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# An analytic and symbolic analysis of a coupled thermo-neutronic problem<sup>☆</sup>

François Dubois<sup>a,b,c</sup>, Olivier Lafitte<sup>a,d</sup>

<sup>a</sup>International Research Laboratory 3457 du Centre National de la Recherche Scientifique, Centre de Recherches Mathématiques, Université de Montréal, Montréal, QC, Canada.

<sup>b</sup>Conservatoire National des Arts et Métiers, LMSSC laboratory, Paris, France.

<sup>c</sup>Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, France.

<sup>d</sup>Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), 99 Avenue J.-B. Clément F-93430, Villetaneuse Cedex, France.

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## 1. Introduction

In this note, we recall briefly the construction of an analytic solution of the low Mach number thermo-hydraulic model obtained by Dellacherie *et al.* detailed in [5]. The thermohydraulic-neutronics problem, stated in a 1d setting reads

$$\frac{d}{dz}(D(h(z))\frac{d\phi}{dz}) + \Sigma_a(h(z))\phi(z) = \frac{\nu\Sigma_f(h(z))}{k_{eff}}\phi(z), h'(z) = K\nu\Sigma_f(h)\phi(z), z \in [0, L], \quad (1)$$

where the unknowns are  $\phi$ , the neutron density,  $h$ , the enthalpy (a measure of the temperature) and  $k_{eff}$  is called the multiplication factor of neutrons. The physical data are the functions  $\Sigma_a$  and  $\nu\Sigma_f$  (respectively the absorption and fission cross section of the fissile material) and the constants  $D$  (diffusion coefficient) and  $K$  (heat transfer constant). This system is supplemented with the boundary conditions

$$\phi(0) = \phi(L) = 0, \phi \geq 0, h(0) = h_e, h(L) = h_s \quad (2)$$

Recall that in [5], we constructed an analytical solution of (1)(2) for any set of continuous positive functions  $D, \Sigma_a, \nu\Sigma_f$ , as a generalisation of Dellacherie and Lafitte [3]. Let  $\mathcal{X}$  and  $\mathcal{Y}$  be uniquely defined by:

$$-\frac{d}{dh}(D(h)\nu\Sigma_f(h)\frac{d\mathcal{X}}{dh}) = \frac{\Sigma_a(h)}{\nu\Sigma_f(h)}, -\frac{d}{dh}(D(h)\nu\Sigma_f(h)\frac{d\mathcal{Y}}{dh}) = 1, \mathcal{X}(h_e) = \mathcal{X}(h_s) = \mathcal{Y}(h_e) = \mathcal{Y}(h_s) = 0.$$

Denote by  $\Psi(h, k_{eff}) = 2(\frac{\mathcal{Y}(h)}{k_{eff}} - \mathcal{X}(h))$ . Let  $z \rightarrow h(z, k_{eff})$  is the unique solution of:

$$h'(z, k_{eff}) = \sqrt{\Psi(h(z, k_{eff}), k_{eff})}, h(0) = h_e.$$

**Lemma 1.1.** Equation  $h(L, k_{eff}) = h_s$  has a unique solution  $k_{eff}^* \in I := (0, (\max_{[h_e, h_s]} \frac{\mathcal{Y}}{\mathcal{X}})^{-1})$  which provides the unique solution of the system (1), (2).

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Email addresses: francois.dubois@lecnam.net (François Dubois), lafitte@math.univ-paris13.fr (Olivier Lafitte)

This result is the consequence of  $k_{eff} \rightarrow h(L, k_{eff})$  decreasing from  $I$  to  $(+\infty, 0)$  (see [3], [5]).

In this extended abstract, we shall, without losing the generality, simplify the problem (assuming that  $D$  and  $\Sigma_a$  are constant functions), we shall choose  $h_e = 0, h_s = 1$ , denote by  $\lambda = \frac{1}{k_{eff}}$  and consider the totally adimensionnalized problem ( $L = 1$  (unit of length),  $\Sigma_a = 1, D = 1, K = 1$  (physical constants),  $h_e = 0, h_s = 1$  (normalizing the temperature) ):

$$-(D\phi)' + \Sigma_a\phi = \lambda\Sigma(h)\phi, h' = K\phi, z \in [0, 1]$$

supplemented by  $\phi(0) = \phi(1) = 0, h(0) = h_e, h(1) = h_s$ , where  $\Sigma$  is a given continuous function.

