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To cite this version:

Stephane Kirsch, Paul Laurain. An Obstruction to the Existence of Immersed Curves of Prescribed Curvature. Potential Analysis, 2009, 32 (1), pp.29 - 39. 10.1007/s11118-009-9142-8 . hal-03857679

HAL Id: hal-03857679

https://hal.science/hal-03857679v1

Submitted on 17 Nov 2022

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AN OBSTRUCTION TO THE EXISTENCE OF IMMERSED CURVES OF PRESCRIBED CURVATURE

STEPHANE KIRSCH AND PAUL LAURAIN

Abstract. We prove in this paper that there exists no immersed closed curve in $\mathbb{R}^2$ with curvature increasing in a given direction. This generalizes in the 1-dimensional setting a result of the first author for embedded hypersurfaces in $\mathbb{R}^{n+1}$.

1. Introduction

The problem of prescribing curvature of manifolds has been intensively studied in the last decades. In this paper we consider the problem of prescribing the mean curvature of a hypersurface in the Euclidean space $\mathbb{R}^n$. More precisely, considering a function $H : \mathbb{R}^n \to \mathbb{R}$, we can ask whether or not there exists a compact closed (embedded or immersed) hypersurface whose mean curvature at any point x is given by $H(x)$. We are interested here in finding an obstruction to the existence of such a hypersurface, in the spirit of the Kazdan-Warner obstruction for scalar curvature on the 2-sphere. We look for conditions on H ensuring that the problem of prescribed mean curvature has no solution. For example the first author proved in his thesis the following theorem:

Theorem 1.1. *Let $n \geq 2$ and $H : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function such that there exists $\vec{e} \in \mathbb{R}^n$ verifying*

$$\langle \nabla H(x), \vec{e} \rangle > 0 \quad \textit{for almost every} \;\; x \in \mathbb{R}^n,$$

then there exists no C^1-closed hypersurface embedded in $\mathbb{R}^n$ of mean curvature H.

In fact this theorem is a little bit more general since it still holds if we replace the mean curvature by any k-th mean curvature (the k-th symmetric function of the principal curvatures). Theorem 1.1 means that an embedded hypersurface can not have a mean curvature which is monotone in some direction. This obstruction is very similar to the obstruction found by Kazdan-Warner for the scalar curvature on the 2-sphere (see section 8 of [8] and also [4]): there exists no metric on S^2 whose scalar curvature is monotone in a fixed direction of $\mathbb{R}^3$. An analogue of the Kazdan-Warner obstruction was recently proved by Ammann, Humbert and Ould Ahmedou in the mean curvature context (see [2]): if one sees the mean curvature as a function on S^n instead of the ambient space $\mathbb{R}^{n+1}$, there exists no conformal immersion of S^n in $\mathbb{R}^{n+1}$ such that the mean curvature $H : S^n \to \mathbb{R}$ is monotone in one direction.

In this paper we investigate a generalization of theorem 1.1 in the immersed case. The immersed world is much richer and more complicated than the embedded world. For instance, Alexandrov proved (see [1]) that the only closed compact

Date: March 2009.

hypersurfaces of constant mean curvature embedded in $\mathbb{R}^n$ are the spheres. Though for many years it was conjectured that the embedding assumption could be removed, Wente constructed in the 80's (see [10]) immersed tori of constant mean curvature which are not embedded in any Euclidean space of dimension $n \geq 3$. We can then expect the extension of theorem 1.1 to be tricky (maybe even false) in the immersed case. Nevertheless the Alexandrov theorem holds for immersed spheres (see [7]) so we can expect some rigidity of spheres with respect to their mean curvature. In particular, the Alexandrov theorem is true for immersed curved. This suggests to look first at an extension of theorem 1.1 in the 1-dimensional case. In this paper we prove the following:

Theorem 1.2. *Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a positive Lipschitz function such that there exists $\vec{e} \in \mathbb{R}^2$ verifying*

$$\langle \nabla H(x), \vec{e} \rangle > 0 \quad \textit{for almost every } x \in \mathbb{R}^2.$$

Then there exists no C^1-closed curve immersed in $\mathbb{R}^2$ whose curvature at every point x is $H(x)$.

