



HAL
open science

An analytic and symbolic analysis of a coupled thermo-neutronic problem

François Dubois, Olivier Lafitte

► **To cite this version:**

François Dubois, Olivier Lafitte. An analytic and symbolic analysis of a coupled thermo-neutronic problem. IEEE, pp.61-65, 2021, 10.1109/synasc54541.2021.00022 . hal-03858927

HAL Id: hal-03858927

<https://hal.science/hal-03858927>

Submitted on 25 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An analytic and symbolic analysis of a coupled thermo-neutronic problem[☆]

François Dubois^{a,b,c}, Olivier Lafitte^{a,d}

^aInternational Research Laboratory 3457 du Centre National de la Recherche Scientifique, Centre de Recherches Mathématiques, Université de Montréal, Montréal, QC, Canada.

^bConservatoire National des Arts et Métiers, LMSSC laboratory, Paris, France.

^cLaboratoire de Mathématiques d'Orsay, Université Paris-Saclay, France.

^dUniversité Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), 99 Avenue J.-B. Clément F-93430, Villetaneuse Cedex, France.

Keywords:

Ordinary differential equation, analytic functions, Crank-Nicolson numerical scheme

1. Introduction

In this note, we recall briefly the construction of an analytic solution of the low Mach number thermo-hydraulic model obtained by Dellacherie *et al.* detailed in [5]. The thermohydraulic-neutronics problem, stated in a 1d setting reads

$$\frac{d}{dz}(D(h(z))\frac{d\phi}{dz}) + \Sigma_a(h(z))\phi(z) = \frac{\nu\Sigma_f(h(z))}{k_{eff}}\phi(z), h'(z) = K\nu\Sigma_f(h)\phi(z), z \in [0, L], \quad (1)$$

where the unknowns are ϕ , the neutron density, h , the enthalpy (a measure of the temperature) and k_{eff} is called the multiplication factor of neutrons. The physical data are the functions Σ_a and $\nu\Sigma_f$ (respectively the absorption and fission cross section of the fissile material) and the constants D (diffusion coefficient) and K (heat transfer constant). This system is supplemented with the boundary conditions

$$\phi(0) = \phi(L) = 0, \phi \geq 0, h(0) = h_e, h(L) = h_s \quad (2)$$

Recall that in [5], we constructed an analytical solution of (1)(2) for any set of continuous positive functions $D, \Sigma_a, \nu\Sigma_f$, as a generalisation of Dellacherie and Lafitte [3]. Let \mathcal{X} and \mathcal{Y} be uniquely defined by:

$$-\frac{d}{dh}(D(h)\nu\Sigma_f(h)\frac{d\mathcal{X}}{dh}) = \frac{\Sigma_a(h)}{\nu\Sigma_f(h)}, -\frac{d}{dh}(D(h)\nu\Sigma_f(h)\frac{d\mathcal{Y}}{dh}) = 1, \mathcal{X}(h_e) = \mathcal{X}(h_s) = \mathcal{Y}(h_e) = \mathcal{Y}(h_s) = 0.$$

Denote by $\Psi(h, k_{eff}) = 2(\frac{\mathcal{Y}(h)}{k_{eff}} - \mathcal{X}(h))$. Let $z \rightarrow h(z, k_{eff})$ is the unique solution of:

$$h'(z, k_{eff}) = \sqrt{\Psi(h(z, k_{eff}), k_{eff})}, h(0) = h_e.$$

Lemma 1.1. Equation $h(L, k_{eff}) = h_s$ has a unique solution $k_{eff}^* \in I := (0, (\max_{[h_e, h_s]} \frac{\mathcal{Y}}{\mathcal{X}})^{-1})$ which provides the unique solution of the system (1), (2).

[☆]Contribution submitted to the 23th International Symposium on SYmbolic and Numeric Algorithms for Scientific Computing, 7–10 December 2021, Timisoara, Romania. Edition 28 June 2021.

