SOME OPERATOR IDEAL PROPERTIES OF VOLTERRA OPERATORS ON BERGMAN AND BLOCH SPACES

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JOELLE JREIS PASCAL LEFÈVRE *

ABSTRACT. We characterize the integration operators V_g with symbol g for which V_g acts as an absolutely summing operator on weighted Bloch spaces $\mathcal{B}^\beta$ and on weighted Bergman spaces $\mathcal{A}^p_\alpha$. We show that V_g is r-summing on $\mathcal{A}^p_\alpha$, $1 \le p < \infty$, if and only if g belongs to a suitable Besov space. We also show that there is no non trivial nuclear Volterra operators V_g on Bloch spaces and on Bergman spaces.

Keywords: r-summing operators; Nuclear operators; Volterra operators; Bergman spaces; Bloch spaces.

1. INTRODUCTION

The aim of this note is to investigate absolutely summing and nuclear operators of Volterra type operator defined on certain classical spaces of analytic functions on the open unit disk $\mathbb{D}$ of the complex plane.
For an analytic function g on the unit disk, we recall that the integration operator V_g is formally defined by:

$$V_g(f)(z) = \int_0^z f(w)\, g'(w)\, dw, \qquad z \in \mathbb{D},$$

acting on the space $\mathcal{H}(\mathbb{D})$ of all analytic functions on $\mathbb{D}$. We say that g is the symbol of this operator.
These transformations appeared under various names in the literature: Volterra operators, generalized Cesàro operators or integration operators.
This class of operators includes the classical integration operator,

$$f(z) \longmapsto \int_0^z f(w)\, dw$$

obtained with $g(z) = z$.

It also includes the Cesàro operator

$$C(f)(z) = \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} \sum_{k=0}^{n} a_k \Big) z^{n+1},$$

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which is defined for every $f \in \mathcal{H}(\mathbb{D})$, whose Taylor expansion is $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This is the operator V_g where $g(z) = \log\left(\frac{1}{1-z}\right)$.

The Volterra operator for holomorphic functions on the unit disk was introduced by Pommerenke [13]. He showed that V_g is bounded on the Hardy space H^2 if and only if g belongs to the space $BMOA$ of analytic functions of bounded mean oscillation. Moreover, his proof can be adapted to obtain that V_g is compact on H^2 if and only if g belongs to the space $VMOA$ of analytic functions of vanishing mean oscillation. In [1], Aleman and Siskakis extended these results to Hardy spaces H^p with $1 \leq p < \infty$. Moreover, the membership to Schatten classes has been studied by these authors. They proved that V_g belongs to the Schatten class $S^p(H^2)$ if and only if g lies in the Besov space B_p for $p > 1$. They also proved that if $V_g \in S^1$ then $g' = 0$.
Later on, they considered the case of weighted Bergman spaces [2]: for a large class of weights, they showed that V_g is bounded on the weighted Bergman space if and only if g belongs to the Bloch space $\mathcal{B}$. This class includes the classical weights $(1-|z|^2)^\alpha$ for $\alpha > -1$. Additionally, they obtained a similar result about compactness of V_g: in this case the function g must belong to the little Bloch space $\mathcal{B}_0$. Furthermore, Rättyä [14], found a necessary and sufficient condition for g such that V_g is bounded or compact between two different weighted Bergman spaces.
In 2014, Anderson, Jovovic, and Smith [3] conjectured that this operator is bounded on H^∞ if and only if the symbol g belongs to the space BRV of holomorphic functions on the unit disk of bounded radial variation. In 2017, Smith, Stolyarov and Volberg [16] gave a counterexample to this conjecture, but they showed it is true when g is univalent. The characterization of the boundedness of V_g on H^∞ remains an open problem in full generality.
In [20], Yoneda studied the boundedness and compactness of the operator V_g defined on the classical Bloch space $\mathcal{B}$ and the weighted Bloch space $\mathcal{B}^\beta$ with $\beta > 1$ (see definition below in section 2): when $\beta > 1$, he showed that V_g is bounded (resp. compact) on $\mathcal{B}^\beta$ if and only if $g \in \mathcal{B}$ (resp. $g \in \mathcal{B}_0$). For a survey regarding Volterra operators defined on spaces of analytic functions, see [15].

Motivated by the works [6] and [10], we continue this line of research on Volterra-type operators to completely characterize the absolutely summing and nuclear Volterra type integration operators V_g acting on Bergman spaces and on Bloch spaces. Actually, it turns out that (1-) absolutely summing Volterra operators are nuclear operators on the Bloch spaces, so that a specific study of nuclear operators is made for Bergman spaces only. Absolutely summing operators on spaces of analytic functions were investigated in many ways since the late 60's (see for instance the monograph of Pełczyński [12]).

The first section is an introduction to the framework of this paper and the second one, gives some preliminary results and recall some definitions and well known usual facts. The main results of the paper appear in Section 3 and 4, where we characterize the absolutely summing and nuclear Volterra operators on weighted Bloch spaces and on weighted Bergman spaces.

Throughout the paper, $A \lesssim B$, for two real numbers A and B, means that there exists a constant C such that $A \leq C\,B$. Furthermore, the notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. In this case, we say that A and B are comparable. If X is a Banach space, we denote its closed unit ball by B_X. Finally, if $1 \leq p < +\infty$ we define the conjugate index p' by $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Definitions and Preliminary results

In this section, we collect some basic definitions and facts about Bergman, Bloch and Besov spaces. We introduce some background and results that will be used throughout the paper.