We assume throughout this paper

$$(H) \quad \Sigma(h) \geq \Sigma_* > 0, 0 \leq h \leq 1.$$

and we define  $\psi_\lambda$  the unique solution of  $\psi_\lambda''(h) = 2 - 2\lambda\Sigma(h)$  with  $\psi_\lambda(0) = \psi_\lambda(1) = 0$ . We intend to present a numerical method which solves the coupled problem without using the numerical methods traditionally used for solving each equation but rather concentrating on solving the equation  $h(1) = 1$ . This numerical method, as well as analytic and symbolic methods, are implemented when one knows only three values of  $\Sigma$ , and for simplicity again one assumes that one knows  $\Sigma(0) = \sigma_0, \Sigma(\frac{1}{2}) = \sigma_{\frac{1}{2}}, \Sigma(1) = \sigma_1$ . The analytic and symbolic methods are consequences of the finding of **exact** solutions using the incomplete Jacobi functions (which are its solutions) when  $\Psi$  is a polynomial of degree 3 or 4.

This numerical method is compared with an adapted numerical scheme, described in the first paragraph of Section ???. This numerical scheme uses the Crank-Nicholson discretization of the ODE  $h' = \phi$  in order to recover, from the numerical scheme, the property that the vector  $(\phi_j)_j$  is a generalized eigenvector of the discretized version of  $-\phi'' + \phi = \lambda\Sigma(h)\phi$ , hence keeping what researchers working in the neutronics computations call the spectral problem for the neutronics.

We want to construct different relevant approximations of  $\Sigma(h)$  (namely constant, affine, polynomial of order 2) in order to see what is the influence on the value of  $k_{eff}$  in order to be able to study the sensitivity of the value of  $k_{eff}$  for these different approximations, and furthermore to understand the number of significative figures in  $k_{eff}$ . Usually, the benchmarks in realistic cases are using sensitivity of  $10^{-5}$  around  $k_{eff} = 1$ , which is a magnitude or two smaller than the results we obtain here, see Section 4.

## 2. Analytical approach

Following *e.g.* Abramowitz and Stegun [1], we define the incomplete elliptic integral of the first kind  $K(m, \varphi)$  by the relation  $K(m, \varphi) = \int_0^\varphi \frac{d\theta}{\sqrt{1-m \sin^2 \theta}}$ ,  $0 \leq \varphi \leq \frac{\pi}{2}$ ,  $m < 1$ . The complete elliptic integral of the first kind  $K(m)$  is defined by  $K(m) \equiv K(m, \frac{\pi}{2})$ .

**Lemma 2.1.** (i) Given two positive reals  $a$  and  $b$  such that  $0 < a < b$ , we have

$$\int_{-a}^a \frac{dT}{\sqrt{(a^2-T^2)(b^2-T^2)}} = \frac{2}{b} K(m), \quad m = \frac{a^2}{b^2} \quad \text{and} \quad \int_a^b \frac{dT}{\sqrt{(T^2-a^2)(b^2-T^2)}} = \frac{1}{b} K(m), \quad m = 1 - \frac{a^2}{b^2}.$$

(ii) Given two reals  $a$  and  $b$  with  $a > 0$  and  $b \neq 0$ , we have  $\int_{-a}^a \frac{dT}{\sqrt{(a^2-T^2)(b^2+T^2)}} = \frac{2}{|b|} K(m)$ ,  $m = -\frac{a^2}{b^2}$ .

