf{u}_{n} \tag{3.3}
\end{equation*}
$$

Since $\mathbf{u}_{n} \neq 0$, we can assume that $\left\|\mathbf{u}_{n}\right\|_{\mathbf{H}^{2}\left(\Omega_{e}\right)}=1$. Thus we can extract subsequences still denoted $\left(\varepsilon_{n}\right),\left(\xi_{n}\right)$ and $\left(u_{n}\right)$ such that $\xi_{n}$ tends to $\xi_{\infty} \in \mathcal{C}\left(\lambda_{0}, \eta\right)$ and $u_{n}$ tends to $u_{\infty}$ weakly in $\mathrm{H}^{2}\left(\Omega_{e}\right)$ and strongly in $\mathrm{H}^{1}\left(\Omega_{e}\right)$ by compactness of the embedding $\mathrm{H}^{2}\left(\Omega_{e}\right) \subset \mathrm{H}^{1}\left(\Omega_{e}\right)$.

By taking the weak limit in (3.3) and using (3.2), we obtain that

$$
\begin{equation*}
\mathrm{u}_{\infty} \in \mathrm{H}^{2}\left(\Omega_{e}\right) \cap \mathrm{H}_{0}^{1}\left(\Omega_{e}\right) \text { and }-\Delta \mathrm{u}_{\infty}=\xi_{\infty} \mathrm{u}_{\infty} . \tag{3.4}
\end{equation*}
$$

In addition, writing that $-\Delta \mathbf{u}_{n}=\xi_{n} \mathbf{u}_{n}+\mathcal{K}^{\varepsilon_{n}} \mathbf{u}_{n}$, using again (3.2), we obtain that $-\Delta \mathbf{u}_{n}$ tends strongly in $\mathrm{L}^{2}\left(\Omega_{e}\right)$ to $-\Delta \mathrm{u}_{\infty}$, so that $\mathrm{u}_{n}$ tends strongly to $\mathrm{u}_{\infty}$ in $\mathrm{H}^{2}\left(\Omega_{e}\right)$. So, $\left\|\mathrm{u}_{\infty}\right\|_{\mathrm{H}^{2}\left(\Omega_{e}\right)}=1$, thus $u_{\infty} \neq 0$ and by (3.4), we obtain that $\xi_{\infty} \in \operatorname{Spec}\left(\mathcal{L}^{0}\right) \cap \mathcal{C}\left(\lambda_{0}, \eta\right)$, which is false.

With the previous lemma, for $\varepsilon>0$ small enough, for all $\xi \in \mathcal{C}\left(\lambda_{0}, \eta\right), \mathcal{L}^{\varepsilon}-\xi \mathbb{I}$ is inversible. We give a uniform estimate of the resolvent operator in the following lemma:
Lemma 3.2. There exists $\varepsilon_{0}>0$ and $C$ such that for all $\xi \in \mathcal{C}\left(\lambda_{0}, \eta\right)$, for all $\left.\varepsilon \in\right] 0, \varepsilon_{0}[$, for all $f \in \mathrm{~L}^{2}\left(\mathcal{U}_{\varepsilon}\right)$, there exists a unique $u \in \mathrm{H}^{2}\left(\mathcal{U}_{\varepsilon}\right) \cap \mathrm{H}_{0}^{1}\left(\mathcal{U}_{\varepsilon}\right)$ such that $-\Delta u-\xi u=f$ in $\mathcal{U}_{\varepsilon}$. Moreover,

$$
\|u\|_{\mathrm{H}^{2}\left(\mathcal{U}_{\varepsilon}\right)} \leq C\|f\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)} .
$$

Proof. If it is not the case, there exists a sequence $\left(\varepsilon_{n}\right)$ tending to 0 , a sequence $\left(\xi_{n}\right)_{n}$ of points of $\mathcal{C}\left(\lambda_{0}, \eta\right)$, a sequence $\left(f_{n}\right)$ with $f_{n} \in \mathrm{~L}^{2}\left(\mathcal{U}_{\varepsilon_{n}}\right)$ and $u_{n} \in \mathrm{H}^{2}\left(\mathcal{U}_{\varepsilon_{n}}\right) \cap \mathrm{H}_{0}^{1}\left(\mathcal{U}_{\varepsilon_{n}}\right)$ such that $-\Delta u_{n}$ $\xi_{n} u_{n}=f_{n}$ in $\mathcal{U}_{\varepsilon_{n}}$ and:

$$
\left\|u_{n}\right\|_{\mathrm{H}^{2}\left(\mathcal{U}_{\varepsilon_{n}}\right)}>n\left\|f_{n}\right\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon_{n}}\right)} .
$$

We denote $\mathbf{u}_{n}=u_{n} \circ\left(\phi_{\varepsilon_{n}}\right)^{-1}$ and $f_{n}=f_{n} \circ\left(\phi_{\varepsilon_{n}}\right)^{-1}$. We have:

$$
-\Delta \mathbf{u}_{n}-\mathcal{K}^{\varepsilon_{n}} \mathbf{u}_{n}-\xi_{n} \mathbf{u}_{n}=\mathrm{f}_{n} \text { in } \Omega_{e}
$$

and:

$$
\left\|\mathbf{u}_{n}\right\|_{\mathrm{H}^{2}\left(\Omega_{e}\right)}>C n\left\|\mathrm{f}_{n}\right\|_{\mathrm{L}^{2}\left(\Omega_{e}\right)} .
$$

Since $\mathrm{u}_{n} \neq 0$, we can divide by its $\mathrm{H}^{2}\left(\Omega_{e}\right)$-norm. We denote

$$
\tilde{\mathrm{u}}_{n}=\frac{\mathrm{u}_{n}}{\left\|\mathrm{u}_{n}\right\|_{\mathrm{H}^{2}\left(\Omega_{e}\right)}} \quad \text { and } \quad \tilde{\mathrm{f}}_{n}=\frac{\mathrm{f}_{n}}{\left\|\mathrm{u}_{n}\right\|_{\mathrm{H}^{2}\left(\Omega_{e}\right)}} .
$$