Remark 1.1. *Without the positivity assumption on H the result does not hold, whereas it still holds in the embedded case. Indeed, a simple counterexample is given by the lemniscate of Bernoulli.*

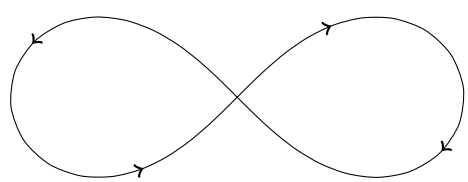


FIGURE 1. Lemniscate of Bernouilli

The cartesian equation of the lemniscate is

$$(x_1^2 + x_2^2)^2 = a^2(x_1^2 - x_2^2),$$

where $a > 0$ is a parameter. A straightforward computation gives the curvature $c(x_1)$ of the lemniscate:

$$c(x_1) = sign(x_1)\frac{3}{a^2}\sqrt{\frac{a\sqrt{8x_1^2 + a^2} - a^2}{2}},$$

which is a C^1 increasing function. The lemniscate of Bernoulli is thus a solution of the prescribed curvature problem where $H(x_1, x_2) = c(x_1)$, with

$$\langle \nabla H(x), e_1 \rangle = c'(x_1) > 0 \textit{ for every } x \in \mathbb{R}^2.$$

Our proof relies on the fact that a self-intersection of a curve is very simple: it is a point. In higher dimension self-intersections of a hypersurface may be very complicated, so our proof seems hard to generalize, and in that case, the problem remains widely open.

The paper is organized as follows: in part 2, we give some definitions, notations, we state a preliminary lemma whose proof is given in the appendix, and we give a

short sketch of the proof of theorem 1.2, and in part 3 we prove theorems 1.1 and 1.2.

2. Notations and preliminaries

All curves are assumed to be at least in $C^2_p \cap C^0$ (i.e continuous, with a piecewise continuous first and second derivative). Moreover, thanks to the rescaling properties of our problem, we assume also that all curves have length 2π and are parametrized by arc length on $\mathbb{S}^1 \simeq [0, 2\pi[$. We denote by Γ a curve parametrized by $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$. Let $\vec{t}(s) = \frac{d\gamma}{ds}(s)$ and $\vec{n}(s)$ be such that $(\gamma(s), \vec{t}(s), \vec{n}(s))$ is a positive orthonormal frame for every s such that $\vec{t}(s)$ exists. Then for a C^2_p curve, we can write the equation of prescribed curvature on every C^2 piece of γ as follows:

$$\frac{d\vec{t}}{ds}(s) = H(\gamma(s))\vec{n}(s). \tag{1}$$

We define here the order of a self-intersection point:

Definition 2.1. *Let x be a point of Γ, we define the order of x, which we denote $ord(x)$, by*

$$ord(x) := \#\{s \in \mathbb{S}^1 \text{ such that } \gamma(s) = x\}.$$

It is obvious that x is a self-intersection point if and only if $ord(x) \geq 2$.

Remark 2.1. *If $H : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function, the Cauchy-Lipschitz theorem insures that every non-periodic C^2-solution of (1) has isolated self-intersection of finite order. Moreover, if two parts of the curve intersect tangentially then they must have opposite orientation.*

Before going any further we need to define the index of a closed curve and some properties of it.

Definition 2.2. *Let $\gamma \in C^0(\mathbb{S}^1, \mathbb{R}^2)$ and $a \in \mathbb{R}^2 \setminus \Gamma$, we define the index of γ with respect to a as the degree of the map from S^1 into S^1 defined by*

$$s \mapsto \frac{\gamma(s) - a}{\|\gamma(s) - a\|},$$

which we denote by $Ind_\gamma(a)$.

