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An analytic and symbolic analysis of a coupled thermo-neutronic problem[☆]

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1. Introduction

In this note, we recall briefly the construction of an analytic solution of the low Mach number thermo-hydraulic model obtained by Dellacherie *et al.* detailed in [5]. The thermohydraulic-neutronics problem, stated in a 1d setting reads

$$\frac{d}{dz}(D(h(z))\frac{d\phi}{dz}) + \Sigma_a(h(z))\phi(z) = \frac{\nu\Sigma_f(h(z))}{k_{eff}}\phi(z), h'(z) = K\nu\Sigma_f(h)\phi(z), z \in [0, L], \quad (1)$$

where the unknowns are ϕ , the neutron density, h , the enthalpy (a measure of the temperature) and k_{eff} is called the multiplication factor of neutrons. The physical data are the functions Σ_a and $\nu\Sigma_f$ (respectively the absorption and fission cross section of the fissile material) and the constants D (diffusion coefficient) and K (heat transfer constant). This system is supplemented with the boundary conditions

$$\phi(0) = \phi(L) = 0, \phi \geq 0, h(0) = h_e, h(L) = h_s \quad (2)$$

Recall that in [5], we constructed an analytical solution of (1)(2) for any set of continuous positive functions $D, \Sigma_a, \nu\Sigma_f$, as a generalisation of Dellacherie and Lafitte [3]. Let \mathcal{X} and \mathcal{Y} be uniquely defined by:

$$-\frac{d}{dh}(D(h)\nu\Sigma_f(h)\frac{d\mathcal{X}}{dh}) = \frac{\Sigma_a(h)}{\nu\Sigma_f(h)}, -\frac{d}{dh}(D(h)\nu\Sigma_f(h)\frac{d\mathcal{Y}}{dh}) = 1, \mathcal{X}(h_e) = \mathcal{X}(h_s) = \mathcal{Y}(h_e) = \mathcal{Y}(h_s) = 0.$$

Denote by $\Psi(h, k_{eff}) = 2(\frac{\mathcal{Y}(h)}{k_{eff}} - \mathcal{X}(h))$. Let $z \rightarrow h(z, k_{eff})$ is the unique solution of:

$$h'(z, k_{eff}) = \sqrt{\Psi(h(z, k_{eff}), k_{eff})}, h(0) = h_e.$$

Lemma 1.1. Equation $h(L, k_{eff}) = h_s$ has a unique solution $k_{eff}^* \in I := (0, (\max_{[h_e, h_s]} \frac{\mathcal{Y}}{\mathcal{X}})^{-1})$ which provides the unique solution of the system (1), (2).

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This result is the consequence of $k_{eff} \rightarrow h(L, k_{eff})$ decreasing from I to $(+\infty, 0)$ (see [3], [5]).

In this extended abstract, we shall, without losing the generality, simplify the problem (assuming that D and Σ_a are constant functions), we shall choose $h_e = 0, h_s = 1$, denote by $\lambda = \frac{1}{k_{eff}}$ and consider the totally adimensionnalized problem ($L = 1$ (unit of length), $\Sigma_a = 1, D = 1, K = 1$ (physical constants), $h_e = 0, h_s = 1$ (normalizing the temperature)):

$$-(D\phi)' + \Sigma_a\phi = \lambda\Sigma(h)\phi, h' = K\phi, z \in [0, 1]$$

supplemented by $\phi(0) = \phi(1) = 0, h(0) = h_e, h(1) = h_s$, where Σ is a given continuous function.

We assume throughout this paper

$$(H) \quad \Sigma(h) \geq \Sigma_* > 0, 0 \leq h \leq 1.$$

and we define ψ_λ the unique solution of $\psi_\lambda''(h) = 2 - 2\lambda\Sigma(h)$ with $\psi_\lambda(0) = \psi_\lambda(1) = 0$. We intend to present a numerical method which solves the coupled problem without using the numerical methods traditionally used for solving each equation but rather concentrating on solving the equation $h(1) = 1$. This numerical method, as well as analytic and symbolic methods, are implemented when one knows only three values of Σ , and for simplicity again one assumes that one knows $\Sigma(0) = \sigma_0, \Sigma(\frac{1}{2}) = \sigma_{\frac{1}{2}}, \Sigma(1) = \sigma_1$. The analytic and symbolic methods are consequences of the finding of **exact** solutions using the incomplete Jacobi functions (which are its solutions) when Ψ is a polynomial of degree 3 or 4.