We will denote by $\mathcal{A}^p_\alpha$, $1 \leq p < +\infty$ and $-1 < \alpha < +\infty$, the (weighted) Bergman space consisting of analytic functions f on the open unit disk satisfying

$$\|f\|_{\mathcal{A}^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) \right)^{1/p} < \infty \,,$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ and A denotes the normalized area measure on $\mathbb{D}$. When $\alpha = 0$, we recover the classical Bergman space $\mathcal{A}^p$.

Let us begin now with the definition of the weighted Bloch space, $\mathcal{B}^\beta$ where β is a real number. An analytic function f on $\mathbb{D}$ belongs to $\mathcal{B}^\beta$ if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta \, |f'(z)| < +\infty \,.$$

Equipped with the norm

$$\|f\|_{\mathcal{B}^\beta} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta \, |f'(z)| \,,$$

it is easy to check that $\mathcal{B}^\beta$ is a Banach space. When $\beta = 1$, we recover the classical Bloch space and we shall write $\mathcal{B}$ instead of $\mathcal{B}^1$ in this case. Let us remark that when $\beta < 0$, $\mathcal{B}^\beta$ contains only the constant functions.

We will also be interested in the little Bloch space $\mathcal{B}_0$ consisting of functions f in $\mathcal{B}$ such that

$$\lim_{|z| \to 1^-} (1 - |z|^2) |f'(z)| = 0.$$

Thus, we let $\mathcal{B}^\beta_0$ denote the subspace of $\mathcal{B}^\beta$ consisting of functions f such that

$$\lim_{|z| \to 1^-} (1 - |z|^2)^\beta |f'(z)| = 0.$$

It is well known that $\mathcal{B}^\beta_0$ is a closed subspace of $\mathcal{B}^\beta$ (see [22] for instance). Actually, when $\beta \leq 0$, $\mathcal{B}^\beta_0$ is also trivially reduced to the constant functions.

We introduce the space $B_{p,\gamma}$ with $1 \leq p < \infty$ and $\gamma \geq 0$ which consists of analytic functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'(z)|^p \, (1 - |z|^2)^\gamma \, d\lambda(z) < \infty$$

where $d\lambda(z) = \dfrac{dA(z)}{(1-|z|^2)^2}$ is the Möbius invariant measure on $\mathbb{D}$.
Two special cases are worth mentioning. When $\gamma = p + \alpha + 2$, we recover the weighted Bergman space $\mathcal{A}^p_\alpha$ and when $\gamma = p$, we recover the classical Besov space B_p consisting of analytic functions f on $\mathbb{D}$ such that

$$\|f\|^p_{B_p} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \, d\lambda(z) < \infty .$$

Note that for $p = 2$, B_2 is actually the classical Dirichlet space $\mathcal{D}^2$ which is the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^2_{\mathcal{D}^2} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty.$$

The following crucial proposition is certainly well known from the specialists. Strangely we did not find this result explicitly in the literature (even for $\gamma = p$, i.e for Besov spaces), so we give a proof for the sake of completeness.

Proposition 2.1. *Let $\gamma > 0$ and $1 \le p \le q < +\infty$. Then we have*

$$B_{p,\,p\gamma} \subset B_{q,\,q\gamma} .$$

In particular, when $\gamma = 1$ we have

$$B_p \subset B_q .$$

Proof. Using [17, Lemma 5] (with say $r = \frac{1}{2}$), we have

$$|f'(z)|^p (1-|z|^2)^{p\gamma} \lesssim \int_{\mathbb{D}} |f'(w)|^p \, (1-|w|^2)^{p\gamma} \, d\lambda(w) \tag{2.1}$$

where the underlying constant depends only on p and γ.

Now, let $f \in B_{p,\,p\gamma}$ and $q \ge p$, then

$$\begin{aligned} \int_{\mathbb{D}} |f'(z)|^q \, (1-|z|^2)^{q\gamma} \, d\lambda(z) &\le \sup_{z\in\mathbb{D}} |f'(z)|^{q-p} \, (1-|z|^2)^{(q-p)\gamma} \int_{\mathbb{D}} |f'(z)|^p \, (1-|z|^2)^{p\gamma} \, d\lambda(z) \\ &= \sup_{z\in\mathbb{D}} \big(|f'(z)|^p (1-|z|^2)^{p\gamma}\big)^{\frac{q-p}{p}} \int_{\mathbb{D}} |f'(z)|^p \, (1-|z|^2)^{p\gamma} \, d\lambda(z) \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we get

$$\begin{aligned} \int_{\mathbb{D}} |f'(z)|^q \, (1-|z|^2)^{q\gamma} \, d\lambda(z) &\lesssim \Big(\int_{\mathbb{D}} |f'(w)|^p \, (1-|w|^2)^{p\gamma} \, d\lambda(w) \Big)^{\frac{q-p}{p}} \int_{\mathbb{D}} |f'(z)|^p \, (1-|z|^2)^{p\gamma} \, d\lambda(z) \\ &\lesssim \Big(\int_{\mathbb{D}} |f'(z)|^p \, (1-|z|^2)^{p\gamma} \, d\lambda(z) \Big)^{q/p} \end{aligned}$$

Hence $f \in B_{q,\,q\gamma}$. □

There are many excellent references concerning these spaces, we refer the reader to [8], [21], [22] and [23].

The notion of absolutely summing operator will play a crucial role in our work, so we recall

Definition 2.2. *Let $r \geq 1$ and $T : X \to Y$ be a linear operator between Banach spaces. We say that T is r-summing if there is a constant $C \geq 0$ such that regardless of the natural number n and regardless of the choice of $x_1, x_2, \ldots, x_n$ in X, we have*

$$\left(\sum_{k=1}^{n} \|Tx_k\|_Y^r \right)^{1/r} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^{n} |x^*(x_k)|^r \right)^{1/r} = C \sup_{a \in B_{l^{r'}}} \left\| \sum_{k=1}^{n} a_k x_k \right\|_X$$

where X^* denotes the dual space of X. The best constant C for which inequality always holds is denoted by $\pi_r(T)$.

