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# A new integral boundary control for de Saint-Venant Partial Differential Equations

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**Abstract:** The paper deals with output feedback regulation of exponentially stable systems by an integral controller. We have recently proposed an appropriate Lyapunov functional to prove exponential stability of the closed-loop system. The approach is dedicated in this paper to hyperbolic systems and especially to the de Saint-Venant equations giving explicitly the gains to ensure an exponentially stabilized integral controller: the parameters expression is deduced directly of the Lyapunov functional based on the Forwarding approach. Numerical simulations illustrate this approach.

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**Keywords:** de Saint-Venant equations, Hyperbolic PDE, Lyapunov, Forwarding, Integral controller

## 1. INTRODUCTION

Output regulation is one of the most popular problems in control theory. The PI-controller has been introduced in the last century and has shown some fantastic behavior to reject constant disturbances or to reach a prescribed constant reference. It is nowadays the most popular control strategy.

The purpose of the control in engineering problems is not only to find an optimal control but also to find a control which stabilizes and regulates the system so that it behaves in a robust way against perturbations (see Corriu (2018) and Oustaloup (1994)). A solution of the control problem in finite dimensional theory has been given by Davison in Davison (1976) where an algorithm has been presented to tune the controller's integral part. The solution has been generalized to some infinite-dimensional systems by Pohjolainen in Pohjolainen (1982) by using semigroup theory. The use of integral action to achieve output regulation and cancel constant disturbances for multivariable systems has been proven efficient by widespread industrial controllers as described in Astrom (1995) and in Bastin and Coron (2016a). However extending robust multivariable control theory to infinite-dimensional systems is not a simple task. For example, the design of PI controllers has been extended in a series of papers by Pohjolainen and others to infinite-dimensional systems governed by partial differential equations (PDE) always by considering **bounded** control operators and by following a spectral approach (see Pohjolainen (1982), Pohjolainen (1985), Xu and Jerbi (1995), Paunonen and Pohjolainen (2010), and Xu and Sallet (2014)). However the spectral approach alone does not allow to deal with stabilization of nonlinear infinite-dimensional systems. On the contrary, in the last two decades Lyapunov approaches have allowed to consider a large class of boundary control problems (see for instance Bastin and Coron (2016a)). Previously,

following a Lyapunov approach, a robust output regulation problem has been solved by using integral controllers. More precisely, the algorithm tuning the integral controller has been extended to more general infinite-dimensional systems compared with the existing literature (Terrand-Jeanne et al., 2019, 2020; Terrand-Jeanne et al., 2018). Moreover the proofs based on the Lyapunov direct approach are simpler and potentially suitable to deal with nonlinearities.

This paper aim is to illustrate those theoretical results on a real and physical system that is modeled by de Saint-Venant equations which has been widely studied (Coron et al., 2008; Dos Santos et al., 2008; Dos Santos and Prieur, 2008; Coron, 2007; Trinh et al., 2017a).

In the first part, theoretical results are recalled, to be easily transposed to the shallow water equations, in the second part. The last part is dedicated to the simulations with this new controller.

**Notation:** subscripts  $t, s, tt, \dots$  denote the first or second derivative w.r.t. the variable  $t$  or  $s$  superscripts  $T$  denote the transposed element. For an integer  $n$ ,  $I_{d_n}$  is the identity matrix in  $\mathbb{R}^{n \times n}$ . Given an operator  $\mathcal{A}$  over a Hilbert space,  $\mathcal{A}^*$  denotes the adjoint operator.  $\mathcal{D}_n$  is the set of diagonal matrices in  $\mathbb{R}^{n \times n}$ .

## 2. BOUNDARY REGULATION FOR HYPERBOLIC PDES

In this section we adapt the framework developed in (Terrand-Jeanne et al., 2019, 2020) to hyperbolic PDE systems with boundary control. The state space is extended from  $[0, 1]$  to  $[0, L]$ .