This numerical method is compared with an adapted numerical scheme, described in the first paragraph of Section ???. This numerical scheme uses the Crank-Nicholson discretization of the ODE $h' = \phi$ in order to recover, from the numerical scheme, the property that the vector $(\phi_j)_j$ is a generalized eigenvector of the discretized version of $-\phi'' + \phi = \lambda\Sigma(h)\phi$, hence keeping what researchers working in the neutronics computations call the spectral problem for the neutronics.

We want to construct different relevant approximations of $\Sigma(h)$ (namely constant, affine, polynomial of order 2) in order to see what is the influence on the value of k_{eff} in order to be able to study the sensitivity of the value of k_{eff} for these different approximations, and furthermore to understand the number of significative figures in k_{eff} . Usually, the benchmarks in realistic cases are using sensitivity of 10^{-5} around $k_{eff} = 1$, which is a magnitude or two smaller than the results we obtain here, see Section 4.

2. Analytical approach

Following *e.g.* Abramowitz and Stegun [1], we define the incomplete elliptic integral of the first kind $K(m, \varphi)$ by the relation $K(m, \varphi) = \int_0^\varphi \frac{d\theta}{\sqrt{1-m \sin^2 \theta}}$, $0 \leq \varphi \leq \frac{\pi}{2}$, $m < 1$. The complete elliptic integral of the first kind $K(m)$ is defined by $K(m) \equiv K(m, \frac{\pi}{2})$.

Lemma 2.1. (i) Given two positive reals a and b such that $0 < a < b$, we have

$$\int_{-a}^a \frac{dT}{\sqrt{(a^2-T^2)(b^2-T^2)}} = \frac{2}{b} K(m), \quad m = \frac{a^2}{b^2} \quad \text{and} \quad \int_a^b \frac{dT}{\sqrt{(T^2-a^2)(b^2-T^2)}} = \frac{1}{b} K(m), \quad m = 1 - \frac{a^2}{b^2}.$$

(ii) Given two reals a and b with $a > 0$ and $b \neq 0$, we have $\int_{-a}^a \frac{dT}{\sqrt{(a^2-T^2)(b^2+T^2)}} = \frac{2}{|b|} K(m)$, $m = -\frac{a^2}{b^2}$.

Recall that we are studying the model described by the following set of equations, where ϕ needs (physically) to be positive because it is a density flux of neutrons:

$$\begin{cases} -\frac{d^2\phi}{dz^2} + \phi(z) = \lambda\Sigma(h(z))\phi(z), \frac{dh}{dz} = \phi(z), 0 < z < 1, \\ h(0) = 0, h(1) = 1, \phi(0) = 0, \phi(1) = 0, \lambda \geq 0, \phi(z) > 0 \text{ if } 0 < z < 1. \end{cases} \quad (3)$$

Here Σ is defined by three positive real numbers $\sigma_0, \sigma_{1/2}$ and σ_1 which represent the values of Σ at, respectively, $0, \frac{1}{2}, 1$. We describe in what follows five different modellings of the function Σ from these

values which lead to analytical approaches for determining the scalar parameter $\lambda > 0$ and deduce secondly the functions $z \mapsto \varphi(z)$ and $z \mapsto h(z)$.

Introduce

Definition 1. We define μ , α and δ through $\mu = \frac{1}{2}(\sigma_0 + \sigma_1)$, $\alpha = 1 - \frac{\sigma_0}{\mu}$ and $\delta = -\frac{\sigma_0 - 2\sigma_{1/2} + \sigma_1}{\sigma_0 + \sigma_1}$. They satisfy the inequalities $\mu > 0$, $|\alpha| < 1$ and $\delta > -1$. Furthermore $\gamma \equiv 1 - \frac{1}{3}|\alpha| + \frac{2}{3}\delta > 0$.

• Zero-th case: Σ is constant ($\alpha = \delta = 0$). Then the system modeled by the previous set of equations is totally decoupled and an exact solution can be provided with elementary arguments.