Email addresses: francois.dubois@lecnam.net (François Dubois), lafitte@math.univ-paris13.fr (Olivier Lafitte)

This result is the consequence of $k_{eff} \rightarrow h(L, k_{eff})$ decreasing from I to $(+\infty, 0)$ (see [3], [5]).

In this extended abstract, we shall, without losing the generality, simplify the problem (assuming that D and Σ_a are constant functions), we shall choose $h_e = 0, h_s = 1$, denote by $\lambda = \frac{1}{k_{eff}}$ and consider the totally adimensionnalized problem ($L = 1$ (unit of length), $\Sigma_a = 1, D = 1, K = 1$ (physical constants), $h_e = 0, h_s = 1$ (normalizing the temperature)):

$$-(D\phi)' + \Sigma_a\phi = \lambda\Sigma(h)\phi, h' = K\phi, z \in [0, 1]$$

supplemented by $\phi(0) = \phi(1) = 0, h(0) = h_e, h(1) = h_s$, where Σ is a given continuous function.

We assume throughout this paper

$$(H) \quad \Sigma(h) \geq \Sigma_* > 0, 0 \leq h \leq 1.$$

and we define ψ_λ the unique solution of $\psi_\lambda''(h) = 2 - 2\lambda\Sigma(h)$ with $\psi_\lambda(0) = \psi_\lambda(1) = 0$. We intend to present a numerical method which solves the coupled problem without using the numerical methods traditionally used for solving each equation but rather concentrating on solving the equation $h(1) = 1$. This numerical method, as well as analytic and symbolic methods, are implemented when one knows only three values of Σ , and for simplicity again one assumes that one knows $\Sigma(0) = \sigma_0, \Sigma(\frac{1}{2}) = \sigma_{\frac{1}{2}}, \Sigma(1) = \sigma_1$. The analytic and symbolic methods are consequences of the finding of **exact** solutions using the incomplete Jacobi functions (which are its solutions) when Ψ is a polynomial of degree 3 or 4.

This numerical method is compared with an adapted numerical scheme, described in the first paragraph of Section ???. This numerical scheme uses the Crank-Nicholson discretization of the ODE $h' = \phi$ in order to recover, from the numerical scheme, the property that the vector $(\phi_j)_j$ is a generalized eigenvector of the discretized version of $-\phi'' + \phi = \lambda\Sigma(h)\phi$, hence keeping what researchers working in the neutronics computations call the spectral problem for the neutronics.

We want to construct different relevant approximations of $\Sigma(h)$ (namely constant, affine, polynomial of order 2) in order to see what is the influence on the value of k_{eff} in order to be able to study the sensitivity of the value of k_{eff} for these different approximations, and furthermore to understand the number of significative figures in k_{eff} . Usually, the benchmarks in realistic cases are using sensitivity of 10^{-5} around $k_{eff} = 1$, which is a magnitude or two smaller than the results we obtain here, see Section 4.

2. Analytical approach

Following *e.g.* Abramowitz and Stegun [1], we define the incomplete elliptic integral of the first kind $K(m, \varphi)$ by the relation $K(m, \varphi) = \int_0^\varphi \frac{d\theta}{\sqrt{1-m \sin^2 \theta}}$, $0 \leq \varphi \leq \frac{\pi}{2}$, $m < 1$. The complete elliptic integral of the first kind $K(m)$ is defined by $K(m) \equiv K(m, \frac{\pi}{2})$.

Lemma 2.1. (i) Given two positive reals a and b such that $0 < a < b$, we have

$$\int_{-a}^a \frac{dT}{\sqrt{(a^2-T^2)(b^2-T^2)}} = \frac{2}{b} K(m), \quad m = \frac{a^2}{b^2} \quad \text{and} \quad \int_a^b \frac{dT}{\sqrt{(T^2-a^2)(b^2-T^2)}} = \frac{1}{b} K(m), \quad m = 1 - \frac{a^2}{b^2}.$$

(ii) Given two reals a and b with $a > 0$ and $b \neq 0$, we have $\int_{-a}^a \frac{dT}{\sqrt{(a^2-T^2)(b^2+T^2)}} = \frac{2}{|b|} K(m)$, $m = -\frac{a^2}{b^2}$.

