

HYBRID SINGULARITY FOR THE OBLIQUE INCIDENCE RESPONSE OF A COLD PLASMA

OLIVIER LAFITTE

ABSTRACT. In this paper, we study the electromagnetic response of a cold plasma with dissipation $\nu > 0$ in a general slab geometry (plasma density $n_0(x)$ and background magnetic field $B_0(x)\vec{e}_3$). We express the behavior of the electromagnetic field (\vec{E}, \vec{H}) (at a given frequency ω) in the neighborhood of the hybrid resonance x_h solution of $\omega = \omega_h(x)$. This generalizes to any incident wave vector (k_1, k_2, k_3) the result obtained by Despres, Imbert-Gerard and Lafitte in the normal incidence case (the incoming electromagnetic field has its wave vector belonging to the orthogonal plane to \vec{e}_3 , *id est* $k_3 = 0$), hence treating a fully coupled system of ODEs of order 4. This is done by deducing from the complete system of ODEs an integro-differential system which differential part contains a regular singular point at $\nu \rightarrow 0_+$. This new system is then solved in the neighborhood of this regular singular point using the variation of parameters method.

In the oblique incidence case at the limit $\nu \rightarrow 0_+$, we observe that (E_2, E_3, H_2, H_3) are integrable in the neighborhood of x_h uniformly in $\nu \rightarrow 0_+$, while the most singular term of the limit of $\varepsilon_0 E_1$ when $\nu \rightarrow 0_+$ is proportional to

$$[P.V.(\frac{1}{x - x_h}) - i\pi \text{sign}(\partial_x \omega_h(x_h))\delta_{x_h} + a \ln(x - x_h)^+].$$

1. INTRODUCTION

Following the seminal work of White and Chen [?], we study the electromagnetic field in a cold plasma under the influence of a strong magnetic field $\vec{B}_0 = B_0(x)\vec{e}_z$ (such that $\nabla \cdot \vec{B}_0 = 0$). This plasma is inhomogeneous, the density of electrons is $n_0(x)$, depending only also on x . The functions $n_0(x), B_0(x)$ are assumed to be locally analytic.

From the given functions $B_0(x), n_0(x)$ and the constants e, m, ε_0 , ($-e$ is the charge of the electron, m is its mass and ε_0 is the permittivity of the vacuum) one observes that two quantities of interest appear in this equation, which have both the dimension of a frequency:

$$(1.1) \quad \omega_c(x) = \frac{eB_0(x)}{m}, \omega_p(x) = \sqrt{\frac{e^2 n_0(x)}{\varepsilon_0 m}},$$

These two frequencies are respectively called the cyclotron frequency and the plasma frequency.

This work was completed while the author was invited professor at IU Mathematics Department, Bloomington (Spring 2017).

The author wants to thank the anonymous referee for all his comments for improving the present paper.

From these two frequencies, one defines the upper hybrid frequency

$$(1.2) \quad \omega_h(x) = \sqrt{\omega_p^2(x) + \omega_c^2(x)}.$$

The aim of the present paper is to study, in the framework of the cold plasma model, the response of the plasma under an electromagnetic wave of frequency ω , that is the electromagnetic field in the neighborhood of a point x_h such that $\omega_h(x_h) = \omega$, generalizing previous works as [?], [?].

Let us review the model used throughout the paper. The cold plasma model assumes that the ions and the electrons follow the classical law of motion (acceleration is equal to the magnetic force). Collisions are, nevertheless, taken into account through a dissipation parameter ν (expressed in s^{-1} and small with respect to the characteristic time of motion). Further, electroneutrality of the plasma is assumed. As the mass of the ions is much larger than the mass of electrons, one also neglects the velocity of ions in the plasma density current \vec{j} .

The motion of electrons is thus characterized by the linearized equation (hence the name of 'cold model')

$$(1.3) \quad \partial_t \vec{v} + \nu \vec{v} = -\frac{e}{m} (\vec{E} + \vec{v} \wedge \vec{B}_0),$$

and the current density is given through

$$(1.4) \quad \vec{j} = -en_0(x)\vec{v}.$$

The electromagnetic field $(\vec{E}, \vec{B}_0 + \vec{B})$ is given through the Maxwell equations with a source current density

$$(1.5) \quad \nabla \wedge \vec{E} = -\partial_t \vec{B}, \nabla \wedge \vec{B} = \varepsilon_0 \mu_0 \partial_t \vec{E} + \mu_0 \vec{j}.$$

We assume that the electromagnetic field has a given frequency $\omega > 0$:

$$(\vec{E}, \vec{B})(x, y, z, t) = e^{-i\omega t}(\vec{E}(x, y, z), \vec{B}(x, y, z)),$$

and we introduce $\omega_\nu = \omega + i\nu$.

Throughout the paper, we shall use \vec{E} , \vec{B} (and \vec{H}) for the components of the electromagnetic field or for related fields.

Usual studies of this problem rely on the construction of the **effective dielectric tensor** $\underline{\underline{\varepsilon}}^\nu$ (see Section 2.1). The resulting PDE on \vec{E} writes

$$(1.6) \quad \nabla \wedge \nabla \wedge \vec{E} = \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}}^\nu(x) \vec{E}.$$

As (n_0, B_0) depend only on x , and as it is done in [?], [?], one assumes that the incident electromagnetic field outside the plasma is

$$(1.7) \quad (\vec{E}, \vec{B})(x, y, z) = e^{i(k_2 y + k_3 z)}(\vec{E}(x), \vec{B}(x)),$$

where (k_2, k_3) are the (y, z) components of the wave vector \vec{k} of the electromagnetic field in the vacuum, such that $k_2^2 + k_3^2 \leq \varepsilon_0 \mu_0 \omega^2$. This assumption is in fact a consequence of the theorem of propagation of singularities for the Maxwell system of equations in $\mathbb{R}^3 \times \mathbb{R}_t$.

In a previous paper, Despres, Imbert-Gérard and Lafitte [?] studied the so called ordinary and extraordinary modes of this plasma in the case $k_3 = 0$. This case is called the normal incidence case because $\vec{k} = (k_1, k_2, 0)$ is normal to \vec{B}_0 . This work followed a first study of Despres, Imbert-Gérard and Weder [?] on slab geometry, where a particular form of $n_0(x)$ was assumed.

One of the main advantages of the normal incidence case is that the resulting system of ODEs consists in two decoupled systems of ODE of order 2. The system on (E_2, B_3) contains the extraordinary mode, and the system on (E_3, B_2) corresponds to the ordinary mode.

One of the novelties of [?] was to bypass the computation of the effective dielectric tensor, hence avoiding a singularity for the ordinary mode (there is amplitude growth but not a mathematical singularity), still observing the singularity for the extraordinary mode. Bypassing this computation was done by expressing the current \vec{j} and the first component of the electric field E_1 in terms of other components of the electromagnetic field. This trick applies here as well.

Two assumptions are used in this paper, for ω given:

(A0) There exists a unique $x_h \in \mathbb{R}$ such that $\omega_h(x) = \omega \Leftrightarrow x = x_h$,

(A1) $\partial_x \omega_h(x_h) \neq 0$.

Introduce

$$(1.8) \quad D_\nu(x) = \omega_c^2(x) + \frac{\omega_\nu}{\omega} \omega_p^2(x) - \omega_\nu^2.$$

Under assumptions (A0),(A1), the unique solution of $D_0(x) = 0$ is x_h and $\partial_x D_0(x_h) \neq 0$. The point x_h is the unique **resonance**. Moreover Despres, Imbert-Gérard and L. proved in [?] the local existence and uniqueness of $X_\nu \in \mathbb{C}$, satisfying $D_\nu(X_\nu) = 0$, and, for $k_2 = 0$, the existence of three analytic functions $a_1^\nu, b_1^\nu, c_1^\nu$ such that $U = (E_2, i\omega B_3)^T$ satisfies:

$$U' = \frac{1}{x - X_\nu} \begin{pmatrix} a_1^\nu(x) & b_1^\nu(x) \\ c_1^\nu(x) & -a_1^\nu(x) \end{pmatrix} U,$$

where the matrix $\begin{pmatrix} a_1^\nu(x) & b_1^\nu(x) \\ c_1^\nu(x) & -a_1^\nu(x) \end{pmatrix}$ is smooth in the neighborhood of X_ν , with

the relation $(a_1^\nu(X_\nu))^2 + b_1^\nu(X_\nu)c_1^\nu(X_\nu) = 0$ (the matrix $\begin{pmatrix} a_1^\nu(x) & b_1^\nu(x) \\ c_1^\nu(x) & -a_1^\nu(x) \end{pmatrix}$ is of rank 1 at $x = X_\nu$).

Note that $X_\nu = x_h + i\theta\nu + O(\nu^2)$ and θ is of the sign of $\partial_x \omega_h(x_h)$.

All solutions were then calculated in a neighborhood of x_h , using the Bessel function of the first kind J_0 and the Bessel function of the second kind Y_0 , both being solutions of $zJ'' + J' + zJ = 0$, given by

$$J_0(z) = \sum_{p \geq 0} \frac{(-\frac{z^2}{4})^p}{(p!)^2}, Y_0(z) = \frac{2}{\pi} \ln\left(\frac{z}{2}\right) J_0(z) + T_0(z),$$

T_0 being an even analytical function described in [?], $\ln z$ being given classically by $\ln|z| + i \arg z$. Observe that $z(Y_0' J_0 - J_0' Y_0) = \frac{2}{\pi}$. These special functions are used also in the present paper.

We treat in the present paper the case called Oblique Incidence, namely $k_2 k_3 \neq 0$. In this case, the system on $(\vec{E}, \vec{H}, \vec{j})$, $\vec{H} = \mu_0^{-1} \vec{B}$, is transformed, using (1.7), into a system of ordinary differential equations on $\mathcal{U} = (\varepsilon_0 E_2, ik_2 H_3 - ik_3 H_2, \varepsilon_0(ik_2 E_3 - ik_3 E_2), H_3)^T$:

$$(1.9) \quad \mathcal{U}' = \mathcal{M}^\nu(x) \mathcal{U},$$

supplemented with relations calculating E_1, E_3, H_1, H_3 (as well as \vec{j}).

When considering the limit $\nu \rightarrow 0_+$, the system becomes a system of singular ODEs

of order 4, which is not separable into ordinary and extraordinary modes, as it is mentioned in paragraph (c) of the Introduction of [?]. It is stated in [?] that one needs to take into account Landau damping via the Vlasov equation, and that one needs to study the normal modes of the system. Furthermore, the authors mention that the problem of oblique incidence is not interesting because no new effects of wave amplification occur if $k_3 \neq 0$. However, the rigorous proof of this assertion is not present in [?] or in the references therein.

We prove that the authors are correct by noticing that no new effects of wave amplification occur at other points and provide a rigorous mathematical proof of the amplification result in this more complicated set-up. In addition, we can, thanks to the present paper,

- confirm that the point x_h is a singularity of the system and wave amplification occurs at this point,
- extend some of the results of [?] in the case (k_2, k_3) , $k_2 k_3 \neq 0$,
- give a precise representation of the electromagnetic field in the neighborhood of x_h .

For $\nu > 0$ small, one has

$$(1.10) \quad \mathcal{M}^\nu(x) = \frac{1}{x - X_\nu} \begin{pmatrix} P^\nu & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{R}^\nu(x),$$

where P^ν is a constant matrix of rank 1, \mathcal{R}^ν is analytic in the neighborhood of X_ν , the projection of $\mathcal{R}^\nu(X_\nu)$ on the image of P^ν and on the image of the matrix $(P^\nu)^T$ being non zero (which ensures that the system is coupled). In the limit $\nu \rightarrow 0_+$, the system has a singular point x_h .

Introduce $b^\nu(x) = (x - X_\nu)\mathcal{M}_{12}^\nu(x)$ and $a^\nu(x) = (x - X_\nu)\mathcal{M}_{11}^\nu(x)$ and obtain δ^ν through (3.10). Let

$$(1.11) \quad \begin{aligned} Be_1(x) &= \begin{pmatrix} \sqrt{b^\nu(x)}J_0(\lambda\sqrt{x - X_\nu}) \\ \delta^\nu(x)J_0(\lambda\sqrt{x - X_\nu}) + \frac{\lambda\sqrt{x - X_\nu}}{2\sqrt{b^\nu(x)}}J'_0(\lambda\sqrt{x - X_\nu}) \end{pmatrix} \\ \tilde{Be}_2(x) &= \begin{pmatrix} \sqrt{b^\nu(x)}Y_0(\lambda\sqrt{x - X_\nu}) \\ \delta^\nu(x)Y_0(\lambda\sqrt{x - X_\nu}) + \frac{\lambda\sqrt{x - X_\nu}}{2\sqrt{b^\nu(x)}}Y'_0(\lambda\sqrt{x - X_\nu}) \end{pmatrix}. \end{aligned}$$

Define, for $\lambda = \lambda_\nu$, of positive real part, given below by (3.3):

$$(1.12) \quad M_2^\nu(x) = \begin{pmatrix} Be_1^\nu(x) & Be_2^\nu(x) \end{pmatrix},$$

with

$$(1.13) \quad Be_2^\nu(x) = \begin{pmatrix} \sqrt{b^\nu(x)}Y_0^*(\lambda_\nu, \sqrt{x - X_\nu}) \\ \delta^\nu(x)Y_0^*(\lambda_\nu, \sqrt{x - X_\nu}) + \frac{\lambda\sqrt{x - X_\nu}}{2\sqrt{b^\nu(x)}}(Y_0^*)'(\lambda_\nu, \sqrt{x - X_\nu}) \end{pmatrix},$$

where $Y_0^*(\lambda_\nu, \sqrt{x - X_\nu}) = \pi^{-1} \ln(x - X_\nu)J_0(\lambda_\nu\sqrt{x - X_\nu}) + T_0(\lambda_\nu\sqrt{x - X_\nu})$. It is important to use this function for representing the singular solutions because of the fact that **one does not have the equality** $2 \ln(\lambda_\nu\sqrt{x - X_\nu}) = 2 \ln \lambda_\nu + \ln(x - X_\nu)$. Observe that the relation $z(Y'_0(z)J_0(z) - J'_0(z)Y_0(z)) = \frac{2}{\pi}$ (which is also valid for Y_0^* because Y_0^* differs from Y_0 by a function proportional to J_0) implies $\det M_2^\nu(x) = \pi^{-1}$.

As the equation (1.10) is singular at x_h for $\nu = 0$, we have to modify the Cauchy data at $x = x_h$ for (1.9) in order to have an uniform in ν result. Instead of writing

the Cauchy data for the components (E_2, E_3, H_2, H_3) , we consider the following condition

$$(1.14) \quad \begin{pmatrix} \varepsilon_0 E_2 \\ ik_2 H_3 - ik_3 H_2 \\ \varepsilon_0(ik_2 E_3 - ik_3 E_2) \\ H_2 \end{pmatrix} (x_h) = \begin{pmatrix} M_2^\nu(x_h) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \\ C \\ D \end{pmatrix},$$

where (A_0, B_0, C, D) is given in \mathbb{C}^4 . Note that one chooses the notations (A_0, B_0) for quantities which are **not** values of the fields $\varepsilon_0 E_2, ik_2 H_3 - ik_3 H_2$ at any point, while C, D are the actual values at x_h of the quantities $(\varepsilon_0(ik_2 E_3 - ik_3 E_2), H_3)$. Note in addition that B_0 has nothing to do with the imposed magnetic field, and is an arbitrary constant.

We prove in the present paper the

Theorem 1.1. *There exists $\delta > 0$ such that, for all $0 < |\nu| \leq \delta$, the unique solution $(\vec{E} \wedge \vec{e}_1, \vec{H} \wedge \vec{e}_1)$ of (1.9) in $[x_h - \delta, x_h + \delta]$ satisfying the partial Cauchy data (1.14) is given through*

$$\begin{pmatrix} \varepsilon_0 E_2 \\ ik_2 H_3 - ik_3 H_2 \\ \varepsilon_0(ik_2 E_3 - ik_3 E_2) \\ H_2 \end{pmatrix} (x) = \begin{pmatrix} M_2^\nu(x) & 0 \\ 0 & I \end{pmatrix} \mathcal{R}(x, \nu) \begin{pmatrix} A_0 \\ B_0 \\ C \\ D \end{pmatrix},$$

where \mathcal{R} is a C^1 matrix for $x \in [x_h - \delta, x_h + \delta]$, for all $\nu \in [-\delta, 0[\cup]0, \delta]$ satisfying $\mathcal{R}(x_h, \nu) = Id$ for all $\nu \in [-\delta, \delta]$. In addition, \mathcal{R} is continuous at $(x_h, 0)$.