Proposition 2.1. If $\sigma_0 = \sigma_{1/2} = \sigma_1 \equiv \mu > 0$, the previous problem (3) admits a unique decoupled solution: we have $\lambda\mu = 1 + \pi^2$, $\varphi(z) = \frac{\pi}{2} \sin(\pi z)$ and $h(z) = \frac{1}{2}(1 - \cos(\pi z))$.

If the function Σ is no-more constant, it has been proven in [2, 3] that the unknown of the problem can be obtained with the following process. First integrate twice the function Σ and obtain a convex negative function V such that $\frac{d^2V}{dh^2} = \Sigma(h)$, $V(0) = V(1) = 0$. Second define $\psi_\lambda(h) \equiv h(h-1) - 2\lambda V(h)$. Then the equation for the function $z \mapsto h(z)$ becomes $\frac{dh}{dz} = \sqrt{\psi_\lambda(h)}$ and the condition $h(1) = 1$ gives a scalar equation for the unknown $\lambda > 0$: $\int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}} = 1$.

A first difficulty is to compute the integral $I_\lambda \equiv \int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}}$. Then we can solve easily the equation $I_\lambda = 1$ with a Newton-like algorithm. Once λ is determined, the explicitation of the functions $z \mapsto \varphi(z)$ and $z \mapsto h(z)$ is not difficult. Thus the analytical method we propose is found on an analytical determination of the integral I_λ . We focus our attention to this question in the next sub-sections.

• First case: $\Sigma \in P_1$.

In this case, the function Σ is a positive affine function on the interval $(0, 1)$. Hence $\delta = 0$ and $\sigma_0 = \mu(1 - \alpha)$, $\sigma_{1/2} = \mu$ and $\sigma_1 = \mu(1 + \alpha)$ with $\mu > 0$ and $|\alpha| < 1$ to satisfy the constraint of positivity. We introduce the notation $\xi \equiv \lambda\mu$.

Proposition 2.2. With the notations introduced previously, the functions V and ψ_λ admit the algebraic expressions $V(h) = \mu h(h-1)\left(\frac{1}{2} - \frac{\alpha}{6} + \frac{\alpha}{3}h\right)$ and $\psi_\lambda(h) = h(h-1)\left(\xi\left(1 + \frac{\alpha}{3}\right) - 1 - \frac{2\alpha\xi}{3}h\right)$. Then we have $I_\lambda = \frac{2}{\sqrt{\xi\left(1 + \frac{|\alpha|}{3}\right) - 1}} K(m)$ with $m = \frac{2|\alpha|\xi}{3\xi + |\alpha|\xi - 3}$.

• Second case: $\Sigma \in P_2$.

In this case, the polynomial ψ_λ is a polynomial of degree 4 with real coefficients and ψ_λ is positive on the interval $(0, 1)$. We have also $\psi_\lambda(0) = \psi_\lambda(1) = 0$. Recall that $\psi_\lambda(h) = h(h-1) - 2\lambda V(h)$ with $V'' \equiv \Sigma$ the Lagrange interpolate polynomial such that $\Sigma(0) = \sigma_0$, $\Sigma(\frac{1}{2}) = \sigma_{1/2}$ and $\Sigma(1) = \sigma_1$. All these coefficients are supposed positive: $\sigma_0 > 0$, $\sigma_{1/2} > 0$, $\sigma_1 > 0$. One has $\sigma_0 = \mu(1 - \alpha)$, $\sigma_{1/2} = \mu(1 + \delta)$ and $\sigma_1 = \mu(1 + \alpha)$. In this sub-section, we exclude the case of a linear interpolation, *id est* $\delta \neq 0$, in coherence with the hypothesis that the degree of the polynomial ψ_λ is exactly equal to 4.

Lemma 2.2. With the above notations and properties, introduce the notation $\xi \equiv \lambda\mu$; then $\xi > 0$. Introduce also the two other roots p and g of the polynomial ψ_λ . We can write $\psi_\lambda(h) = a_0 h(h-1)(h-p)(h-g)$. Then $a_0 = \frac{2}{3}\xi\delta$. Denote by $\sigma \equiv p+g$ the sum and $\pi \equiv pg$ the product and the two roots p and g . We have $a_0\sigma = \frac{2}{3}\xi(\delta + \alpha)$ and $a_0\pi = 1 - \xi + \frac{1}{3}\alpha\xi - \frac{2}{3}\delta\xi$.

