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► **To cite this version:**

Arnaud Duran, Maria Kazakova, Yen-Chung Hung, Rémy Baraille, Frédéric Couderc, et al..  
Développement d'un modèle dispersif hyperbolique pour les vagues côtières et implémentation dans  
Tolosa. Derivation of a dispersive-hyperbolic model for coastal waves et its numerical implementation.  
18e Journées Hydrodynamique, Nov 2022, Poitiers, France. hal-03857979v1

**HAL Id: hal-03857979**

**<https://hal.science/hal-03857979v1>**

Submitted on 17 Nov 2022 (v1), last revised 25 Jan 2023 (v2)

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# Développement d'un modèle dispersif hyperbolique pour les vagues côtières et implémentation dans Tolosa

## *Derivation of a dispersive-hyperbolic model for coastal waves et its numerical implementation*

Y.-C. Hung<sup>(1)</sup>, M. Kazakova<sup>(1)</sup>, G. Richard<sup>(2)</sup>, J. Chauchat<sup>(3)</sup>,  
R. Baraille<sup>(5,6)</sup>, F. Couderc<sup>(5)</sup>, A. Duran<sup>(4)</sup>, J.-P. Vila<sup>(5)</sup>  
*yen-chung.hung@univ-smb.fr*

<sup>(1)</sup> Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, Chambéry

<sup>(2)</sup> Université Grenoble Alpes, INRAE, UR ETNA, Grenoble

<sup>(3)</sup> Univ. Grenoble Alpes, CNRS, Grenoble INP, LEGI, Grenoble

<sup>(4)</sup> Univ. Claude Bernard Lyon 1, ICJ, Lyon

<sup>(5)</sup> Univ. de Toulouse, CNRS, IMT, INSA, Toulouse

<sup>(6)</sup> SHOM - Service Hydrographique et Océanographique de la Marine, France.

### Summary

In the present study we propose a new hyperbolic model with the same dispersive properties as the classical Serre-Green-Naghdi capable to capture wave breaking phenomenon.

### Résumé

Dans cette étude, nous proposons un nouveau modèle hyperbolique avec les mêmes propriétés dispersives que le modèle classique de Serre-Green-Naghdi capable de capturer le phénomène de déferlement des vagues.

In the current environmental context, in view of the issues related to the increase in the frequency of extreme events such as severe storms and flooding, the precise and operational description of the the coastal flows is essential. Similarly, the phenomenon of coastal erosion, aggravated by climate change, requires the development of effective prevention models and tools. In spite of constant technical progress, the direct resolution of the Navier-Stokes equations is still out of reach from an operational point of view. In this context, the interest is focused on simplified models, less expensive numerically and allowing to get closer to real time simulations. This is where depth-average models such as the Shallow Water equations, or more precise models of type weakly dispersive Boussinesq or Serre-Green Naghdi (SGN) come into play. The latter are often used in coastal oceanography because they allow to capture the dispersion and the strong nonlinearities induced by the bottom variations at the approach of the shore. However, the obstacles associated with the numerical implementation of these models are numerous. One of the main difficulties associated with the dispersive equations is the presence of an elliptic-type operator, which requires the inversion of a global linear system. This is a serious obstacle since takes a large part of the computation time. Furthermore, the treatment of the elliptical operator requires a specific choice of discrete formulations in order to ensure the stability and the well-posedness of the problem.

In the last few years, new methods have emerged that allow to tackle these difficulties in part. They are based on hyperbolic models consisting of relaxed versions of the SGN equations. One can for instance refer to the recent work [5] where the derivation of the equations relies on an extended Lagrangian variational principle, or [9] where a compressible version of dispersive model for water waves has been derived. The hyperbolic nature of those model provides an appropriate framework for the construction of a numerical scheme and allows to get rid of most of the limitations due to the existing strategies for SGN equations.

However those models (dispersive or hyperbolic) are derived in conservative contexte, meaning that they conserve mechanical energy, and therefore not capable to capture the dissipation when breaking of waves occurs. The different methodologies proposed in the literature to describe wave-breaking fall into two categories. The first strategy consists in adding an artificial viscosity term in the mass and/or momentum equations (see e.g. [8]). The second strategy (used for exemple in [11], [2]), commonly called switching, consists in suppressing the dispersive terms in the vicinity of breaking waves, leaving only the hyperbolic part of the model (Shallow Water) locally resolved. This approach allows to model the breaking waves as shocks and to use the natural dissipation through the shock. Although these strategies allow to describe the breaking process they requires a calibration of a large number of parameters, moreover, the discontinuities introduced in the model by the addition or deletion of terms generate transition instabilities, which can propagate and considerably deteriorate the quality of the approximations. A new approach has recently been proposed in [7], [3] : it generalizes the classical SGN equations by taking into account wave-breaking through a new variable (scalar in 1D and tensor in 2d) related to the turbulent energy, called enstrophy. However, the proposed models have a similar structure to the SGN equations and implies therefore a time-consuming treatment of the elliptic step.