This Theorem proves that the singular contribution when $\nu \rightarrow 0_+$ (i.e. a non C^0 solution at $\nu = 0$ and $x \rightarrow x_h$) appears only through the action of M_2^ν .

Remark 1.1. *The matrix $M_2^\nu(x_h)$ contains terms of the form $\frac{1}{\pi} \ln(x_h - X_\nu) = \frac{1}{\pi} [\ln|x_h - X_\nu| + i \arg(x_h - X_\nu)]$, hence is singular when $\nu \rightarrow 0_+$. As a consequence,*

$\begin{pmatrix} A_0 \\ B_0 \\ C \\ D \end{pmatrix}$ being given independent on ν does not ensure that (E_2, E_3, H_2, H_3) is

bounded uniformly in ν at x_h (even worse E_2, H_2, H_3 are merely in L^1). We need to impose the modified Cauchy data (1.14) in order to track, uniformly in ν , continuous

quantities, such as $\mathcal{R}(x, \nu) \begin{pmatrix} A_0 \\ B_0 \\ C \\ D \end{pmatrix}$.

This Remark leads to the following definition:

Definition 1.1. *One calls*

$$\ln(x - x_h)^+ = \ln(x - x_h), x > x_h, \ln(x_h - x) + i\pi \text{sign}(i\partial_\nu X_\nu|_{\nu=0}), x_h > x.$$

In a similar way one calls

$$\ln(x_h - x)^- = \ln(x_h - x), x < x_h, \ln(x - x_h) - i\pi \text{sign}(i\partial_\nu X_\nu|_{\nu=0}), x_h < x.$$

It will be observed in Lemma 3.7 that the limit of $\ln(x - X_\nu)$ when $\nu \rightarrow 0_+$ is $\ln(x - x_h)^+$ and the limit of $\ln(X_\nu - x)$ when $\nu \rightarrow 0_+$ is $\ln(x_h - x)^-$.

We deduce the following

Theorem 1.2. *Under the same hypotheses as in Theorem 1.1, for each $\nu > 0$ and for each (C, D) there exists $\zeta_\nu \in \mathbb{R}$ given by (3.37), $\zeta_\nu = O(\nu \ln \nu)$, $(K_\nu, e_\nu) \in \mathbb{R}^2$ given in (3.38) which limit is non zero when $\nu \rightarrow 0_+$, $\delta_\nu \in \mathbb{R}$, $\delta_\nu = O(\nu^{\frac{3}{2}})$ and two $C^0(B(x_h, \delta))$ functions R_1^ν and R_0^ν such that, for all $x \in [x_h - \delta, x_h + \delta]$, $\nu \in]0, \delta]$*

$$\varepsilon_0 E_1(x) = \frac{K_\nu(B_0 - \zeta_\nu) + \delta_\nu}{x - X_\nu} + [(B_0 - \zeta_\nu)e_\nu + R_0^\nu(x)] \ln(x - X_\nu) + R_1^\nu(x).$$

This second result allows us to use the absorption principle limit when $\nu \rightarrow 0_+$ for $(E_2, ik_2 H_3 - ik_3 H_2, ik_2 E_3 - ik_3 E_2, H_2, E_1)$. This is stated in Theorem 4.1, and this result is similar to the one obtained in [?].

This singular behavior can also being captured through the value of the electromagnetic field at a point x_1 , such that $|x_1 - x_h| \geq \frac{\delta}{2}$:

Proposition 1.1. *Let $x_1 \in [x_h + \frac{\delta}{2}, x_h + \delta]$. Assume $(A^1, B^1, C^1, D^1) \in \mathbb{C}^4$, independent of ν and assume that*

$$(1.15) \quad \begin{pmatrix} \varepsilon_0 E_2 \\ ik_2 H_3 - ik_3 H_2 \\ \varepsilon_0(ik_2 E_3 - ik_3 E_2) \\ H_3 \end{pmatrix} (x_1) = \begin{pmatrix} A^1 \\ B^1 \\ C^1 \\ D^1 \end{pmatrix}.$$

The quantities $(A_\nu, B_\nu, C_\nu, D_\nu)$ given by

$$\begin{pmatrix} A_\nu \\ B_\nu \\ C_\nu \\ D_\nu \end{pmatrix} = \begin{pmatrix} M_2^\nu(x_1)^{-1} & 0 \\ 0 & I \end{pmatrix} \mathcal{R}(x_1, \nu)^{-1} \begin{pmatrix} A^1 \\ B^1 \\ C^1 \\ D^1 \end{pmatrix}$$

have a finite limit when $\nu \rightarrow 0_+$.

One has also

Proposition 1.2. *A similar result holds for $x_1 \in [x_h - \delta, x_h - \frac{\delta}{2}]$.*

Observe that

Remark 1.2. *The behavior, in the neighborhood of x_h , of the electromagnetic field satisfying the Cauchy condition (1.15) is deduced from the given value of the field at a point x_1 at finite distance of x_h .*

The method that we develop in this paper for obtaining Theorems 1.1 and 1.2 amounts to studying, in the neighborhood of the point x_h , a system of ODEs whose coefficients have a singularity at x_h for $\nu = 0$. This type of studies dates back to Budden [?], p476 or Weyl [?] where for both authors, the problem boils down to an ODE of order 2 (the Budden problem).

We were able, in previous studies, to reduce rigorously a system of ODEs of order greater than 2 to a block-diagonal equivalent system of ODEs. For example, using the fact that the singularity at x_h can be considered as a high frequency limit through a suitable change of variable, we obtained in [?] (for the study of the instability of a detonation profile) a linearized system of the form $hU' = \Theta U$ that we studied in the high frequency regime $h \rightarrow 0$. This was performed by block-diagonalizing Θ to deduce a block-diagonal equivalent system after further conjugation. In the study of the ablation growth rate in [?], one of the key arguments was to prove that, in the neighborhood of a singularity at $-\infty$ of the same

type, one could boil down the problem to a representation using a solution of a singular ODE (namely the confluent hypergeometric equation).

In the present paper, we observe that the eigenvalues of the matrix \mathcal{M}^ν , depending on $k_2, k_3, x, \omega, \omega_\nu$, are $\pm k^s(x), \pm k^r(x)$, where $k^s(x)$ has an infinite limit for $(x, \nu) \rightarrow (x_h, 0)$, while k^r is regular in a complex neighborhood of $(x_h, 0)$. This renders the case studied here more complicated than the case studied in [?], because the eigenvector matrix is both complicated and singular.

We propose here to treat successively the regular then the singular part of the system. The regular part of the system gives $(\varepsilon_0(ik_2E_3 - ik_3E_2), H_3)$ in terms of $(\varepsilon_0E_2, ik_2H_3 - ik_3H_2)$. Plugging this relation into the singular part of the system leads to an integro-differential equation on $(\varepsilon_0E_2, ik_2H_3 - ik_3H_2)$.

We use in this paper the variation of parameters method initiated in [?], namely constructing approximate solutions with the same singular behavior as the solutions obtained in [?] but we apply this method to an **integro-differential** system, which is a powerful tool, as this method applies for example for all system of ODEs of the form (1.10). Note that the numerical analysis of the hybrid mode with a manufactured solution (see the talk of A. Nicolopoulos at WAFU, 2017 or at Waves, 2017 [?]) is presented through a toy model that may be treated also with the methods of the present paper.

The cold plasma model in the oblique incidence case appears already in [?], and the authors assert that the TM mode has **not** an accessible resonance. They prove this assertion only in the limit B_0 large, reverting to an ODE where they neglect terms small with respect to B_0 even if they may be singular at x_h (see equations (59) and (65) of [?]). Note that the case B_0 large cannot be addressed in the present paper because it would lead to a point \tilde{x}_h close to a cyclotron frequency.

In order for this paper to be self-contained, we derive this system of equations for $(\vec{E}, \vec{H}, \vec{j})$ in Section 2 and deduce from it a system of ODEs of the form (1.10). We then choose an approximate fundamental solution for studying this integro-differential system and obtain in Section 3 the representation of all solutions of the system of ODEs of order 4.

2. COLD PLASMA MODEL

We consider here the real physical problem (namely coupling the equation for the current \vec{j} with Maxwell equations in the cold plasma approximation).

2.1. Physical equations. In this section, we assume that the frequency is known, equal to ω . Indeed, all quantities shall depend on (x, ω) . The equation on the electric current \vec{j} reads:

$$(-i\omega + \nu)\vec{j} = \frac{e^2 n_0(x)}{m} \vec{E} - \frac{eB_0(x)}{m} \vec{j} \wedge \vec{e}_3,$$

that is

$$(2.1) \quad (-i\omega + \nu)\vec{j} = \omega_p^2(x)\varepsilon_0 \vec{E} - \omega_c(x)\vec{j} \wedge \vec{e}_3.$$

In previous papers ([?], [?]) one studied points of interest in the plasma, namely the cyclotron resonances (points x_c such that $\omega_c(x_c) = \omega$) and the hybrid resonances (points x_h such that $\omega_c^2(x_h) + \omega_p^2(x_h) = \omega^2$). We proved that, in the limit $\nu \rightarrow 0_+$, the electromagnetic field is not singular at x_c (hence it is not a resonance but an artificial singularity created by the dielectric tensor which links the electric current

to the electric field (see forthcoming studies [?]). The equation for the electric current is coupled with Maxwell's equations

$$(2.2) \quad \nabla \wedge \vec{E} = i\omega\mu_0\vec{H}, \nabla \wedge \vec{H} = \vec{j} - i\omega\varepsilon_0\vec{E}.$$

One considers electromagnetic waves of the form (1.7), and $(\vec{E}(x), \vec{H}(x), \vec{j}(x))$ is a solution of a system of first order differential equations in x . For convenience, we still keep the notation ∂_x for the derivative with respect to x and we omit the dependency on x in ω_p and in ω_c .

2.2. Physics revisited: cut-off points, resonances and turning points. This work gives more precise ideas on the notions of cut-off and resonances in the case of B_0 and n_0 depending only on x . Indeed, cut-off frequencies and resonances are, by definition in the Physics literature, obtained for B_0 , n_0 constants, **not taking into account the spatial dependency**. Let us observe for the moment that the cut-off points described in the Physics literature do not correspond to the cut-off points of the system of ODEs obtained after Fourier transform in (y, z) , while Physics resonances correspond to resonances also for this system of ODEs, as we see below.

Most of the textbooks in Plasma Physics use the **dielectric tensor** approach, in which the current \vec{j} is expressed in terms of \vec{E} through the relation $\vec{j} = \underline{\underline{\sigma}}\vec{E}$ ($\underline{\underline{\sigma}}$ being called the conductivity tensor). See for example Stix [?]. One thus deduces $\nabla \wedge \vec{H} = (\underline{\underline{\sigma}} - i\omega\varepsilon_0 Id)\vec{E}$. Let the dielectric tensor $\underline{\underline{\varepsilon}}$ being equal to $Id + \frac{i}{\varepsilon_0\omega}\underline{\underline{\sigma}}$ and introduce

$$S = 1 - \frac{\omega_p\omega_p^2}{\omega(\omega_p^2 - \omega_c^2)}, P = 1 - \frac{\omega_p^2}{\omega_p^2}, D = -\frac{\omega_c\omega_p^2}{\omega(\omega_p^2 - \omega_c^2)}.$$

Denoting by

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{pmatrix},$$

the system on \vec{E} writes, when \vec{E} is expressed in a basis of the form $(\vec{a}, \vec{e}_3 \wedge \vec{a}, \vec{e}_3)$, with $\vec{a} \cdot \vec{e}_3 = 0$

$$(2.3) \quad \nabla \wedge \nabla \wedge \vec{E} = \frac{\omega^2}{c^2} \begin{pmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \vec{E}.$$

Physics textbooks begin by assuming that ω_p and ω_c are independent on x . System (2.3) is a constant coefficient system, hence one can perform the Fourier transform (replacing ∇ by \vec{k}). The relation on \vec{k} for which this system has a non trivial solution yields:

$$F(n_0, B_0, k_1, k_2, k_3, \omega) := \det(\vec{k} \wedge \vec{k} \wedge \cdot + \frac{\omega^2}{c^2} \begin{pmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{pmatrix}) = 0.$$

This is called the **dispersion relation**.

Physics textbooks give the following expression: $F(n_0, B_0, k_1, k_2, k_3, \omega) = (\frac{\omega^2}{c^2}P - k_1^2)[(\frac{\omega^2}{c^2}S - |\vec{k}|^2)(\frac{\omega^2}{c^2}S - k_3^2) - (\frac{\omega^2}{c^2}D)^2] - \frac{\omega^4}{c^4}k_1^2k_3^2(\frac{\omega^2}{c^2}S - |\vec{k}|^2)$, from which one deduces the definitions of cut-off points and resonance points (see [?] for example). When n_0 and B_0 depend only on x , this dispersion relation is a relation between x, k_1, k_2, k_3, ω .

Definition 2.1. A cut-off point x^0 for the plasma characterized by $(n_0(x), B_0(x), \omega)$ is a solution of $F(n_0(x^0), B_0(x^0), 0, 0, 0, \omega) = 0$.

A resonance for the plasma characterized by $(n_0(x), B_0(x), \omega)$ is a point x_0 such that there exists a sequence $(x_0^n, k_1^n, k_2^n, k_3^n)$ satisfying $F(n_0(x_0^n), B_0(x_0^n), k_1^n, k_2^n, k_3^n, \omega) = 0$, $x_0^n \rightarrow x_0$ and $\|k_1^n, k_2^n, k_3^n\| \rightarrow +\infty$.

Note that we used a dispersion relation obtained in the case of constant coefficients to deduce definitions and properties in the case n_0, B_0 depending on x .

One then performs the Fourier transform in y, z of the system on (E, B, j) , to deduce the system of ODEs (stated below in (2.13)). We observe that the eigenvalues $k_1(x)$ solve

$$F(n_0(x), B_0(x), k_1(x), k_2, k_3, \omega) = 0.$$

For each (k_2, k_3, ω) , we call also **solution of the dispersion relation** a value of (x, k_1) such that $F(n_0(x), B_0(x), k_1, k_2, k_3, \omega) = 0$. Note that

$$D_\nu = (\omega_c^2 - \omega_\nu^2)S.$$

Definition 2.2. Let k_2, k_3, ω be given and consider the limit $\nu = 0$.

Assume x_0 is a point such that there exist (x_0^n, k_1^n) , solution of the dispersion relation, with $x_0^n \rightarrow x_0$ and $|k_1^n| \rightarrow \infty$. It is a resonance for the problem.

Assume x^0 is a point such that $(x^0, 0)$ is a solution of the dispersion relation. It is not a cut-off for the problem except when $(k_2, k_3) = (0, 0)$. Points $(x_0, 0)$ solution of the dispersion relation are **turning points** for the system of ODEs.

2.3. Derivation of the system of ordinary differential equations. Let us rewrite all the equations which are needed, in the suitable order, after using the relation $\vec{j} = \vec{j}(x)e^{ik_2y+ik_3z}$ as well:

$$(2.4) \quad \begin{cases} ik_3E_1 - \partial_x E_3 = i\omega\mu_0 H_2 \\ \partial_x E_2 - ik_2E_1 = i\omega\mu_0 H_3 \\ ik_3H_1 - \partial_x H_3 = j_2 - i\omega\varepsilon_0 E_2 \\ \partial_x H_2 - ik_2H_1 = j_3 - i\omega\varepsilon_0 E_3 \\ \\ ik_2E_3 - ik_3E_2 = i\omega\mu_0 H_1 \\ -i\omega_\nu j_3 = \varepsilon_0\omega_p^2 E_3 \\ \\ j_1 - i\omega\varepsilon_0 E_1 = ik_2H_3 - ik_3H_2 \\ -i\omega_\nu j_1 + \omega_c j_2 = \varepsilon_0\omega_p^2 E_1 \\ -\omega_c j_1 - i\omega_\nu j_2 = \varepsilon_0\omega_p^2 E_2. \end{cases}$$

Observe that only four of the nine equations exhibit a derivative in x , (H'_2, E'_2, H'_3, E'_3) . We express j_1, j_2, E_1 through the last three equations of the system (2.4), and replace the resulting quantities in the first four equations, leading to a system of ODEs on E_2, E_3, H_2, H_3 . After a straightforward but tedious calculation, one obtains, using Notation (1.8)

$$(2.5) \quad \begin{aligned} D_\nu j_1 &= \frac{\omega_\nu\omega_p^2}{\omega}(ik_2H_3 - ik_3H_2) - \omega_c\omega_p^2\varepsilon_0 E_2 \\ D_\nu j_2 &= -\frac{\omega_c\omega_p^2}{i\omega}(ik_2H_3 - ik_3H_2) - (i\omega_\nu + \frac{\omega_p^2}{i\omega})\omega_p^2\varepsilon_0 E_2 \\ D_\nu\varepsilon_0 E_1 &= -\frac{\omega_c^2 - \omega_\nu^2}{i\omega}(ik_2H_3 - ik_3H_2) - \frac{\omega_c\omega_p^2}{i\omega}\varepsilon_0 E_2. \end{aligned}$$

The singularity at X_ν for the electric current which was identified as the key tool for the study of the problem is common to the framework of this paper and

to the framework of [?]. No other singularity occurs here in the matrix, hence no other resonance is present.