Recall that we are studying the model described by the following set of equations, where  $\phi$  needs (physically) to be positive because it is a density flux of neutrons:

$$\begin{cases} -\frac{d^2\phi}{dz^2} + \phi(z) = \lambda\Sigma(h(z))\phi(z), \frac{dh}{dz} = \phi(z), 0 < z < 1, \\ h(0) = 0, h(1) = 1, \phi(0) = 0, \phi(1) = 0, \lambda \geq 0, \phi(z) > 0 \text{ if } 0 < z < 1. \end{cases} \quad (3)$$

Here  $\Sigma$  is defined by three positive real numbers  $\sigma_0, \sigma_{1/2}$  and  $\sigma_1$  which represent the values of  $\Sigma$  at, respectively,  $0, \frac{1}{2}, 1$ . We describe in what follows five different modellings of the function  $\Sigma$  from these

values which lead to analytical approaches for determining the scalar parameter  $\lambda > 0$  and deduce secondly the functions  $z \mapsto \varphi(z)$  and  $z \mapsto h(z)$ .

Introduce

**Definition 1.** We define  $\mu$ ,  $\alpha$  and  $\delta$  through  $\mu = \frac{1}{2}(\sigma_0 + \sigma_1)$ ,  $\alpha = 1 - \frac{\sigma_0}{\mu}$  and  $\delta = -\frac{\sigma_0 - 2\sigma_{1/2} + \sigma_1}{\sigma_0 + \sigma_1}$ . They satisfy the inequalities  $\mu > 0$ ,  $|\alpha| < 1$  and  $\delta > -1$ . Furthermore  $\gamma \equiv 1 - \frac{1}{3}|\alpha| + \frac{2}{3}\delta > 0$ .

• Zero-th case:  $\Sigma$  is constant ( $\alpha = \delta = 0$ ). Then the system modeled by the previous set of equations is totally decoupled and an exact solution can be provided with elementary arguments.

**Proposition 2.1.** If  $\sigma_0 = \sigma_{1/2} = \sigma_1 \equiv \mu > 0$ , the previous problem (3) admits a unique decoupled solution: we have  $\lambda\mu = 1 + \pi^2$ ,  $\varphi(z) = \frac{\pi}{2} \sin(\pi z)$  and  $h(z) = \frac{1}{2}(1 - \cos(\pi z))$ .

If the function  $\Sigma$  is no-more constant, it has been proven in [2, 3] that the unknown of the problem can be obtained with the following process. First integrate twice the function  $\Sigma$  and obtain a convex negative function  $V$  such that  $\frac{d^2V}{dh^2} = \Sigma(h)$ ,  $V(0) = V(1) = 0$ . Second define  $\psi_\lambda(h) \equiv h(h-1) - 2\lambda V(h)$ . Then the equation for the function  $z \mapsto h(z)$  becomes  $\frac{dh}{dz} = \sqrt{\psi_\lambda(h)}$  and the condition  $h(1) = 1$  gives a scalar equation for the unknown  $\lambda > 0$ :  $\int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}} = 1$ .

A first difficulty is to compute the integral  $I_\lambda \equiv \int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}}$ . Then we can solve easily the equation  $I_\lambda = 1$  with a Newton-like algorithm. Once  $\lambda$  is determined, the explicitation of the functions  $z \mapsto \varphi(z)$  and  $z \mapsto h(z)$  is not difficult. Thus the analytical method we propose is found on an analytical determination of the integral  $I_\lambda$ . We focus our attention to this question in the next sub-sections.

• First case:  $\Sigma \in P_1$ .

In this case, the function  $\Sigma$  is a positive affine function on the interval  $(0, 1)$ . Hence  $\delta = 0$  and  $\sigma_0 = \mu(1 - \alpha)$ ,  $\sigma_{1/2} = \mu$  and  $\sigma_1 = \mu(1 + \alpha)$  with  $\mu > 0$  and  $|\alpha| < 1$  to satisfy the constraint of positivity. We introduce the notation  $\xi \equiv \lambda\mu$ .