### 2.1 System description

The hyperbolic partial differential equations case is considered as studied in Coron et al. (2008) but with a domain

$[0, L]$  where  $L$  is the length. More precisely, the system is given by a one dimensional  $n \times n$  hyperbolic system

$$\begin{aligned} \phi_t(s, t) + \Lambda_0(s)\phi_s(s, t) + \Lambda_1(s)\phi(s, t) = 0 \\ s \in (0, L), t \in [0, +\infty), \end{aligned} \quad (1)$$

where  $\phi : [0, +\infty) \times [0, L] \rightarrow \mathbb{R}^n$

$$\begin{aligned} \Lambda_0(s) &= \text{diag}\{\lambda_1(s), \dots, \lambda_n(s)\} \\ \lambda_i(s) &> 0 \quad \forall i \in \{1, \dots, \ell\} \\ \lambda_i(s) &< 0 \quad \forall i \in \{\ell + 1, \dots, n\}, \end{aligned}$$

where the maps  $\Lambda_0$  is in  $C^1([0, L]; \mathcal{D}_n)$  and  $\Lambda_1$  is in  $C^1([0, L]; \mathbb{R}^{n \times n})$  with the initial condition  $\phi(0, s) = \phi_0(s)$  for  $s$  in  $[0, L]$  where  $\phi_0 : [0, L] \rightarrow \mathbb{R}^n$  and with the boundary conditions

$$\begin{bmatrix} \phi_+(t, 0) \\ \phi_-(t, L) \end{bmatrix} = K \begin{bmatrix} \phi_+(t, L) \\ \phi_-(t, 0) \end{bmatrix} + Bu(t) + w_b \quad (2)$$

$$= \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \phi_+(t, L) \\ \phi_-(t, 0) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) + w_b \quad (3)$$

where  $\phi = [\phi_+ \ \phi_-]^T$  with  $\phi_+ \in \mathbb{R}^\ell$ ,  $\phi_- \in \mathbb{R}^{n-\ell}$  and where  $w_b \in \mathbb{R}^n$  is an unknown disturbance,  $u(t)$  is a control input taking values in  $\mathbb{R}^m$  and  $K, B$  are matrices of appropriate dimensions.

The output to be regulated to a prescribed value denoted by  $y_{\text{ref}}$ , is given as a disturbed linear combination of the boundary conditions. Namely, the outputs to regulate are in  $\mathbb{R}^m$  given as

$$y(t) = L_1 \begin{bmatrix} \phi_+(t, 0) \\ \phi_-(t, L) \end{bmatrix} + L_2 \begin{bmatrix} \phi_+(t, L) \\ \phi_-(t, 0) \end{bmatrix} + w_y, \quad (4)$$

where  $L_1$  and  $L_2$  are two matrices in  $\mathbb{R}^{m \times n}$  and  $w_y$  is an unknown constant disturbance in  $\mathbb{R}^m$ . Applying the same methodology as in (Terrand-Jeanne et al., 2019, 2020), the aim is to find a positive real number  $k_i$  and a full rank matrix  $K_i$  such that

$$u(t) = k_i K_i z(t), \quad z_t(t) = y(t) - y_{\text{ref}}, \quad z(0) = z_0 \quad (5)$$

where  $z(t)$  takes value in  $\mathbb{R}^m$  and  $z_0 \in \mathbb{R}^m$  solves the regulation problem for all  $y_{\text{ref}} \in \mathbb{R}^m$ .

The state space denoted by  $\mathcal{X}_e$  of the system (1)-(2) in closed loop with the control law (5) is the Hilbert space  $\mathcal{X}_e = (L^2(0, L), \mathbb{R}^n) \times \mathbb{R}^m$ , equipped with the norm defined for  $\varphi_e = (\phi, z) \in \mathcal{X}_e$  as:

$$\|\varphi_e\|_{\mathcal{X}_e} = \|\phi\|_{L^2((0, L), \mathbb{R}^n)} + |z|.$$

A smoother state space is also introduced defined as:  $\mathcal{X}_{e1} = (H^1(0, L), \mathbb{R}^n) \times \mathbb{R}^m$ .

## 2.2 Output regulation

In this section, we give a set of sufficient conditions allowing to solve the regulation problem. Our approach follows what we have done in Terrand-Jeanne et al. (2019, 2020). Following (Bastin and Coron, 2016a, Proposition 5.1, p161) we consider the following assumption.

*Assumption 1.* (Input-to-State Exponential Stability).