Remark 2.2. *For a given curve Γ the index is defined up to a choice of orientation.*

It is well known that the map $x \mapsto Ind_\gamma(x)$ is constant on each connected component of $\mathbb{R}^2 \setminus \Gamma$. Moreover, thanks to the following proposition, we can quantify the variation of index passing from a component to another.

Proposition 2.1. *Let $\gamma \in C^1_p(\mathbb{S}^1, \mathbb{R}^2) \cap C^0(\mathbb{S}^1, \mathbb{R}^2)$ and $a \in \mathbb{R}^2 \setminus \Gamma$. Let $p \in \Gamma$ be a regular point; by regular point we mean that p is not a self-intersection point of Γ and γ is C^1 in a neighborhood of $\gamma^{-1}(p)$; then for r sufficiently small, $B(p, r) \setminus \Gamma = U \cup V$, where U and V are two connected open subsets of $\mathbb{R}^2$ such that $U \cap V = \emptyset$ and $\vec{n}$ points into U. Moreover $\forall (u, v) \in U \times V$ we have*

$$Ind_\gamma(u) - Ind_\gamma(v) = 1.$$

A proof can be found in Lemma 9.2.5 of [3]; moreover it is obviously false if p is a point of self-intersection as shown by a circle parametrized twice.
We are now able to give the following lemma, which is a generalization of the Stokes theorem:

Lemma 2.1. *Let $\Gamma \subset \mathbb{R}^2$ be a continuous C^1_p-closed curve with isolated self-intersections of finite order, and $\vec{V}$ be a Lipschitz vector field on $\mathbb{R}^2$, then we have the following formula:*

$$\int_{\mathbb{R}^2} div(\vec{V}) Ind_\gamma(x)dx = -\oint_\Gamma \left\langle \vec{V}, \vec{n} \right\rangle d\sigma, \tag{2}$$

where $d\sigma$ is the restriction of the Lebesgue measure to Γ and $\vec{n}$ the unit normal vector field on Γ.

The proof of this lemma is postponed to the appendix.

Remark 2.3. *There are two possible choices for the orientation of Γ, but (2) obviously holds in both cases, because turning $\vec{n}$ into $-\vec{n}$ turns $Ind_\gamma(x)$ into $-Ind_\gamma(x)$.*

Remark 2.4. *If Γ has no self-intersection, we have the classical Stokes formula. Indeed, let $\vec{n}$ be the outward unit normal vector, then $Ind_\gamma(x)$ is -1 if x is in the area enclosed by the curve, which we denote Ω, and 0 otherwise, which gives*

$$-\int_\Omega div(\vec{V}) = -\int_{\partial\Omega} \left\langle \vec{V}, \vec{n} \right\rangle.$$

Formula (2) enables us to prove theorem 1.2 in the case where the index is everywhere non negative (resp. everywhere non positive).

Let Γ be a C^1-closed curve of curvature H, and let $\vec{V} = H\vec{e}$ be such that $\langle \nabla H, \vec{e} \rangle > 0$, then we get

$$\begin{aligned} \int_{\mathbb{R}^2} \langle \nabla H, \vec{e} \rangle \, Ind_\gamma(x)dx &= -\oint_\Gamma \langle \vec{e}, H\vec{n} \rangle \, d\sigma \\ &= -\left\langle \vec{e}, \oint_\Gamma \frac{d\vec{t}}{ds} d\sigma \right\rangle \\ &= 0, \end{aligned} \tag{3}$$

leading to $Ind_\gamma = 0$ almost everywhere in $\mathbb{R}^2$, which is impossible.

In order to prove the general case, we consider some closed subpart of the curve whose own index is non negative almost everywhere in $\mathbb{R}^2$. The idea is to cut off a loop of the curve, as the bold one in figure 2. Of course this subcurve may not be regular and then

$$\oint_\Gamma \frac{d\vec{t}}{ds} d\sigma \neq 0,$$

but to get a contradiction we just need to have the right-hand side of (3) negative. For that purpose the singularities of our loop have to be "well oriented" (left side on figure 2). The following section is devoted to the proof of theorem 1.1 and theorem 1.2 that we have just sketched.