Let us mention that (1-)absolutely summing operators are the operators $T : X \to Y$ which do take unconditionally summable sequences $\{x_n\}$ in X to absolutely summable sequences $\{Tx_n\}$ in Y.

We shall also focus on a particular family of compact operators, the nuclear operators:

Definition 2.3. *Let X, Y be two Banach spaces. An operator $T : X \longrightarrow Y$ is said to be nuclear if there are two sequences $(x_n^*) \subset X^*$ and $(y_n) \subset Y$ such that $\sum_n \|x_n^*\| \|y_n\| < \infty$ and*

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$$

where $x_n^* \otimes y_n : X \to Y$ is defined to be the mapping $x \mapsto x_n^*(x) y_n$.

In other words, nuclear operators are absolutely convergent series of rank one operators. Therefore they are compact. It is easy to see that any nuclear operator is absolutely summing but the converse is not true since the canonical injection from $C([0,1])$ into $L^1([0,1])$ is 1-summing but not even compact. See [4] and [19] to know more on these notions. Nevertheless absolutely summing and nuclear operators coincide in the case of Bloch spaces (see [6, Lemma 2.1]). It is due to the fact that Bloch spaces are isomorphic to ℓ^∞ (see [11]).

The unit disk $\mathbb{D}$ is endowed with a hyperbolic metric (also called the Bergman metric) given by

$$\beta(z,w) = \frac{1}{2} \log \frac{1+\rho(z,w)}{1-\rho(z,w)}$$

where

$$\rho(z,w) = \frac{|z-w|}{|1-z\overline{w}|}$$

is the pseudohyperbolic metric. For $z \in \mathbb{D}$ and $r > 0$, $D(z,r) = \{w \in \mathbb{D};\, \beta(z,w) < r\}$ is called hyperbolic disk.

Given a positive Borel measure μ on $\mathbb{D}$, we can consider the identity operator J_μ viewed from $\mathcal{A}_\alpha^p$ into $L^q(\mu)$ with $1 \leq p, q < \infty$, formally defined until now by

$$\begin{aligned} J_\mu : \mathcal{A}_\alpha^p &\longrightarrow L^q(\mu) \\ f &\longmapsto f \end{aligned}$$

Actually, we are interested in the case $1 \leq p \leq q < \infty$:
It is defined and bounded precisely when μ is a $(2+\alpha)\frac{q}{p}$ - Carleson measure (see for example [18]) i.e when

$$\sup_{a\in\mathbb{D}} \frac{\mu(D(a,r))}{(1-|a|^2)^{(2+\alpha)\frac{q}{p}}} < +\infty\,.$$

And J_μ is compact if and only if μ is a vanishing Carleson measure (in the hyperbolic sense), that is

$$\lim_{|a|\to 1^-} \frac{\mu(D(a,r))}{(1-|a|^2)^{(2+\alpha)\frac{q}{p}}} = 0\,.$$

For simplicity, we shall use the terminology Carleson measure instead of "$(2+\alpha)\frac{q}{p}$-Carleson" or "$\mathcal{A}^p_\alpha$-Carleson" measure and we will write D_z instead of $D(z, r_0)$ for some fixed $r_0 \in (0,1)$.

In [9], the main result about the Carleson embedding J_μ viewed on weighted Bergman space $\mathcal{A}^p_\alpha$ states the following:

Theorem 2.4. [9, Theorem 10.1]
Let $\alpha > -1$ and μ be a Carleson measure on $\mathcal{A}^p_\alpha$. Then

(1) Let $1 < p \leq 2$ and $r \geq 1$. Then $J_\mu : \mathcal{A}^p_\alpha \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}} \frac{(\mu(D_z))^{2/p}}{(1-|z|^2)^{(\alpha+2)(1+2/p)}}\, dA_\alpha(z) < \infty\,.$$

Moreover, we have

$$\pi_2(J_\mu) \approx \left(\int_{\mathbb{D}} \frac{(\mu(D_z))^{2/p}}{(1-|z|^2)^{(\alpha+2)(1+2/p)}}\, dA_\alpha(z)\right)^{1/2}. \tag{2.3}$$

(2) Let $p \geq 2$ and $1 \leq r \leq p'$. Then $J_\mu : \mathcal{A}^p_\alpha \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}} \frac{(\mu(D_z))^{p'/p}}{(1-|z|^2)^{(\alpha+2)p'}}\, dA_\alpha(z).$$

Moreover, we have

$$\pi_1(J_\mu) \approx \pi_r(J_\mu) \approx \left(\int_{\mathbb{D}} \frac{(\mu(D_z))^{p'/p}}{(1-|z|^2)^{(\alpha+2)p'}}\, dA_\alpha(z)\right)^{1/p'}. \tag{2.4}$$

(3) Let $p \geq 2$ and $p' \leq r \leq p$. Then $J_\mu : \mathcal{A}^p_\alpha \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}} \frac{(\mu(D_z))^{r/p}}{(1-|z|^2)^{(\alpha+2)(1+r/p)}}\, dA_\alpha(z).$$

Moreover, we have

$$\pi_r(J_\mu) \approx \left(\int_{\mathbb{D}} \frac{(\mu(D_z))^{r/p}}{(1-|z|^2)^{(\alpha+2)(1+r/p)}}\, dA_\alpha(z)\right)^{1/r}. \tag{2.5}$$

(4) Let $p \geq 2$ and $r \geq p$. Then $J_\mu : \mathcal{A}^p_\alpha \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^{2+\alpha}} d\mu(z) < \infty .$$

Moreover, we have

$$\pi_r(J_\mu) \approx \left(\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^{2+\alpha}} d\mu(z) \right)^{1/p}. \tag{2.6}$$

3. The case of Bergman spaces

In this section, we characterize r-summing and nuclear Volterra operators acting on weighted Bergman spaces for any $r \geq 1$. But first, let us recall the results on the boundedness and the compactness of the Volterra operators on Bergman spaces.