There exist a  $C^1$  function  $P : [0, L] \rightarrow \mathcal{D}_n$ , positive real numbers  $\mu, \underline{P}, \overline{P}$  and a positive definite matrix  $S$  in  $\mathbb{R}^{n \times n}$  such that the Lyapunov function

$$V(t) = \int_0^L \phi(t, s)^\top P(s) \phi(t, s) ds$$

where  $\phi(t, s)$  is the solution of (1) with  $w_b = 0$ , and

$$(P(s)\Lambda_0(s))_s - P(s)\Lambda_1(s) - \Lambda_1^\top(s)P(s) \leq -\mu P(s), \quad (6)$$

$$\underline{P} \mathbf{I}_{d_n} \leq P(s) \leq \overline{P} \mathbf{I}_{d_n}, \quad \forall s \in [0, L], \quad (7)$$

and

$$-K_L^\top P(L)\Lambda_0(L)K_L + K_0^\top P(0)\Lambda_0(0)K_0 \leq -S \quad (8)$$

where

$$K_L = \begin{bmatrix} \mathbf{I}_{d_\ell} & 0 \\ K_{21} & K_{22} \end{bmatrix}, \quad K_0 = \begin{bmatrix} K_{11} & K_{12} \\ 0 & \mathbf{I}_{d_{n-\ell}} \end{bmatrix}. \quad (9)$$

This assumption is a sufficient condition for exponential stability of the equilibrium of the open loop system. It can be found in (Bastin and Coron, 2016a, Prop. 5.1, p. 161) in the case in which  $S$  may be semi-definite positive. The positive definiteness of  $S$  is fundamental to get an input-to-state stability (ISS) property of the open loop system with respect to the disturbances on the boundary. More general results are given in Prieur and Mazenc (2012).

The second and third assumption are related to the rank condition. Let  $\Phi$  and  $\Psi : [0, L] \rightarrow \mathbb{R}^{n \times n}$  be the matrix function solution of the systems

$$\begin{aligned} \Phi_s(s) &= -\Lambda_0(s)^{-1} \Lambda_1(s) \Phi(s), \\ \Phi(0) &= \mathbf{I}_{d_n}. \end{aligned} \quad (10)$$

and respectively

$$\begin{aligned} \Psi_s(s) &= \Psi(s) (\Lambda_1(s) - \Lambda_0'(s)) \Lambda_0(s)^{-1}, \\ \Psi(0) &= \mathbf{I}_{d_n}. \end{aligned} \quad (11)$$

We denote  $\Phi(s) = \begin{bmatrix} \Phi_{11}(s) & \Phi_{12}(s) \\ \Phi_{21}(s) & \Phi_{22}(s) \end{bmatrix}$  and

$$\Phi_+ = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ 0 & \mathbf{I}_{d_{n-\ell}} \end{bmatrix}, \quad \Phi_- = \begin{bmatrix} \mathbf{I}_{d_\ell} & 0 \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

*Assumption 2.* (Rank condition 1). The matrix in  $\mathbb{R}^{n \times n}$   $\Phi_-(L) - K\Phi_+(L)$  is full rank and so is the matrix  $T_1$  defined as

$$T_1 = [L_1\Phi_-(L) + L_2\Phi_+(L)] [\Phi_-(L) - K\Phi_+(L)]^{-1} B. \quad (12)$$

*Assumption 3.* (Rank condition 2). The matrix in  $\mathbb{R}^{n \times n}$

$$\Psi(L)\Lambda_0(L)K_L - \Lambda_0(0)K_0 \quad (13)$$

is full rank and so is the matrix

$$T_2 = -L_1 B + M \left( \Lambda_0(0) \begin{bmatrix} B_1 \\ 0 \end{bmatrix} - \Psi(L)\Lambda_0(L) \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right)$$

where

$$M = (L_1 K + L_2) (\Lambda_0(0)K_0 - \Psi(L)\Lambda_0(L)K_L)^{-1}. \quad (14)$$

With these assumptions, the following result has been stated (Terrand-Jeanne et al., 2019, 2020):

*Theorem 1.* (Regulation for hyperbolic PDE systems).