The above relations have been computed with the help of the SageMath [6] formal calculus software. Observe that with the above notations and properties, if the integral I_λ is a positive real number, the roots p and g cannot be equal to 0 or 1.

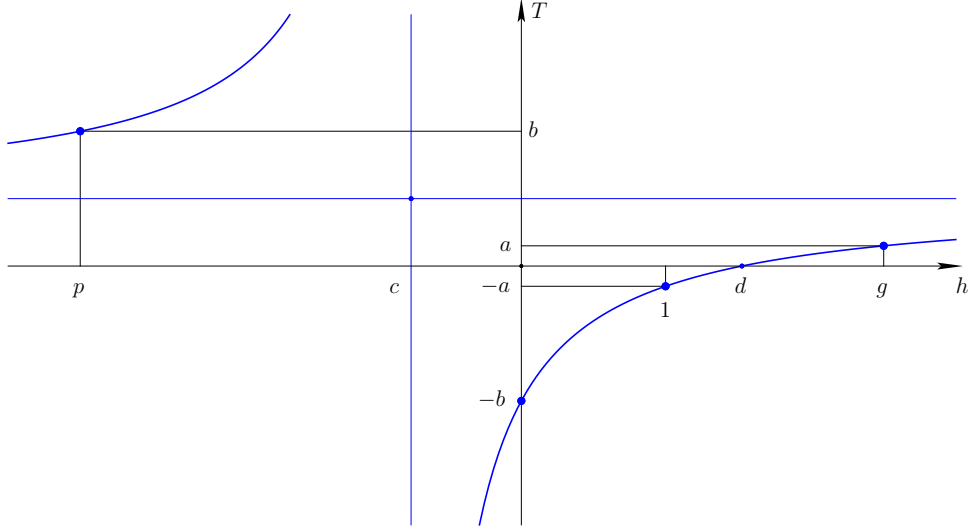


Figure 1: Homographic transformation T for the computation of the integral I_λ ; case $p < 0 < 1 < g$. It is defined by the constraints $T(p) = b$, $T(0) = -b$, $T(1) = -a$ and $T(g) = a$. The coefficients $0 < a < 1 < b$ are determined in Proposition 2.4.

Proposition 2.3. *We keep active the above notations.*

(i) *If the discrete positive family $(\sigma_0, \sigma_{1/2}, \sigma_1)$ is the trace of a concave function, id est if $\sigma_{1/2} - \sigma_0 > \sigma_1 - \sigma_{1/2}$, we have $\delta > 0$, $a_0 > 0$ and $\gamma\xi > 1$. The two roots of the function ψ_λ are real with opposite signs. To fix the ideas, $p < 0 < 1 < g$. If we set $\Delta \equiv \sigma^2 - 4\pi = \frac{1}{a_0^2}\xi[(5\delta^2 + \alpha^2 + 6\delta)\xi - 6\delta]$, we have $p = \frac{1}{2}(\sigma - \sqrt{\Delta})$ and $g = \frac{1}{2}(\sigma + \sqrt{\Delta})$ with σ and π introduced in Lemma (2.2).*

(ii) *Conversely, if the function ψ_λ are real zeros with opposite signs, then $\sigma_{1/2} - \sigma_0 > \sigma_1 - \sigma_{1/2}$.*

Proposition 2.4. *We suppose as in Proposition (2.3) that the function $\psi_\lambda(h)$ has two real zeros p and g that satisfy $p < 0 < 1 < g$. Then the integral $I_\lambda \equiv \int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}}$ can be computed with the help of the following*

formulas: $A = -\sqrt{\frac{g(g-p)}{1-p}}$, $B = \sqrt{\frac{(1-p)(g-p)}{g}}$, $a = \frac{A+1}{A-1}$, $b = \frac{B+1}{B-1}$, $m = 1 - \frac{a^2}{b^2}$, $I_\lambda = \frac{1}{\sqrt{a_0}}(1 - \frac{a}{b}) \frac{1}{\sqrt{g-p}} K(m)$. Moreover, we have $0 < a < 1 < b$ and $0 < m < 1$.