Define A_ν , B_ν , C_ν through

$$(2.6) \quad A_\nu(x) = -\frac{\omega_c(x)\omega_p^2(x)}{i\omega D_\nu(x)}, B_\nu(x) = \frac{\omega_\nu^2 - \omega_c^2(x)}{i\omega D_\nu(x)}, C_\nu(x) = \frac{\omega_p^2(x)(i\omega_\nu + \frac{\omega_p^2(x)}{i\omega})}{D_\nu(x)}.$$

We deduce from (2.5) the equalities

$$(2.7) \quad \varepsilon_0 E_1 = A_\nu \varepsilon_0 E_2 + B_\nu (ik_2 H_3 - ik_3 H_2), j_2 = -C_\nu \varepsilon_0 E_2 + A_\nu (ik_2 H_3 - ik_3 H_2).$$

Lemma 2.1. *The coefficients A_ν , B_ν , C_ν are singular at $x = X_\nu$. They depend only on x, ω, ν . Moreover $(A_\nu^2 + B_\nu C_\nu) \times (x - X_\nu)$ is bounded for (x, ν) in a neighborhood \mathcal{V} of $(x_h, 0)$ in \mathbb{C}^2 .*

Introduce the values

$$a_\nu = -\frac{\omega_c \omega_p^2 k_2}{\omega D'_\nu}, b_\nu = \frac{k_2((\omega_\nu)^2 - \omega_c^2)}{\omega D'_\nu}$$

evaluated at $x = X_\nu$. The boundedness of $(x - X_\nu)((A_\nu(x))^2 + B_\nu(x)C_\nu(x))$ is a consequence of

$$-\omega_c^2 \omega_p^4 \omega^{-2} + \frac{\omega_p^2}{\omega}(\omega_\nu - \frac{\omega_p^2}{\omega})(\omega_\nu^2 - \omega_c^2) = \frac{\omega_p^2}{\omega^2}[-\omega_c^2 \omega_p^2 + (\omega \omega_\nu - \omega_p^2)(\omega_\nu^2 - \omega_c^2)],$$

equal to $\frac{\omega_p^2}{\omega^2} \omega \omega_\nu [\omega_\nu^2 - \omega_p^2 \frac{\omega_\nu}{\omega} - \omega_c^2]$, which is equal to 0 when $x = X_\nu$. The assertions of Lemma 2.1 are proven.

The system of equations (2.4) implies the

Lemma 2.2. *Let T, R being given respectively by $ik_2 H_3 - ik_3 H_2, ik_2 E_3 - ik_3 E_2$. There exists three matrices $G_{11}(x, \nu), G_{12}(x, \nu), G_{22}(x, \nu)$, depending on x, ν and the constants k_2, k_3, ω , given by (2.12) below, bounded in \mathcal{V} such that the system of ODEs deduced from (2.4) is*

$$(2.8) \quad \begin{pmatrix} \varepsilon_0 E_2 \\ T \\ \varepsilon_0 R \\ H_2 \end{pmatrix}' = \begin{pmatrix} ik_2 \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & -A_\nu \end{pmatrix} + G_{11}(x, \nu) & G_{12}(x, \nu) \\ -G_{12}(x, \nu) & G_{22}(x, \nu) \end{pmatrix} \begin{pmatrix} \varepsilon_0 E_2 \\ T \\ \varepsilon_0 R \\ H_2 \end{pmatrix}.$$

The matrix \mathcal{M}^ν is thus

$$(2.9) \quad \mathcal{M}^\nu(x) = \begin{pmatrix} ik_2 \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & -A_\nu \end{pmatrix} + G_{11}(x, \nu) & G_{12}(x, \nu) \\ -G_{12}(x, \nu) & G_{22}(x, \nu) \end{pmatrix}.$$

Proof. Using (2.7) in the first four equation of (2.4), one obtains

$$(2.10) \quad \begin{cases} \partial_x(\varepsilon_0 E_2) = ik_2 A_\nu \varepsilon_0 E_2 + ik_2 B_\nu (ik_2 H_3 - ik_3 H_2) + i\omega \mu_0 \varepsilon_0 H_3 \\ \partial_x H_3 = \frac{k_3}{\omega \mu_0} (ik_2 E_3 - ik_3 E_2) + C_\nu \varepsilon_0 E_2 - A_\nu (ik_2 H_3 - ik_3 H_2) + i\omega \varepsilon_0 E_2 \\ \partial_x(\varepsilon_0 E_3) = ik_3 A_\nu \varepsilon_0 E_2 + ik_3 B_\nu (ik_2 H_3 - ik_3 H_2) - i\omega \mu_0 \varepsilon_0 H_2 \\ \partial_x H_2 = -(i\omega + \frac{\omega_p^2}{i\omega_\nu}) \varepsilon_0 E_3 + \frac{k_2}{\omega \mu_0} (ik_2 E_3 - ik_3 E_2). \end{cases}$$

This system is equivalent to

$$(2.11) \quad \begin{cases} \partial_x(\varepsilon_0 E_2) = ik_2 A_\nu \varepsilon_0 E_2 + ik_2 B_\nu T + i \frac{\omega}{c^2} H_3 \\ \partial_x T = C_\nu ik_2 \varepsilon_0 E_2 - A_\nu ik_2 T + i\omega \varepsilon_0 ik_2 E_2 + (i\omega + \frac{\omega_p^2}{i\omega_\nu}) \varepsilon_0 ik_3 E_3 \\ \partial_x(\varepsilon_0 R) = \frac{\omega}{c^2} (k_2 H_2 + k_3 H_3) \\ \partial_x H_2 = -(i\omega + \frac{\omega_p^2}{i\omega_\nu}) \varepsilon_0 E_3 + \frac{k_2}{\omega \mu_0} R, \end{cases}$$

expressing H_3, E_3 with, respectively, T, H_2 and R, E_2 . For simplicity, denote by

$$(2.12) \quad \begin{aligned} G_{11}(x) &= \begin{pmatrix} 0 & -\frac{i\omega}{k_2^2 c^2} \\ (i\omega + \frac{\omega_p^2}{i\omega_\nu}) \frac{k_3^2}{k_2^2} + i\omega & 0 \end{pmatrix}, \\ G_{12}(x) &= \begin{pmatrix} 0 & i\omega \mu_0 \varepsilon_0 \frac{k_3}{k_2} \\ \frac{k_3}{k_2} (i\omega + \frac{\omega_p^2(x)}{i\omega_\nu}) & 0 \end{pmatrix}, \\ G_{22}(x) &= \begin{pmatrix} 0 & \frac{k_2^2 + k_3^2}{k_2} \omega \mu_0 \varepsilon_0 \\ \frac{k_2}{\omega \mu_0 \varepsilon_0} - \frac{\omega}{k_2} + \frac{\omega_p^2}{k_2 \omega_\nu} & 0 \end{pmatrix}. \end{aligned}$$

One deduces

$$(2.13) \quad \begin{cases} \partial_x \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} = [ik_2 \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & -A_\nu \end{pmatrix} + G_{11}(x)] \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} + G_{12}(x) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix}, \\ \partial_x \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} = -G_{12}(x) \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} + G_{22}(x) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix}. \end{cases}$$

This ends the proof of Lemma 2.2 hence the expression of the matrix \mathcal{M}^ν .

Introduce

$$(2.14) \quad \begin{pmatrix} a^\nu(x) & b^\nu(x) \\ c^\nu(x) & -a^\nu(x) \end{pmatrix} = (x - X_\nu) [ik_2 \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & -A_\nu \end{pmatrix} + G_{11}(x, \nu)],$$

that is

$$(2.15) \quad \begin{aligned} a^\nu(x) &= ik_2 A_\nu(x)(x - X_\nu), b^\nu(x) = ik_2(x - X_\nu) B_\nu(x) + \frac{\omega}{c^2 k_2} (x - X_\nu), \\ c^\nu(x) &= ik_2(x - X_\nu) [C_\nu + (i\omega + \frac{\omega_p^2}{i\omega_\nu}) \frac{k_3^2}{k_2^2} + i\omega]. \end{aligned}$$

The last assertion of Lemma 2.1 is equivalent to $a_\nu^2 + b_\nu c^\nu(X_\nu) = 0$. One has

Lemma 2.3. *The point x_h is the unique resonance of the oblique incidence cold plasma model for $\nu = 0$ according to Definition 2.2.*

Proof of Lemma 2.3. Assume $\nu = 0$ and $x \neq x_h$. It is equivalent to say that (x, k_1, k_2, k_3) is solution of the dispersion relation and to say that

$$\det \left[\begin{pmatrix} ik_2 \begin{pmatrix} A_0 & B_0 \\ C_0 & -A_0 \end{pmatrix} + G_{11}(x, 0) & G_{12}(x, 0) \\ -G_{12}(x, 0) & G_{22}(x, 0) \end{pmatrix} - ik_1 Id_4 \right] = 0.$$

Indeed, a solution (x, k_1, k_2, k_3) of the dispersion relation is a value of (k_1, k_2, k_3) for which the system (2.4) with $\nu = 0$ and where the coefficients ω_c and ω_p are frozen at x , ∂_x being replaced by ik_1 has a non trivial solution.

The modal system (where ∂_x is replaced by ik_1) is

$$(2.16) \quad \begin{cases} ik_3 E_1 - ik_1 E_3 = i\omega\mu_0 H_2 \\ ik_1 E_2 - ik_2 E_1 = i\omega\mu_0 H_3 \\ ik_3 H_1 - ik_1 H_3 = j_2 - i\omega\varepsilon_0 E_2 \\ ik_1 H_2 - ik_2 H_1 = j_3 - i\omega\varepsilon_0 E_3 \\ \\ ik_2 E_3 - ik_3 E_2 = i\omega\mu_0 H_1 \\ -i\omega j_3 = \varepsilon_0 \omega_p^2 E_3 \\ \\ j_1 - i\omega\varepsilon_0 E_1 = ik_2 H_3 - ik_3 H_2 \\ -i\omega j_1 + \omega_c j_2 = \varepsilon_0 \omega_p^2 E_1 \\ -\omega_c j_1 - i\omega j_2 = \varepsilon_0 \omega_p^2 E_2. \end{cases}$$

It is equivalent to

$$(2.17) \quad \begin{cases} ik_3 E_1 - ik_1 E_3 = i\omega\mu_0 H_2 \\ ik_1 E_2 - ik_2 E_1 = i\omega\mu_0 H_3 \\ ik_3 H_1 - ik_1 H_3 = j_2 - i\omega\varepsilon_0 E_2 \\ ik_1 H_2 - ik_2 H_1 = j_3 - i\omega\varepsilon_0 E_3 \\ \\ ik_2 E_3 - ik_3 E_2 = i\omega\mu_0 H_1 \\ -i\omega j_3 = \varepsilon_0 \omega_p^2 E_3 \\ \\ D_0 j_1 = \frac{\omega\omega_p^2}{\omega} (ik_2 H_3 - ik_3 H_2) - \omega_c \omega_p^2 \varepsilon_0 E_2 \\ D_0 j_2 = -\frac{\omega_c \omega_p^2}{i\omega} (ik_2 H_3 - ik_3 H_2) - (i\omega + \frac{\omega_p^2}{i\omega}) \omega_p^2 \varepsilon_0 E_2 \\ D_0 \varepsilon_0 E_1 = -\frac{\omega_c^2 - \omega^2}{i\omega} (ik_2 H_3 - ik_3 H_2) - \frac{\omega_c \omega_p^2}{i\omega} \varepsilon_0 E_2. \end{cases}$$

As $x \neq x_h$, one deduces j_1, j_2, E_1 from the last three equations, and, consequently (2.16) is equivalent to

$$(2.18) \quad \begin{cases} ik_1 \begin{pmatrix} \varepsilon_0 E_2 \\ T \\ \varepsilon_0 R \\ H_2 \end{pmatrix} = \begin{pmatrix} ik_2 \begin{pmatrix} A_0 & B_0 \\ C_0 & -A_0 \end{pmatrix} + G_{11}(x, 0) & G_{12}(x, 0) \\ -G_{12}(x, 0) & G_{22}(x, 0) \end{pmatrix} \begin{pmatrix} \varepsilon_0 E_2 \\ T \\ \varepsilon_0 R \\ H_2 \end{pmatrix} \\ ik_2 E_3 - ik_3 E_2 = i\omega\mu_0 H_1 \\ -i\omega j_3 = \varepsilon_0 \omega_p^2 E_3 \\ \\ D_0 j_1 = \frac{\omega\omega_p^2}{\omega} (ik_2 H_3 - ik_3 H_2) - \omega_c \omega_p^2 \varepsilon_0 E_2 \\ D_0 j_2 = -\frac{\omega_c \omega_p^2}{i\omega} (ik_2 H_3 - ik_3 H_2) - (i\omega + \frac{\omega_p^2}{i\omega}) \omega_p^2 \varepsilon_0 E_2 \\ D_0 \varepsilon_0 E_1 = -\frac{\omega_c^2 - \omega^2}{i\omega} (ik_2 H_3 - ik_3 H_2) - \frac{\omega_c \omega_p^2}{i\omega} \varepsilon_0 E_2. \end{cases}$$

If the system

$$ik_1 \begin{pmatrix} \varepsilon_0 E_2 \\ T \\ \varepsilon_0 R \\ H_2 \end{pmatrix} = \begin{pmatrix} ik_2 \begin{pmatrix} A_0 & B_0 \\ C_0 & -A_0 \end{pmatrix} + G_{11}(x, 0) & G_{12}(x, 0) \\ -G_{12}(x, 0) & G_{22}(x, 0) \end{pmatrix} \begin{pmatrix} \varepsilon_0 E_2 \\ T \\ \varepsilon_0 R \\ H_2 \end{pmatrix}$$

has only a trivial solution, then $(E_2, ik_2 H_3 - ik_3 H_2, H_2, ik_2 E_3 - ik_3 E_2)$ is zero. As k_3 is not zero, $E_3 = 0$ and $H_3 = 0$. As $D_0(x) \neq 0$ thanks to $x \neq x_h$, $j_1 = j_2 = 0$,

$E_1 = 0$. Finally, using $ik_2E_3 - ik_3E_2 = i\omega\mu_0H_1$, $-i\omega j_3 = \varepsilon_0\omega_p^2E_3$, $j_3 = 0$ and $H_1 = 0$, hence a trivial solution. When $k_3 \neq 0$, the non trivial solutions of (2.16) are deduced from non trivial solutions of

$$ik_1 \begin{pmatrix} \varepsilon_0 E_2 \\ T \\ \varepsilon_0 R \\ H_2 \end{pmatrix} = \begin{pmatrix} ik_2 \begin{pmatrix} A_0 & B_0 \\ C_0 & -A_0 \end{pmatrix} + G_{11}(x, 0) & G_{12}(x, 0) \\ -G_{12}(x, 0) & G_{22}(x, 0) \end{pmatrix} \begin{pmatrix} \varepsilon_0 E_2 \\ T \\ \varepsilon_0 R \\ H_2 \end{pmatrix}.$$

Resonances (as defined by Definition 2.2) are then points x^0 such that there exists a sequence (k_1^n, x_n^0) such that ik_1^n is an eigenvalue of

$$\mathcal{M}^0(x_n^0) = \begin{pmatrix} ik_2 \begin{pmatrix} A_0(x_n^0) & B_0(x_n^0) \\ C_0(x_n^0) & -A_0(x_n^0) \end{pmatrix} + G_{11}(x_n^0, 0) & G_{12}(x_n^0, 0) \\ -G_{12}(x_n^0, 0) & G_{22}(x_n^0, 0) \end{pmatrix},$$

where $x_n^0 \rightarrow x^0$ and $|k_1^n| \rightarrow +\infty$.