Assume that Assumptions 1, 2 and 3 are satisfied, then with  $K_i = T_2^{-1}$  there exists  $k_i^* > 0$

$$k_i^* = \frac{\sqrt{\mu \overline{P}}}{|M| \overline{\Psi} \sqrt{\gamma_V} |T_2^{-1}|} \quad (15)$$

such that for all  $0 < k_i < k_i^*$  the output regulation is obtained, where  $\overline{\Psi} > 0$  be such that

$$|\Psi(s)| \leq \overline{\Psi}, \quad \forall s \in [0, L].$$

and  $\mu$  and  $\gamma_V$  such that this inequality issue from forwarding techniques

$$\dot{V}(t) \leq -\mu V(t) + \gamma_V |u(t)|^2 \quad (16)$$

is also satisfied.

The proof of this theorem follows the same steps as in Terrand-Jeanne et al. (2019), taking the  $s \in [0, L]$  instead of  $s \in [0, 1]$ . This result is applied to the hyperbolic PDE describing flows in shallow waters.

Note that the result remains for any  $K_i$  such that

$$T_2 K_i + K_i^T T_2^T > 0. \quad (17)$$

### 3. ILLUSTRATION IN DE SAINT-VENANT EQUATIONS

Theorem 1 generalizes many available results on output regulation via integral action for hyperbolic PDEs available in the literature. For instance, the case of  $2 \times 2$  linear hyperbolic systems has been considered in Trinh et al. (2017b), Dos Santos et al. (2008), (see also (Bastin and Coron, 2016a, Section 2.2.4)). Note also that in Terrand-Jeanne et al. (2020), this procedure is applied on a Drilling model which is composed of a  $2 \times 2$  linear hyperbolic PDE coupled with a linear ordinary differential equation.

In order to compare the way we improve existing results, the same example as in Dos Santos et al. (2008) is considered.

#### 3.1 Non Linear System

A prismatic open channel with a constant rectangular section and a constant slope is considered. The flow dynamics are described by the de Saint-Venant equations de Saint Venant (1871):

$$\partial_t H + \partial_s(Q/\hat{b}) = 0, \quad (18)$$

$$\partial_t Q + \partial_s \left( \frac{Q^2}{\hat{b}H} + \frac{1}{2} g \hat{b} H^2 \right) - g \hat{b} H (I - J) = 0, \quad (19)$$

$$H(s, 0) = H_0(s), Q(s, 0) = Q_0(s), \quad (20)$$

for all  $s \in \Omega = (0, L)$ , where  $H(s, t)$  represents the water level and  $Q(s, t)$  the water flow rate,  $\hat{b}$  the channel width and  $g$  the gravitation constant.  $I$  is the bottom slope and  $J$  is the friction slope expressed with the Manning-Strickler expression:

$$J(H, Q) = \frac{n_M^2 Q^2}{[S(H)]^2 [R(H)]^{4/3}},$$

with  $n_M$  the Manning coefficient while  $S(H) = \hat{b}H$  is the wet surface and  $R(H)$  is the hydraulic radius given by:

$$R(H) = \frac{S(H)}{P(H)}, P(H) = \hat{b} + 2H := \text{wet perimeter.}$$

$L$  is the length of the reach from the upstream  $x = 0$  to the downstream  $x = L$ ,  $U_{up} = U_0(t)$ ,  $U_{do} = U_L(t)$  are the opening of the gates at upstream and downstream respectively. A linear model with variable coefficients can be deduced from the non-linear PDE, in order to describe the variation of the water level and flow for an open channel.

The boundary conditions considered here are the multi-variable case,  $\forall s \in \Gamma = \partial\Omega$  the boundary of  $\Omega$ , with for an underflow gate:

$$Q(s, t) = U(t) \hat{b} \kappa_j \sqrt{2g(H_{up} - H_{do})} \quad (21)$$

and for an overflow gate (spillway):

$$Q(s, t) = (\kappa_j \hat{b})^3 [2g(H_{up} - U(t))]^{3/2} \quad (22)$$

$H_{up}$   $H_{do}$  are the water height at the upstream, resp. at downstream, of the considered gate,  $\kappa_j$  is the water flow rate coefficient of the gate considered,  $U(t)$  is the control of the considered gate.

The variables to control are the height of water at downstream  $H(L, t)$  and the water flow at upstream  $Q(0, t)$ , considering two underflow or overflow gates.