In the present study we aim to derived a hyperbolic dispersive model capable to describe breaking waves. We note that the approach proposed in [7] to describe wave breaking can be used without compromising the derivation proposed in [9], and therefore the hyperbolic structure of the system can be obtain. This model is intended to be implemented in the TOLOSA code developed for the forecasting and prevention of the risk of marine submersion in association with the SHOM and Météo France. Also, the consideration of turbulence also opens up prospects for coupling with morphodynamic models for the description of erosion and sediment transport. This constitutes a subject of a future research.

## I – Setup and derivation

### I – 1 Setup

Consider the two-dimensional flow over a time-independent bottom  $b$ . The velocity components are denoted as  $u, w$  for  $Ox, Oz$  direction, respectively. Let  $Z = Z(x, t)$  be the free surface and  $h = Z - b$  as the depth. Figure 1 gives a demonstration of the setup.

### I – 2 Derivation

Following [7] we begin with mimicking large-eddy simulation method that decomposes the velocity field  $\mathbf{v} = (u, w)^\top$  into a filtered velocity field  $\bar{\mathbf{v}}$  and a residual term  $\mathbf{v}^r$

$$\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}^r.$$

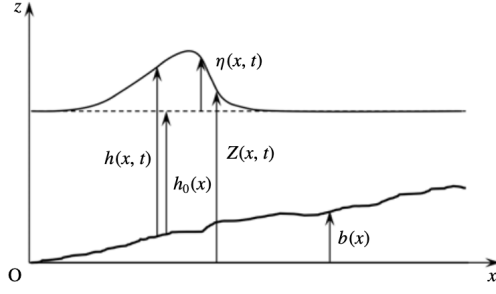


FIGURE 1 – Setup of the model

The filtered velocity field  $\bar{\mathbf{v}}$  here includes a large scale turbulence. The filter is then applied to the Navier-Stokes equation of incompressible Newtonian fluid of density  $\rho$  and kinematic viscosity  $\nu$ .

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0, \quad (1a)$$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{w}}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \left( \frac{\partial A_{xx}^r}{\partial x} + \frac{\partial A_{xz}^r}{\partial z} \right) + \nu \left( \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial z^2} \right), \quad (1b)$$

$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{u}\bar{w}}{\partial x} + \frac{\partial \bar{w}^2}{\partial z} = -g - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{1}{\rho} \left( \frac{\partial A_{xz}^r}{\partial x} + \frac{\partial A_{zz}^r}{\partial z} \right) + \nu \left( \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial z^2} \right), \quad (1c)$$

where  $A_{xx}^r$ ,  $A_{xz}^r$ ,  $A_{zz}^r$  are part of the stress tensor that has the form

$$A_{xx}^r = 2\nu_T \frac{\partial u}{\partial x}, \quad A_{zz}^r = -A_{xx}^r, \quad A_{xz}^r = \nu_T \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \quad (2)$$

with  $\nu_T$  be the turbulence viscosity.

The model is derived by averaging the filtered equation in the vertical direction and neglecting all the terms of high order with respect to the shallowness parameter

$$\varepsilon = \frac{h_0}{L} \ll 1,$$

where  $L$ ,  $h_0$  are the horizontal and vertical characteristic length, respectively. The boundary conditions we impose are

- No-penetration condition at the bottom

$$w(b) = u(b) \frac{db}{dx} \quad (3)$$

- Kinematic and dynamic condition at the free surface

$$w(Z) = \frac{\partial h}{\partial t} + u(Z) \frac{\partial Z}{\partial x} \quad \text{and} \quad (\boldsymbol{\sigma} \cdot \mathbf{n})(Z) = 0, \quad (4)$$

In the derivation of our model, we further assume that there is no surface tension and shear stress at the free surface. We decompose the filtered horizontal velocity as

$$\bar{u}(x, z, t) = U(x, t) + u'(x, z, t), \quad (5)$$

where  $U$  is a mean velocity and  $u'$  represents the deviation part. The hypothesis of weakly turbulent flow from Teshukov [10] gives the order of the vertical variation  $u'$  as  $O(\varepsilon)$  to

include both shearing and dispersion. Furthermore, the pressure  $p$  is separated into the sum of hydrostatic pressure  $p_H$  and non-hydrostatic pressure  $p_N$

$$p = p_H + p_N = \rho g(Z - z) + p_N. \quad (6)$$

Then, we apply the following scaling to nondimensionalize the system of filtered equations