Proof. The general ideas of the proof are given in the book [1]. A detailed proof and a complete description of the other cases $p < g < 0 < 1$, $0 < 1 < p < g$ and p and g conjugate complex numbers can be founded in [4]. \square

If the sequence $\sigma_0, \sigma_{1/2}, \sigma_1$ is convex, id est if $\sigma_{1/2} - \sigma_0 < \sigma_1 - \sigma_{1/2}$, we have $\delta < 0$, $a_0 < 0$ and the discriminant Δ introduced in Proposition 2.3 can be negative or positive. In the case $\Delta < 0$ and we have two conjugate roots for the polynomial ψ_λ . In the case $\Delta > 0$, we have of course the case $p < 0 < 1 < g$ studied at Proposition 2.4; this case must be completed by the two sub-cases $p < g < 0 < 1$ and $0 < 1 < p < g$. All these cases are detailed in [4].

• **Third case:** Σ is a continuous positive function, affine in each interval $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$.

The function Σ is defined by its values $\Sigma(0) = \sigma_0$, $\Sigma(\frac{1}{2}) = \sigma_{1/2}$ and $\Sigma(1) = \sigma_1$. We introduce new parameters, still denoted by α and β to represent the data: $\sigma_0 = \sigma_{1/2}(1 - \alpha)$ and $\sigma_1 = \sigma_{1/2}(1 + \beta)$. Then the inequalities $\alpha < 1$ and $\beta > -1$ express the constraints $\sigma_0 > 0$ and $\sigma_1 > 0$. Moreover, $\sigma_{1/2}$ is positive.

Lemma 2.3. *If the continuous function Σ defined on $(0, 1)$ by its values $\Sigma(0) = \sigma_0$, $\Sigma(\frac{1}{2}) = \sigma_{1/2}$ and $\Sigma(1) = \sigma_1$ is affine in each interval $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, the function ψ_λ defined by the conditions*

$$\frac{d^2\psi_\lambda}{dh^2} = 1 - 2\lambda\Sigma \text{ and } \psi_\lambda(0) = \psi_\lambda(1) = 0$$

admits the following expression

$$\psi_\lambda(h) = \begin{cases} \psi^0(h) = -\frac{2}{3}\alpha\xi h^3 + (\alpha\xi - \xi + 1)h^2 + (\frac{\beta-5\alpha}{12}\xi + \xi - 1)h & \text{if } h \leq \frac{1}{2} \\ \psi^1(h) = \frac{2}{3}\beta\xi(1-h)^3 + (-\beta\xi - \xi + 1)(1-h)^2 + (\frac{5\beta-\alpha}{12}\xi + \xi - 1)(1-h) & \text{if } h \geq \frac{1}{2}, \end{cases}$$

with $\xi \equiv \lambda\sigma_{1/2}$. We have in particular $\psi_\lambda(\frac{1}{2}) = \frac{\beta-\alpha}{24}\xi + \frac{1}{4}(\xi - 1)$. Moreover, if ψ_λ is positive on the interval $(0, 1)$, $\frac{d\psi_\lambda}{dh}(0) = \frac{\beta-5\alpha}{12}\xi + \xi - 1 > 0$ and $\frac{d\psi_\lambda}{dh}(1) = -(\frac{5\beta-\alpha}{12}\xi + \xi - 1) < 0$.

Remark 1. *We observe that the expression of ψ^1 is obtained from the expression of ψ^0 by making the transformations $\alpha \longleftrightarrow (-\beta)$ and $h \longleftrightarrow (1-h)$.*

Proof. The function Σ admits the algebraic expressions: $\Sigma(h) = (2\alpha h - \alpha + 1)\sigma_{1/2}$ if $h \leq \frac{1}{2}$ and $\Sigma(h) = (2\beta h - \beta + 1)\sigma_{1/2}$ if $h \geq \frac{1}{2}$. We integrate two times, enforce the conditions $\psi_\lambda(0) = \psi_\lambda(1) = 0$ and impose the continuity of ψ_λ and $\frac{d\psi_\lambda}{dh}$ at the specific value $h = \frac{1}{2}$. The result follows. \square

We have to compute the integral $I_\lambda \equiv \int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}}$. We have the following calculus: $I_\lambda = \int_0^{1/2} \frac{dh}{\sqrt{\psi^0(h)}} + \int_{1/2}^1 \frac{dh}{\sqrt{\psi^1(h)}} = \int_0^{1/2} \frac{dh}{\sqrt{\psi^0(h)}} + \int_0^{1/2} \frac{dh}{\sqrt{\psi^1(1-h)}}$. Due to the Remark 1, the determination of the second term relative to ψ^1 is very analogous to the term associated to ψ^0 .