3.1. **Boundedness and compactness.**

Rättyä [14] characterized the boundedness and the compactness of Volterra operator acting between different weighted Bergman spaces. Let us recall the results in the case $p \leq q$ and we will give a direct proof using the point of view of Carleson embeddings (we could also use the same point of view to treat the case $q < p$, but we only work with $p = q$ in the sequel).

Theorem 3.1. [14, Th. 2 and Cor. 7]
Let $\gamma = 1 + \frac{2+\alpha'}{q} - \frac{2+\alpha}{p}$ with $-1 < \alpha, \alpha' < \infty$ and $1 \leq p \leq q < \infty$. Let g be an analytic function on $\mathbb{D}$, then we have

(1) $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^q_{\alpha'}$ is bounded if and only if $g \in \mathcal{B}^\gamma$.

(2) $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^q_{\alpha'}$ is compact if and only if $g \in \mathcal{B}^\gamma_0$.

Let us remark that:

(1) If $\gamma < 0$, V_g is never bounded unless $g' = 0$.

(2) If $\gamma \leq 0$, V_g is never compact unless $g' = 0$.

(3) In particular, $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^p_{\alpha'}$ is bounded if and only if $g \in \mathcal{B}^{1+\frac{\alpha'-\alpha}{p}}$ and V_g is compact if and only if $g \in \mathcal{B}_0^{1+\frac{\alpha'-\alpha}{p}}$.

Proof of Theorem 3.1. Using [8, Prop.1.11], we can see easily that $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^q_{\alpha'}$ is bounded (resp. compact) if and only if

$$\begin{aligned} J_\mu : \mathcal{A}^p_\alpha &\longrightarrow L^q(\mathbb{D}, \mu) \\ f &\longmapsto f \end{aligned} \tag{3.1}$$

is bounded (resp. compact) where $d\mu(z) = |g'(z)|^q (1-|z|^2)^{q+\alpha'} dA(z)$.
Therefore, (see section 2), it is equivalent to the fact that μ is a Carleson measure (resp.

vanishing Carleson measure), i.e

$$\mu(D(w,\rho)) \lesssim (1-|w|^2)^{(2+\alpha)\frac{q}{p}}$$

resp.

$$\mu(D(w,\rho)) = o((1-|w|^2)^{(2+\alpha)\frac{q}{p}})$$

where $D(w,\rho)$ is the Bergman disk with $0<\rho<1$ (fixed).

Hence, V_g is bounded if and only if

$$\int_{D(w,\rho)} |g'(z)|^q (1-|z|^2)^{q+\alpha'} dA(z) \lesssim (1-|w|^2)^{(2+\alpha)\frac{q}{p}}$$

and V_g is compact if and only if

$$\lim_{|w|\to 1} \frac{1}{(1-|w|^2)^{(2+\alpha)\frac{q}{p}}} \int_{D(w,\rho)} |g'(z)|^q (1-|z|^2)^{q+\alpha'} dA(z) = 0$$

Using [23, Prop. 4.13] and the fact that $1-|z|^2 \approx 1-|w|^2$ whenever $z \in D(w,\rho)$, we get that V_g is bounded if and only if

$$|g'(w)|^q (1-|w|^2)^{q+\alpha'+2} \lesssim (1-|w|^2)^{(2+\alpha)\frac{q}{p}}$$

and V_g is compact if and only if

$$\lim_{|w|\to 1} |g'(w)|^q (1-|w|^2)^{q(1+\frac{2+\alpha'}{q}-\frac{2+\alpha}{p})} = 0\,.$$

Finally, we conclude that V_g is bounded (resp.compact) if and only if $g \in \mathcal{B}^\gamma$ (resp. $g \in \mathcal{B}^\gamma_0$). □

Remark 3.2. From now on, we focus on the case $q = p$. Moreover, thanks to the remark after Th.3.1, we shall only consider the non trivial case of compactness where $\gamma > 0$, i.e $\alpha' - \alpha > -p$.

3.2. **Absolutely summing operators.**

We focus now on absolutely summing operators between Bergman spaces. Actually, let us point out that in this framework absolutely summing operators are compact.

Let $\alpha, \alpha' > -1$ and $p \geq 1$.

When $p = 1$, assume that $T : \mathcal{A}^1_\alpha \to \mathcal{A}^1_{\alpha'}$ is absolutely summing then T is weakly compact. But, $\mathcal{A}^1_{\alpha'}$ is isomorphic to the space ℓ^1, since we have $\mathcal{A}^1_{\alpha'} \sim (\mathcal{B}_0)^*$ (see [23, Th. 5.15]) and $\mathcal{B}_0 \sim c_0$ (see [11, Th. 1.1]). The space ℓ^1 has the Schur property, hence T is compact.

When $p > 1$, and $T : \mathcal{A}^p_\alpha \to \mathcal{A}^p_{\alpha'}$ is absolutely summing then T is a Dunford Pettis operator (see for instance [4, Th. 2.17]). Since $\mathcal{A}^p_\alpha$ is reflexive, compact operator and Dunford Pettis operator coincide. Hence T is compact.

That is why, thanks to remark 3.2, we assume in the sequel that $\alpha' - \alpha > -p$.

Theorem 2.4 will play a key role in this study in the case $p > 1$, but we first need several technical lemmas.