**Proposition 2.2.** With the notations introduced previously, the functions  $V$  and  $\psi_\lambda$  admit the algebraic expressions  $V(h) = \mu h(h-1)\left(\frac{1}{2} - \frac{\alpha}{6} + \frac{\alpha}{3}h\right)$  and  $\psi_\lambda(h) = h(h-1)\left(\xi\left(1 + \frac{\alpha}{3}\right) - 1 - \frac{2\alpha\xi}{3}h\right)$ . Then we have  $I_\lambda = \frac{2}{\sqrt{\xi\left(1 + \frac{|\alpha|}{3}\right) - 1}} K(m)$  with  $m = \frac{2|\alpha|\xi}{3\xi + |\alpha|\xi - 3}$ .

• Second case:  $\Sigma \in P_2$ .

In this case, the polynomial  $\psi_\lambda$  is a polynomial of degree 4 with real coefficients and  $\psi_\lambda$  is positive on the interval  $(0, 1)$ . We have also  $\psi_\lambda(0) = \psi_\lambda(1) = 0$ . Recall that  $\psi_\lambda(h) = h(h-1) - 2\lambda V(h)$  with  $V'' \equiv \Sigma$  the Lagrange interpolate polynomial such that  $\Sigma(0) = \sigma_0$ ,  $\Sigma(\frac{1}{2}) = \sigma_{1/2}$  and  $\Sigma(1) = \sigma_1$ . All these coefficients are supposed positive:  $\sigma_0 > 0$ ,  $\sigma_{1/2} > 0$ ,  $\sigma_1 > 0$ . One has  $\sigma_0 = \mu(1 - \alpha)$ ,  $\sigma_{1/2} = \mu(1 + \delta)$  and  $\sigma_1 = \mu(1 + \alpha)$ . In this sub-section, we exclude the case of a linear interpolation, *id est*  $\delta \neq 0$ , in coherence with the hypothesis that the degree of the polynomial  $\psi_\lambda$  is exactly equal to 4.

**Lemma 2.2.** With the above notations and properties, introduce the notation  $\xi \equiv \lambda\mu$ ; then  $\xi > 0$ . Introduce also the two other roots  $p$  and  $g$  of the polynomial  $\psi_\lambda$ . We can write  $\psi_\lambda(h) = a_0 h(h-1)(h-p)(h-g)$ . Then  $a_0 = \frac{2}{3}\xi\delta$ . Denote by  $\sigma \equiv p+g$  the sum and  $\pi \equiv pg$  the product and the two roots  $p$  and  $g$ . We have  $a_0\sigma = \frac{2}{3}\xi(\delta + \alpha)$  and  $a_0\pi = 1 - \xi + \frac{1}{3}\alpha\xi - \frac{2}{3}\delta\xi$ .

The above relations have been computed with the help of the SageMath [6] formal calculus software. Observe that with the above notations and properties, if the integral  $I_\lambda$  is a positive real number, the roots  $p$  and  $g$  cannot be equal to 0 or 1.

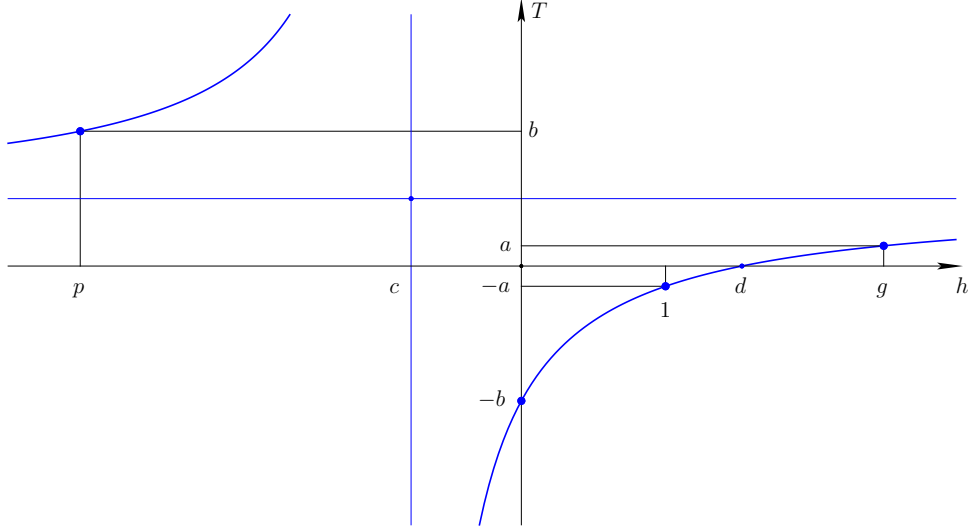


Figure 1: Homographic transformation  $T$  for the computation of the integral  $I_\lambda$ ; case  $p < 0 < 1 < g$ . It is defined by the constraints  $T(p) = b$ ,  $T(0) = -b$ ,  $T(1) = -a$  and  $T(g) = a$ . The coefficients  $0 < a < 1 < b$  are determined in Proposition 2.4.