#### 3.2 Linearized system

An equilibrium state ( $\partial_t(\cdot) \equiv 0$ ) of the system (18)-(19), i.e.  $H(s, t) = H_e(s)$ ,  $Q(s, t) = Q_e \forall t$  and  $\forall s$  without any assumptions on  $I, J$ , satisfies the following equations (Dos Santos and Bastin, 2007; Dos Santos et al., 2008):

$$\partial_s Q_e = 0, \partial_s H_e = g \hat{b} H_e \frac{I + 2J_e + \frac{4}{3} J_e \frac{1}{1 + 2H_e/\hat{b}}}{g \hat{b} H_e - Q_e^2 / (\hat{b} H_e^2)} \quad (23)$$

The fluvial case is considered and it follows that:

$$H_e > \sqrt[3]{Q_e^2 / (g \hat{b}^2)} \quad (24)$$

A linearized model is used to describe the variations around this equilibrium profile. The following notations are introduced:

$$h(s, t) \hat{=} H(s, t) - H_e(s), q(s, t) \hat{=} Q(s, t) - Q_e.$$

The linearized model around  $(H_e, Q_e)$  is written as Dos Santos and Prieur (2008), Dos Santos et al. (2008)

$$\partial_t \Phi(s, t) + \Lambda_{0,N}(s) \partial_s \Phi(s, t) + \Lambda_{1,N}(s) \Phi(s, t) = 0 \quad (25)$$

$$\text{with} \quad \partial_t \hat{b} h(s, t) + \partial_s q(s, t) = 0, \quad (26)$$

$$\partial_t q(s, t) + cd \partial_s \hat{b} h(s, t) + (c - d) \partial_s q(s, t) + \gamma \hat{b} h(s, t) + \delta q(s, t) = 0, \quad (27)$$

and  $c = \sqrt{g H_e} + \frac{Q_e}{H_e \hat{b}}$ ,  $d = \sqrt{g H_e} - \frac{Q_e}{H_e \hat{b}}$ ,

$$\gamma = -g \left( I + 2J_e(s) + \frac{\frac{4}{3} J_e(s)}{1 + 2H_e(s)/\hat{b}} \right) = -cd \frac{\partial_s H_e(s)}{H_e(s)},$$

$$\delta = \frac{2g J_e(s) \hat{b} H_e(s)}{Q_e}, \quad \Phi = \begin{bmatrix} \hat{b} h(s, t) \\ q(s, t) \end{bmatrix}.$$

and

$$\Lambda_{0,N}(s) = \begin{bmatrix} 0 & 1 \\ cd & c - d \end{bmatrix}, \quad \Lambda_{1,N}(s) = \begin{bmatrix} 0 & 0 \\ \gamma & \delta \end{bmatrix}$$

In order to explicit the control laws, the gate characteristics (22-21) are linearized around the steady-state  $(H_e, Q_e)$ :

$$q(0, t) = B_{n,0} \hat{b} h(0, t) + B_{u,0} u_0(t), \quad (28a)$$

$$q(L, t) = B_{n,L} \hat{b} h(L, t) + B_{u,L} u_L(t), \quad (28b)$$

For the underflow gates, the coefficients are

$$B_{u,0} = \kappa_0 \hat{b} \sqrt{2g(H_{up} - H_e(0))}, \quad (29)$$

$$B_{h,0} = \frac{-Q_e(0)}{2\sqrt{2g(H_{up} - H_e(0))}}, \quad (30)$$

$$B_{u,L} = \kappa_L \hat{b} \sqrt{2g(H_e(L) - H_{do})}, \quad (31)$$

$$B_{h,L} = \frac{Q_e(L)}{2\sqrt{2g(H_e(L) - H_{do})}} \quad (32)$$

where  $\kappa_0$  and  $\kappa_L$  are the gate water flow coefficients, while  $u_0$  and  $u_L$  denote the variations of the control signals at the upstream and downstream gates respectively, around the equilibrium. For spillways, the coefficients are:

$$B_{u,0} = 3(\kappa_j \hat{b})^2 Q_e(0)^{1/3}, \quad (33)$$

$$B_{h,0} = 0, \quad (34)$$

$$B_{u,L} = -3(\kappa_j \hat{b})^2 Q_e(L)^{1/3}, \quad (35)$$

$$B_{h,L} = 3\hat{b}(\kappa_j)^2 Q_e(L)^{1/3} \quad (36)$$

Writing the system (25) in Riemann coordinates, (Dos Santos et al., 2008; Bastin and Coron, 2016b), one gets, with

$$\phi = \begin{bmatrix} q(s, t) + \hat{d}bh(s, t) \\ q(t, s) - \hat{c}bh(s, t) \end{bmatrix}$$

$$\partial_t \phi(s, t) + \Lambda_0(s) \partial_s \phi(s, t) + \Lambda_1(s) \phi(s, t) = 0 \quad (37)$$

$$\Lambda_0(s) = \begin{bmatrix} c & 0 \\ 0 & -d \end{bmatrix},$$

$$\Lambda_1(s) = \frac{1}{c+d} \begin{bmatrix} \gamma + c\delta - cd' & -\gamma + d\delta + cd' \\ \gamma + c\delta - c'd & -\gamma + d\delta + c'd \end{bmatrix} \quad (38)$$

The boundary conditions (2) of the system (37) terms are:

$$K = \begin{bmatrix} 0 & k_0 \\ k_L & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_0 & 0 \\ 0 & b_L \end{bmatrix}, \quad (39)$$

with  $b_0 \neq 0$  and  $b_L \neq 0$ .

For the system (1)-(2) with these parameters, it is shown in Dos Santos et al. (2008) that the output of dimension  $m = 2$  defined in (4) with

$$L_1 = \begin{bmatrix} \frac{c}{c+d} & 0 \\ 0 & \frac{-1}{c+d} \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 0 & \frac{d}{c+d} \\ 1 & 0 \end{bmatrix} \quad (40)$$

can be regulated with a proportional controller provided the condition on the gain and the Lyapunov function below, Dos Santos et al. (2008)

$$U(t) = \frac{A}{c} \int_0^L (\phi_+)^2 e^{-\mu x/c} + \frac{B}{d} \int_0^L (\phi_-)^2 e^{\mu x/d} dx \quad (41)$$

$$|k_0 k_L| < 1, \mu \text{ sufficiently small} \quad (42)$$

It has been shown that adding an integral term preserves the stability provided that the gain coefficients  $m_0$  and  $m_L$  of the integral part satisfy, to control  $q(0, t)$  and  $h(L, t)$ :

$$m_0 < 0, m_L > 0, \frac{d}{c} < 1 \quad (43)$$

$$|k_0 k_L| < 1, |k_0| < 1, |k_L| < \frac{c}{d} \quad (44)$$

On another hand, employing (Coron and Hayat (2019)-Dos Santos and Prieur (2008)), Assumption 1 is satisfied

assuming that  $|k_0 k_L| < 1 - \epsilon$ ,  $\epsilon$  depending on the time-delay and properties of the Riemann coordinates. The links between the control law and the boundary conditions (29)-(32) or (33)-(36) can be done as follows:

$$u_0(t) = \frac{q(0)}{B_{u,0}} + q(0) \frac{B_{h,0}}{B_{u,0}} \frac{(1 - k_0)}{(d(0) + c(0) * k_0)} \quad (45)$$

$$u_L(t) = \frac{\hat{b}h(L)}{B_{u,L}} \left[ \left( \frac{c(L) + k_L d(L)}{1 - k_L} \right) - B_{h,L} \right] \quad (46)$$

$$+ \frac{b_L k_i}{B_{u,L}(1 - k_L)} \int K_{i,21} q(0, \tau) + K_{i,22} \hat{b}h(L, \tau) d\tau$$

### 3.3 Tuning the control gain

Taking into account the definition of the function  $\gamma$  and  $\delta$ , both systems (10) and (11) can be solved. Indeed,

$$\Phi_s(s) = -\Lambda_0(s)^{-1} \Lambda_1(s) \Phi(s), \quad (47)$$

$$= \left[ \frac{3}{4} \frac{\partial_s H_e(s)}{H_e(s)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{\delta}{(c+d)} \begin{pmatrix} -1 & -\frac{d}{c} \\ \frac{c}{d} & 1 \end{pmatrix} \right] \Phi(s)$$

$$= \Theta(s) \Phi(s) \quad (48)$$

So  $\Phi$  can be evaluated, and the same is done for  $\Psi(s)$ .