Recall that we are studying the model described by the following set of equations, where ϕ needs (physically) to be positive because it is a density flux of neutrons:

$$\begin{cases} -\frac{d^2\phi}{dz^2} + \phi(z) = \lambda\Sigma(h(z))\phi(z), \quad \frac{dh}{dz} = \phi(z), \quad 0 < z < 1, \\ h(0) = 0, \quad h(1) = 1, \quad \phi(0) = 0, \quad \phi(1) = 0, \quad \lambda \geq 0, \quad \phi(z) > 0 \text{ if } 0 < z < 1. \end{cases} \quad (3)$$

Here Σ is defined by three positive real numbers $\sigma_0, \sigma_{1/2}$ and σ_1 which represent the values of Σ at, respectively, $0, \frac{1}{2}, 1$. We describe in what follows five different modellings of the function Σ from these

values which lead to analytical approaches for determining the scalar parameter $\lambda > 0$ and deduce secondly the functions $z \mapsto \varphi(z)$ and $z \mapsto h(z)$.

Introduce

Definition 1. We define μ , α and δ through $\mu = \frac{1}{2}(\sigma_0 + \sigma_1)$, $\alpha = 1 - \frac{\sigma_0}{\mu}$ and $\delta = -\frac{\sigma_0 - 2\sigma_{1/2} + \sigma_1}{\sigma_0 + \sigma_1}$. They satisfy the inequalities $\mu > 0$, $|\alpha| < 1$ and $\delta > -1$. Furthermore $\gamma \equiv 1 - \frac{1}{3}|\alpha| + \frac{2}{3}\delta > 0$.

• Zero-th case: Σ is constant ($\alpha = \delta = 0$). Then the system modeled by the previous set of equations is totally decoupled and an exact solution can be provided with elementary arguments.

Proposition 2.1. If $\sigma_0 = \sigma_{1/2} = \sigma_1 \equiv \mu > 0$, the previous problem (3) admits a unique decoupled solution: we have $\lambda\mu = 1 + \pi^2$, $\varphi(z) = \frac{\pi}{2} \sin(\pi z)$ and $h(z) = \frac{1}{2}(1 - \cos(\pi z))$.

If the function Σ is no-more constant, it has been proven in [2, 3] that the unknown of the problem can be obtained with the following process. First integrate twice the function Σ and obtain a convex negative function V such that $\frac{d^2V}{dh^2} = \Sigma(h)$, $V(0) = V(1) = 0$. Second define $\psi_\lambda(h) \equiv h(h-1) - 2\lambda V(h)$. Then the equation for the function $z \mapsto h(z)$ becomes $\frac{dh}{dz} = \sqrt{\psi_\lambda(h)}$ and the condition $h(1) = 1$ gives a scalar equation for the unknown $\lambda > 0$: $\int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}} = 1$.

A first difficulty is to compute the integral $I_\lambda \equiv \int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}}$. Then we can solve easily the equation $I_\lambda = 1$ with a Newton-like algorithm. Once λ is determined, the explicitation of the functions $z \mapsto \varphi(z)$ and $z \mapsto h(z)$ is not difficult. Thus the analytical method we propose is found on an analytical determination of the integral I_λ . We focus our attention to this question in the next sub-sections.

• First case: $\Sigma \in P_1$.

In this case, the function Σ is a positive affine function on the interval $(0, 1)$. Hence $\delta = 0$ and $\sigma_0 = \mu(1 - \alpha)$, $\sigma_{1/2} = \mu$ and $\sigma_1 = \mu(1 + \alpha)$ with $\mu > 0$ and $|\alpha| < 1$ to satisfy the constraint of positivity. We introduce the notation $\xi \equiv \lambda\mu$.