**Proposition 2.3.** *We keep active the above notations.*

(i) *If the discrete positive family  $(\sigma_0, \sigma_{1/2}, \sigma_1)$  is the trace of a concave function, id est if  $\sigma_{1/2} - \sigma_0 > \sigma_1 - \sigma_{1/2}$ , we have  $\delta > 0$ ,  $a_0 > 0$  and  $\gamma\xi > 1$ . The two roots of the function  $\psi_\lambda$  are real with opposite signs. To fix the ideas,  $p < 0 < 1 < g$ . If we set  $\Delta \equiv \sigma^2 - 4\pi = \frac{1}{a_0^2} \xi [(5\delta^2 + \alpha^2 + 6\delta)\xi - 6\delta]$ , we have  $p = \frac{1}{2}(\sigma - \sqrt{\Delta})$  and  $g = \frac{1}{2}(\sigma + \sqrt{\Delta})$  with  $\sigma$  and  $\pi$  introduced in Lemma (2.2).*

(ii) *Conversely, if the function  $\psi_\lambda$  are real zeros with opposite signs, then  $\sigma_{1/2} - \sigma_0 > \sigma_1 - \sigma_{1/2}$ .*

**Proposition 2.4.** *We suppose as in Proposition (2.3) that the function  $\psi_\lambda(h)$  has two real zeros  $p$  and  $g$  that satisfy  $p < 0 < 1 < g$ . Then the integral  $I_\lambda \equiv \int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}}$  can be computed with the help of the following*

*formulas:  $A = -\sqrt{\frac{g(g-p)}{1-p}}$ ,  $B = \sqrt{\frac{(1-p)(g-p)}{g}}$ ,  $a = \frac{A+1}{A-1}$ ,  $b = \frac{B+1}{B-1}$ ,  $m = 1 - \frac{a^2}{b^2}$ ,  $I_\lambda = \frac{1}{\sqrt{a_0}} (1 - \frac{a}{b}) \frac{1}{\sqrt{g-p}} K(m)$ . Moreover, we have  $0 < a < 1 < b$  and  $0 < m < 1$ .*

*Proof.* The general ideas of the proof are given in the book [1]. A detailed proof and a complete description of the other cases  $p < g < 0 < 1$ ,  $0 < 1 < p < g$  and  $p$  and  $g$  conjugate complex numbers can be founded in [4].  $\square$

If the sequence  $\sigma_0, \sigma_{1/2}, \sigma_1$  is convex, id est if  $\sigma_{1/2} - \sigma_0 < \sigma_1 - \sigma_{1/2}$ , we have  $\delta < 0$ ,  $a_0 < 0$  and the discriminant  $\Delta$  introduced in Proposition 2.3 can be negative or positive. In the case  $\Delta < 0$  and we have two conjugate roots for the polynomial  $\psi_\lambda$ . In the case  $\Delta > 0$ , we have of course the case  $p < 0 < 1 < g$  studied at Proposition 2.4; this case must be completed by the two sub-cases  $p < g < 0 < 1$  and  $0 < 1 < p < g$ . All these cases are detailed in [4].

• **Third case:**  $\Sigma$  is a continuous positive function, affine in each interval  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$ .