Proposition 2.5. *Recall that $\psi^0(h) = -\frac{2}{3}\alpha\xi h^3 + (\alpha\xi - \xi + 1)h^2 + (\frac{\beta-5\alpha}{12}\xi + \xi - 1)h$.*

(i) If the polynomial $\psi^0(h)$ has two complex roots, we write it under the form $\psi^0(h) \equiv a_0 h(h^2 + \mu h + \zeta^4)$ with $a_0 = \frac{2}{3}|\alpha|\xi > 0$ and we have the inequality $\mu^2 - 4\zeta^4 < 0$; we introduce $m = \frac{1}{2} - \frac{\mu}{4\zeta^2}$ that satisfies $0 < m < 1$ and $\varphi_0 = 2 \arctan(\frac{1}{\zeta\sqrt{2}})$. Then we have $I_0 = \frac{1}{\zeta\sqrt{a_0}} K(\varphi_0, m)$.

(ii) If the polynomial $\psi^0(h)$ has three real roots, we set $\psi^0(h) \equiv a_0 h(r_- - h)(r_+ - h)$ with $a_0 = \frac{2}{3}|\alpha|\xi > 0$ and $0 < r_- < r_+$; we consider $m = \frac{r_-}{r_+} \in (0, 1)$ and $\varphi_0 = \arcsin(\frac{1}{\sqrt{2}r_-})$. The integral I_0 is computed with the following expression: $I_0 = \frac{2}{\sqrt{a_0}r_+} K(\varphi_0, m)$.

• Fourth case and fifth cases: Σ is a P2 polynomial or a continuous positive function, piecewise affine. The function $h \mapsto \frac{V(h)}{h(1-h)}$ is approximated by an affine function, as described in [3].

3. Numerical method

After adimensionalization, the problem can be formulated with the formulation (3). We introduce a nonregular meshing $0 = z_0 < z_1 < \dots < z_{N-1} < z_N = 1$ of the interval $[0, 1]$ and we set $\Delta z_{j+1/2} \equiv z_{j+1} - z_j$ for $j = 1, \dots, N$. We integrate the differential equation $\frac{dh}{dz} = \varphi(z)$ with the Crank-Nicolson scheme:

$$\frac{h_{j+1} - h_j}{\Delta z_{j+1/2}} = \frac{1}{2}(\varphi_j + \varphi_{j+1}) \text{ for } j = 0, \dots, N-1.$$

Then after two integrations, the first equation can be written as $\varphi_j = \sqrt{\psi_\lambda(h_j)}$ and the Crank-Nicolson scheme takes the form $\frac{h_{j+1} - h_j}{\frac{1}{2}(\sqrt{\psi_\lambda(h_j)} + \sqrt{\psi_\lambda(h_{j+1})})} = \Delta z_{j+1/2}(\lambda)$.

We impose the values $h_j \equiv \sin^2(\frac{\pi j}{2N})$ in order to take into account the singularities and two boundary conditions of the problem at $z = 0$ and $z = 1$. The notation $\Delta z_{j+1/2}(\lambda)$ in the right hand side of the previous relation is justified by the fact that if the numbers h_j are given, the left hand side is a simple function of the

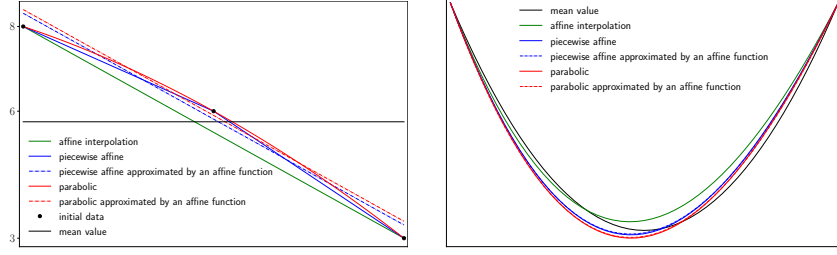


Figure 2: Function $h \mapsto \Sigma(h)$ on the left, $h \mapsto V(h)$ on the right.