Lemma 3.3. *Let $a \in \mathbb{R}$, $b > 0$, $\alpha > -1$ and h be an analytic function on $\mathbb{D}$. Then we have:*

$$\int_{\mathbb{D}} \frac{|h(z)|^b}{(1-|z|^2)^a}\, dA_\alpha(z) \approx \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^a} \sup_{w\in D_z} |h(w)|^b\, dA_\alpha(z) \approx \int_{\mathbb{D}} \sup_{w\in D_z} \frac{|h(w)|^b}{(1-|w|^2)^a}\, dA_\alpha(z)$$

where the underlying constants depend only on a and r_0.

Let us recall that r_0 is involved in the definition of the hyperbolic disk $D_z = D(z, r_0)$.

Proof. Obviously the second quantity in the statement is equivalent to the third one because

$$1-|z|^2 \approx 1-|w|^2 \qquad \text{for } w \in D_z\,.$$

It is clear that

$$\frac{|h(z)|^b}{(1-|z|^2)^a} \le \sup_{w\in D_z} \frac{|h(w)|^b}{(1-|w|^2)^a}\,.$$

Integrating over $\mathbb{D}$, we get

$$\int_{\mathbb{D}} \frac{|h(z)|^b}{(1-|z|^2)^a}\, dA_\alpha(z) \le \int_{\mathbb{D}} \sup_{w\in D_z} \frac{|h(w)|^b}{(1-|w|^2)^a}\, dA_\alpha(z)$$

To see the reverse inequality, we fix $z \in \mathbb{D}$. By [23, Lemma 4.13], we have:

$$|h(w)|^b \lesssim \frac{1}{(1-|w|^2)^{2+\alpha}} \int_{D_w} |h(u)|^b\, dA_\alpha(u)$$

for every $w \in D_z$.
Since $D_w \subset D(z, 2r_0)$ and $1-|z|^2 \approx 1-|w|^2$ whenever $w \in D_z$, we get

$$|h(w)|^b \lesssim \frac{1}{(1-|z|^2)^{2+\alpha}} \int_{D(z,2r_0)} |h(u)|^b\, dA_\alpha(u).$$

Once again since $1-|z|^2 \approx 1-|u|^2$ whenever $u \in D(z, 2r_0)$:

$$\frac{|h(w)|^b}{(1-|z|^2)^a} \lesssim \frac{1}{(1-|z|^2)^{(2+\alpha)+a}} \int_{D(z,2r_0)} |h(u)|^b\, dA_\alpha(u) \approx \int_{D(z,2r_0)} \frac{|h(u)|^b}{(1-|u|^2)^{(2+\alpha)+a}}\, dA_\alpha(u)\,.$$

It follows that for $z \in \mathbb{D}$, we have:

$$\sup_{w\in D_z} \frac{|h(w)|^b}{(1-|w|^2)^a} \lesssim \int_{D(z,2r_0)} \frac{|h(u)|^b}{(1-|u|^2)^{(2+\alpha)+a}}\, dA_\alpha(u)\,.$$

Finally, integrating over $\mathbb{D}$ and using Fubini's Theorem, we get

$$\begin{aligned}
\int_{\mathbb{D}} \sup_{w\in D_z} \frac{|h(w)|^b}{(1-|w|^2)^a}\, dA_\alpha(z) &\lesssim \int_{\mathbb{D}} \int_{D(z,2r_0)} \frac{|h(u)|^b}{(1-|u|^2)^{(2+\alpha)+a}}\, dA_\alpha(u)\, dA_\alpha(z) \\
&= \int_{\mathbb{D}} \frac{|h(u)|^b}{(1-|u|^2)^{(2+\alpha)+a}}\, A_\alpha(D(u, 2r_0))\, dA_\alpha(u) \\
&\approx \int_{\mathbb{D}} \frac{|h(u)|^b}{(1-|u|^2)^a}\, dA_\alpha(u)\,.
\end{aligned}$$

□

Now, we prove an L^p-version of the previous lemma.

Lemma 3.4. *Let α, $\alpha' > -1$, $a \in \mathbb{R}$, $b > 0$ and h be an analytic function on $\mathbb{D}$. Let $p > 0$, then we have*

$$\int_{\mathbb{D}} \Big(\frac{|h(z)|^p}{(1-|z|^2)^{a-(2+\alpha')}} \Big)^b dA_\alpha(z) \approx \int_{\mathbb{D}} \bigg(\int_{D_z} \frac{|h(w)|^p}{(1-|w|^2)^a} \, dA_{\alpha'}(w) \bigg)^b dA_\alpha(z)$$

$$\approx \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^{ab}} \bigg(\int_{D_z} |h(w)|^p \, dA_{\alpha'}(w) \bigg)^b dA_\alpha(z)$$

where the underlying constants depend only on a, b and r_0.

Proof. Once again, the second quantity in the statement is equivalent to the third one. Now, from [23, Lemma 4.13], we have for every $z \in \mathbb{D}$

$$|h(z)|^p \lesssim \frac{1}{(1-|z|^2)^{2+\alpha'}} \int_{D_z} |h(w)|^p \, dA_{\alpha'}(w) \, .$$

Since $1-|z|^2 \approx 1-|w|^2$ whenever $w \in D_z$, we get

$$|h(z)|^p (1-|z|^2)^{(2+\alpha')-a} \lesssim \int_{D_z} \frac{|h(w)|^p}{(1-|w|^2)^a} \, dA_{\alpha'}(w).$$

Integrating over $\mathbb{D}$ and since $b > 0$, it follows that

$$\int_{\mathbb{D}} \Big(\frac{|h(z)|^p}{(1-|z|^2)^{a-(2+\alpha')}} \Big)^b dA_\alpha(z) \lesssim \int_{\mathbb{D}} \bigg(\int_{D_z} \frac{|h(w)|^p}{(1-|w|^2)^a} \, dA_{\alpha'}(w) \bigg)^b dA_\alpha(z).$$