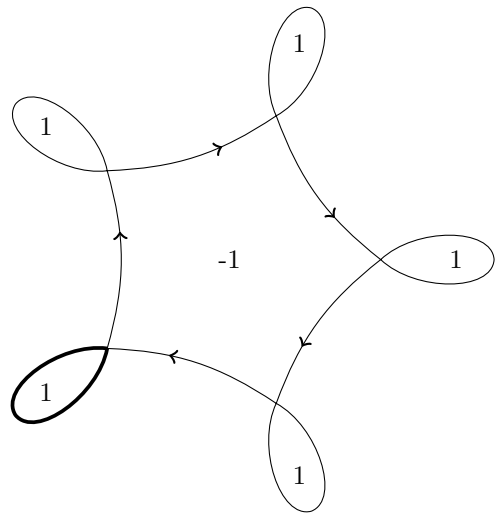


FIGURE 2. Flower, a curve whose index has different signs

3. Proof of the theorems

3.1. **Proof of theorem 1.1.** Suppose by contradiction that there exists a closed compact C^1-hypersurface S embedded in $\mathbb{R}^n$ whose mean curvature is H. Then using the first variation formula with a constant vector field we easily find the following identity:

$$\int_S H(x)\vec{n}(x)dS = 0, \tag{4}$$

where $\vec{n}$ is the unit normal vector field on S, given by H, and $dS = dH^{n-1}|_S$. In the case of the k-th mean curvature the identity (4) still holds (see [5]). Since S is embedded we can use the generalized Jordan-curve theorem (see [9]) to find $\Omega \subset \mathbb{R}^n$ a bounded open set such that $S = \partial\Omega$. We have then, using the Stokes formula,

$$\int_S H\vec{n}dS = \int_\Omega \nabla H(x)dx = 0,$$

which contradicts the assumption $\langle \nabla H(x), \vec{e}\rangle > 0$, and thus proves theorem 1.1. □

3.2. **Proof of theorem 1.2.** We prove the theorem by contradiction. We assume that there exists a closed C^2-curve Γ of curvature H. Without loss of generality, we may assume that $\vec{e} = (1, 0)$.

We first give some definitions when the curve Γ we consider is only in $C^0 \cap C^2_p$.

We say that x is a **singularity** with respect to s if $x = \gamma(s)$ and

$$\Delta\vec{t}(s) := \lim_{u\to 0^+} \vec{t}(s-u) - \vec{t}(s+u) \neq \vec{0}.$$

Moreover we say that the singularity is **well-oriented** if

$$\begin{cases} \left\langle \lim\limits_{u\to 0^+} \vec{t}(s-u), \vec{e} \right\rangle & > & 0, \\ \left\langle \lim\limits_{u\to 0^+} \vec{t}(s+u), \vec{e} \right\rangle & < & 0. \end{cases}$$

Thus, for a well-oriented singularity, we have $\langle \Delta\vec{t}(s), \vec{e}\rangle > 0$.

As announced at the end of the previous section, in order to get a contradiction, we need to find $\Gamma' \subset \Gamma$ such that

$$Ind_{\gamma'} \geq 0 \text{ and } \left\langle \vec{e}, \oint_{\Gamma'} \frac{d\vec{t}}{ds} d\sigma \right\rangle > 0.$$

For that purpose, we construct by induction a decreasing sequence of closed subcurves Γ_n of Γ, such that all the singularities of Γ_n are well-oriented.