Proposition 2.2. With the notations introduced previously, the functions V and ψ_λ admit the algebraic expressions $V(h) = \mu h(h-1)(\frac{1}{2} - \frac{\alpha}{6} + \frac{\alpha}{3}h)$ and $\psi_\lambda(h) = h(h-1)(\xi(1 + \frac{\alpha}{3}) - 1 - \frac{2\alpha\xi}{3}h)$. Then we have $I_\lambda = \frac{2}{\sqrt{\xi(1 + \frac{|\alpha|}{3}) - 1}} K(m)$ with $m = \frac{2|\alpha|\xi}{3\xi + |\alpha|\xi - 3}$.

• Second case: $\Sigma \in P_2$.

In this case, the polynomial ψ_λ is a polynomial of degree 4 with real coefficients and ψ_λ is positive on the interval $(0, 1)$. We have also $\psi_\lambda(0) = \psi_\lambda(1) = 0$. Recall that $\psi_\lambda(h) = h(h-1) - 2\lambda V(h)$ with $V'' \equiv \Sigma$ the Lagrange interpolate polynomial such that $\Sigma(0) = \sigma_0$, $\Sigma(\frac{1}{2}) = \sigma_{1/2}$ and $\Sigma(1) = \sigma_1$. All these coefficients are supposed positive: $\sigma_0 > 0$, $\sigma_{1/2} > 0$, $\sigma_1 > 0$. One has $\sigma_0 = \mu(1 - \alpha)$, $\sigma_{1/2} = \mu(1 + \delta)$ and $\sigma_1 = \mu(1 + \alpha)$. In this sub-section, we exclude the case of a linear interpolation, *id est* $\delta \neq 0$, in coherence with the hypothesis that the degree of the polynomial ψ_λ is exactly equal to 4.

Lemma 2.2. With the above notations and properties, introduce the notation $\xi \equiv \lambda\mu$; then $\xi > 0$. Introduce also the two other roots p and g of the polynomial ψ_λ . We can write $\psi_\lambda(h) = a_0 h(h-1)(h-p)(h-g)$. Then $a_0 = \frac{2}{3}\xi\delta$. Denote by $\sigma \equiv p+g$ the sum and $\pi \equiv pg$ the product and the two roots p and g . We have $a_0\sigma = \frac{2}{3}\xi(\delta + \alpha)$ and $a_0\pi = 1 - \xi + \frac{1}{3}\alpha\xi - \frac{2}{3}\delta\xi$.

The above relations have been computed with the help of the SageMath [6] formal calculus software. Observe that with the above notations and properties, if the integral I_λ is a positive real number, the roots p and g cannot be equal to 0 or 1.

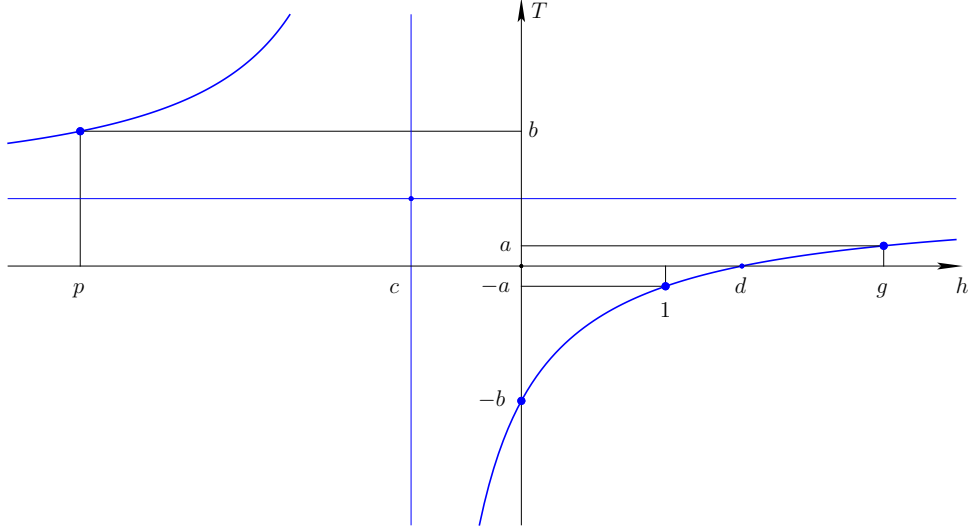


Figure 1: Homographic transformation T for the computation of the integral I_λ ; case $p < 0 < 1 < g$. It is defined by the constraints $T(p) = b$, $T(0) = -b$, $T(1) = -a$ and $T(g) = a$. The coefficients $0 < a < 1 < b$ are determined in Proposition 2.4.