To see the reverse inequality: since $A_{\alpha'}(D_z) \approx (1-|z|^2)^{2+\alpha'} \approx (1-|w|^2)^{2+\alpha'}$ whenever $w \in D_z$, we get

$$\begin{aligned} \int_{\mathbb{D}} \bigg(\int_{D_z} \frac{|h(w)|^p}{(1-|w|^2)^a} \, dA_{\alpha'}(w) \bigg)^b dA_\alpha(z) &\lesssim \int_{\mathbb{D}} \Big(\sup_{w \in D_z} \frac{|h(w)|^p}{(1-|w|^2)^a} \, A_{\alpha'}(D_z) \Big)^b dA_\alpha(z) \\ &\lesssim \int_{\mathbb{D}} \sup_{w \in D_z} \frac{|h(w)|^{pb}}{(1-|w|^2)^{b(a-(2+\alpha'))}} \, dA_\alpha(z) \end{aligned}$$

From Lemma 3.3, we have that

$$\int_{\mathbb{D}} \sup_{w \in D_z} \frac{|h(w)|^{pb}}{(1-|w|^2)^{b(a-(2+\alpha'))}} \, dA_\alpha(z) \approx \int_{\mathbb{D}} \frac{|h(z)|^{pb}}{(1-|z|^2)^{b(a-(2+\alpha'))}} \, dA_\alpha(z)$$

and the reverse inequality is proved. □

The following lemma is a weighted version of [5, Lemma 7 p.49].

Lemma 3.5. *Let $\alpha > -1$, $t \in (0,1)$, f and g be analytic functions on $\mathbb{D}$. Then, we have*

$$\int_{\mathbb{D}} f(tz) \, \overline{g(tz)} \, dA_\alpha(z) = \frac{t^2}{\alpha+1} \int_{\mathbb{D}} W(f)(tz) \, \overline{g'(tz)} \, (1-|z|^2) \, dA_\alpha(z) + \, f(0)\overline{g(0)} \tag{3.2}$$

where $W(f)$ is the analytic function defined by $W(f)(z) = \dfrac{f(z) - f(0)}{z}$.
In particular, if $f \in H^\infty$ with $f(0) = 0$ and $g \in \mathcal{A}^1_\alpha$, then we have

$$(\alpha + 1) \int_{\mathbb{D}} f(z)\, \overline{g(z)}\, dA_\alpha(z) = \int_{\mathbb{D}} \frac{f(z)}{z}\, \overline{g'(z)}\, (1 - |z|^2)\, dA_\alpha(z) \tag{3.3}$$

Proof. We compute both integrals by using Taylor expansion of involved functions and integrate with polar coordinates:
If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$, we have on one hand

$$\begin{aligned}
\int_{\mathbb{D}} f(tz)\, \overline{g(tz)}\, dA_\alpha(z) &= 2(\alpha + 1) \sum_{n=0}^{\infty} t^{2n}\, a_n\, \overline{b_n} \int_0^1 r^{2n+1} (1 - r^2)^\alpha dr \\
&= \sum_{n=0}^{\infty} (\alpha + 1)\, t^{2n}\, a_n\, \overline{b_n}\, B(n + 1, \alpha + 1) \\
&= \sum_{n=0}^{\infty} t^{2n} \frac{n!\, \Gamma(2 + \alpha)}{\Gamma(n + 2 + \alpha)}\, a_n \overline{b_n}
\end{aligned} \tag{3.4}$$

where Γ is the classical Euler function and B is the beta function.

On the other hand,

$$\begin{aligned}
\int_{\mathbb{D}} W(f)(tz)\, \overline{g'(tz)}\, (1 - |z|^2)\, dA_\alpha(z) &= 2(\alpha + 1) \sum_{n=1}^{\infty} t^{2n-2}\, n\, a_n\, \overline{b_n} \int_0^1 r^{2(n-1)+1} (1 - r^2)^{\alpha+1} dr \\
&= (\alpha + 1) \sum_{n=1}^{\infty} t^{2n-2} \frac{n!\, \Gamma(2 + \alpha)}{\Gamma(n + 2 + \alpha)} a_n \overline{b_n}\,.
\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we get (3.2).

In particular, for any $f \in H^\infty$ with $f(0) = 0$ and $g \in \mathcal{A}^1_\alpha$, $\lim_{t\to 1} \int_{\mathbb{D}} f(tz)\, \overline{g(tz)}\, dA_\alpha(z)$ exists, and is equal to $\int_{\mathbb{D}} f(z)\, \overline{g(z)}\, dA_\alpha(z)$. In the same way, $\lim_{t\to 1} \int_{\mathbb{D}} \frac{f(tz)}{z}\, \overline{g'(tz)}\, (1 - |z|^2)\, dA_\alpha(z)$ exists, and is equal to $\int_{\mathbb{D}} \frac{f(z)}{z}\, \overline{g'(z)}\, (1 - |z|^2)\, dA_\alpha(z)$. From (3.2), we get (3.3). □

Now we can state the main theorem in this section.