Starting from $\Gamma_0 = \Gamma$, we obtain Γ_{n+1} from Γ_n as follows : Consider

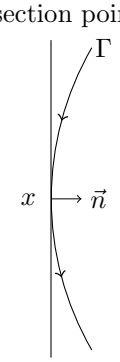
$$\Omega_n := \{x \in \mathbb{R}^2 \backslash \Gamma_n \text{ s.t. } Ind_{\gamma_n}(x) < 0\}.$$

If $\Omega_n = \emptyset$ then $\Gamma_{n+1} = \Gamma_n$, otherwise Ω_n is bounded and we cut Γ_n at the "left extremity" of Ω_n. In order to do so, we proceed in several steps:

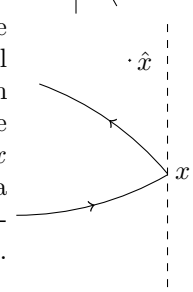
Step 1: isolate the "left extremity" of Ω_n

Let $x = (x_1, x_2) \in \overline{\Omega_n}$ be such that x_1 is minimal. We claim that x is a point of self-intersection of Γ_n. Otherwise it is either a regular point, or a singularity created at a previous step of the algorithm which is not a self-intersection point.

Suppose it is a regular point, then thanks to remark 2.1, there is no self-intersection of Γ, and consequently of Γ_n, in a small ball centered at x. Moreover, the tangent line to Γ_n at x has to be vertical and the curve lies locally on the right-hand side of the tangent line, because of the minimality of the abscissa of x. Now, because of the orientation imposed by $H > 0$ and proposition 2.1, the index has to decrease when we cross Γ_n at the point x from the right to the left, which contradicts again the minimality of the abscissa of x. Hence x can not be a regular point.



Now, suppose x is a singularity created at a previous step of the algorithm and not a self-intersection point. Since, as we shall see, the algorithm only creates well-oriented singularities—which means that the curve arrives transversally from the left (i.e $\{(x, y) \text{ s.t. } x \leq x_1\}$) on the vertical line passing through x and goes back transversally to the left—it is easy to find a point $\hat{x}$ whose abscissa is smaller than the abscissa of x and verifying $Ind_{\gamma_n}(\hat{x}) < 0$, which is a contradiction and proves the claim.



Step 2: picture of the curve around x

By remark 2.1, there exists a neighborhood of x such that $\Gamma_n \setminus \{x\}$ has no self-intersection. Then for $r > 0$ small enough, the different parts of the curve divide the disk $D(x, r)$ into several sectors (see figure 3).

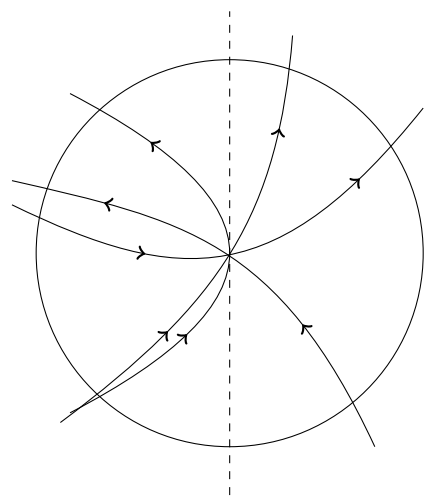


FIGURE 3. Picture of the curve around x

Let (S_i) be an indexation of this partition of $D(x, r)\backslash\Gamma_n$. Let S_0 be the sector with the smallest index. First, we remark that this sector lies on the right-hand side of x, because its index is negative. This ensures that it cannot be edged by a singularity, because this one would not be well-oriented. Since the index has to increase crossing out of S_0, the normal vector, imposed by the orientation and $H > 0$, points outward of the sector, as shown by figure 4.