Proposition 2.3. *We keep active the above notations.*

(i) *If the discrete positive family $(\sigma_0, \sigma_{1/2}, \sigma_1)$ is the trace of a concave function, id est if $\sigma_{1/2} - \sigma_0 > \sigma_1 - \sigma_{1/2}$, we have $\delta > 0$, $a_0 > 0$ and $\gamma\xi > 1$. The two roots of the function ψ_λ are real with opposite signs. To fix the ideas, $p < 0 < 1 < g$. If we set $\Delta \equiv \sigma^2 - 4\pi = \frac{1}{a_0^2}\xi[(5\delta^2 + \alpha^2 + 6\delta)\xi - 6\delta]$, we have $p = \frac{1}{2}(\sigma - \sqrt{\Delta})$ and $g = \frac{1}{2}(\sigma + \sqrt{\Delta})$ with σ and π introduced in Lemma (2.2).*

(ii) *Conversely, if the function ψ_λ are real zeros with opposite signs, then $\sigma_{1/2} - \sigma_0 > \sigma_1 - \sigma_{1/2}$.*

Proposition 2.4. *We suppose as in Proposition (2.3) that the function $\psi_\lambda(h)$ has two real zeros p and g that satisfy $p < 0 < 1 < g$. Then the integral $I_\lambda \equiv \int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}}$ can be computed with the help of the following*

formulas: $A = -\sqrt{\frac{g(g-p)}{1-p}}$, $B = \sqrt{\frac{(1-p)(g-p)}{g}}$, $a = \frac{A+1}{A-1}$, $b = \frac{B+1}{B-1}$, $m = 1 - \frac{a^2}{b^2}$, $I_\lambda = \frac{1}{\sqrt{a_0}}(1 - \frac{a}{b})\frac{1}{\sqrt{g-p}}K(m)$. Moreover, we have $0 < a < 1 < b$ and $0 < m < 1$.

Proof. The general ideas of the proof are given in the book [1]. A detailed proof and a complete description of the other cases $p < g < 0 < 1$, $0 < 1 < p < g$ and p and g conjugate complex numbers can be founded in [4]. \square

If the sequence $\sigma_0, \sigma_{1/2}, \sigma_1$ is convex, id est if $\sigma_{1/2} - \sigma_0 < \sigma_1 - \sigma_{1/2}$, we have $\delta < 0$, $a_0 < 0$ and the discriminant Δ introduced in Proposition 2.3 can be negative or positive. In the case $\Delta < 0$ and we have two conjugate roots for the polynomial ψ_λ . In the case $\Delta > 0$, we have of course the case $p < 0 < 1 < g$ studied at Proposition 2.4; this case must be completed by the two sub-cases $p < g < 0 < 1$ and $0 < 1 < p < g$. All these cases are detailed in [4].

• **Third case:** Σ is a continuous positive function, affine in each interval $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$.

The function Σ is defined by its values $\Sigma(0) = \sigma_0$, $\Sigma(\frac{1}{2}) = \sigma_{1/2}$ and $\Sigma(1) = \sigma_1$. We introduce new parameters, still denoted by α and β to represent the data: $\sigma_0 = \sigma_{1/2}(1 - \alpha)$ and $\sigma_1 = \sigma_{1/2}(1 + \beta)$. Then the inequalities $\alpha < 1$ and $\beta > -1$ express the constraints $\sigma_0 > 0$ and $\sigma_1 > 0$. Moreover, $\sigma_{1/2}$ is positive.