Theorem 3.6. ***Characterization of absolutely summing Volterra operators.***
Let $\alpha', \alpha > -1$, $\gamma = 1 + \frac{\alpha' - \alpha}{p} > 0$ and g be an analytic function. Then we have the following statements:

(1) Let $1 \le p \le 2$ and $r \ge 1$. Then $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^p_{\alpha'}$ is r-summing if and only if $g \in B_{2,2\gamma}$. Moreover,

$$\pi_2(V_g) \approx \left(\int_{\mathbb{D}} |g'(z)|^2 \, (1-|z|^2)^{\frac{2}{p}(\alpha'-\alpha)} dA(z) \right)^{1/2}. \tag{3.6}$$

(2) Let $p \ge 2$ and $1 \le r \le p'$. Then $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^p_{\alpha'}$ is r-summing if and only if $g \in B_{p',p'\gamma}$.
Moreover,

$$\pi_r(V_g) \approx \left(\int_{\mathbb{D}} |g'(z)|^{p'} \, (1-|z|^2)^{\frac{p'}{p}(\alpha'-\alpha)+p'-2} \, dA(z) \right)^{1/p'}. \tag{3.7}$$

(3) Let $p \ge 2$ and $p' \le r \le p$. Then V_g is r-summing if and only if $g \in B_{r,r\gamma}$.
Moreover,

$$\pi_r(V_g) \approx \left(\int_{\mathbb{D}} |g'(z)|^{r} \, (1-|z|^2)^{\frac{r}{p}(\alpha'-\alpha)+r-2} \, dA(z) \right)^{1/r}. \tag{3.8}$$

(4) Let $p \ge 2$ and $r \ge p$. Then V_g is r-summing if and only if $g \in B_{p,p\gamma}$.
Moreover,

$$\pi_r(V_g) \approx \left(\int_{\mathbb{D}} |g'(z)|^{p} \, (1-|z|^2)^{(\alpha'-\alpha)+p-2} \, dA(z) \right)^{1/p}. \tag{3.9}$$

Proof. We use the same viewpoint as in the proof of Theorem 3.1. Using [8, Prop. 1.11] and since $V_g(f)(0) = 0$, we have:

$$\|V_g f\|_{\mathcal{A}^p_{\alpha'}} \approx \left(\int_{\mathbb{D}} |f(z)|^p \, |g'(z)|^p \, (1-|z|^2)^p \, dA_{\alpha'}(z) \right)^{1/p} = \|f\|_{L^p(\mu)}$$

where $d\mu(z) = |g'(z)|^p (1-|z|^2)^p dA_{\alpha'}(z)$ is a positive measure on $\mathbb{D}$.
Therefore, V_g is r-summing if and only if J_μ is r-summing from $\mathcal{A}^p_\alpha$ to $L^p(\mu)$. Point out that μ is finite if and only if g belongs to $\mathcal{A}^p_{\alpha'}$.
We need to split the proof into different cases according to the value of p:

- The case $p > 1$.

In order to use Theorem 2.4, we need to justify that μ is finite, either when one of the integrals in (3.6), (3.7), (3.8) and (3.9) is finite or when V_g is r-summing for some $r \ge 1$. Let us justify this fact.
If V_g is r-summing then it is bounded from $\mathcal{A}^p_\alpha$ into $\mathcal{A}^p_{\alpha'}$ and by testing the function 1, we get that $V_g(1) = g - g(0)$ belongs to $\mathcal{A}^p_{\alpha'}$ and then $g \in \mathcal{A}^p_{\alpha'}$. Hence, μ is a finite measure.
Now, when $p \ge 2$ if (3.9) is satisfied, i.e $g \in B_{p,p\gamma}$, then $g \in \mathcal{A}^p_{\alpha'}$ since $\alpha' + p > (\alpha'-\alpha) + p - 2$ and μ is finite. If (3.7) or (3.8) is satisfied then (3.9) is satisfied as well since by Prop. 2.1: $B_{p',p'\gamma} \subset B_{r,r\gamma} \subset B_{p,p\gamma}$ when $p' \le r \le p$, therefore μ is also finite in this case.

When $p \leq 2$, we have

$$\begin{aligned}\int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{\alpha'+p}\, dA(z) &= \int_{\mathbb{D}} |g'(z)|^p \,(1-|z|^2)^{(\alpha'-\alpha)+p+\alpha}\, dA(z)\\ &\leq \int_{\mathbb{D}} |g'(z)|^p \,(1-|z|^2)^{(\alpha'-\alpha)}\, dA(z)\\ &\leq \left(\int_{\mathbb{D}} |g'(z)|^2 \,(1-|z|^2)^{\frac{2}{p}(\alpha'-\alpha)} dA(z)\right)^{p/2} < \infty\end{aligned}$$

where we used in the last equality the fact that $\|(1-|z|^2)^{\frac{\alpha'-\alpha}{p}}\, g'\|_{L^p(\mathbb{D},A)} \leq \|(1-|z|^2)^{\frac{\alpha'-\alpha}{p}}\, g'\|_{L^2(\mathbb{D},A)}$ since $p \leq 2$.
Therefore, we conclude that if one of the conditions (3.6), (3.7), (3.8) and (3.9) is fulfilled, μ is finite.

From now on, we can assume that μ is finite and we can apply Theorem 2.4 to the measure μ. Moreover, Lemma 3.4 (applied separately according to the different values of a and b) leads to our characterizations.

Let us clarify the first case and we could proceed in the same way for the other cases: When $1 < p \leq 2$ and $r \geq 1$, we have from Theorem 2.4

$$\begin{aligned}\pi_r(V_g) \approx \pi_2(V_g) &\approx \left(\int_{\mathbb{D}} \frac{(\mu(D_z))^{2/p}}{(1-|z|^2)^{(\alpha+2)(1+2/p)}}\, dA_\alpha(z)\right)^{1/2}\\ &\approx \left(\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^{(\alpha+2)(1+2/p)}} \left(\int_{D_z} |g'(w)|^p\, dA_{\alpha'+p}(w)\right)^{2/p} dA_\alpha(z)\right)^{1/2}\end{aligned} \tag{3.10}$$

Using Lemma 3.4 applied with $h = g'$, $a = (\alpha+2)(\frac{p}{2}+1)$, $b = \frac{2}{p}$ and $\alpha'+p$ instead of α', we get:

$$\begin{aligned}\pi_2(V_g) &\approx \left(\int_{\mathbb{D}} \frac{(|g'(z)|^p)^{2/p}}{(1-|z|^2)^{(\alpha+2)(1+2/p)-\frac{2}{p}(\alpha'+p+2)}} dA_\alpha(z)\right)^{1/2}\\ &\approx \left(\int_{\mathbb{D}} |g'(z)|^2 \,(1-|z|^2)^{\frac{2}{p}(\alpha'-\alpha)} dA(z)\right)^{1/2}.\end{aligned} \tag{3.11}$$

The case $p > 1$ is proved.