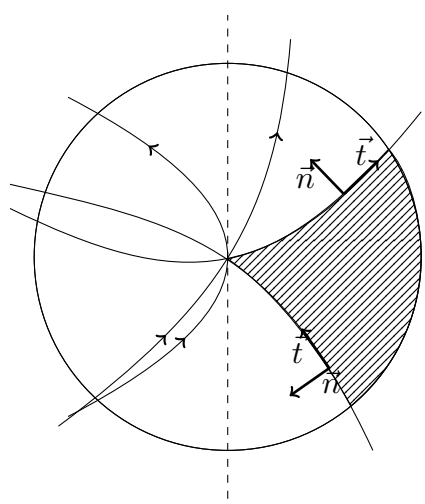


FIGURE 4. Sector of least index

Step 3: cut the curve at x

We choose an arc length parametrization $\gamma_n : [0, 2\pi[\to \mathbb{R}^2$ of Γ_n, such that $\gamma_n(0) \neq x$. Then there exist $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $s_1 < s_2 \in [0, 2\pi[$ such that the boundary of the sector of least index (where we do not consider the boundary of $D(x, r)$) is parametrized by two curves: $\gamma_n(]s_1-\epsilon_1, s_1])$ and $\gamma_n([s_2, s_2+\epsilon_2])$. Thanks to the fact that $x = \gamma_n(s_1) = \gamma_n(s_2)$ is not a singularity with respect to s_1 or s_2, we have $s_1 \neq s_2$. We set $\Gamma_{n+1} = \gamma_n(]s_1, s_2])$, and we easily check that the set of singularities of Γ_{n+1} is the singularity we created at x, which is well-oriented by construction, plus a subset of the singularities of Γ_n, which are all well-oriented by induction hypothesis. To end the algorithm, we rescale in order to have Γ_{n+1} of length 2π.

End of the proof:

We have a decreasing sequence of curves Γ_n. Let C_n be the number of connected components of $\mathbb{R}^2 \setminus \Gamma_n$. C_0 is finite because Γ has a finite number of self-intersection points which are all of finite order. Moreover C_n is decreasing, since at every step we cut off a part of the curve which encloses at least one component. But Γ_n is never empty, because at each step we keep a part of the curve, so the cutting process has to stop. This means that for n large enough Ind_{γ_n} is nonnegative on $\mathbb{R}^2 \setminus \Gamma_n$, with $\Gamma_n \neq \emptyset$. For such a curve we get

$$\left\langle \vec{e}, \oint_{\Gamma_n} \frac{d\vec{t}}{ds} d\sigma \right\rangle = \sum_{i=1}^{k} \langle \Delta\vec{t}(s_i), \vec{e} \rangle > 0,$$

where $\gamma_n(s_i)$ are the singularity of Γ_n. As already remarked at the beginning of this section, this ends the proof. □

Appendix A. Proof of Lemma 2.1

Here we give a proof of lemma 2.1 using elementary tools, though this can be proved shortly using degree theory, see for example [6].

We denote $\Omega := \mathbb{R}^2 \setminus \Gamma$, which is an open subset of $\mathbb{R}^2$, and $(\Omega_i)_{i=0..n}$ its connected components, where Ω_0 is the unbounded one. We remark that on every Ω_i, the index function is constant, because it changes only when we cross Γ, so we can define $Ind_\gamma(\Omega_i)$ to be $Ind_\gamma(x)$ for every $x \in \Omega_i$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^2} div(\vec{V}) Ind_\gamma(x) dx &= \sum_{i=0}^{n} \int_{\Omega_i} div(\vec{V}) Ind_\gamma(x) dx, \\
&= \sum_{i=0}^{n} Ind_\gamma(\Omega_i) \int_{\Omega_i} div(\vec{V}) dx, \\
&= \sum_{i=0}^{n} Ind_\gamma(\Omega_i) \int_{\partial\Omega_i} \left\langle \vec{V}, \vec{n}_i \right\rangle dx,
\end{aligned}$$

where $\vec{n}_i$ is the outward normal vector to Ω_i. The last equality is a consequence of the standard Stokes's theorem.

We can remark that Ω_0 is not bounded and has infinite Lebesgue measure, but $Ind_\gamma(\Omega_0) = 0$, so the term for $i = 0$ in the right-hand side of the equation is 0.