The function  $\Sigma$  is defined by its values  $\Sigma(0) = \sigma_0$ ,  $\Sigma(\frac{1}{2}) = \sigma_{1/2}$  and  $\Sigma(1) = \sigma_1$ . We introduce new parameters, still denoted by  $\alpha$  and  $\beta$  to represent the data:  $\sigma_0 = \sigma_{1/2}(1 - \alpha)$  and  $\sigma_1 = \sigma_{1/2}(1 + \beta)$ . Then the inequalities  $\alpha < 1$  and  $\beta > -1$  express the constraints  $\sigma_0 > 0$  and  $\sigma_1 > 0$ . Moreover,  $\sigma_{1/2}$  is positive.

**Lemma 2.3.** *If the continuous function  $\Sigma$  defined on  $(0, 1)$  by its values  $\Sigma(0) = \sigma_0$ ,  $\Sigma(\frac{1}{2}) = \sigma_{1/2}$  and  $\Sigma(1) = \sigma_1$  is affine in each interval  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$ , the function  $\psi_\lambda$  defined by the conditions*

$$\frac{d^2\psi_\lambda}{dh^2} = 1 - 2\lambda\Sigma \text{ and } \psi_\lambda(0) = \psi_\lambda(1) = 0$$

*admits the following expression*

$$\psi_\lambda(h) = \begin{cases} \psi^0(h) = -\frac{2}{3}\alpha\xi h^3 + (\alpha\xi - \xi + 1)h^2 + (\frac{\beta-5\alpha}{12}\xi + \xi - 1)h & \text{if } h \leq \frac{1}{2} \\ \psi^1(h) = \frac{2}{3}\beta\xi(1-h)^3 + (-\beta\xi - \xi + 1)(1-h)^2 + (\frac{5\beta-\alpha}{12}\xi + \xi - 1)(1-h) & \text{if } h \geq \frac{1}{2}, \end{cases}$$

*with  $\xi \equiv \lambda\sigma_{1/2}$ . We have in particular  $\psi_\lambda(\frac{1}{2}) = \frac{\beta-\alpha}{24}\xi + \frac{1}{4}(\xi - 1)$ . Moreover, if  $\psi_\lambda$  is positive on the interval  $(0, 1)$ ,  $\frac{d\psi_\lambda}{dh}(0) = \frac{\beta-5\alpha}{12}\xi + \xi - 1 > 0$  and  $\frac{d\psi_\lambda}{dh}(1) = -(\frac{5\beta-\alpha}{12}\xi + \xi - 1) < 0$ .*

**Remark 1.** *We observe that the expression of  $\psi^1$  is obtained from the expression of  $\psi^0$  by making the transformations  $\alpha \longleftrightarrow (-\beta)$  and  $h \longleftrightarrow (1-h)$ .*

*Proof.* The function  $\Sigma$  admits the algebraic expressions:  $\Sigma(h) = (2\alpha h - \alpha + 1)\sigma_{1/2}$  if  $h \leq \frac{1}{2}$  and  $\Sigma(h) = (2\beta h - \beta + 1)\sigma_{1/2}$  if  $h \geq \frac{1}{2}$ . We integrate two times, enforce the conditions  $\psi_\lambda(0) = \psi_\lambda(1) = 0$  and impose the continuity of  $\psi_\lambda$  and  $\frac{d\psi_\lambda}{dh}$  at the specific value  $h = \frac{1}{2}$ . The result follows.  $\square$

We have to compute the integral  $I_\lambda \equiv \int_0^1 \frac{dh}{\sqrt{\psi_\lambda(h)}}$ . We have the following calculus:  $I_\lambda = \int_0^{1/2} \frac{dh}{\sqrt{\psi^0(h)}} + \int_{1/2}^1 \frac{dh}{\sqrt{\psi^1(h)}} = \int_0^{1/2} \frac{dh}{\sqrt{\psi^0(h)}} + \int_0^{1/2} \frac{dh}{\sqrt{\psi^1(1-h)}}$ . Due to the Remark 1, the determination of the second term relative to  $\psi^1$  is very analogous to the term associated to  $\psi^0$ .