Lemma 2.3. *If the continuous function Σ defined on $(0, 1)$ by its values $\Sigma(0) = \sigma_0$, $\Sigma(\frac{1}{2}) = \sigma_{1/2}$ and $\Sigma(1) = \sigma_1$ is affine in each interval $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, the function ψ_λ defined by the conditions*

$$\frac{d^2\psi_\lambda}{dh^2} = 1 - 2\lambda\Sigma \text{ and } \psi_\lambda(0) = \psi_\lambda(1) = 0$$

admits the following expression

$$\psi_\lambda(h) = \begin{cases} \psi^0(h) = -\frac{2}{3}\alpha\xi h^3 + (\alpha\xi - \xi + 1)h^2 + (\frac{\beta-5\alpha}{12}\xi + \xi - 1)h & \text{if } h \leq \frac{1}{2} \\ \psi^1(h) = \frac{2}{3}\beta\xi(1-h)^3 + (-\beta\xi - \xi + 1)(1-h)^2 + (\frac{5\beta-\alpha}{12}\xi + \xi - 1)(1-h) & \text{if } h \geq \frac{1}{2}, \end{cases}$$

with $\xi \equiv \lambda\sigma_{1/2}$. We have in particular $\psi_\lambda(\frac{1}{2}) = \frac{\beta-\alpha}{24}\xi + \frac{1}{4}(\xi - 1)$. Moreover, if ψ_λ is positive on the interval $(0, 1)$, $\frac{d\psi_\lambda}{dh}(0) = \frac{\beta-5\alpha}{12}\xi + \xi - 1 > 0$ and $\frac{d\psi_\lambda}{dh}(1) = -(\frac{5\beta-\alpha}{12}\xi + \xi - 1) < 0$.

Remark 1. *We observe that the expression of ψ^1 is obtained from the expression of ψ^0 by making the transformations $\alpha \longleftrightarrow (-\beta)$ and $h \longleftrightarrow (1-h)$.*

Proof. The function Σ admits the algebraic expressions: $\Sigma(h) = (2\alpha h - \alpha + 1)\sigma_{1/2}$ if $h \leq \frac{1}{2}$ and $\Sigma(h) = (2\beta h - \beta + 1)\sigma_{1/2}$ if $h \geq \frac{1}{2}$. We integrate two times, enforce the conditions $\psi_\lambda(0) = \psi_\lambda(1) = 0$ and impose the continuity of ψ_λ and $\frac{d\psi_\lambda}{dh}$ at the specific value $h = \frac{1}{2}$. The result follows. \square

We have to compute the integral $I_\lambda \equiv \int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}}$. We have the following calculus: $I_\lambda = \int_0^{1/2} \frac{dh}{\sqrt{\psi^0(h)}} + \int_{1/2}^1 \frac{dh}{\sqrt{\psi^1(h)}} = \int_0^{1/2} \frac{dh}{\sqrt{\psi^0(h)}} + \int_0^{1/2} \frac{dh}{\sqrt{\psi^1(1-h)}}$. Due to the Remark 1, the determination of the second term relative to ψ^1 is very analogous to the term associated to ψ^0 .

Proposition 2.5. *Recall that $\psi^0(h) = -\frac{2}{3}\alpha\xi h^3 + (\alpha\xi - \xi + 1)h^2 + (\frac{\beta-5\alpha}{12}\xi + \xi - 1)h$.*

(i) If the polynomial $\psi^0(h)$ has two complex roots, we write it under the form $\psi^0(h) \equiv a_0 h(h^2 + \mu h + \zeta^4)$ with $a_0 = \frac{2}{3}|\alpha|\xi > 0$ and we have the inequality $\mu^2 - 4\zeta^4 < 0$; we introduce $m = \frac{1}{2} - \frac{\mu}{4\zeta^2}$ that satisfies $0 < m < 1$ and $\varphi_0 = 2 \arctan(\frac{1}{\zeta\sqrt{2}})$. Then we have $I_0 = \frac{1}{\zeta\sqrt{a_0}} K(\varphi_0, m)$.