$$\tilde{x} = \frac{x}{L}, \quad \tilde{z} = \frac{z}{h_0}, \quad \tilde{t} = \varepsilon t \sqrt{\frac{g}{h_0}}, \quad \tilde{h} = \frac{h}{h_0}, \quad \tilde{b} = \frac{b}{h_0}, \quad \tilde{u} = \frac{\bar{u}}{\sqrt{gh_0}}, \quad \tilde{w} = \frac{\bar{w}}{\varepsilon \sqrt{gh_0}}, \quad \tilde{p} = \frac{p}{\rho gh_0},$$

$$\tilde{Z} = \frac{Z}{h_0}, \quad \tilde{A}_{xx}^r = \frac{A_{xx}^r}{\varepsilon^2 \rho gh_0}, \quad \tilde{A}_{zz}^r = \frac{A_{zz}^r}{\varepsilon^2 \rho gh_0}, \quad \tilde{A}_{xz}^r = \frac{A_{xz}^r}{\varepsilon \rho gh_0}.$$

By taking depth-average to the dimensionless filtered equation (1), the continuity equation takes form

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \tilde{h} \tilde{U}}{\partial \tilde{x}} = 0, \quad (7)$$

and the momentum equation  $Ox$  (1b) becomes

$$\frac{\partial \tilde{h} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left( \tilde{h} \tilde{U}^2 + \frac{\tilde{h}^2}{2} + \varepsilon^2 \tilde{h} \langle \tilde{u}'^2 \rangle + \varepsilon^2 \int_{\tilde{b}}^{\tilde{Z}} \tilde{p}_N d\tilde{z} - \varepsilon^2 2\tilde{\nu}_T \tilde{h} \frac{\partial \tilde{U}}{\partial \tilde{x}} \right) = -\tilde{p}(b) \frac{\partial \tilde{b}}{\partial \tilde{x}} \quad (8)$$

Then, following [7] we define the enstrophy as

$$\tilde{\varphi} := \frac{\langle \tilde{u}'^2 \rangle}{\tilde{h}^2}, \quad (9)$$

and following [9] the depth-averaged non-hydrostatic pressure as

$$\tilde{P} = \frac{1}{\tilde{h}} \int_{\tilde{b}}^{\tilde{Z}} \tilde{p}_N d\tilde{z}. \quad (10)$$

This definition of  $P$  allows the whole system to be hyperbolic. Note that to keep the enstrophy terms in our model, we should keep all the terms up to the order of  $O(\varepsilon^2)$ . In this case, the momentum equation  $Oz$  (1c) becomes

$$\frac{\partial \tilde{h} \tilde{W}}{\partial \tilde{t}} + \frac{\partial \tilde{h} \tilde{U} \tilde{W}}{\partial \tilde{x}} = \tilde{p}_N(b) \quad (11)$$

To obtain  $\tilde{p}_N(b)$ , we first use the continuity equation (1a)

$$\frac{\partial \tilde{w}}{\partial \tilde{z}} = -\frac{\partial \tilde{u}}{\partial \tilde{x}} = -\frac{\partial \tilde{U}}{\partial \tilde{x}} - \varepsilon \frac{\partial \tilde{u}'}{\partial \tilde{x}} \quad (12)$$

Integrating (12) from  $\tilde{b}$  to  $\tilde{z}$  with respect to  $\tilde{z}$ , we have

$$\tilde{w} = (\tilde{b} - \tilde{z}) \frac{\partial \tilde{U}}{\partial \tilde{x}} + \tilde{U} \frac{\partial \tilde{b}}{\partial \tilde{x}} + O(\varepsilon). \quad (13)$$

Then the depth-averaged vertical velocity reads

$$\tilde{W} = -\frac{\tilde{h}}{2} \frac{\partial \tilde{U}}{\partial \tilde{x}} + \tilde{U} \frac{\partial \tilde{b}}{\partial \tilde{x}} + O(\varepsilon). \quad (14)$$

Substituting (13) and (14) into the momentum equation Oz (1c) and integrating from  $\tilde{z}$  to  $\tilde{Z}$  yields

$$\begin{aligned} \tilde{p}_N(\tilde{z}) = & \left( \frac{(\tilde{b} - \tilde{z})^2}{2} - \frac{\tilde{h}^2}{2} \right) \left( \frac{\partial^2 \tilde{U}}{\partial \tilde{t} \partial \tilde{x}} + \frac{\partial}{\partial \tilde{x}} \left( \tilde{U} \frac{\partial \tilde{U}}{\partial \tilde{x}} \right) - 2 \left( \frac{\partial \tilde{U}}{\partial \tilde{x}} \right)^2 \right) \\ & + (\tilde{Z} - \tilde{z}) \ddot{\tilde{b}} + \tilde{A}_{zz}^r + O(\varepsilon), \end{aligned}$$