On the other side, values of (k_1, x) such that $P(n_0(x), B_0(x), k_1, k_2, k_3) = 0$ are given by a quadratic equation on k_1^2 which coefficients depend on (x, k_2, k_3, ω) , the coefficient of $(k_1^2)^2$ being $D_0(x)$. This expression extends to the case $\nu \neq 0$.

As a consequence, there exists two functions, smooth in \mathcal{V} , non zero at X_ν , denoted respectively by r_ν and m_ν , such that the solutions of $P(n_0(x), B_0(x), k_1, k_2, k_3) = 0$ are $\pm k^r(x) = \pm \sqrt{r_\nu(x)}$ (regular) and $\pm k^s(x) = \pm \sqrt{\frac{m_\nu(x)}{D_\nu(x)}}$ (singular at X_ν). As eigenvalues of (2.11) are for $x \neq x_h$ and $\nu = 0$ are solutions of $P = 0$, the eigenvalues of $\mathcal{M}^0(x)$ are thus, for $x \neq x_h$, equal to $\pm \sqrt{r_0(x)}, \pm \sqrt{\frac{m_0(x)}{D_0(x)}}$. Hence, for any sequence of points x_n^0 converging to x_h , the sequence $\sqrt{\frac{m_0(x_n^0)}{D_0(x_n^0)}}$ goes to infinity, hence x_h is a resonance of the problem. No other point x^0 is associated to an infinite limit of an eigenvalue, hence Lemma 2.3 is proven.

Remark 2.1. Thanks to (2.6), the functions $ik_2(x - X_\nu)A_\nu(x), ik_2(x - X_\nu)B_\nu(x)$ have a_ν, b_ν as limits when $x \rightarrow X_\nu$, hence

$$ik_2A_\nu(x) - \frac{a_\nu}{x - X_\nu}, ik_2B_\nu(x) - \frac{b_\nu}{x - X_\nu}$$

are bounded in a complex neighborhood of X_ν (excluding X_ν) and can be extended at X_ν . System (2.13) is of the form¹ (1.9) where P^ν is the constant coefficient matrix of rank 1

$$(2.19) \quad P^\nu = \begin{pmatrix} a_\nu & b_\nu & 0 & 0 \\ -a_\nu \frac{a_\nu}{b_\nu} & -a_\nu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As the eigenvalue-eigenvector decomposition does not lead to a simple behavior of the solution near X_ν , we develop a new method, treating successively the regular part and the singular part. This is done in Section 3.

¹The procedure described in [?] does not apply to this system in the neighborhood of the point X_ν , because there is no splitting of the eigenvalues in the neighborhood of this point (the unique eigenvalue of P^ν is zero). One could have used block-diagonalization techniques, separating the singular and regular eigenvalues in the neighborhood of x_h but, after inspection, the eigenvectors are extremely complicated.

3. RESOLUTION OF SYSTEM (2.8)

One observes that, in Lemma 2.2, the ordinary differential equations on $(\varepsilon_0 E_2, T)$ have coefficients A_ν, B_ν, C_ν which have a singularity at X_ν , while the ordinary differential equations on $(\varepsilon_0 R, H_2)$ have regular (uniformly bounded) coefficients. Hence one calls in the sequel **singular unknowns** the quantities $(\varepsilon_0 E_2, T)$ and **regular unknowns** the quantities $(\varepsilon_0 R, H_2)$.

3.1. Reduction to a system of two integro-differential equations. Previous work of the author([?]) use this transformation into an integro-differential system. We express $(\varepsilon_0 R, H_2)$ in terms of $(\varepsilon_0 E_2, T)$, and we use the result for obtaining an integrodifferential system on $(\varepsilon_0 E_2, T)$.

Denote by U_{x_h} the fundamental solution of

$$\partial_x \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} = G_{22}(x) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix}$$

such that $U_{x_h}(x_h) = Id$. Its expression is obtained in Lemma 5.1.

System (2.13) implies

$$(3.1) \quad \begin{pmatrix} \frac{d(\varepsilon_0 E_2)}{dx} \\ T \end{pmatrix} = [ik_2 \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & -A_\nu \end{pmatrix} + G_{11}(x)] \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} - G_{12}(x) \int_{x_h}^x (U_{x_h}(x))(U_{x_h}(y))^{-1} G_{12}(y) \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} (y) dy - G_{12}(x) U_{x_h}(x) W^0.$$

The operator U_{x_h} is explicit in the case of a toy model presented in the literature (for example [?], following closely the ideas of the slab geometry model of [?]), on which numerics are possible. Let us describe this model. One chooses, in equation (1.6), the following **effective dielectric tensor**:

$$\underline{\underline{\varepsilon}}^\nu = \begin{pmatrix} x + i\nu & -i\delta & 0 \\ i\delta & x + i\nu & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

Note that this model is suitable only for small values of x .

One thus considers (1.6) with $\omega^2 \mu_0 = 1$, $\varepsilon_0 = 1$ and the above dielectric tensor.

Introducing \vec{B} given by $i^{-1} \nabla \wedge \vec{E}$, the Maxwell equations write:

$$(3.2) \quad \begin{cases} ik_3 E_1 - \partial_x E_3 = iB_2 \\ \partial_x E_2 - ik_2 E_1 = iB_3 \\ ik_3 B_1 - \partial_x B_3 = -i(i\delta E_1 + (x + i\nu)E_2) \\ \partial_x B_2 - ik_2 B_1 = -i\gamma E_3 \\ ik_2 E_3 - ik_3 E_2 = iB_1 \\ ik_2 B_3 - ik_3 B_2 = -i((x + i\nu)E_1 - i\delta E_2) \end{cases}$$

Two relations do not involve derivatives here, hence we reduce to a system of 4 equations with 4 unknowns, that we choose to be $T = ik_2 B_3 - ik_3 B_2$, E_2, B_1, B_2 .

The system on B_1, B_2 is

$$(3.3) \quad \begin{aligned} \partial_x B_1 &= -ik_2 B_2 - ik_3 B_3 = -ik_2 B_2 - \frac{k_3}{k_2} (T + ik_3 B_2) \\ \partial_x B_2 &= ik_2 B_1 + \gamma E_3 = ik_2 B_1 - \frac{\gamma}{k_2} (ik_3 E_2 + iB_1). \end{aligned}$$

It is a constant coefficient linear system on (B_1, B_2) with source terms expressed in terms of (T, E_2) . In this simple case, U_{x_h} is explicit and one expresses B_1, B_2 in

terms of T, E_2 and of $B_1(0), B_2(0)$. (Lemma 3.1, of which the proof is left to the reader). This result leads to an integrodifferential system similar to (3.1), it is then solved using the general method.

Introduce $\theta = \sqrt{k_2^2 + k_3^2} \sqrt{\frac{\gamma}{k_2^2} - 1}$ for $\gamma > k_2^2$ (which is always true if $\gamma = 1$ because $k_2^2 + k_3^2 \leq 1$).

Lemma 3.1. *Let $\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -(\theta k_2)^{-1} & (\gamma - k_2^2)^{-1} \\ (\theta k_2)^{-1} & (k_2^2 - \gamma)^{-1} \end{pmatrix} \begin{pmatrix} B_1(0) \\ B_2(0) \end{pmatrix}$.*

The solution of (3.3) is equal to

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = C_+ e^{i\theta x} \begin{pmatrix} -\theta k_2 \\ \gamma - k_2^2 \end{pmatrix} + C_- e^{-i\theta x} \begin{pmatrix} \theta k_2 \\ \gamma - k_2^2 \end{pmatrix} \\ + \frac{1}{2} i \frac{k_3}{k_2} \int_0^x e^{i\theta(x-y)} \left(\frac{\gamma}{\theta k_2} E_2(y) + \frac{i}{\gamma - k_2^2} T(y) \right) dy \begin{pmatrix} -\theta k_2 \\ \gamma - k_2^2 \end{pmatrix} \\ + \frac{1}{2} i \frac{k_3}{k_2} \int_0^x e^{-i\theta(x-y)} \left(-\frac{\gamma}{\theta k_2} E_2(y) + \frac{i}{\gamma - k_2^2} T(y) \right) dy \begin{pmatrix} \theta k_2 \\ \gamma - k_2^2 \end{pmatrix}.$$

3.2. An approximate fundamental matrix for (3.1). In this Section, we construct an approximate fundamental matrix $M_2^\nu(x)$, from which one deduces a Volterra equation obtained for $\vec{C} = (M_2^\nu(x))^{-1} \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix}$. Recall that $(a^\nu(x), b^\nu(x), c^\nu(x))$

are given in (2.14). The fundamental matrix $M_2^\nu(x)$ that shall be used in the proof of the main result below is given by (1.12). We present its construction and show that it is associated with a C^0 kernel in the Volterra equation.

Let λ be any complex number and let γ^ν be given by (3.12).

Lemma 3.2. *Let C, S be two constants. The functions $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = C B e_1(x) + S B e_2(x)$ are solutions of the system*

$$\frac{d}{dx} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \frac{1}{x - X_\nu} \begin{pmatrix} a^\nu(x) & b^\nu(x) \\ c^\nu(x) & -a^\nu(x) \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \gamma^\nu(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}.$$

Proof of Lemma 3.2. As the second order singular system writes

$$(3.4) \quad (x - X_\nu)u' = a^\nu(x)u + b^\nu(x)v, (x - X_\nu)v' = c^\nu(x)u - a^\nu(x)v,$$

one deduces

$$(3.5) \quad v = \frac{(x - X_\nu)u' - a^\nu(x)u(x)}{b^\nu(x)}.$$

We replace into the equation for v , one obtains

$$(x - X_\nu) \frac{d}{dx} \left(\frac{(x - X_\nu)u'}{b^\nu(x)} \right) = [c^\nu(x) + \frac{(a^\nu(x))^2}{b^\nu(x)} + \frac{d}{dx} \frac{a^\nu(x)}{b^\nu(x)}] u.$$

To avoid complicatedness, assume $\Re b^\nu(X_\nu) > 0$ (this can be achieved by considering $-v$ instead of v in the system). If one writes $u = \sqrt{b^\nu(x)}U$, the equation on U is

$$(3.6) \quad (x - X_\nu) \frac{d}{dx} ((x - X_\nu)U') + (x - X_\nu)U' + k^\nu(x)U = 0$$

where

$$k^\nu(x) = \sqrt{b^\nu(x)}(x - X_\nu) \frac{d}{dx} \left(\frac{x - X_\nu}{b^\nu} \frac{d}{dx} (\sqrt{b^\nu}) \right) + b^\nu c^\nu(x) + (a^\nu(x))^2 + b^\nu \frac{d}{dx} \left(\frac{a^\nu(x)}{b^\nu(x)} \right).$$

Equation (3.6) has a regular singular point at X_ν hence an approximate solution can be expressed by means of the Bessel functions. The aim of what follows is to construct the approximate solutions of (3.6) and to perform the variation of parameters method.

For this purpose, consider \tilde{u} and U_0 given by

$$(3.7) \quad \tilde{u}(x) = \sqrt{b^\nu(x)}[CJ_0(\lambda\sqrt{x-X_\nu}) + SY_0(\lambda\sqrt{x-X_\nu})] := \sqrt{b^\nu(x)}U_0(x).$$

We have the identity

$$\frac{d}{dx}\left(\frac{(x-X_\nu)\tilde{u}'}{b^\nu(x)}\right) = \frac{d}{dx}\left(\frac{(x-X_\nu)(\sqrt{b^\nu(x)})'}{b^\nu(x)}\right)U_0 + \frac{1}{\sqrt{b^\nu(x)}}((x-X_\nu)U_0'' + U_0').$$

As U_0 solves

$$(x-X_\nu)U_0'' + U_0' = -\frac{\lambda^2}{4}U_0,$$

one deduces

$$(3.8) \quad (x-X_\nu)\frac{d}{dx}\left(\frac{(x-X_\nu)\tilde{u}'}{b^\nu(x)}\right) = (x-X_\nu)\left[\frac{d}{dx}\left(\frac{(x-X_\nu)(\sqrt{b^\nu(x)})'}{b^\nu(x)}\right) - \frac{\lambda^2}{4\sqrt{b^\nu(x)}}\right]U_0.$$

Use then the relation (3.5) to define the function \tilde{v} :

$$(3.9) \quad \tilde{v} = \frac{1}{\sqrt{b^\nu(x)}}(x-X_\nu)U_0' + \frac{(x-X_\nu)(\sqrt{b^\nu(x)})' - a^\nu(x)}{b^\nu(x)}U_0.$$

If one introduces the quantity

$$(3.10) \quad \delta^\nu(x) = \frac{(x-X_\nu)(b^\nu)'(x)}{2(b^\nu)^{\frac{3}{2}}(x)} - \frac{a^\nu(x)}{(b^\nu)^{\frac{1}{2}}(x)},$$

one checks that

$$\begin{aligned} \tilde{u}(x) &= \sqrt{b^\nu(x)}[CJ_0(\lambda\sqrt{x-X_\nu}) + SY_0(\lambda\sqrt{x-X_\nu})] \\ \tilde{v}(x) &= \delta^\nu(x)[CJ_0(\lambda\sqrt{x-X_\nu}) + SY_0(\lambda\sqrt{x-X_\nu})] \\ &\quad + (x-X_\nu)\sqrt{b^\nu(x)}\frac{\lambda}{2\sqrt{x-X_\nu}}[CJ_0'(\lambda\sqrt{x-X_\nu}) + SY_0'(\lambda\sqrt{x-X_\nu})], \end{aligned}$$

which rewrites

$$(3.11) \quad \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = CBe_1(x) + SBe_2(x),$$

where Be_1 , Be_2 are the fundamental solutions standing for the Bessel solution of the first of second kind, introduced in (1.11), where λ is to be chosen below.

Compute then

$$I = (x-X_\nu)\frac{d}{dx}(\tilde{v}(x)) + a^\nu\tilde{v}(x).$$

One has

$$\begin{aligned} I &= (x-X_\nu)\frac{d}{dx}\left(\frac{(x-X_\nu)\tilde{u}'}{b^\nu(x)} - \frac{a^\nu(x)}{b^\nu(x)}\tilde{u}\right) + a^\nu\left(\frac{(x-X_\nu)\tilde{u}'}{b^\nu(x)} - \frac{a^\nu(x)}{b^\nu(x)}\tilde{u}\right) \\ &= (x-X_\nu)\frac{d}{dx}\left(\frac{(x-X_\nu)\tilde{u}'}{b^\nu(x)}\right) - [(x-X_\nu)\left(\frac{a^\nu(x)}{b^\nu(x)}\right)' + \frac{(a^\nu(x))^2}{b^\nu(x)}]\tilde{u}. \end{aligned}$$

Use (3.8) to deduce

$$I = (x-X_\nu)\left[\frac{d}{dx}\left(\frac{(x-X_\nu)(\sqrt{b^\nu(x)})'}{b^\nu(x)}\right) - \sqrt{b^\nu(x)}\left(\frac{a^\nu(x)}{b^\nu(x)}\right)' - \frac{\lambda^2}{4\sqrt{b^\nu(x)}}\right]U_0 - \frac{(a^\nu(x))^2}{b^\nu(x)}\sqrt{b^\nu(x)}U_0.$$

Observe then that the relation $((a^\nu)^2 + b^\nu c^\nu)(X_\nu) = 0$ implies that there exists $\beta^\nu(x)$ such that

$$c^\nu(x) + \frac{(a^\nu(x))^2}{b^\nu(x)} = (x - X_\nu)\beta^\nu(x).$$

Define

(3.12)

$$\gamma^\nu(x) = \frac{1}{\sqrt{b^\nu(x)}} \left[\frac{d}{dx} \left(\frac{(x - X_\nu)(\sqrt{b^\nu(x)})'}{b^\nu(x)} \right) - \sqrt{b^\nu(x)} \left(\frac{a^\nu(x)}{b^\nu(x)} \right)' - \frac{\lambda^2}{4\sqrt{b^\nu(x)}} \right] - \beta^\nu(x).$$

One has $I = (x - X_\nu)\gamma^\nu(x)\sqrt{b^\nu(x)}U_0(x)$, hence $\frac{d\tilde{v}}{dx} + \frac{a^\nu\tilde{v} - c^\nu\tilde{u}}{x - X_\nu} = \gamma^\nu\tilde{u}$. Along with (3.7) and (3.9), this proves Lemma 3.2.