We have, for all $n$ :

$$
\left\|\tilde{u}_{n}\right\|_{\mathrm{H}^{2}\left(\Omega_{e}\right)}=1>n\left\|\tilde{\mathfrak{f}}_{n}\right\|_{\mathrm{L}^{2}\left(\Omega_{e}\right)}
$$

and

$$
\begin{equation*}
-\Delta \tilde{\mathrm{u}}_{n}-\mathcal{K}^{\varepsilon_{n}} \tilde{\mathrm{u}}_{n}-\xi_{n} \tilde{\mathrm{u}}_{n}=\tilde{\mathrm{f}}_{n} \text { in } \Omega_{e} . \tag{3.5}
\end{equation*}
$$

By extracting a sub sequence, we can assume that $\xi_{n}$ tends to $\xi_{\infty} \in \mathcal{C}\left(\lambda_{0}, \eta\right)$, and $\tilde{\mathrm{u}}_{n}$ tends to a limit $\tilde{u}_{\infty}$ weakly in $\mathrm{H}^{2}\left(\Omega_{e}\right)$ and strongly in $\mathrm{H}^{1}\left(\Omega_{e}\right)$. In addition, we remark that $\tilde{f}_{n}$ tends to zero strongly in $\mathrm{L}^{2}\left(\Omega_{e}\right)$. Taking the weak limit in (3.5), we obtain that:

$$
\begin{equation*}
\tilde{\mathrm{u}}_{\infty} \in \mathrm{H}^{2}\left(\Omega_{e}\right) \cap \mathrm{H}_{0}^{1}\left(\Omega_{e}\right) \quad \text { and } \quad-\Delta \tilde{\mathrm{u}}_{\infty}-\xi_{\infty} \tilde{\mathrm{u}}_{\infty}=0 \text { in } \Omega_{e} . \tag{3.6}
\end{equation*}
$$

In addition, using (3.2) and (3.5), we obtain that $\Delta \tilde{u}_{n}$ converges strongly in $\mathrm{L}^{2}\left(\Omega_{e}\right)$. Since its limit is the weak limit $\Delta \tilde{\mathrm{u}}_{\infty}$, then $\tilde{\mathrm{u}}_{n}$ converges strongly in $\mathrm{H}^{2}\left(\Omega_{e}\right)$ to $\tilde{u}_{\infty}$. So $\left\|\tilde{u}_{\infty}\right\|_{\mathrm{H}^{2}\left(\Omega_{e}\right)}=1$, and $\tilde{u}_{\infty} \neq 0$. Therefore, with (3.6), $\xi_{\infty}$ is in $\operatorname{Spec}\left(\mathcal{L}^{0}\right)$, which leads to a contradiction.

## Proof of Lemma 2.4

Inequality (2.46) is proved by duality arguments in [3] (see Lemma 2.1, on page 95). We use the same method and uniform estimates proved in Lemma 3.2 to establish (2.47). Let $\varepsilon_{0}$ given by Lemma 3.2. Let $\varepsilon \in\left(0, \varepsilon_{0}\right)$ and $\xi \in \mathcal{C}\left(\lambda_{0}, \eta\right)$. We fix $v \in \mathrm{H}^{1}\left(\mathcal{U}_{\varepsilon}\right)$ such that $-\Delta v-\xi v$ belongs to $\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)$. Let $f \in \mathrm{~L}^{2}\left(\mathcal{U}_{\varepsilon}\right)$. By Lemma 3.2, there exists $w \in \mathrm{H}^{2}\left(\mathcal{U}_{\varepsilon}\right) \cap \mathrm{H}_{0}^{1}\left(\mathcal{U}_{\varepsilon}\right)$ such that $-\Delta w-\xi w=f$ in $\mathcal{U}_{\varepsilon}$. Moreover,

$$
\|w\|_{\mathrm{H}^{2}\left(\mathcal{U}_{\varepsilon}\right)} \leq C\|f\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)},
$$

where $C$ does not depend on $\varepsilon, f$ or $\xi$. We have:

$$
\int_{\mathcal{U}_{\varepsilon}} v f=\int_{\mathcal{U}_{\varepsilon}} v(-\Delta w-\xi w)=\int_{\mathcal{U}_{\varepsilon}}(-\Delta v-\xi v) w+\int_{\Gamma_{\varepsilon}} v \frac{\partial w}{\partial \nu}
$$

Therefore:

$$
\begin{aligned}
\left|\int_{\mathcal{U}_{\varepsilon}} v f\right| & \leq\|-\Delta v-\xi v\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}\|w\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}+\|v\|_{\mathrm{L}^{2}\left(\Gamma_{\varepsilon}\right)}\left\|\frac{\partial w}{\partial \nu}\right\|_{\mathrm{L}^{2}\left(\Gamma_{\varepsilon}\right)} \\
& \leq C\left(\|-\Delta v-\xi v\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}+\|v\|_{\mathrm{L}^{2}\left(\Gamma_{\varepsilon}\right)}\right)\|f\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}
\end{aligned}
$$

This is true for all $f \in \mathrm{~L}^{2}\left(\mathcal{U}_{\varepsilon}\right)$ so:

$$
\|v\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)} \leq C\left(\|-\Delta v-\xi v\|_{\mathrm{L}^{2}\left(\mathcal{U}_{\varepsilon}\right)}+\|v\|_{\mathrm{L}^{2}\left(\Gamma_{\varepsilon}\right)}\right) .
$$

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