where  $\dot{b} = \frac{Db}{Dt} = \frac{\partial b}{\partial t} + U \frac{\partial b}{\partial x}$  and  $\ddot{b} = \frac{D\dot{b}}{Dt}$ . Then, we have the following relation between  $\tilde{P}$  and  $\tilde{p}_N(b)$

$$\tilde{p}_N(b) = \frac{3}{2} \tilde{P} + \frac{\tilde{h}}{4} \ddot{\tilde{b}} + 3\tilde{\nu}_T \frac{\partial \tilde{U}}{\partial \tilde{x}} + O(\varepsilon). \quad (15)$$

Combining (7), (8), and (11) with the definitions of  $\varphi$  and  $P$  ((9), (10)) and relation (15), we finally have the three dimensional equations

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0 \quad (16a)$$

$$\begin{aligned} \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left( hU^2 + \frac{gh^2}{2} + h^3\varphi + hP - 2\nu_T h \frac{\partial U}{\partial x} \right) \\ = -gh \frac{\partial b}{\partial x} - \left( \frac{3}{2} P + \frac{h}{4} \ddot{b} + 3\nu_T \frac{\partial U}{\partial x} \right) \frac{\partial b}{\partial x} \end{aligned} \quad (16b)$$

$$\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2} P + \frac{h}{4} \ddot{b} + 3\nu_T \frac{\partial U}{\partial x} \quad (16c)$$

Note that we introduce two additional variables  $P$  and  $\varphi$ . To close the system, we consider the acoustic energy equation

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} + \frac{\rho w^2}{2} + \rho g z + \rho e_a \right) \\ + \frac{\partial}{\partial x} \left[ \left( \frac{\rho u^2}{2} + \frac{\rho w^2}{2} + \rho g z + \rho e_a + p - A_{xx}^r \right) u - A_{xz}^r w \right] \\ + \frac{\partial}{\partial z} \left[ \left( \frac{\rho u^2}{2} + \frac{\rho w^2}{2} + \rho g z + \rho e_a + p - A_{zz}^r \right) w - A_{xz}^r u \right] = -P^r, \end{aligned} \quad (17)$$

where  $e_a$  is the acoustic energy and  $P^r$  is a dissipative term. By depth-averaging (17) over the depth, we end up having the coupled equation for enstrophy and the mean acoustic energy

$$\begin{aligned} \frac{h^2}{2} \left( \frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} \right) + \frac{\partial h\langle e_a \rangle}{\partial t} + \frac{\partial hU\langle e_a \rangle}{\partial x} \\ = -h\langle P^r \rangle - \left( hP - 2\nu_T h \frac{\partial U}{\partial x} \right) \frac{\partial U}{\partial x} - 2 \left( P + 2\nu_T \frac{\partial U}{\partial x} \right) (W - \dot{b}). \end{aligned} \quad (18)$$

We apply the same postulation used in [9] for the averaged acoustic energy

$$\langle e_a \rangle = r^2 \frac{P^2}{2a^2}, \quad (19)$$

where  $r^2$  is a shape factor of the distribution of the non-hydrostatic pressure in the fluid depth, and  $a$  is the constant sound velocity. Following [7], we introduce the empirical expressions for the dissipation term  $\langle P^r \rangle$  and the turbulent viscosity  $\nu_T$

$$\langle P^r \rangle = \frac{C_r}{2} h^2 \varphi^{3/2}, \quad \nu_T = \frac{h^2 \sqrt{\varphi}}{R}, \quad (20)$$

where  $C_r$  is a dimensionless quantity and  $R$  can be interpreted as the a turbulent Reynolds number.

For shoaling zone, where wave breaking does not occur, the assumption  $\varphi \equiv 0$  is appropriate, then the equation for averaged non-hydrostatics pressure  $P$  reads

$$\frac{\partial hP}{\partial t} + \frac{\partial hUP}{\partial x} = -\frac{a^2}{r^2} \left( h \frac{\partial U}{\partial x} + 2W - 2\dot{b} \right). \quad (21)$$

Thus, we can obtain the equation for enstrophy  $\varphi$

$$\frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = -C_r h \varphi^{3/2} + \frac{4h\sqrt{\varphi}}{R} \left( \frac{\partial U}{\partial x} \right)^2 - \frac{8\sqrt{\varphi}}{R} \frac{\partial U}{\partial x} (W - \dot{b}). \quad (22)$$

### I – 3 Final system

The equations (16) as well as (21) and (22) are the final system of equations. Under the mild slope condition (we neglect the terms involving  $\partial_x b$  of order larger than  $O(\varepsilon^3)$ ), we have :