Consider now

$$(3.13) \quad r(x) = \frac{\gamma^\nu(x)}{x - X_\nu}$$

One has thus the two identities:

$$(3.14) \quad \begin{aligned} \frac{d\tilde{u}}{dx} &= \frac{a^\nu(x)\tilde{u} + b^\nu(x)\tilde{v}}{x - X_\nu} \\ \frac{d\tilde{v}}{dx} &= \frac{c^\nu(x)\tilde{u} - a^\nu(x)\tilde{v}}{x - X_\nu} + (x - X_\nu)r(x)\tilde{u}(x). \end{aligned}$$

It is then fundamental to observe

Lemma 3.3. *Let λ_ν being given by (3.15) below. There exists a constant Λ such that for all $|\nu| \leq \Lambda$ the function r defined by (3.13) is continuous in the complex neighborhood $B(x_h, \delta)$.*

The function r has a finite limit when $x \rightarrow X_\nu$ (by taking the limit in the complex plane) if and only if $\lambda = \lambda_\nu$ such that $\gamma^\nu(X_\nu) = 0$, that is

$$(3.15) \quad \frac{\lambda_\nu^2}{4} = \frac{(b^\nu)'}{2b^\nu}(X_\nu) - \left(\frac{a^\nu}{b^\nu} \right)'(X_\nu) - \lim_{x \rightarrow X_\nu} \frac{b^\nu c^\nu + (a^\nu)^2}{x - X_\nu}.$$

One then deduces that r is smooth. This ends the proof of Lemma 3.3.

From now on, we will omit the subscript ν in λ_ν except when considering the limit $\nu \rightarrow 0$.

Using (3.11) and (2.14), one observes that the ODE satisfied by Be_1 is

(3.16)

$$(Be_1)'(x) - [ik_2 \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & -A_\nu \end{pmatrix} + G_{11}(x)]Be_1(x) = (x - X_\nu)r(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} Be_1(x).$$

Observe that Be_2 satisfies also (3.16). One deduces that $M_2^\nu = \begin{pmatrix} Be_1(x) & Be_2(x) \end{pmatrix}$ satisfies

$$(M_2^\nu)'(x) - [ik_2 \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & -A_\nu \end{pmatrix} + G_{11}(x)]M_2^\nu(x) = (x - X_\nu)r(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_2^\nu(x).$$

Remark 3.1. *A simple fundamental solution associated with the reduced system*

$$\frac{d}{dx} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{x - X_\nu} \begin{pmatrix} a_\nu & b_\nu \\ -\frac{(a_\nu)^2}{b_\nu} & -a_\nu \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

$$\text{is } M_1^\nu(x) = \begin{pmatrix} b_\nu \ln(x - X_\nu) & b_\nu \\ 1 - a_\nu \ln(x - X_\nu) & -a_\nu \end{pmatrix}.$$

This matrix is used in the computation of manufactured solutions of the cold plasma equations (see [?], [?]), and it is another approximate solution of (3.4).

Let Θ_ν^* be the matrix

$$(3.17) \quad ik_2 \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & -A_\nu \end{pmatrix} - \frac{1}{x - X_\nu} \begin{pmatrix} a_\nu & b_\nu \\ -\frac{(a_\nu)^2}{b_\nu} & -a_\nu \end{pmatrix} + G_{11}(x) = \Theta_\nu^*(x).$$

It is bounded in the neighborhood of X_ν , uniformly in ν . System (3.4) rewrites

$$\frac{d}{dx} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{x - X_\nu} \begin{pmatrix} a_\nu & b_\nu \\ -\frac{(a_\nu)^2}{b_\nu} & -a_\nu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \Theta_\nu^*(x) \begin{pmatrix} u \\ v \end{pmatrix}.$$

From the identity

$$(M_1^\nu)'(x) = \frac{1}{x - X_\nu} \begin{pmatrix} a_\nu & b_\nu \\ -\frac{(a_\nu)^2}{b_\nu} & -a_\nu \end{pmatrix} M_1^\nu(x),$$

M_1^ν is an approximate solution for (3.4).

The additional feature of the present paper is that we replace Θ_ν^* by $(x - X_\nu)r(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ which, in addition to being bounded, is 0 at the singular point.

3.3. Contraction argument. Define the integral operator \mathcal{T}^ν on $(C^1(\mathbb{R}))^2$

$$(3.18) \quad \begin{aligned} \mathcal{T}^\nu(f)(x) &= \int_{x_h}^x (y - X_\nu)(M_2(y))^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_2(y)r(y)f(y)dy \\ &\quad - \int_{x_h}^x (M_2(y))^{-1}G_{12}(y)U_{x_h}(y) \int_{x_h}^y (U_{x_h}(z))^{-1}G_{12}(z)M_2(z)f(z)dzdy. \end{aligned}$$

A key ingredient of our study is the following contraction result:

Proposition 3.1. *Let $|\nu| \leq \Lambda$, Λ small enough.*

There exists $\delta > 0$ such that \mathcal{T}^ν is a contraction on $(C^1([x_h - \delta, x_h + \delta]))^2$, with a constant independent on ν .

Proof of Proposition 3.1. We observe that the entries of

$$(x - X_\nu)(M_2^\nu(y))^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_2^\nu(y)$$

are continuous functions for $(x, \nu) \in [x_h - \delta, x_h + \delta] \times [-\delta, \delta]$, (the most singular terms which appear here are $(x - X_\nu)(\ln(x - X_\nu))^2$). The contraction lemma is thus a classical result for the first part of the operator (involving r). The C^1 behavior of the solutions comes from the fact that, uniformly in $\nu \geq 0$, $f \rightarrow \int_{x_h}^x k^\nu(y)f(y)dy$ maps C^1 to C^1 when one considers a function k^ν which is continuous for $(x, \nu) \in V(x_h, 0)$. It is classical to get that the derivative of $T(f) := \int_{x_h}^x k^\nu(y)f(y)dy$ is $(T(f))'(x) = k^\nu(x)f(x)$ hence $(T(f))'$ is continuous.

One has

Lemma 3.4. *There exists $\delta > 0$, such that for all $p, p_1 > 1$ there exists a constant C_* with, for $x \in [x_h - \delta, x_h + \delta]$*

$$|\mathcal{T}^\nu(x)| \leq C_* |x - x_h|^{\frac{1}{p}} (|x_h - X_\nu| + |x - x_h|^{\frac{1}{p_1}}).$$

Proof of Lemma 3.4. Assume f in C^0 . The first estimate deals with $\int_{x_h}^x (y - X_\nu) \ln(y - X_\nu) M(y) f(y) dy$, because one reduces the problem to such integrals. Note that

$$\int_{x_h}^x (y - X_\nu) \ln(y - X_\nu) f(y) dy = \int_{x_h}^x (y - x_h) \ln(y - X_\nu) f(y) dy + (x_h - X_\nu) \int_{x_h}^x \ln(y - X_\nu) f(y) dy.$$

Using the Holder inequality, it is thus bounded by

$$C \|f\|_\infty \left[\left(\int_{x_h}^x |y - x_h|^p dy \right)^{\frac{1}{p}} + |x_h - X_\nu| \left(\int_{x_h}^x dy \right)^{\frac{1}{p}} \right] \leq C' |x_h - x|^{\frac{1}{p}} [|x_h - X_\nu| + |x_h - x|].$$

The second term of the estimate is the estimate of the contribution of the regular term. It deals with integrals of the form

$$\int_{x_h}^x \ln(y - X_\nu) \int_{x_h}^y \ln(z - X_\nu) dz dy := J.$$

Using the Holder inequality for (p, q) such that $\frac{1}{p} + \frac{1}{q} = 1$, one has

$$\left| \int_{x_h}^y \ln(z - X_\nu) dz \right| \leq \left(\int_{x_h}^y |\ln(z - X_\nu)|^q dz \right)^{\frac{1}{q}} \left(\int_{x_h}^y 1^p dz \right)^{\frac{1}{p}} \leq C |y - x_h|^{\frac{1}{p}},$$

hence one gets

$$\left| \int_{x_h}^x \ln(y - X_\nu) \int_{x_h}^y \ln(z - X_\nu) dz dy \right| \leq C \left| \int_{x_h}^x |\ln(y - X_\nu)| |y - x_h|^{\frac{1}{p}} dy \right|.$$

Applying again the Holder inequality with (p_1, q_1) such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$ and integrating, we obtain the estimate, thanks to $\int_{x_h}^x (y - x_h)^{\frac{p_1-1}{p}} dy = (1 + \frac{p_1}{p})^{-1} (x - x_h)^{1 + \frac{p_1}{p}}$ for $x_h < x$ and a similar equality for $x < x_h$:

$$\left| \int_{x_h}^x \ln(y - X_\nu) \int_{x_h}^y \ln(z - X_\nu) dz dy \right| \leq C C_1 |x - x_h|^{\frac{1}{p} + \frac{1}{p_1}}.$$

Combining the two inequalities, for all $p, p_1 > 1$ there exists a constant C_* such that

$$C_* |x - x_h|^{\frac{1}{p}} [|x - x_h|^{\frac{1}{p_1}} + |x_h - X_\nu|]$$

Note that the the entries of matrix used for constructing \mathcal{T}^ν , id est the continuity of $(x - X_\nu) \ln(y - X_\nu)$, allows us to consider C^1 solutions of $(I - \mathcal{T}^\nu)^{-1}(f_0)$. **It is an improvement of the Volterra operator that one would obtain with the manufactured solution** when f_0 is a function belonging to C^1 locally in the neighborhood of x_h , the coupling term being automatically in C^1 thanks to the C^0 behavior of M_2^ν .

As the point x_h is a singular point for the system when $\nu = 0$, even if the system has analytic coefficients for $x \in [x_h - \delta, x_h + \delta]$ for all $\nu > 0$, one constructs in this paper another Volterra operator \mathcal{T}_*^ν , which corresponds to selecting the values of $(\varepsilon_0 E_2, T)$ at $x^* \neq x_h$ in the variation of parameters method. This is done in the next subsection.

3.4. Variation of parameters methods. The aim of this Section is to prove the following Proposition. Let us fix a point x^* , $|x^* - x_h| \geq \frac{\delta}{4}$. Using the contraction \mathcal{T}_*^ν defined by (3.19) (similar to \mathcal{T}^ν), the inverse of $(Id - \mathcal{T}_*^\nu)$ being $\sum_{n \geq 0} (\mathcal{T}_*^\nu)^n$, one can find all the components of the electromagnetic field for $\nu > 0$ in terms of $(\varepsilon_0 R, H_2)$ evaluated at x_h and $(\varepsilon_0 E_2, T)$ evaluated at x^* .

One deduces that $(M_2^\nu(x_h))^{-1} \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} (x_h)$ is bounded for $0 < \nu < \delta$ and has a finite limit when $\nu \rightarrow 0_+$, even if $(M_2^\nu(x_h))^{-1}$ is singular when $\nu \rightarrow 0_+$.

This leads to the unique solution of the system of ODEs of Lemma 2.2,

$$(M_2^\nu(x))^{-1} \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix}$$

being given at x_h .

Consider $x^* \in B(x_h, \delta)$ and define

(3.19)

$$\begin{aligned} \mathcal{T}_*^\nu(f)(x) &= \int_{x^*}^x (y - X_\nu)(M_2(y))^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_2(y) r(y) f(y) dy \\ &\quad - \int_{x^*}^x (M_2(y))^{-1} G_{12}(y) U_{x_h}(y) \int_{x_h}^y (U_{x_h}(z))^{-1} G_{12}(z) M_2(z) f(z) dz dy. \end{aligned}$$

The operator \mathcal{T}_*^ν is also a contraction on $C^1([x_h - \delta, x_h + \delta])$ for $|\nu| \leq \Lambda$. From now on, we replace Λ and δ by $\min(\Lambda, \delta)$ and one denotes it by δ .

Proposition 3.2. *Denote by $(A^*, B^*)^T = (M_2^\nu(x^*))^{-1}((\varepsilon_0 E_2(x^*), T(x^*))^T$. Assume $(\varepsilon_0 R, H_2)$ is given at x_h .*

(1) *The singular unknowns in the system of Lemma 2.2 are given by*

$$\begin{aligned} \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} &= M_2^\nu(x)(I - \mathcal{T}_*^\nu)^{-1} \left(\begin{pmatrix} A^* \\ B^* \end{pmatrix} \right)(x) \\ &\quad - M_2^\nu(x)(I - \mathcal{T}_*^\nu)^{-1} \left(\int_{x^*}^x (M_2^\nu)^{-1}(z) G_{12}(z) U_{x_h}(z) dz \right) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h). \end{aligned}$$

(2) *The regular unknowns in the system of Lemma 2.2 are then given by*

$$\begin{aligned} \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} &= U_{x_h}(x) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h) \\ &\quad - U_{x_h}(x) \int_{x_h}^x (U_{x_h})^{-1}(y) M_2^\nu(y) (I - \mathcal{T}_*^\nu)^{-1} dy \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \\ &\quad - U_{x_h}(x) \int_{x_h}^x ((U_{x_h})^{-1} G_{12} M_2^\nu)(y) (I - \mathcal{T}_*^\nu)^{-1} \int_{x^*}^y ((M_2^\nu)^{-1} G_{12} U_{x_h})(z) dz dy \left(\begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h) \right) (z) dz (x). \end{aligned}$$

Proof. One uses the variation of parameters on the approximate fundamental solutions. Introduce

$$(3.20) \quad \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} (x) = M_2^\nu(x) \begin{pmatrix} A(x) \\ B(x) \end{pmatrix}.$$

The first step is to obtain

(3.21)

$$\begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x) = U_{x_h}(x) \left[\begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h) - \int_{x_h}^x (U_{x_h})^{-1} G_{12} M_2^\nu(y) \begin{pmatrix} A \\ B \end{pmatrix} (y) dy \right].$$

Introduce $\Theta^\nu(x) = (x - X_\nu) r(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

The system on (A, B) becomes:

$$\begin{pmatrix} A'(x) \\ B'(x) \end{pmatrix} = (M_2^\nu(x))^{-1} \Theta^\nu(x) M_2^\nu(x) \begin{pmatrix} A(x) \\ B(x) \end{pmatrix} \\ - (M_2^\nu(x))^{-1} G_{12}(x) \int_{x_h}^x (U_{x_h}(x))(U_{x_h}(y))^{-1} G_{12}(y) M_2^\nu(y) \begin{pmatrix} A(y) \\ B(y) \end{pmatrix} dy \\ - (M_2^\nu(x))^{-1} G_{12}(x) U_{x_h}(x) \begin{pmatrix} \varepsilon_0 R(x_h) \\ H_2(x_h) \end{pmatrix}.$$

Using the operator \mathcal{T}_*^ν defined above, one obtains

$$\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = \begin{pmatrix} A^* \\ B^* \end{pmatrix} + \mathcal{T}_*^\nu \left(\begin{pmatrix} A \\ B \end{pmatrix} \right) (x) - \int_{x^*}^x ((M_2^\nu)^{-1} G_{12} U_{x_h})(z) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h) dz.$$

The operator \mathcal{T}_*^ν is a contraction, hence this system has a unique solution in $C^1([x_h - \delta, x_h + \delta])$ given by

$$\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = (Id - \mathcal{T}_*^\nu)^{-1} \left[\begin{pmatrix} A^* \\ B^* \end{pmatrix} - \int_{x^*}^x ((M_2^\nu)^{-1} G_{12} U_{x_h})(z) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h) dz \right] (x),$$

hence obtaining

$$(3.22) \quad \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} (x) = M_2^\nu(x) [(Id - \mathcal{T}_*^\nu)^{-1} \left[\begin{pmatrix} A^* \\ B^* \end{pmatrix} - \int_{x^*}^x (M(z))^{-1} G_{12}(z) U_{x_h}(z) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h) dz \right] (x)]$$

for $x \in [x_h - \delta, x_h + \delta]$.