**Proposition 2.5.** *Recall that  $\psi^0(h) = -\frac{2}{3}\alpha\xi h^3 + (\alpha\xi - \xi + 1)h^2 + (\frac{\beta-5\alpha}{12}\xi + \xi - 1)h$ .*

*(i) If the polynomial  $\psi^0(h)$  has two complex roots, we write it under the form  $\psi^0(h) \equiv a_0 h(h^2 + \mu h + \zeta^4)$  with  $a_0 = \frac{2}{3}|\alpha|\xi > 0$  and we have the inequality  $\mu^2 - 4\zeta^4 < 0$ ; we introduce  $m = \frac{1}{2} - \frac{\mu}{4\zeta^2}$  that satisfies  $0 < m < 1$  and  $\varphi_0 = 2 \arctan(\frac{1}{\zeta\sqrt{2}})$ . Then we have  $I_0 = \frac{1}{\zeta\sqrt{a_0}} K(\varphi_0, m)$ .*

*(ii) If the polynomial  $\psi^0(h)$  has three real roots, we set  $\psi^0(h) \equiv a_0 h(r_- - h)(r_+ - h)$  with  $a_0 = \frac{2}{3}|\alpha|\xi > 0$  and  $0 < r_- < r_+$ ; we consider  $m = \frac{r_-}{r_+} \in (0, 1)$  and  $\varphi_0 = \arcsin(\frac{1}{\sqrt{2}r_-})$ . The integral  $I_0$  is computed with the following expression:  $I_0 = \frac{2}{\sqrt{a_0}r_+} K(\varphi_0, m)$ .*

• Fourth case and fifth cases:  $\Sigma$  is a P2 polynomial or a continuous positive function, piecewise affine. The function  $h \mapsto \frac{V(h)}{h(1-h)}$  is approximated by an affine function, as described in [3].

### 3. Numerical method

After adimensionalization, the problem can be formulated with the formulation (3). We introduce a nonregular meshing  $0 = z_0 < z_1 < \dots < z_{N-1} < z_N = 1$  of the interval  $[0, 1]$  and we set  $\Delta z_{j+1/2} \equiv z_{j+1} - z_j$  for  $j = 1, \dots, N$ . We integrate the differential equation  $\frac{dh}{dz} = \varphi(z)$  with the Crank-Nicolson scheme:

$$\frac{h_{j+1} - h_j}{\Delta z_{j+1/2}} = \frac{1}{2}(\varphi_j + \varphi_{j+1}) \text{ for } j = 0, \dots, N-1.$$

Then after two integrations, the first equation can be written as  $\varphi_j = \sqrt{\psi_\lambda(h_j)}$  and the Crank-Nicolson scheme takes the form  $\frac{h_{j+1} - h_j}{\frac{1}{2}(\sqrt{\psi_\lambda(h_j)} + \sqrt{\psi_\lambda(h_{j+1})})} = \Delta z_{j+1/2}(\lambda)$ .

We impose the values  $h_j \equiv \sin^2(\frac{\pi j}{2N})$  in order to take into account the singularities and two boundary conditions of the problem at  $z = 0$  and  $z = 1$ . The notation  $\Delta z_{j+1/2}(\lambda)$  in the right hand side of the previous relation is justified by the fact that if the numbers  $h_j$  are given, the left hand side is a simple function of the

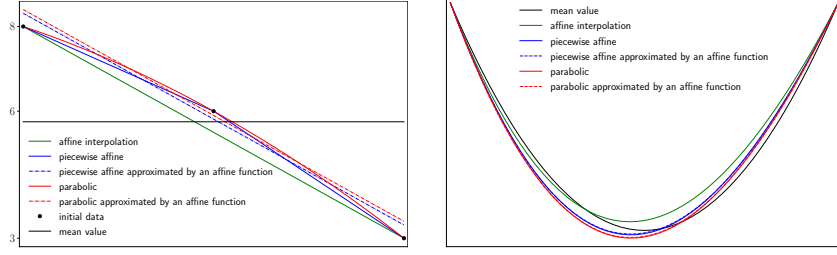


Figure 2: Function  $h \mapsto \Sigma(h)$  on the left,  $h \mapsto V(h)$  on the right.