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} &= 0 \\ \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left( hU^2 + \frac{gh^2}{2} + h^3\varphi + hP \right) &= \frac{\partial}{\partial x} \left( \frac{2h^3\sqrt{\varphi}}{R} \frac{\partial U}{\partial x} \right) - gh \frac{\partial b}{\partial x} \\ \frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} &= \frac{3}{2}P - \frac{6hW\sqrt{\varphi}}{R} \\ \frac{\partial hP}{\partial t} + \frac{\partial hUP}{\partial x} &= -\frac{a^2}{r^2} \left( h \frac{\partial U}{\partial x} + 2W \right) \\ \frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} &= -C_r h \varphi^{3/2} + \frac{4h\sqrt{\varphi}}{R} \left( \frac{\partial U}{\partial x} \right)^2 + \frac{16W^2\sqrt{\varphi}}{hR} \end{aligned} \quad (23)$$

Where we have used the relation

$$W = -\frac{h}{2} \frac{\partial U}{\partial x} + \text{small terms},$$

for the source terms in the third and the last equations.

## II – Model analysis

### II – 1 Eigenstructure

Using the primitive form of the system (23), we define the Jacobian matrix as

$$\begin{pmatrix} U & h & 0 & 0 & 0 \\ g + 3h\varphi + \frac{P}{h} & U & 0 & 1 & h^2 \\ 0 & 0 & U & 0 & 0 \\ 0 & \frac{a^2}{r^2} & 0 & U & 0 \\ 0 & 0 & 0 & 0 & U \end{pmatrix} \quad (24)$$

The eigenvalues are  $\lambda = U$  (triple roots),  $U \pm \sqrt{gh + 3h^2\varphi + P + \frac{a^2}{r^2}}$ . We note that the corresponding eigenvectors form a basis. This makes the matrix (24) diagonalizable and hence guarantees the hyperbolicity of the system (23).

### II – 2 Dispersive properties

To derive the linear dispersive relation associated to the system (23) without dissipation and over the flat bottom, we linearize the equations around the equilibrium state  $(h_0, U_0 = 0, W_0 = 0, P_0 = 0, \varphi_0)$  with respect to small perturbations  $(h_1, U_1, W_1, P_1, \varphi_1)$ :

$$h = h_0 + \varepsilon h_1, \quad U = \varepsilon U_1, \quad W = \varepsilon W_1, \quad P = \varepsilon P_1, \quad \varphi = \varphi_0 + \varepsilon \varphi_1.$$

We are searching for the plain wave solution in the following form

$$(h_1, U_1, W_1, P_1, \varphi_1)^\top = (A_1, A_2, A_3, A_4, A_5)^\top e^{i(kx - \omega t)}. \quad (25)$$

Substituting (25) into linearised equation leads to the dispersive relation

$$\frac{r^2 h_0^2}{3a^2} \omega^4 - \omega^2 \left[ 1 + \frac{h_0^2 k^2}{3} \left( 1 + gh_0 \frac{r^2}{a^2} \right) \right] + gh_0 k^2 = 0. \quad (26)$$

The complete dispersive relation can be solved explicitly as

$$\omega^2 = \frac{1}{2r^2} \left[ \frac{3a^2}{h_0^2} + a^2 k^2 \left( 1 + gh_0 \frac{r^2}{a^2} \right) \pm \sqrt{\left( \frac{3a^2}{h_0^2} + a^2 k^2 \left( 1 + gh_0 \frac{r^2}{a^2} \right) \right)^2 - \frac{12r^2 a^2 g k^2}{h_0}} \right]. \quad (27)$$

Note that as  $a \rightarrow \infty$ , the dispersive relation of (26) approaches to that of Serre-Green-Naghdi equation

$$\omega^2 = \frac{gh_0 k^2}{1 + \frac{h_0^2 k^2}{3}}. \quad (28)$$

Another important characteristic of dispersive systems for water waves is the existence of solitary wave solutions. A solitary wave is a self-similar solution which propagates at a constant velocity  $c$ . The system of equation (23) admits a solitary wave solution assuming no dissipation and a flat bottom. Supposing the following conditions at infinity

$$h(x = \infty) = h_\infty, \quad U(x = \infty) = 0, \quad W(x = \infty) = 0, \quad P(x = \infty) = 0, \quad \varphi(x = \infty) = \varphi_\infty,$$



we shown that the solitary wave of speed  $c$  for (23) satisfies

$$U = c \left( 1 - \frac{h_\infty}{h} \right), \quad P = \frac{C_2}{h} - h^2 \varphi_\infty - \frac{gh}{2} - \frac{C_1^2}{h^2}, \quad \varphi = \varphi_\infty$$

where  $C_1 = -ch_\infty$  and  $C_2 = \frac{C_1^2}{h_\infty} + \frac{gh_\infty^2}{2} + h_\infty^3 \varphi_\infty$  and  $(h, W)$  solves the ODE

$$\begin{cases} h' = \frac{2W}{C_1} \left( \frac{1}{h} - \frac{r^2}{a^2} \frac{dP}{dh} \right)^{-1} \\ W' = -\frac{3}{2C_1} P \end{cases} \quad (29)$$

### III – Numerical scheme

To obtain a second order accuracy, the second order in space is implemented with finite volume MUSCL scheme. For the second order in time, we use the IMEX ARS2(2,2,2) scheme.