scalar λ . The number λ is *a priori* not known, but we have the natural relation $\sum_{j=0}^{N-1} \Delta z_{j+1/2}(\lambda) = 1$ that takes the form

$$\sum_{j=0}^{N-1} \frac{h_{j+1} - h_j}{\frac{1}{2}(\sqrt{\psi_\lambda(h_j)} + \sqrt{\psi_\lambda(h_{j+1})})} = 1. \quad (4)$$

Lemma 3.1. Recall that V is the unique solution of $V'' = \Sigma$, $V(0) = V(1) = 0$. We have observed previously that $\psi_\lambda(h) = h(h-1) - 2\lambda V(h)$. Define $\lambda_* = \max_{[0,1]} \frac{h(h-1)}{V(h)}$.

i) For $\lambda > \lambda_*$, $\psi_\lambda(h_j) > 0$

ii) Equation (4) has a unique solution $\lambda \in (\lambda_*, +\infty)$ when $\sum_{j=0}^{N-1} \frac{h_{j+1} - h_j}{\frac{1}{2}(\sqrt{\psi_{\lambda_*}(h_j)} + \sqrt{\psi_{\lambda_*}(h_{j+1})})} \in \bar{\mathbb{R}}_+ > 1$.

For a fixed discretization with N intervals, a Newton algorithm can be implemented without difficulty. With this procedure, we recover on one hand an approximated value λ_N of the unknown λ and on the other hand the entire approximate solution of the problem $h_j \approx h(x_j)$ and $\varphi_j \approx \sqrt{\psi_\lambda(h_j)}$. Observe that at convergence of the Newton algorithm, the abscissas x_j are a function of the solution λ_N and the converged space mesh is a result of the problem. This coupled problem can be reduced to a single equation with only one real variable even after discretization!

4. Numerical results

They are presented in Figures 2 and 3. They correspond to the decreasing data $\sigma_0 = 8$, $\sigma_{1/2} = 6$ and $\sigma_1 = 3$. We obtain the following exact values for the scalar parameter: $\lambda = 1.89036$ in the decoupled case (case 0), $\lambda = 1.99533$ in the affine case (case 1), $\lambda = 1.86593$ in the parabolic case (case 2), $\lambda = 1.89454$ in the piecewise affine case (case 3), $\lambda = 1.85769$ in the parabolic case approached by an affine function (case 4) and $\lambda = 1.88614$ in the piecewise affine case projected on affine functions (case 5). For each of these six cases, our analytical-numerical approach gives converging results at second order accuracy for the parameter λ , hence showing the powerfulness and precision of the explicit calculations that one describes using the elliptic integrals.

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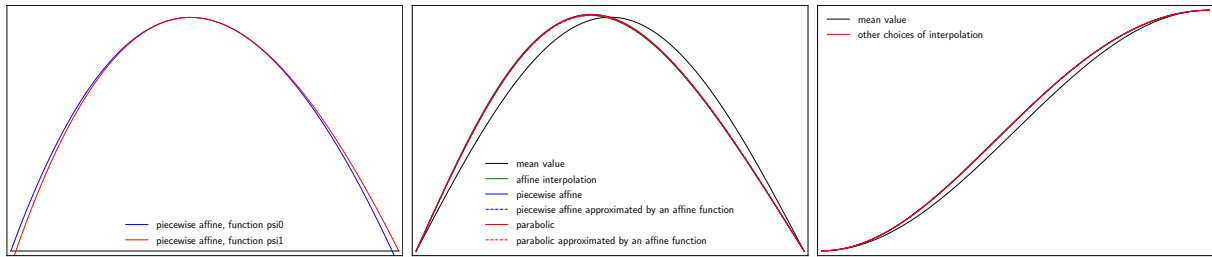


Figure 3: Functions $h \mapsto \psi^0(h)$ and $h \mapsto \psi^1(h)$ on the left, $z \mapsto \varphi(z)$ in the middle and $z \mapsto h(z)$ on the right.

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