(ii) If the polynomial $\psi^0(h)$ has three real roots, we set $\psi^0(h) \equiv a_0 h(r_- - h)(r_+ - h)$ with $a_0 = \frac{2}{3}|\alpha|\xi > 0$ and $0 < r_- < r_+$; we consider $m = \frac{r_-}{r_+} \in (0, 1)$ and $\varphi_0 = \arcsin(\frac{1}{\sqrt{2}r_-})$. The integral I_0 is computed with the following expression: $I_0 = \frac{2}{\sqrt{a_0}r_+} K(\varphi_0, m)$.

• Fourth case and fifth cases: Σ is a P2 polynomial or a continuous positive function, piecewise affine. The function $h \mapsto \frac{V(h)}{h(1-h)}$ is approximated by an affine function, as described in [3].

3. Numerical method

After adimensionalization, the problem can be formulated with the formulation (3). We introduce a nonregular meshing $0 = z_0 < z_1 < \dots < z_{N-1} < z_N = 1$ of the interval $[0, 1]$ and we set $\Delta z_{j+1/2} \equiv z_{j+1} - z_j$ for $j = 1, \dots, N$. We integrate the differential equation $\frac{dh}{dz} = \varphi(z)$ with the Crank-Nicolson scheme:

$$\frac{h_{j+1} - h_j}{\Delta z_{j+1/2}} = \frac{1}{2}(\varphi_j + \varphi_{j+1}) \text{ for } j = 0, \dots, N-1.$$

Then after two integrations, the first equation can be written as $\varphi_j = \sqrt{\psi_\lambda(h_j)}$ and the Crank-Nicolson scheme takes the form $\frac{h_{j+1} - h_j}{\frac{1}{2}(\sqrt{\psi_\lambda(h_j)} + \sqrt{\psi_\lambda(h_{j+1})})} = \Delta z_{j+1/2}(\lambda)$.

We impose the values $h_j \equiv \sin^2(\frac{\pi j}{2N})$ in order to take into account the singularities and two boundary conditions of the problem at $z = 0$ and $z = 1$. The notation $\Delta z_{j+1/2}(\lambda)$ in the right hand side of the previous relation is justified by the fact that if the numbers h_j are given, the left hand side is a simple function of the

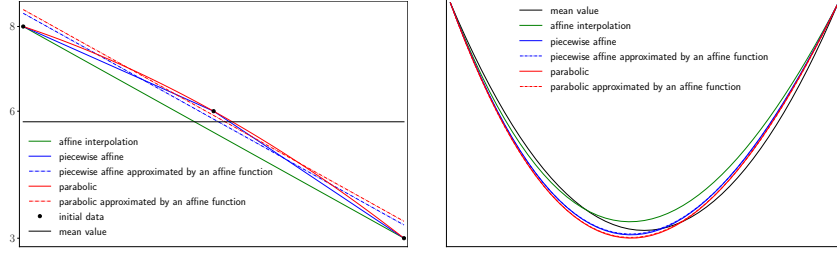


Figure 2: Function $h \mapsto \Sigma(h)$ on the left, $h \mapsto V(h)$ on the right.

scalar λ . The number λ is *a priori* not known, but we have the natural relation $\sum_{j=0}^{N-1} \Delta z_{j+1/2}(\lambda) = 1$ that takes the form

$$\sum_{j=0}^{N-1} \frac{h_{j+1} - h_j}{\frac{1}{2}(\sqrt{\psi_\lambda(h_j)} + \sqrt{\psi_\lambda(h_{j+1})})} = 1. \quad (4)$$

Lemma 3.1. *Recall that V is the unique solution of $V'' = \Sigma$, $V(0) = V(1) = 0$. We have observed previously that $\psi_\lambda(h) = h(h-1) - 2\lambda V(h)$. Define $\lambda_* = \max_{[0,1]} \frac{h(h-1)}{V(h)}$.*

i) For $\lambda > \lambda_*$, $\psi_\lambda(h_j) > 0$

ii) Equation (4) has a unique solution $\lambda \in (\lambda_*, +\infty)$ when $\sum_{j=0}^{N-1} \frac{h_{j+1} - h_j}{\frac{1}{2}(\sqrt{\psi_{\lambda_*}(h_j)} + \sqrt{\psi_{\lambda_*}(h_{j+1})})} \in \bar{\mathbb{R}}_+ > 1$.