Let  $\mathbf{q} = (h, hU, hW, hP, h\varphi)^\top$ . System (23) can be written in semi-discrete form as a sum of a slow part  $s(\mathbf{q})$  and a fast part  $f(\mathbf{q})$  :

$$\mathbf{q}_t = s(\mathbf{q}) + f(\mathbf{q}), \quad (30)$$

where

$$s(\mathbf{q}) = -F(\mathbf{q})_x + \begin{pmatrix} 0 \\ \frac{\partial}{\partial x} \left( \frac{2h^3 \sqrt{\varphi}}{R} \frac{\partial U}{\partial x} \right) - (hP)_x \\ -\frac{6hW \sqrt{\varphi}}{R} \\ 0 \\ -C_r h \varphi^{3/2} + \frac{4h \sqrt{\varphi}}{R} \left( \frac{\partial U}{\partial x} \right)^2 + \frac{16W^2 \sqrt{\varphi}}{hR} \end{pmatrix},$$

$$f(\mathbf{q}) = \left( 0, 0, \frac{3}{2}P, -\frac{a^2}{r^2} \left( h \frac{\partial U}{\partial x} + 2W \right), 0 \right)^\top,$$

$$F(\mathbf{q}) = \left( hU, hU^2 + \frac{gh^2}{2} + h^3 \varphi, hUW, hUP, hU\varphi, \right)^\top.$$

Let  $\mathbf{Q}(t)$  be the approximation of the cell averages of  $\mathbf{q}$ . The IMEX ARS2(2,2,2) scheme consists of two explicit and two implicit stages. The first explicit stage is written as

$$\mathbf{Q}_1^\dagger = \mathbf{Q}^n + \gamma \Delta t s(\mathbf{Q}^n), \quad (31)$$

where  $\gamma = 1 - \sqrt{2}/2$  and  $\mathbf{Q}^n = \mathbf{Q}(t_n)$ . Then, the intermediate state  $\mathbf{Q}_1$  is solved implicitly from  $\mathbf{Q}_1^\dagger$  by the first implicit stage

$$\mathbf{Q}_1 = \mathbf{Q}_1^\dagger + \gamma \Delta t f(\mathbf{Q}_1). \quad (32)$$

The second explicit stage is written as

$$(\mathbf{Q}^{n+1})^\dagger = \mathbf{Q}^n + \delta \Delta t s(\mathbf{Q}^n) + (1 - \delta) \Delta t s(\mathbf{Q}_1) + (1 - \gamma) \Delta t f(\mathbf{Q}_1), \quad (33)$$

where  $\delta = 1 - 1/(2\gamma)$ . Finally, the update of the solution to next time step  $\mathbf{Q}^{n+1}$  is solved implicitly from  $(\mathbf{Q}^{n+1})^\dagger$  by the second implicit stage

$$\mathbf{Q}^{n+1} = (\mathbf{Q}^{n+1})^\dagger + \gamma\Delta t f(\mathbf{Q}^{n+1}). \quad (34)$$

Using the analytical solution from the previous section we confirm the convergence order of the proposed numerical algorithm.

### III – 1 Breaking criteria

The enstrophy  $\varphi$  plays an important role in determining the moment of wave breaking as it propagates close to the shore. The breaking point is characterized by a sudden increase of the enstrophy. This idea allows us to determine an appropriate threshold value  $\varphi_0$  for the maximal value of enstrophy in order to determine the moment when turbulent viscosity should be activated.

For solitary wave we introduce the nonlinearity parameter  $\delta = a^*/h_0^*$  where  $a^*$  is the amplitude of the initial wave and  $h_0^*$  is the still water level. Note that when  $\delta < 0.05$ , the model (23) can be used directly *without any breaking criterion* ( $\varphi_0 = 0$ ). However, for solitary waves with a nonlinearity  $\delta > 0.05$ , the enstrophy is still a relevant quantity for breaking but its values are not small enough before breaking to be entirely negligible and the turbulent viscosity should be activated only when the breaking is likely to occur. In this case the choice of the threshold depends on the nonlinearity parameter (as in [7]) :

$$\tilde{\varphi}_0 = \left(0.1 + \frac{0.031}{\delta}\right), \quad \varphi_0 = \frac{g}{h_0^*} \tilde{\varphi}_0$$

Following [7], we chose the parameters  $C_r$  and  $R$  as follow

$$C_r = 0.48, \quad R = \begin{cases} 1.7, & \delta > 0.05 \\ 6, & \delta < 0.05 \end{cases}$$

Note that here the threshold value  $\varphi_0$  and turbulent Reynold's number  $R$  are depending on the initial nonlinearity parameter  $\delta$ , which may not be robust in the operational context. A new local criterion is under development.