The boundary of each Ω_i is a C^1_p curve that we can decompose as follows

$$\partial\Omega_i = \bigcup_{j=0}^{m_i} \Gamma_{i,j},$$

where each $\Gamma_{i,j}$ is a C^1_p curve, and the self-intersections of Γ belonging to $\partial\Omega_i$ are the points where we switch from a $\Gamma_{i,j}$ to another.

Coming back to our computations we get

$$\int_{\mathbb{R}^2} div(\vec{V}) Ind_\gamma(x)dx = \sum_{i=0}^{n}\sum_{j=0}^{m_i} Ind_\gamma(\Omega_i) \int_{\Gamma_{i,j}} \left\langle \vec{V}, \vec{n}_{i,j} \right\rangle dx, \tag{5}$$

where $\vec{n}_{i,j}$ is the outward normal vector to Ω_i on $\Gamma_{i,j}$.

We remark that in (5), each $\Gamma_{i,j}$ appears exactly twice because it is on the boundary of exactly two Ω_i. Thus for each couple (i, j) there exists a unique couple (i', j'), with $i \neq i'$ such that $\Gamma_{i,j} = \Gamma_{i',j'}$.

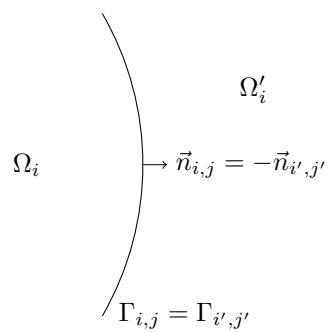


FIGURE 5. Γ between two connected components

We can now change the indices of the sum and take a sum over k in order to count each $\Gamma_{i,j}$ only once.

$$\int_{\mathbb{R}^2} div(\vec{V}) Ind_\gamma(x)dx = \sum_k \int_{\Gamma_k} Ind_\gamma(\Omega_i) \left\langle \vec{V}, \vec{n}_{i,j} \right\rangle + Ind_\gamma(\Omega_{i'}) \left\langle \vec{V}, \vec{n}_{i',j'} \right\rangle.$$

Moreover, we have in the previous formula that $\vec{n}_{i,j} = -\vec{n}_{i',j'}$ because the outward normal for the open on one side of Γ_k is the inward normal for the open on the other side. We get then

$$\int_{\mathbb{R}^2} div(\vec{V}) Ind_\gamma(x)dx = \sum_k \int_{\Gamma_k} (Ind_\gamma(\Omega_i) - Ind_\gamma(\Omega_{i'})) \left\langle \vec{V}, \vec{n}_{i,j} \right\rangle.$$

Finally, by definition of $\vec{n}$, we have that $\vec{n}$ points in the direction of increasing index. Moreover, since we assume only pointwise self-intersections for γ, we have

$Ind_\gamma(\Omega_i) - Ind_\gamma(\Omega_{i'}) = \pm 1$. More precisely we have

$$\begin{cases} Ind_\gamma(\Omega_i) - Ind_\gamma(\Omega_{i'}) & = & -1 & \text{if} & \vec{n}_{i,j} & = & \vec{n}, \\ Ind_\gamma(\Omega_i) - Ind_\gamma(\Omega_{i'}) & = & 1 & \text{if} & \vec{n}_{i,j} & = & -\vec{n}. \end{cases}$$

Consequently we get

$$(Ind_\gamma(\Omega_i) - Ind_\gamma(\Omega_{i'}))\vec{n}_{i,j} = -\vec{n}.$$

Since $(\Gamma_k)_k$ is a partition of Γ, we obtain

$$\int_{\mathbb{R}^2} div(\vec{V}) Ind_\gamma(x)dx = -\sum_k \int_{\Gamma_k} \left\langle \vec{V}, \vec{n} \right\rangle = -\oint_\Gamma \left\langle \vec{V}, \vec{n} \right\rangle.$$

This ends the proof. □

Acknowledgements : This work started while the second author was staying at PIMS, Vancouver. He wishes to thank all the members of PIMS for their great hospitality. The two authors would like to thank Olivier Druet for interesting and stimulating discussions.

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