The expression of $\begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix}$ is obtained by replacing (3.22) in (5.2). This ends the proof of Proposition 3.2.

We then deduce the unique solution with given data at x^* for $(\varepsilon_0 E_2, T)$ and given data at x_h for $(\varepsilon_0 R, H_2)$. This is expressed in the following fundamental Lemma:

Lemma 3.5. *Let $(\varepsilon_0 R(x_h), H_2(x_h)) = (C, D)$ be given. Define*

$$\begin{pmatrix} A^* \\ B^* \end{pmatrix} = (M_2^\nu(x^*))^{-1} \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} (x^*)$$

and denote by $(\varepsilon_0 E_2, T, \varepsilon_0 R, H_2)$ be the unique solution constructed through Proposition 3.2.

i) *The value $\begin{pmatrix} A_\nu \\ B_\nu \end{pmatrix} = (M_2^\nu(x_h))^{-1} \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} (x_h)$ exist for all $\nu \neq 0$ and has a finite limit when $\nu \rightarrow 0_+$.*

ii) *For all $(A_0, B_0) \in \mathbb{C}^2$, there exists $(A^*, B^*) \in \mathbb{C}^2$ such that $(\varepsilon_0 E_2(x^*), T(x^*))^T = M_2^\nu(x^*)(A^*, B^*)^T$ imply that the limit, when $\nu \rightarrow 0_+$, of $(M_2^\nu(x_h))^{-1} \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} (x_h)$ is (A_0, B_0) (uniform continuity with respect to initial conditions for the problem on $(A, B)^T$ for $\nu \geq 0$).*

Proof. We have the equality

$$\begin{pmatrix} A_\nu \\ B_\nu \end{pmatrix} = (Id - \mathcal{T}_*^\nu)^{-1} \left[\begin{pmatrix} A^* \\ B^* \end{pmatrix} - \int_{x^*}^{x_h} (M_2^\nu(z))^{-1} G_{12}(z) U_{x_h}(z) \begin{pmatrix} C \\ D \end{pmatrix} dz \right] (x_h).$$

Use Lemma 3.7, item i), below.

The limit, when $\nu \rightarrow 0_+$, in L^1 of $\ln(y - X_\nu)$ being $\ln(y - x_h)^+$ and the limit when $\nu \rightarrow 0_+$ in L^1 of $\ln(X_\nu - y)$ being $\ln(x_h - x)^-$ one may compute the limit of the matrix M_2^ν when $\nu \rightarrow 0_+$. In this limit, three types of terms appear: the function $\ln(x - X_\nu)$, the coefficients b^ν, a^ν, δ^ν , and the functions J_0, T_0, R_0, Z_0 . As these

four functions are analytic in z^2 , the limit when $\nu \rightarrow 0$ is expressed in terms of $z^2 = \lambda_0^2(x - x_h)$. The limit of $(b^\nu, a^\nu, \delta^\nu)$ is smooth and yields (b^0, a^0, δ^0) . Call then $M_2^{0,+}$ the limit of M_2^ν when $\nu \rightarrow 0_+$ in $L^1([x_h - \delta, x_h + \delta])$. One deduces the limit of the operator T_*^ν when $\nu \rightarrow 0_+$. It can be denoted by $T_*^{0,+}$. It is a continuous operator from $C^1([x_h - \delta, x_h + \delta])$ to itself. We consider then the limit $\nu \rightarrow 0_+$ in the equality at the beginning of this proof to deduce the limit of (A_ν, B_ν) in terms of A^*, B^*, C, D , called (A_0, B_0) .

Remark 3.2. Let $\ln(x - x_h)^- = \overline{\ln(x - x_h)^+}$ and notice that this is $\lim_{\nu \rightarrow 0_-} \ln(x - X_\nu)$. The same limit is performed when $\nu \rightarrow 0_-$ and one replaces $\ln(x - x_h)^+$ by $\ln(x - x_h)^-$ as well as $\ln(x_h - x)^-$ by $\ln(x_h - x)^+$.

The second item comes from a careful study of this relation, solving the system on (A^*, B^*) , (A_ν, B_ν) being given. Moreover, as the limit when $\nu \rightarrow 0_+$ of this relation is

$$(Id - \mathcal{T}_*^{0,+}) \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} A^* \\ B^* \end{pmatrix} - \int_{x^*}^{x_h} (M_2^{0,+}(z))^{-1} G_{12}(z) U_{x_h}(z) \begin{pmatrix} C \\ D \end{pmatrix} dz,$$

one obtains (A_0, B_0) . As there exists C_0 independent of ν such that $|\mathcal{T}^\nu(f)(x_h)|$ and $|\int_{x^*}^{x_h} (M_2^{0,+}(z))^{-1} G_{12}(z) U_{x_h}(z) \begin{pmatrix} C \\ D \end{pmatrix} dz|$ are bounded by $C_0|x^* - x_h|$, for $C_0|x^* - x_h| < 1$, one deduces (A^*, B^*) depending on ν in terms of (A_ν, B_ν, C, D) .

Proof of Theorem 1.1. Lemma 3.5, item ii) allows us to use, uniformly in $\nu > 0$, this argument to construct the unique solution using the operator \mathcal{T}^ν on $C^1([x_h - \delta, x_h + \delta])$ defined by (3.18) to rewrite Proposition 3.2 as well as Lemma 3.5. to obtain the expression of the solution of (1.10): Consider $(A_0, B_0, C, D) \in \mathbb{C}^4$, and assume $(\varepsilon_0 R, H_2)(x_h) = (C, D)$ and, for $\nu > 0$,

$$(M_2^\nu)^{-1}(x_h) \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} (x_h) = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}.$$

One obtains, through Proposition 3.2 and through the second item of Lemma 3.5, which gives (A^*, B^*) (values of $M_2^\nu(x^*)(\varepsilon_0 E_2, T)^T(x^*)$),

$$\begin{aligned} \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} (x) &= M_2^\nu(x) (I - \mathcal{T}_*^\nu)^{-1} \left(\begin{pmatrix} A^* \\ B^* \end{pmatrix} \right) (x) \\ &\quad - M_2^\nu(x) (I - \mathcal{T}_*^\nu)^{-1} \left(\int_{x_h}^x (M_2^\nu)^{-1}(z) G_{12}(z) U_{x_h}(z) dz \right) (x) \begin{pmatrix} C \\ D \end{pmatrix}. \end{aligned}$$

The regular unknowns are then given by

$$\begin{aligned} (3.23) \quad \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} &= U_{x_h}(x) \begin{pmatrix} C \\ D \end{pmatrix} - U_{x_h}(x) \int_{x_h}^x ((U_{x_h})^{-1} G_{12} M_2^\nu)(y) \begin{pmatrix} A \\ B \end{pmatrix} (y) dy \\ &= U_{x_h}(x) \begin{pmatrix} C \\ D \end{pmatrix} - U_{x_h}(x) \int_{x_h}^x (U_{x_h})^{-1}(y) (I - \mathcal{T}_*^\nu)^{-1} M_2^\nu(y) dy \begin{pmatrix} A^* \\ B^* \end{pmatrix} \\ &\quad - U_{x_h}(x) \int_{x_h}^x ((U_{x_h})^{-1} G_{12} M_2^\nu)(y) (I - \mathcal{T}_*^\nu)^{-1} \left(\int_{x_h}^y ((M_2^\nu)^{-1} G_{12}(z) U_{x_h}(z)) dz \right) dy \begin{pmatrix} C \\ D \end{pmatrix}. \end{aligned}$$

Once this solution is well defined for all $\nu > 0$, revert to expression (3.21). The system on $(A, B)^T$ yields (by integrating from x_h to x instead of integrating from x_* to x and by using $(A, B)^T(x_h) = (A_0, B_0)$):

$$\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} + \mathcal{T}^\nu \left(\begin{pmatrix} A \\ B \end{pmatrix} \right) (x) - \int_{x_h}^x ((M_2^\nu)^{-1} G_{12} U_{x_h})(z) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h) dz.$$

One then deduces the equality

$$\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = (I - \mathcal{T}^\nu)^{-1}(x) \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} - (I - \mathcal{T}^\nu)^{-1} \left[\int_{x_h}^x ((M_2^\nu)^{-1} G_{12} U_{x_h})(z) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h) dz \right] (x).$$

This allows us to introduce the matrices

$$\mathcal{R}_{11}(x, \nu) = (I - \mathcal{T}^\nu)^{-1}(x), \mathcal{R}_{12}(x, \nu) = -(I - \mathcal{T}^\nu)^{-1} \left(\int_{x_h}^x (M_2^\nu)^{-1}(z) G_{12}(z) U_{x_h}(z) dz \right) (x),$$

$$\mathcal{R}_{21}(x, \nu) = -U_{x_h}(x) \int_{x_h}^x (U_{x_h})^{-1}(y) (I - \mathcal{T}^\nu)^{-1} M_2^\nu(y) dy,$$

$$\mathcal{R}_{22}(x, \nu) = U_{x_h}(x) \left(Id - \int_{x_h}^x ((U_{x_h})^{-1} G_{12} M_2^\nu)(y) (I - \mathcal{T}^\nu)^{-1} \left(\int_{x_h}^x ((M_2^\nu)^{-1} G_{12}(z) U_{x_h}(z)) dz \right) dy \right).$$

These are C^1 2×2 matrices on the (complex) ball $B(x_h, \delta)$. One has

$$\begin{pmatrix} \varepsilon_0 E_2 \\ T \\ \varepsilon_0 R \\ H_2 \end{pmatrix} (x) = \begin{pmatrix} M_2^\nu & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \\ C \\ D \end{pmatrix}.$$

Denoting by \mathcal{R} the matrix $\begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{pmatrix}$, this ends the proof of Theorem 1.1.

Proposition 1.1 follows. Let $\begin{pmatrix} A^1 \\ B^1 \\ C^1 \\ D^1 \end{pmatrix}$ be given independent of ν , and let

$$\begin{pmatrix} A_0^\nu \\ B_0^\nu \\ C^\nu \\ D^\nu \end{pmatrix} = \mathcal{R}(x_1, \nu)^{-1} \begin{pmatrix} (M_2^\nu(x_1))^{-1} & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} A^1 \\ B^1 \\ C^1 \\ D^1 \end{pmatrix}.$$

We observe that the limit of $M_2^\nu(x_1)$ and of $\mathcal{R}(x_1, \nu)$ exist when $\nu \rightarrow 0_+$, hence the result.

3.5. Proof of Theorem 1.2. The Bessel functions J_0 and Y_0 have to be studied carefully. One knows that J_0 and T_0 are even analytical functions, hence J'_0 and T'_0 are odd hence there exists Z_0, R_0, H_0 , even analytical functions such that $J'_0(z) = zZ_0(z)$ and $T'_0(z) = zR_0(z)$, $J_0(z) - 1 = z^2 H_0(z)$. Note that $J_0(0) = 1$, $Z_0(0) = -\frac{1}{2}$, $T_0(0) = \frac{\gamma}{2\pi}$, $R_0(0) = \frac{\gamma-1}{\pi}$, where γ is the Euler constant. One has the equality, for $z^2 = \lambda_\nu^2(x - X_\nu)$

$$z(A^* J'_0(z) + B^* (Y_0^*)'(z)) = \frac{1}{\pi} B^* J_0(z) + z^2 \left[\frac{1}{\pi} B^* Z_0(z) \ln(x - X_\nu) + A^* Z_0(z) + B^* R_0(z) \right].$$

In all what follows, remember that all the functions J_0, T_0, R_0, Z_0 are expressed as analytic even sequences $\sum_{p \geq 0} a_p z^{2p}$ and are evaluated at $z^2 = \lambda_\nu^2(x - X_\nu)$. To simplify notations, we will denote these quantities suppressing the argument thereafter. We use the function $Y_0^*(\lambda_\nu, \sqrt{x - X_\nu})$ and modify the analytical functions accordingly.

Our aim is to compute the leading order term of $(\varepsilon_0 E_2, T, \varepsilon R, H_2)$ and, finally, of E_1

through the third equality of (2.5), that is $-i\omega D_\nu(x)\varepsilon_0 E_1 = \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} \cdot \begin{pmatrix} \omega_c \omega_p^2 \\ \omega_c^2 - \omega_\nu^2 \end{pmatrix}$. As $\begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix}$ is given by (3.20), one deduces that

$$-i\omega D_\nu(x)\varepsilon_0 E_1 = \begin{pmatrix} A \\ B \end{pmatrix} \cdot (M_2^\nu(x))^T \begin{pmatrix} \omega_c \omega_p^2 \\ \omega_c^2 - \omega_\nu^2 \end{pmatrix}.$$

We begin with the following Lemma

Lemma 3.6. *Let $\psi(x, \nu)$, analytic in a neighborhood of $(x_h, 0)$. One has the identity*

$$\begin{aligned} \int_{x_h}^x \psi(y, \nu) \ln(y - X_\nu) dy &= \left(\int_{X_\nu}^x \psi(y, \nu) dy \right) \ln(x - X_\nu) \\ &+ \int_{x_h}^x \int_0^1 \partial_x \psi(X_\nu + t(y - X_\nu)) dt dy \\ &- \ln(x_h - X_\nu) \int_{x_h}^{X_\nu} \psi(y, \nu) dy. \end{aligned}$$

This Lemma ensures that all the integrals that occur in our analysis are of the form $\phi_\nu(x) \ln(x - X_\nu) + \mu_\nu(x)$. We may start the estimates. In all what follows, we keep, for simplicity, the notation λ instead of λ_ν given in (3.15). Notice that $\det M_2^\nu(y) = \pi^{-1}$. Introduce $M_2^\nu(z)$, $\hat{M}_2^\nu(z)$, which contain only terms which are regular in z , uniformly for $\nu \geq 0$:

$$\begin{aligned} M_2^\nu(z) &= \frac{1}{\pi} \ln(z - X_\nu) \begin{pmatrix} 0 & \sqrt{b^\nu(z)} J_0 \\ 0 & \delta^\nu(z) J_0 + \frac{\lambda^2}{2\sqrt{b^\nu(z)}} (z - X_\nu) Z_0 \end{pmatrix} + \tilde{M}_2^\nu(z), \\ (M_2^\nu(z))^{-1} &= \ln(z - X_\nu) \begin{pmatrix} \delta^\nu(z) J_0 + \frac{\lambda^2}{2\sqrt{b^\nu(z)}} (z - X_\nu) Z_0 & -\sqrt{b^\nu(z)} J_0 \\ 0 & 0 \end{pmatrix} + \hat{M}_2^\nu(z), \end{aligned}$$

One needs, in addition, the regular term of M_2^ν for the evaluation of E_1 :

$$\tilde{M}_2^\nu(x) = \begin{pmatrix} \sqrt{b^\nu} J_0 & \sqrt{b^\nu} T_0 \\ \delta^\nu J_0 + \frac{\lambda^2(x - X_\nu) Z_0}{2\sqrt{b^\nu}} & \delta^\nu T_0 + \frac{\lambda^2(x - X_\nu) R_0}{2\sqrt{b^\nu}} + \frac{J_0}{\pi\sqrt{b^\nu}} \end{pmatrix}.$$

From (2.6), one deduces

$$\frac{b^\nu(x) \omega_c \omega_p^2 - a^\nu(x) (\omega_c^2 - \omega_\nu^2)}{x - X_\nu} = (B_\nu(x) - i \frac{\omega}{k_2 c^2}) \omega_c \omega_p^2 - A_\nu(x) (\omega_c^2 - \omega_\nu^2) = -i \frac{\omega}{k_2 c^2} \omega_c \omega_p^2,$$

which implies, using (3.10), that there exists ζ^ν , smooth on $B(x_h, \delta)$ such that

$$\omega_c \omega_p^2 \sqrt{b^\nu} + (\omega_c^2 - \omega_\nu^2) \delta^\nu(x) = (x - X_\nu) [(\omega_c^2 - \omega_\nu^2) \frac{b_\nu'(x)}{2b_\nu^{\frac{3}{2}}(x)} - i \frac{\omega \omega_c \omega_p^2}{k_2 c^2 b_\nu^{\frac{1}{2}}(x)}] = (x - X_\nu) \zeta^\nu(x).$$