$$\Psi_s(s) = \Psi(s) (\Lambda_1(s) - \Lambda'_0(s)) \Lambda_0(s)^{-1}, \quad (49)$$

$$= \Psi(s) \left[ \frac{-\partial_s H_e}{4H_e} \begin{pmatrix} 2c + 3d & \frac{3c}{d} \\ \frac{3d}{c} & 3c + 2d \end{pmatrix} + \frac{\delta}{c+d} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \right]$$

Computing all those data, assumptions on the rank 2 and 3 are satisfied if

$$k_0 \neq \frac{\Phi_{22}(L) - k_L \Phi_{12}(L)}{\Phi_{21}(L) - k_L \Phi_{11}(L)} \quad (50)$$

$$\text{and } k_0 \neq \frac{d(0) \Psi_{11}(L)c(L) - \Psi_{12}(L)d(L)k_L}{c(0) \Psi_{21}(L)c(L) - \Psi_{22}(L)d(L)k_L} \quad (51)$$

One can notice that previous stability conditions (41) are emerging from above equations. Indeed taking constants functions  $c$  and  $d$ ,  $k_0 k_L$  is linked to the ratio  $\frac{c}{d}$  as  $\Phi_{21}$  and  $\Phi_{11}$  are linked by the same ratio  $\frac{c}{d}$  example given.

Taking the values of  $k_i$  and  $K_i$  given by previous results (Terrand-Jeanne et al., 2019, 2020),

$$K_i = T_2^{-1} \text{ and } k_i * = \frac{\sqrt{\mu P}}{|M|\Psi\sqrt{\gamma_V}|T_2^{-1}|}, \quad (52)$$

then the shallow water equations can be regulated.

The theoretical equations are too long to be readable here. We implement the de Saint-Venant equations, and calculate the value of each term in the following section. Simulation are done, employing boundary conditions (45)-(46).

#### 4. SIMULATION

To compare with existing results, the case of the Sambre river is taken, as in Dos Santos and Prieur (2008), river located in Belgium. Physical parameters of this river are given in Table 1, and the gates are overflow ones.

For the regulation of the Sambre, a PID controller is actually used on site.

##### 4.1 Data



Fig. 1. Picture of the Sambre river

parameters	$B$ (m)	$L$ (m)	slope $I$ ( $m^1 \cdot s^{-1}$ )	$\mu_0$ $= \mu_L$	$K$ ( $m^{1/3} \cdot s^{-1}$ )
values	40	11239	$7.92e^{-5}$	0.4	33

Table 1. Parameters of a reach of the Sambre river

For all numerical simulations we use the Chang and Cooper theta-scheme of order 2 (Cordier et al., 2004; Dos Santos and Prieur, 2008).

##### 4.2 Simulation results

For an initial condition satisfying our compatibility conditions, we choose  $Q^\#(0) = 10m^3 \cdot s^{-1}$ ,  $H^\#(0) = 3.75m$ ,  $H^\#(L) = 4.65m$ .

In these numerical simulations, we consider the stability problem around the following equilibrium:  $Q_e = 12m^3 \cdot s^{-1}$ ,  $H_e(0) = 3.80m$ ,  $H_e(L) = 4.7m$  for which the flow is fluvial. In figure (2), the proportional part is simulated, showing the offset, then with the integral controller which was deduced initially from (43) of Dos Santos et al. (2008). Using the results developed here, the theoretical gains controller are defined by:

$$K_i = \begin{pmatrix} -1.0810 & -6.7957 \\ -0.9330 & 6.5483 \end{pmatrix} \quad (53)$$

$$k_i^* = 0.0484 \quad (54)$$

For the simulations done here, the value of the gain  $kiK_i$  is

$$K_{i,comm} = \begin{pmatrix} -0.0011 & -0.0059 \\ -0.0009 & 0.0061 \end{pmatrix}$$

in order to compare with previous results. In figure (3), simulations, with integral gains  $m_0 = -0.005$ ,  $m_L = 0.005$ , are recalled and compared with our new controller  $K_{i,comm}$ . As one can notice, the convergence is obtained