### III – 2 Experimental comparison

The new model (23) is validated by comparison with the experimental data from Hsiao *et al.* [6]. A solitary wave of amplitude  $a^*$  propagates over the topography of slope  $\beta$  with  $\tan \beta = 1/60$  with given still water level  $h_0^*$ . In the given experiment wave gauges measure a free surface elevation in time at different locations along the 300 m channel (14 trials with different  $a^*$  and  $h_0^*$  were compared in total). We provide several comparison for this time series data, using parameters given above. The results are presented on Figure 2. In trial 3,  $\delta = 0.048 < 0.05$ , so no breaking criterion is used. Although the numerical breaking point at  $x = 116.4$  m which is very close to the real breaking point at  $x = 116.7$  m, we observe that there's a phase shift in the comparison which indicates that the numerical wave propagates slower than the experimental one. This phase shift is acceptable. In trial 41,  $\delta = 0.137 > 0.05$ , the criterion is applied. In this trial, we have much better results. The numerical breaking point is at  $x = 148.7$  m compare to the real breaking point at  $x = 148$  m, the numerical propagation speed is in a good agreement with experimental

one. We must note that a smaller amplitude than experimental one just before breaking (e.g. see, Trial 3 (x = 116,54m) or Trial 41(x = 150m) is due to a weakness of Green-Naghdi model for description of strongly nonlinear effects. Using the idea introduced in [9], we can improve the dispersive properties.

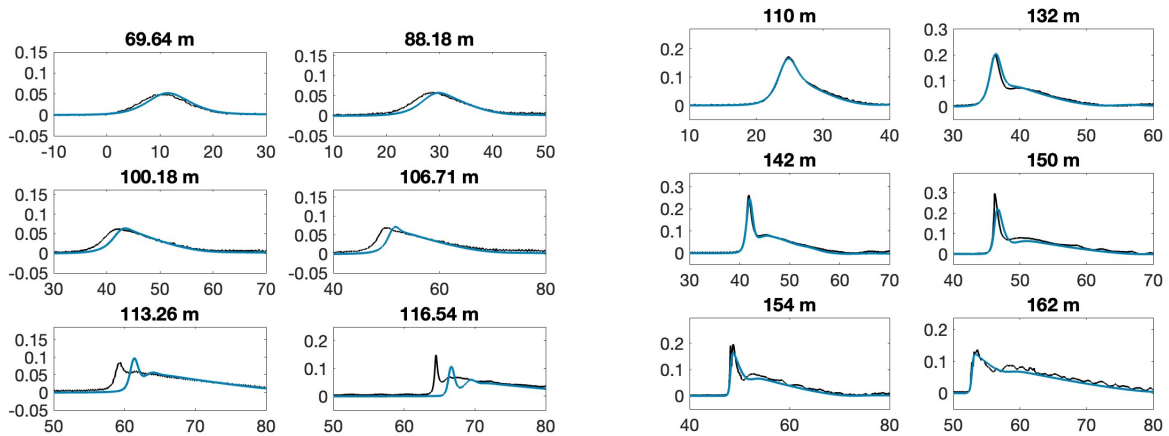


FIGURE 2 – Comparison of numerical (blue line) and experimental (black line) time series Trial 3 (left) and Trial 41 (right).

## IV – Numerical scheme and implementation in Tolosa

We have found that the scheme (31)-(34) proposed for numerical implementation of the new model is not always stable, and a new robust strategy is required. Moreover, we have recently shown that the hyperbolic structure of the model can be exploited to guarantee the stability of the approximations (in the sense of discrete energy decay). The new numerical approach is first studied, on a flat bottom and without taking into account the enstrophy. The model we consider is the following :

$$\begin{cases} \partial_t H + \nabla \cdot (HU) = 0, \\ \partial_t (HU) + \nabla (HU \otimes U + \frac{1}{2}gH^2 + HP) = 0, \\ \partial_t (HW) + \nabla \cdot (HUW) = \frac{3}{2}P, \\ \partial_t (HP) + \nabla \cdot (HUP) = -\frac{1}{A}(2W + H\nabla \cdot U). \end{cases} \quad (35)$$

with the total energy conservation

$$\partial_t E + \nabla \cdot \left( \left( E + \frac{1}{2}gH^2 + \Pi \right) U \right) = 0, \quad (36)$$

where

$$\Pi = HP \quad \text{et} \quad E = \frac{1}{2}H \|U\|^2 + \frac{2}{3}HW^2 + \frac{1}{2}gH^2 + \frac{A}{2}HP^2.$$