Denote by $V^\nu(x)$ the vector

$$(3.25) \quad V^\nu(x) := \begin{pmatrix} v_1^\nu(x) \\ v_2^\nu(x) \end{pmatrix} = \zeta^\nu(x) \begin{pmatrix} J_0 \\ T_0 \end{pmatrix} + \frac{\lambda^2(\omega_c^2 - \omega_\nu^2)}{2\sqrt{b^\nu(x)}} \begin{pmatrix} Z_0 \\ R_0 \end{pmatrix}.$$

One has the identity

$$\begin{aligned} (3.26) \quad (M_2^\nu(x))^T \begin{pmatrix} \omega_c \omega_p^2 \\ \omega_c^2 - \omega_\nu^2 \end{pmatrix} &= \frac{\omega_c^2 - \omega_\nu^2}{\pi \sqrt{b^\nu(x)}} J_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &+ (x - X_\nu) [V^\nu(x) + \frac{1}{\pi} \ln(x - X_\nu) v_1^\nu(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}]. \end{aligned}$$

The second preliminary calculation concerns the term $(M_2^\nu(y))^{-1}G_{12}(y)$. Define

$$(3.27) \quad \alpha^\nu(x) = -\sqrt{b^\nu(z)}\left(1 - \frac{\omega_p^2}{\omega\omega_\nu}\right)J_0, \beta^\nu(x) = \varepsilon_0\mu_0(\delta^\nu(z)J_0 + \frac{\lambda^2}{2\sqrt{b^\nu(z)}}(z - X_\nu)Z_0).$$

One has

$$(3.28) \quad (M_2^\nu(x))^{-1}G_{12}(x) = \hat{M}_2^\nu(x)G_{12}(x) + i\omega\frac{k_3}{k_2}\ln(x - X_\nu)\begin{pmatrix} \alpha^\nu(x) & \beta^\nu(x) \\ 0 & 0 \end{pmatrix},$$

hence introducing the regular matrix Υ such that $U_{x_h}(x) - I = (x - x_h)\Upsilon(x)$

$$\begin{aligned} \int_{x_h}^x ((M_2^\nu)^{-1}G_{12}U_{x_h})(y)dy &= \int_{x_h}^x [(\hat{M}_2^\nu G_{12}U_{x_h})(y)dy \\ &\quad + i\omega\frac{k_3}{k_2}\int_{x_h}^x \ln(y - X_\nu)dy \begin{pmatrix} \alpha^\nu & \beta^\nu \\ 0 & 0 \end{pmatrix}(x_h) \\ &\quad + i\omega\frac{k_3}{k_2}\int_{x_h}^x (y - x_h)\ln(y - X_\nu) \begin{pmatrix} \frac{\alpha^\nu(y) - \alpha^\nu(x_h)}{y - x_h} & \frac{\beta^\nu(y) - \beta^\nu(x_h)}{y - x_h} \\ 0 & 0 \end{pmatrix} dy \\ &\quad + i\omega\frac{k_3}{k_2}\int_{x_h}^x (y - x_h)\ln(y - X_\nu) \begin{pmatrix} \alpha^\nu & \beta^\nu \\ 0 & 0 \end{pmatrix} \Upsilon(y)dy]. \end{aligned}$$

Let

$$\epsilon_a^\nu(x) = \frac{1}{2}\int_{x_h}^x (y - x_h)\ln(y - X_\nu) \left[\begin{pmatrix} \frac{\alpha^\nu(y) - \alpha^\nu(x_h)}{y - x_h} & \frac{\beta^\nu(y) - \beta^\nu(x_h)}{y - x_h} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha^\nu & \beta^\nu \\ 0 & 0 \end{pmatrix} \Upsilon(y) \right] dy.$$

It satisfies the inequality, for $0 \leq \nu \leq \delta$ and $x \in [x_h - \delta, x_h + \delta]$, $|\epsilon_a^\nu(x)| \leq C|x - x_h|^{\frac{3}{2}}$.

Let us define the matrix $m^\nu(x)$, analytic, satisfying $m^\nu(x_h) = 0$:

$$m^\nu(x) = \int_{x_h}^x \hat{M}_2^\nu(x)G_{12}(y)U_{x_h}(y)dy.$$

One has

$$(3.29) \quad \begin{aligned} \int_{x_h}^x (M_2^\nu(y))^{-1}G_{12}(y)U_{x_h}(y)dy &= i\omega\frac{k_3}{k_2}\int_{x_h}^x \ln(y - X_\nu)dy \begin{pmatrix} \alpha^\nu & \beta^\nu \\ 0 & 0 \end{pmatrix}(x_h) \\ &\quad + m^\nu(x) + 2i\omega\frac{k_3}{k_2}\epsilon_a^\nu(x). \end{aligned}$$

The inequality on ϵ_a^ν comes from the Cauchy-Schwartz inequality applied to $\int_{x_h}^x (y - x_h)\ln(y - X_\nu)G_\nu(y)dy$, through

$$\begin{aligned} \int_{x_h}^x |y - x_h||\ln(y - X_\nu)|dy &\leq \left(\int_{x_h}^x |y - x_h|^2 dy\right)^{\frac{1}{2}} \left(\int_{x_h}^x (\ln(y - X_\nu))^2 dy\right)^{\frac{1}{2}} \\ &= 3^{-\frac{1}{2}}|x - x_h|^{\frac{3}{2}} \left(\int_{x_h}^x (\ln(y - X_\nu))^2 dy\right)^{\frac{1}{2}}. \end{aligned}$$

Let $S^\nu = C\alpha^\nu(x_h) + D\beta^\nu(x_h)$. The equality (3.29) yields

$$(3.30) \quad \begin{aligned} \int_{x_h}^x ((M_2^\nu)^{-1}G_{12}U_{x_h})(y)dy \begin{pmatrix} C \\ D \end{pmatrix} &= i\omega\frac{k_3}{k_2}S^\nu \int_{x_h}^x \ln(y - X_\nu)dy \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\quad + [m^\nu(x) + \epsilon_a^\nu(x)] \begin{pmatrix} C \\ D \end{pmatrix}. \end{aligned}$$

Remark that Lemma 3.6 allows us to rewrite (3.29) as

$$(3.31) \quad \int_{x_h}^x (M_2^\nu(y))^{-1}G_{12}(y)U_{x_h}(y)dy = (x - X_\nu)\ln(x - X_\nu) \begin{pmatrix} \tilde{A}^\nu(x) \\ 0 \end{pmatrix} + \tilde{B}^\nu(x),$$

where the vector \tilde{A}^ν and the matrix \tilde{B}^ν are analytic in x (and no longer analytic in ν). In means in particular that all the terms containing $\ln(x - X_\nu)$ have been

collected in the matrix $(x - X_\nu) \ln(x - X_\nu) \begin{pmatrix} \tilde{A}^\nu(x) \\ 0 \end{pmatrix}$. One observes that, however, there exists an analytic function K_ν such that

$$\tilde{B}^\nu(X_\nu) - i\omega \frac{k_3}{k_2} \ln(x - X_\nu) \int_{x_h}^{X_\nu} \begin{pmatrix} \alpha^\nu(y) & \beta^\nu(y) \\ 0 & 0 \end{pmatrix} U_{x_h}(y) dy = \int_{x_h}^x K_\nu(y) dy. \quad (3.32)$$

Recall that

$$\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} - \int_{x_h}^x (M_2^\nu(z))^{-1} G_{12}(z) U_{x_h}(z) \begin{pmatrix} C \\ D \end{pmatrix} dz - \mathcal{T}^\nu \begin{pmatrix} A(x) \\ B(x) \end{pmatrix}.$$

For simplicity, introduce $\varepsilon_1^\nu(x), \varepsilon_2^\nu(x), \varepsilon_3^\nu(x)$ such that

$$\begin{cases} \varepsilon_1^\nu(x) = m^\nu(x) \begin{pmatrix} C \\ D \end{pmatrix}, \\ \varepsilon_2^\nu(x) = \mathcal{T}^\nu \begin{pmatrix} A(x) \\ B(x) \end{pmatrix} + \epsilon_a^\nu(x) \begin{pmatrix} C \\ D \end{pmatrix} := \varepsilon_3^\nu(x) + \epsilon_a^\nu(x) \begin{pmatrix} C \\ D \end{pmatrix}. \end{cases}$$

One finally obtains

$$(3.33) \quad \begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} - i\omega \frac{k_3}{k_2} S^\nu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \int_{x_h}^x \ln(z - X_\nu) dz - (\varepsilon_1^\nu + \varepsilon_2^\nu)(x),$$

with the estimates, in $[x_h - \delta, x_h + \delta]$ uniformly in $\nu \geq 0$

$$|\varepsilon_1^\nu| \leq C|x - x_h|, \frac{\varepsilon_2^\nu(x)}{x - x_h} \text{ belongs to } C^0([x_h - \delta, x_h + \delta]), |\varepsilon_3^\nu(x)| \leq C|x - x_h|^{\frac{3}{2}}.$$

Use $M_2^\nu(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tilde{M}_2^\nu(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. One deduces

$$\begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} = \frac{1}{\pi} \ln(x - X_\nu) B_0 \begin{pmatrix} \sqrt{b^\nu(x)} J_0 \\ \delta^\nu(x) J_0 + \frac{\lambda^2}{2\sqrt{b^\nu(x)}} (x - X_\nu) Z_0 \end{pmatrix} \\ + \tilde{M}_2^\nu(x) \left[\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} - i\omega \frac{k_3}{k_2} S^\nu \int_{x_h}^x \ln(z - X_\nu) dz \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] - M_2^\nu(x) (\varepsilon_1^\nu + \varepsilon_2^\nu)(x).$$

We have a more precise estimate. Using $\tilde{A}^\nu, \tilde{B}^\nu$, define

$$s^\nu(x) = B_0 \begin{pmatrix} \sqrt{b^\nu(x)} J_0 \\ \delta^\nu(x) J_0 + \frac{\lambda^2}{2\sqrt{b^\nu(x)}} (x - X_\nu) Z_0 \end{pmatrix} \\ - \left[\begin{pmatrix} 0 & \sqrt{b^\nu(x)} J_0 \\ 0 & \delta^\nu(x) J_0 + \frac{\lambda^2}{2\sqrt{b^\nu(x)}} (x - X_\nu) Z_0 \end{pmatrix} \tilde{B}^\nu + \pi(x - X_\nu) \tilde{M}_2^\nu(x) \begin{pmatrix} \tilde{A}^\nu \\ 0 \end{pmatrix} \right] \begin{pmatrix} C \\ D \end{pmatrix}.$$

There exists a vector r^ν , analytic in x , such that

$$(3.34) \quad \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} = \frac{1}{\pi} \ln(x - X_\nu) s^\nu(x) + r^\nu(x) - M_2^\nu(x) \epsilon_3^\nu(x).$$

Indeed,

$$\begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} = \frac{1}{\pi} \ln(x - X_\nu) \left[B_0 \begin{pmatrix} \sqrt{b^\nu(x)} J_0 \\ \delta^\nu(x) J_0 + \frac{\lambda^2}{2\sqrt{b^\nu(x)}} (x - X_\nu) Z_0 \end{pmatrix} - \tilde{B}^\nu(x) \begin{pmatrix} C \\ D \end{pmatrix} \right] \\ + \tilde{M}_2^\nu \left[\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} - \tilde{B}^\nu(x) \begin{pmatrix} C \\ D \end{pmatrix} \right] - M_2^\nu \mathcal{T}^\nu \begin{pmatrix} A \\ B \end{pmatrix} (x).$$

The expression of s^ν and of ε_3^ν lead to Equality (3.34). The estimate

$$(3.35) \quad \left| \left(\frac{\varepsilon_0 E_2}{T} \right) - \frac{1}{\pi} s^\nu(X_\nu) \ln(x - X_\nu) - r^\nu(X_\nu) \right| \leq C |x - X_\nu|^{\frac{1}{2}}$$

follows.

Note that

$$s^\nu(X_\nu) = B_0 \begin{pmatrix} \sqrt{b^\nu(X_\nu)} \\ \delta^\nu(X_\nu) \end{pmatrix} - \begin{pmatrix} 0 & \sqrt{b^\nu(X_\nu)} \\ 0 & \delta^\nu(X_\nu) \end{pmatrix} \tilde{B}^\nu(X_\nu) \begin{pmatrix} C \\ D \end{pmatrix}.$$

Observe then that $\begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} = U_{x_h}(x) \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix}(x_h) - U_{x_h}(x) \int_{x_h}^x U_{x_h}^{-1} G_{12}(y) \begin{pmatrix} \varepsilon_0 E_2 \\ T \end{pmatrix} dy$, hence for all $\alpha < 1$, one has

$$(3.36) \quad \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix}(x_h) + O(|x - x_h|^\alpha).$$

Calculation of the super singular term. One is then left with the estimation of the 'super singular term', namely E_1 because one needs to divide by $D_\nu(x)$. For this purpose, we plug (3.33) in (3.26) and we notice that

$$\left[\begin{pmatrix} \tilde{A}^\nu(x) \\ 0 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \right] \cdot \frac{\omega_c^2 - \omega_\nu^2}{\pi \sqrt{b^\nu(x)}} J_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

Hence

$$\begin{aligned} -i\omega D_\nu \varepsilon_0 E_1 &= \frac{\omega_c^2 - \omega_\nu^2}{\pi \sqrt{b^\nu(x)}} [J_0 \cdot B_0 - J_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \tilde{B}_\nu(x) \begin{pmatrix} C \\ D \end{pmatrix}] \\ &\quad + (x - X_\nu) [V^\nu \cdot \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} + \frac{1}{\pi} v_1^\nu(x) \ln(x - X_\nu) B_0] \\ &\quad - (x - X_\nu) \tilde{B}^\nu \begin{pmatrix} C \\ D \end{pmatrix} \cdot [V^\nu(x) + \frac{1}{\pi} \ln(x - X_\nu) v_1^\nu(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}] \\ &\quad - (x - X_\nu)^2 \ln(x - X_\nu) \begin{pmatrix} \tilde{A}^\nu(x) \\ 0 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \cdot [V^\nu(x) + \frac{1}{\pi} \ln(x - X_\nu) v_1^\nu(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}], \\ &\quad - [\frac{\omega_c^2 - \omega_\nu^2}{\pi \sqrt{b^\nu(x)}} J_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (x - X_\nu) (V^\nu(x) + \frac{1}{\pi} \ln(x - X_\nu) v_1^\nu(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix})] \mathcal{T}^\nu \begin{pmatrix} A \\ B \end{pmatrix} (x) \end{aligned}$$

Define $D_1^\nu(x) := \frac{D_\nu(x)}{x - X_\nu} = \int_0^1 D'_\nu(X_\nu + t(x - X_\nu)) dt$ and the constants

$$(3.37) \quad \zeta_\nu = \tilde{B}^\nu(X_\nu) \begin{pmatrix} C \\ D \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$(3.38) \quad K_\nu = \frac{\omega_c^2(X_\nu) - \omega_\nu^2}{\pi D'_\nu(X_\nu) \sqrt{b^\nu(X_\nu)}}, e_\nu = \frac{1}{\pi} (D'_\nu(X_\nu))^{-1} (\zeta_\nu(X_\nu) \frac{\gamma}{2\pi} + \frac{\lambda^2(\omega_c^2 - \omega_\nu^2)}{2\sqrt{b^\nu(X_\nu)}}).$$

One used that $e_\nu = \frac{1}{\pi} (D'_\nu(X_\nu))^{-1} v_1^\nu(X_\nu)$, where v_1^ν is given by (3.25) and that K_ν is the limit, when $x \rightarrow X_\nu$, of $(D_1^\nu(x))^{-1} \cdot \frac{\omega_c^2 - \omega_\nu^2}{\pi \sqrt{b^\nu(x)}}$.