For a fixed discretization with N intervals, a Newton algorithm can be implemented without difficulty. With this procedure, we recover on one hand an approximated value λ_N of the unknown λ and on the other hand the entire approximate solution of the problem $h_j \approx h(x_j)$ and $\varphi_j \approx \sqrt{\psi_\lambda(h_j)}$. Observe that at convergence of the Newton algorithm, the abscissas x_j are a function of the solution λ_N and the converged space mesh is a result of the problem. This coupled problem can be reduced to a single equation with only one real variable even after discretization!

4. Numerical results

They are presented in Figures 2 and 3. They correspond to the decreasing data $\sigma_0 = 8$, $\sigma_{1/2} = 6$ and $\sigma_1 = 3$. We obtain the following exact values for the scalar parameter: $\lambda = 1.89036$ in the decoupled case (case 0), $\lambda = 1.99533$ in the affine case (case 1), $\lambda = 1.86593$ in the parabolic case (case 2), $\lambda = 1.89454$ in the piecewise affine case (case 3), $\lambda = 1.85769$ in the parabolic case approached by an affine function (case 4) and $\lambda = 1.88614$ in the piecewise affine case projected on affine functions (case 5). For each of these six cases, our analytical-numerical approach gives converging results at second order accuracy for the parameter λ , hence showing the powerfulness and precision of the explicit calculations that one describes using the elliptic integrals.

- [1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series - 55, June 1964.
- [2] S. Dellacherie, E. Jamelot, O. Lafitte, M. Riyaz, "Numerical results for the coupling of a simple neutronics diffusion model and a simple hydrodynamics low mach number model without coupling codes, 18th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC 2016), pages 119-124, DOI: 10.1109/SYNASC40771.2016. International series in pure and applied mathematics, McGraw-Hill, 1955.
- [3] S. Dellacherie and O. Lafitte, "Une solution explicite monodimensionnelle d'un modèle simplifié de couplage stationnaire thermohydraulique-neutronique", *Annales mathématiques du Québec*, volume 41, pages 221–264, 2017.
- [4] F. Dubois and O. Lafitte, to appear, 2021.

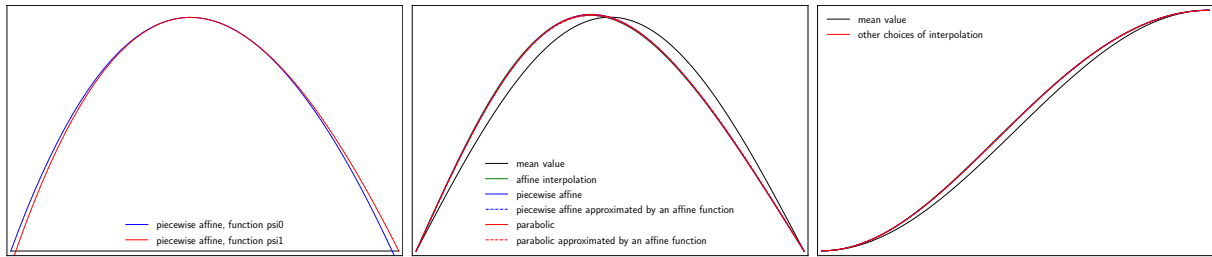


Figure 3: Functions $h \mapsto \psi^0(h)$ and $h \mapsto \psi^1(h)$ on the left, $z \mapsto \varphi(z)$ in the middle and $z \mapsto h(z)$ on the right.

- [5] S. Dellacherie, E. Jamelot and O. Lafitte, “A simple monodimensional model coupling an enthalpy transport equation and a neutron diffusion equation”, *Mathematics Letters*, (62) 2016, pp. 35-41.
- [6] W. Stein and D. Joyner, “SAGE: System for Algebra and Geometry Experimentation”, *ACM SIGSAM Bulletin*, volume 39, number 2, pages 61-64, 2005. Sagemath, with more than 600 contributors around the world, is a free open-source mathematics software system licensed under the GPL, www.sagemath.org.