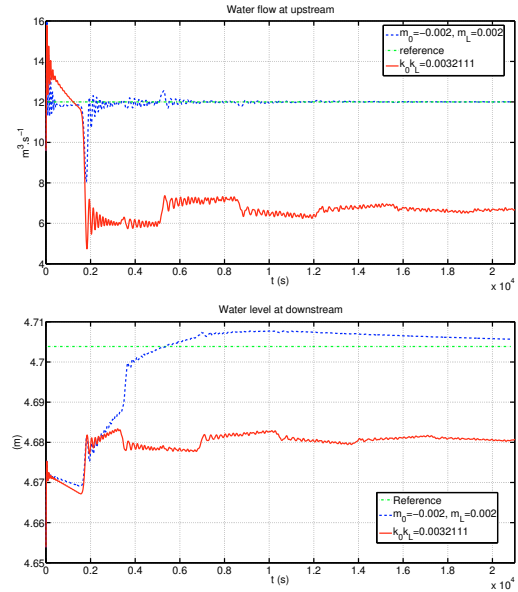


Fig. 2. Water flow at upstream and water level at downstream

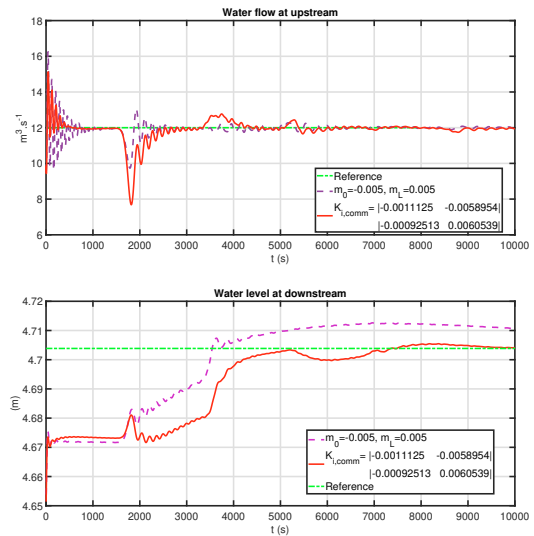


Fig. 3. Water flow at upstream and water level at downstream

around  $t = 8000s$  instead of  $t = 20000s$ . The proportional gains product is  $k_0 * k_L = 0.0128$  instead of 0.041 previously. So, one can notice that with a proportional part reduced, our controller reaches the reference more quickly than in (Dos Santos and Prieur, 2008; Dos Santos et al., 2008). We can underline that the control (gates opening) is physically feasible, figure (4).

The next simulation, figure (5), is obtained with the same proportional gains and  $k_i = 0.001$  (to get the same  $K_{i,comm}$  than above) than simulation (S1) of Dos Santos and Prieur (2008), showing the efficiency and the improvement of this approach.

#### 5. CONCLUSION

For a long time, the regulation problem has been studied for different classes of distributed parameter systems. In

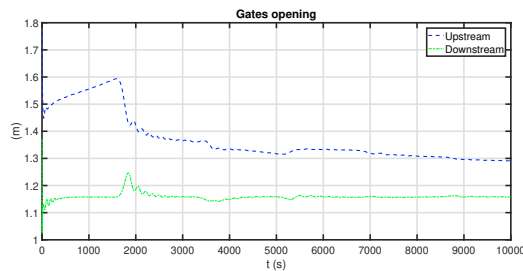


Fig. 4. Gates opening

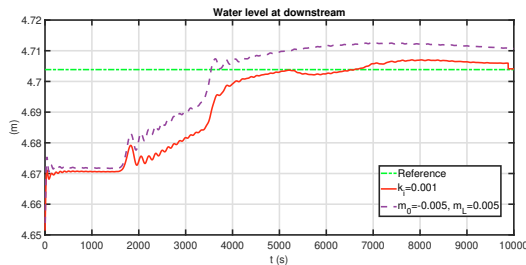


Fig. 5. Water flow at upstream and water level at downstream

this paper we have followed the construction of a Lyapunov functional to address the regulation problem using an integral controller in the case of de Saint-Venant equations. This work gives explicit integral gain and the previous results are clearly improved by our approach. The interest of our approach is that it may be used to the case in which the control and measurements operators are not bounded.

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