Introduce

$$\epsilon^\nu(x) = [\frac{\omega_c^2 - \omega_\nu^2}{\pi \sqrt{b^\nu(x)}} J_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (x - X_\nu) (V^\nu(x) + \frac{1}{\pi} \ln(x - X_\nu) v_1^\nu(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix})] \mathcal{T}^\nu \begin{pmatrix} A \\ B \end{pmatrix} (x).$$

Introduce $\delta_\nu = \epsilon^\nu(X_\nu)$. One has $\delta_\nu = O(|X_\nu - x_h|^{\frac{3}{2}})$. One checks that $\zeta_\nu = O(\nu \ln \nu)$ thanks to (3.32) and $\int_{x_h}^{X_\nu} K_\nu(y) dy = O(\nu)$. There exist an analytic function $R_0^\nu(x)$ such that $R_0^\nu(X_\nu) = 0$ and a C^0 function $R_1^\nu(x)$ (in which one collects

all the terms of order at least $O(|x - X_\nu|^{\frac{1}{2}})$ such that

$$(3.39) \quad -i\omega\varepsilon_0 E_1 = \frac{K_\nu(B_0 - \zeta_\nu) + \delta_\nu}{x - X_\nu} + \frac{1}{\pi}[e_\nu(B_0 - \zeta_\nu) + R_0^\nu(x)] \ln(x - X_\nu) + R_1^\nu(x).$$

This equality is a consequence of $J_0.B_0 - J_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \tilde{B}_\nu(x) \begin{pmatrix} C \\ D \end{pmatrix} (X_\nu) = B_0 - \zeta_\nu$ and of the existence of \tilde{R}_{00}^ν , continuous, such that

$$\frac{\omega_c^2 - \omega_\nu^2}{\pi\sqrt{b^\nu(x)}D_1^\nu(x)}[J_0.B_0 - J_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \tilde{B}_\nu(x) \begin{pmatrix} C \\ D \end{pmatrix}] - K_\nu(B_0 - \zeta_\nu) = (x - X_\nu)\tilde{R}_{00}^\nu(x).$$

Collecting all the terms containing $\ln(x - X_\nu)$, and observing that the leading order term is associated with $v_1^\nu(x)$, given by (3.25), with $v_1^\nu(X_\nu) = \zeta^\nu(X_\nu)\frac{\gamma}{2\pi} + \frac{\lambda^2(\omega_c^2 - \omega_\nu^2)}{2\sqrt{b^\nu(X_\nu)}}$, the value of its coefficient at X_ν being also $B_0 - \zeta_\nu$, one denotes by R_0^ν what is left in the coefficient of $\ln(x - X_\nu)$. One has finally $\frac{\epsilon^\nu(x) - \epsilon^\nu(X_\nu)}{x - X_\nu}$ bounded and continuous, hence is added to the term $R_1^\nu(x)$. Theorem 1.2 is proven.

In the next paragraph, we study the limit of the terms appearing in all the quantities.

Evaluation of the singular integrals. We have the two following Lemmas

Lemma 3.7. *i) The limit in $L^1([x_h - \delta, x_h + \delta])$ of $\ln(x - X_\nu)$ is the function $\ln(x - x_h)$, $x > x_h$, $\ln(x_h - x) + i\pi \text{sign}(i\partial_\nu X_\nu|_{\nu=0})$, $x < x_h$.*

This limit is denoted by $\ln(x - x_h)^+$, $x \neq x_h$ (as in [?]).

ii) One has, in \mathcal{D}' ,

$$\lim_{\nu \rightarrow 0^+} \frac{1}{x - X_\nu} = P.V.(\frac{1}{x - x_h}) - i \text{sign}(i\partial_\nu X_\nu|_{\nu=0})\delta_{x_h}.$$

iii) The limit in \mathcal{D}' of $\frac{\int_{x_h}^x \ln(y - X_\nu) dy}{x - X_\nu}$ is $\ln(x - x_h)^+$. This limit is valid in $L^1([x_h - \delta, x_h + \delta])$.

iv) One has $\lim_{\nu \rightarrow 0^+} \frac{\int_{x_h}^x (y - X_\nu) \ln(y - X_\nu) dy}{x - X_\nu} = O(|x - x_h|^{\frac{1}{p}})$ for all $p > 1$.

Observe in addition that $\mathcal{T}^\nu(f)(x) - \mathcal{T}^\nu(f)(X_\nu) = O(|X_\nu - x_h|^{\frac{1}{p}})$ for all $p > 1$. Proof of Lemma 3.7. Item i) is a consequence of the fact that

$$X_\nu = x_h + i\nu \int_0^1 (-i\partial_\nu X_{\nu\tau} d\tau).$$

Hence, for $x < x_h$, $x - X_\nu = (x_h - x)[-1 + i\frac{\nu}{x_h - x} \int_0^1 (i\partial_\nu X_{\nu\tau} d\tau)]$, hence the result. Item ii) is a classical result of distribution theory, using

$$\frac{1}{x - X_\nu} = \frac{x - \Re X_\nu + i\Im X_\nu}{(x - \Re X_\nu)^2 + (\Im(X_\nu))^2}$$

and using the change of variable $x - \Re X_\nu = |\Im X_\nu|t$ in the integral $< \frac{1}{x - X_\nu}, \psi >$, which leads, with $\psi(x + \Re X_\nu) = \psi_p(x) + x\psi_{imp}(x)$, to the expression $\int_{\mathbb{R}} \frac{x^2 \psi_{imp}(x)}{x^2 + (\Im \nu)^2} dx + \int_{\mathbb{R}} \frac{|\Im X_\nu| |\Im X_\nu| \psi_p(t|\Im X_\nu|)}{|\Im X_\nu|^2 (1+t^2)} dt$, with $\psi_p(0) \rightarrow \psi(x_h)$.

Let us prove the point iii). The derivative of $(y - X_\nu) \ln(y - X_\nu)$ with respect to y is $\ln(y - X_\nu) + 1$, hence one has the identity

$$\ln(x - X_\nu) = \frac{(x_h - X_\nu) \ln(x_h - X_\nu)}{x - X_\nu} + \frac{x - x_h}{x - X_\nu} + \frac{\int_{x_h}^x \ln(y - X_\nu) dy}{x - X_\nu}.$$

The distribution $\frac{1}{x - X_\nu}$ converges to $P.V.(\frac{1}{x - x_h}) + i s \delta_{x_h}$ (s given above), hence, as the sequence of numbers $(x_h - X_\nu) \ln(x_h - X_\nu)$ converges to 0, the first term converges to 0.

The function $\frac{x - x_h}{x - X_\nu}$ converges to 1 in the distribution sense (product of $(x - x_h)$ and of $\frac{1}{x - X_\nu}$, which limit is $P.V.(\frac{1}{x - x_h}) - i \pi \text{sign}(\partial_x \omega_h(x_h)) \delta_{x_h}$).

The point iv) comes from the identity

$$\begin{aligned} \frac{\int_{x_h}^x (y - X_\nu) \ln(y - X_\nu) dy}{x - X_\nu} &= \frac{1}{2} (x - X_\nu) \ln(x - X_\nu) - \frac{1}{2} \frac{(x_h - X_\nu) \ln(x_h - X_\nu)}{x - X_\nu} \\ &\quad - \frac{1}{4} \frac{(x - x_h)(x - x_h - 2X_\nu)}{x - X_\nu}, \end{aligned}$$

hence

$$\lim_{\nu \rightarrow 0+} \frac{\int_{x_h}^x (y - X_\nu) \ln(y - X_\nu) dy}{x - X_\nu} = \frac{1}{2} (x - x_h) \ln(x - x_h)^+ + \frac{1}{4} (x - x_h),$$

hence the result.

4. EXPRESSION OF THE ELECTROMAGNETIC FIELD IN THE NEIGHBORHOOD OF THE HYBRID RESONANCE

Assume that, in the vacuum (which is a region with no surrounding magnetic field and no plasma hence $\omega_c = \omega_p = 0$, assumed to be $x > C$, C large enough), the electromagnetic wave is a plane wave

$$(\vec{E}_0, \vec{B}_0) e^{ik_2 y + ik_3 z + i \sqrt{\frac{\omega^2}{c^2} - k_2^2 - k_3^2} x},$$

with $k_2 \neq 0$ (oblique incidence, not parallel to the direction of the background magnetic field).

Deduce from the Cauchy problem on (E_2, E_3, H_2, H_3) the electromagnetic field

$$\begin{pmatrix} \varepsilon_0 E_2 \\ ik_2 H_3 - ik_3 H_2 \\ \varepsilon_0 (ik_2 E_3 - ik_3 E_2) \\ H_2 \end{pmatrix} \text{ at } x_1 \in [x_h + \delta/2, x_h + \delta], \text{ and assume it is independent}$$

on ν (this is not true and one should perform the proof with a kernel, smooth uniformly in $\nu \geq 0$ when the wave reaches the boundary of the plasma but we skip this step, taking into account the linearity of the equations). Introduce

$$\begin{pmatrix} A_0^\nu \\ B_0^\nu \\ C^\nu \\ D^\nu \end{pmatrix} = \mathcal{R}(x_1, \nu)^{-1} \begin{pmatrix} (M_2^\nu(x_1))^{-1} & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} \varepsilon_0 E_2 \\ ik_2 H_3 - ik_3 H_2 \\ \varepsilon_0 (ik_2 E_3 - ik_3 E_2)(x_1) \\ H_2 \end{pmatrix},$$

which limit at $\nu \rightarrow 0+$ is equal to $\begin{pmatrix} A_0 \\ B_0 \\ C \\ D \end{pmatrix}$. Observe moreover that K_ν and e_ν have a limit when $\nu \rightarrow 0$.

Theorem 4.1. • One has, in $C^0([x_h - \delta, x_h + \delta])$,

$$\lim_{\nu \rightarrow 0_+} \begin{pmatrix} \varepsilon_0(ik_2 E_3 - ik_3 E_2) \\ H_2 \end{pmatrix} (x) = \begin{pmatrix} \varepsilon_0 R \\ H_2 \end{pmatrix} (x_h) + o(|x - x_h|^{\frac{1}{2}}).$$

• One has, in $L^1([x_h - \delta, x_h + \delta])$,

$$\lim_{\nu \rightarrow 0_+} \begin{pmatrix} \varepsilon_0 E_2 \\ (ik_2 H_3 - ik_3 H_2) \end{pmatrix} (x) = \frac{1}{\pi} B_0 \begin{pmatrix} \sqrt{b^0(x_h)} \\ \delta^0(x_h) \end{pmatrix} \ln(x - x_h)^+ + o(1).$$

• One has, in $\mathcal{D}'([x_h - \delta, x_h + \delta])$,

$$\lim_{\nu \rightarrow 0_+} (\varepsilon_0 E_1) = \frac{iB_0}{\omega} [K_0(P.V.(\frac{1}{x - x_h}) - i\pi \text{sign}(\partial_x \omega_h(x_h))\delta_{x_h}) + \frac{2}{\pi} e_0 \ln(x - x_h)^+] + O(1),$$

where $O(1)$ is a $C^0([x_h - \delta, x_h + \delta])$ function.

The proof of this theorem uses Proposition 1.2 as well as the results proven in Section 3 of [?] (Theorem 1) where such limits are thoroughly studied in detail. The limits studied in Lemma 3.7 are used on (3.34) and on (3.36) for (E_2, H_3, E_3, H_2) (and H_1 thanks to (2.4)) to obtain the first limits.

The limit of the expression of $\varepsilon_0 E_1$ obtained in Theorem 1.2 is calculated using item ii) of Lemma 3.7.

This ends the proof of Theorem 4.1.

Observe that $b_0 = \sqrt{-\frac{k_2(\omega_p(x_h))^2}{\omega D_0^2(x_h)}}$, $K_0 = k_2^{-1} \sqrt{b_0}$, $\delta^0(x_h) = -\omega_c(x_h) \sqrt{b_0}$. Denote by $P_1^0 = \frac{iB_0}{\omega} K_0 = \frac{iB_0 \sqrt{b_0}}{\omega k_2}$.

Remark 4.1. There exists $f \in L^1([x_h - \delta, x_h + \delta])$ and $g \in (C^0([x_h - \delta, x_h + \delta]))^2$ such that

$$\lim_{\nu \rightarrow 0_+} (\varepsilon_0 E_1) = P_1^0(P.V.(\frac{1}{x - x_h}) - i\pi \text{sign}(\partial_x \omega_h(x_h))\delta_{x_h}) + f$$

and

$$\lim_{\nu \rightarrow 0_+} \begin{pmatrix} \varepsilon_0 E_2 \\ (ik_2 H_3 - ik_3 H_2) \end{pmatrix} (x) = ik_2 P_1^0 \begin{pmatrix} -1 \\ \omega_c(x_h) \end{pmatrix} \ln(x - x_h)^+ + g.$$

When one captures the principal term (distribution of order 1) in $\varepsilon_0 E_1$, one deduces the principal term (element of L^1) of $E_2, ik_2 H_3 - ik_3 H_2$.

5. ANNEX

The aim of this Section is to write explicitly the fundamental matrix for the system of ODE on $\varepsilon_0 R, T$.

Consider $a_{ij}(x)$ such that

$$G_{22}(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}.$$

Even though in the system considered, $a_{ii}(x) = 0$, we keep the generality in order to show how to construct a fundamental matrix associated with $U' = G_{22}(x)U$.

In all what follows, we shall denote by x^* a point in the interval $[x_h - \delta, x_h + \delta]$, where δ is a small constant. We introduce the following notations:

Definition 5.1. Let \mathcal{K} be the integral operator

$$\mathcal{K}(u)(x) = e^{\int_{x^*}^x a_{22}(z)dz} \int_{x^*}^x u(y) e^{-\int_{x^*}^y a_{22}(z)dz} dy$$

In addition, define $\mathcal{V}(u)$ the inverse Volterra operator given by the unique² solution of

$$\mathcal{V}(u)(x) = u(x) + e^{\int_{x^*}^x a_{11}(s)ds} \int_{x^*}^x e^{-\int_{x^*}^y a_{11}(s)ds} a_{12}(y) \mathcal{K}(a_{21} \mathcal{V}(u))(y) dy.$$

Introduce the functions

$$(5.1) \quad \begin{cases} F = R(x^*) e^{\int_{x^*}^x a_{11}(s)ds} + H_2(x^*) e^{\int_{x^*}^x a_{11}(s)ds} \int_{x^*}^x e^{-\int_{x^*}^y a_{11}(s)ds} a_{12}(y) e^{\int_{x^*}^y a_{22}(s)ds} dy \\ G_1 = \int_{x^*}^x e^{\int_{x^*}^y a_{11}(s)ds} f_1(y) dy \\ G_2 = \int_{x^*}^x e^{\int_{x^*}^y a_{11}(s)ds} [a_{12}(y) \mathcal{K}(f_2)(y)] dy \end{cases}$$

We rely on the following

Lemma 5.1. Denote by U_{x^*} the fundamental solution (which is a matrix) of the system $U' = G_{22}(x)U$ with the Cauchy data $U_{x^*}(x^*) = Id$. Denote by U_1, V_1, U_2, V_2 the entries of U_{x^*} , such that

$$U_{x^*} = \begin{pmatrix} U_1 & V_1 \\ U_2 & V_2 \end{pmatrix}.$$

- (1) Let u_0 be given by $W_1^0 + W_2^0 \int_{x^*}^x a_{12}(y) e^{-\int_{x^*}^y a_{22}(z)dz} dy$. The components U_i, V_i are given by

$$\mathcal{V}(u_0) e^{\int_{x^*}^x a_{11}(y)dy} = W_1^0 U_1(x) + W_2^0 U_2(x),$$

$$\mathcal{K}(a_{21}(W_1^0 U_1 + W_2^0 U_2)) + W_2^0 e^{\int_{x^*}^x a_{22}(y)dy} = W_1^0 V_1(x) + W_2^0 V_2(x).$$

- (2) The system $\frac{dW}{dx} = G_{22}(x)W + f$, such that $W(x^*) = W^0$ complex number given, has a unique solution

$$(5.2) \quad W(x) = U_{x^*}(x)W^0 + \int_{x^*}^x (U_{x^*}(x))(U_{x^*}(y))^{-1} f(y) dy.$$

- (3) The expression of $W = (\varepsilon_0 R, H_2)^T$ is given as follows in terms of $f = (f_1, f_2)^T$:

$$(5.3) \quad \begin{cases} \varepsilon_0 R(x) = \mathcal{V}(F + G_1 + G_2) \\ H_2(x) = H_2(x^*) \exp(\int_{x^*}^x a_{22}(z)dz) + \mathcal{K}(a_{21}R + f_2)(x) \end{cases}$$

LAGA, INSTITUT GALILÉE, UNIVERSITÉ PARIS 13, SORBONNE PARIS CITÉ 99, AVENUE J.B. CLÉMENT, 93430 VILLETANEUSE, FRANCE

²Note that this relation imposes $\mathcal{V}(u)(x^*) = u(x^*)$