scalar  $\lambda$ . The number  $\lambda$  is *a priori* not known, but we have the natural relation  $\sum_{j=0}^{N-1} \Delta z_{j+1/2}(\lambda) = 1$  that takes the form

$$\sum_{j=0}^{N-1} \frac{h_{j+1} - h_j}{\frac{1}{2}(\sqrt{\psi_\lambda(h_j)} + \sqrt{\psi_\lambda(h_{j+1})})} = 1. \quad (4)$$

**Lemma 3.1.** Recall that  $V$  is the unique solution of  $V'' = \Sigma$ ,  $V(0) = V(1) = 0$ . We have observed previously that  $\psi_\lambda(h) = h(h-1) - 2\lambda V(h)$ . Define  $\lambda_* = \max_{[0,1]} \frac{h(h-1)}{V(h)}$ .

i) For  $\lambda > \lambda_*$ ,  $\psi_\lambda(h_j) > 0$

ii) Equation (4) has a unique solution  $\lambda \in (\lambda_*, +\infty)$  when  $\sum_{j=0}^{N-1} \frac{h_{j+1} - h_j}{\frac{1}{2}(\sqrt{\psi_{\lambda_*}(h_j)} + \sqrt{\psi_{\lambda_*}(h_{j+1})})} \in \bar{\mathbb{R}}_+ > 1$ .

For a fixed discretization with  $N$  intervals, a Newton algorithm can be implemented without difficulty. With this procedure, we recover on one hand an approximated value  $\lambda_N$  of the unknown  $\lambda$  and on the other hand the entire approximate solution of the problem  $h_j \approx h(x_j)$  and  $\varphi_j \approx \sqrt{\psi_\lambda(h_j)}$ . Observe that at convergence of the Newton algorithm, the abscissas  $x_j$  are a function of the solution  $\lambda_N$  and the converged space mesh is a result of the problem. This coupled problem can be reduced to a single equation with only one real variable even after discretization!

#### 4. Numerical results

They are presented in Figures 2 and 3. They correspond to the decreasing data  $\sigma_0 = 8$ ,  $\sigma_{1/2} = 6$  and  $\sigma_1 = 3$ . We obtain the following exact values for the scalar parameter:  $\lambda = 1.89036$  in the decoupled case (case 0),  $\lambda = 1.99533$  in the affine case (case 1),  $\lambda = 1.86593$  in the parabolic case (case 2),  $\lambda = 1.89454$  in the piecewise affine case (case 3),  $\lambda = 1.85769$  in the parabolic case approached by an affine function (case 4) and  $\lambda = 1.88614$  in the piecewise affine case projected on affine functions (case 5). For each of these six cases, our analytical-numerical approach gives converging results at second order accuracy for the parameter  $\lambda$ , hence showing the powerfulness and precision of the explicit calculations that one describes using the elliptic integrals.

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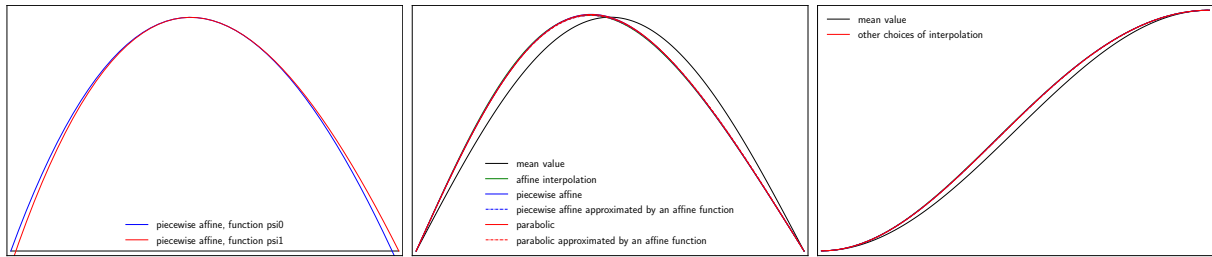


Figure 3: Functions  $h \mapsto \psi^0(h)$  and  $h \mapsto \psi^1(h)$  on the left,  $z \mapsto \varphi(z)$  in the middle and  $z \mapsto h(z)$  on the right.

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