The hyperbolic nature of the model provides an appropriate framework for the construction of a numerical scheme : absence of elliptic phase, existence of adapted methods for hyperbolic problems, propensity for parallelization. As a matter of fact, this structure allows to get rid of most of the limitations due to the existing strategies based on direct

approaches for SGN equations. In particular, this structure can be exploited to guarantee the stability of the approximations in the sense of discrete energy decrease. On this basis, it is actually possible to propose several variants of energetically stable and low-diffusive numerical schemes on unstructured meshes, exhibiting an excellent compromise between accuracy and algorithmic complexity. These results are fundamental in the targeted application contexts. In the perspective of the construction of a numerical scheme, the model (35) can be recast under an appropriate split form. The first step reduces to a simple Shallow Water system with source term :

$$\begin{cases} \partial_t H + \nabla \cdot (H\mathbf{U}) = 0, \\ \partial_t (H\mathbf{U}) + \nabla \cdot (H\mathbf{U} \otimes \mathbf{U} + \frac{1}{2}gH^2\mathbf{U}) = 0, \\ \partial_t (HW) + \nabla \cdot (H\mathbf{U}W) = 0, \\ \partial_t (HP) + \nabla \cdot (H\mathbf{U}P) = 0. \end{cases} \quad (37)$$

admitting the following energy budget :

$$\partial_t E + \nabla \cdot \left( \left( E + \frac{1}{2}gH^2 \right) \mathbf{U} \right) = 0, \quad (38)$$

The second step of the splitting contains the acoustic contributions and can be written :

$$\begin{cases} \partial_t H = 0, \\ \partial_t \mathbf{U} + \frac{1}{H} \nabla (HP) = 0, \\ \partial_t W = \frac{3}{2} \frac{P}{H}, \\ \partial_t P = -\frac{1}{A} (2\bar{W} + H\nabla \cdot \mathbf{U}). \end{cases} \quad (39)$$

The energy equation associated with this second part is :

$$\partial_t E + \nabla \cdot (H\mathbf{U}P) = 0. \quad (40)$$

Regarding the hyperbolic phase (37), modern techniques dedicated to hyperbolic problems [1, 4] can be adapted to ensure a discrete counterpart of (38), while minimizing diffusive losses. At the numerical level, this step results in a state  $\mathbf{Q} = (\bar{H}, \bar{\mathbf{U}}, \bar{W}, \bar{P})$  which will have to evolve through the second stage. The water height being constant through this step, we propose a discretization of the form :

$$\begin{cases} \mathbf{U}_K^{n+1} = \bar{\mathbf{U}}_K - \frac{\Delta t}{\bar{H}_K} \nabla_K [\bar{H} P^{n+1}], \\ W_K^{n+1} = \bar{W}_K + \frac{3}{2} \Delta t \frac{P_K^{n+1}}{\bar{H}_K}, \\ P_K^{n+1} = \bar{P}_K - \frac{\Delta t}{A} \left( 2 \frac{W_K^{n+1}}{\bar{H}_K} + \nabla_K \cdot [\bar{\mathbf{U}}^n] \right). \end{cases} \quad (41)$$

Although appearing in implicit form the quantities  $W_K^{n+1}$  and  $P_K^{n+1}$  can be solved explicitly since they do not intervene in differential form in the source terms of the two last equations. This allows in a second time an explicit treatment of the velocity equation. The discrete operator  $\nabla_K$  is centred and it is actually possible to define the discrete divergence  $\nabla_K$ . in order to control the energy balance (40) of the second part. This approach will be used for implementation in Tolosa-project ([tolosa-project.com](http://tolosa-project.com)).

## V – Conclusions and Perspectives

We have derived a new hyperbolic model which is capable to describe wave breaking in the surf zone. The mechanism of wave breaking is related to the emergence of turbulent structures. This is accounted for by introducing a new variable enstrophy in addition to water depth and average flow velocity. The new model is validated by comparison with the experiment [6]. Numerical results are in good agreement with experimental data.

However, the breaking criterion proposed in [7] and used here is not local and a new strategy is required. Moreover, no attempt is made here to improve the dispersive properties. The model is fully nonlinear and has the same dispersive properties as the classical Serre-Green-Naghdi equations. Using the idea introduced in [9], we can improve the dispersive properties.

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