- The case $p = 1$.

First, we assume that $\displaystyle\int_{\mathbb{D}} |g'(z)|^2 \,(1-|z|^2)^{2(\alpha'-\alpha)} dA(z) < \infty$, and we consider the operator:

$$\begin{aligned}\widetilde{V_g} : \mathcal{A}^1_\alpha &\longrightarrow L^1(\mathbb{D},\nu)\\ f &\longmapsto f.g'(1-|z|^2)^{2\alpha-\alpha'+1}\end{aligned}$$

where $d\nu = (1-|z|^2)^{2(\alpha'-\alpha)} dA$, which can be infinite when $\alpha'-\alpha \leq -\frac{1}{2}$ (here we only know that $\alpha'-\alpha > -1$) but it is not a problem in the sequel.

We can factorize $\widetilde{V_g}$ as follows:

$$\mathcal{A}^1_\alpha \xrightarrow{M_1} L^2(\mathbb{D},\nu) \xrightarrow{M_2} L^1(\mathbb{D},\nu)$$

where M_1 is the multiplier operator by $(1-|z|^2)^{2\alpha-\alpha'+1}$ and M_2 is the multiplier operator by g'.

The operator M_1 is bounded since the point evaluation at point z is a bounded linear functional on $\mathcal{A}^1_\alpha$ with norm equal to $\|\delta_z\|_{(\mathcal{A}^1_\alpha)^*} = \frac{1}{(1-|z|^2)^{2+\alpha}}\cdot$ More precisely, we have

$$\begin{aligned} \|M_1(f)\|^2_{L^2(\mathbb{D},\nu)} &= \int_{\mathbb{D}} |f(z)|^2 \,(1-|z|^2)^{2(\alpha+1)}\, dA(z) \\ &\lesssim \|f\|_{\mathcal{A}^1_\alpha} \int_{\mathbb{D}} |f(z)|\, dA_\alpha(z) \\ &= \|f\|^2_{\mathcal{A}^1_\alpha}. \end{aligned} \tag{3.12}$$

Thanks to Grothendieck's theorem, we get that M_1 is an absolutely summing operator since $\mathcal{A}^1_\alpha \sim \ell^1$.

The operator M_2 is bounded by the Cauchy-Schwarz inequality since $g' \in L^2(\mathbb{D},\nu)$ by assumption.

Hence, $\widetilde{V_g}$ is absolutely summing since $\widetilde{V_g} = M_2 \circ M_1$.

Finally, for every $f \in \mathcal{A}^1_\alpha$, we have $\|V_g(f)\|_{\mathcal{A}^1_{\alpha'}} \approx \|\widetilde{V_g}(f)\|_{L^1(\mathbb{D},\nu)}$, hence V_g is absolutely summing as well.

Now we assume that $V_g : \mathcal{A}^1_\alpha \to \mathcal{A}^1_{\alpha'}$ is absolutely summing, and we want to prove that

$$\int_{\mathbb{D}} |g'(z)|^2\,(1-|z|^2)^{2(\alpha'-\alpha)} dA(z) < \infty\,.$$

Since V_g is 2-summing, V_g factorizes through an Hilbert space H (see for instance [4, Cor. 2.16]). Hence, using the fact that the dual space of $\mathcal{A}^1_\alpha$ can be identified with the Bloch space $\mathcal{B}$ (see [21, Th. 5.3]), $V_g^* : \mathcal{B} \to \mathcal{B}$ factorizes through the Hilbert space $H^* = H'$ as well:

$$V_g^* :\ \mathcal{B} \xrightarrow{u} H' \xrightarrow{v} \mathcal{B}\,.$$

But, we know that $\mathcal{B} \sim \ell_\infty$ (see for instance [11, Th. 1.1]) and we know from [19, Th. 29, p.223], that any operator from ℓ_∞ into an Hilbert space is 2-summing. Hence, V_g^* is a 2-summing operator.

Let us compute $V_g^*(h)$ when $h \in H^\infty$ with $h(0) = 0$. Using [23, Th. 5.3] and Lemma 3.5, we have for every $f \in H^\infty$

$$\begin{aligned} \langle V_g^*(h), f\rangle_{\mathcal{B},\mathcal{A}^1_\alpha} &= \langle h, V_g(f)\rangle_{\mathcal{B},\mathcal{A}^1_{\alpha'}} \\ &= \int_{\mathbb{D}} h(s)\,\overline{V_g(f)(s)}\, dA_{\alpha'}(s) \\ &= \frac{1}{\alpha'+1}\int_{\mathbb{D}} \frac{h(s)}{s}\,\overline{g'(s)\,f(s)}\,(1-|s|^2)\, dA_{\alpha'}(s) \end{aligned} \tag{3.13}$$

In particular, fixing $z \in \mathbb{D}$ and testing $f(s) = k_z(s) = (1 - \overline{z}s)^{-2-\alpha}$, the Bergman reproducing kernel, we get on one hand

$$\langle V_g^*(h), k_z \rangle_{\mathcal{B},\mathcal{A}^1_\alpha} = \frac{1}{\alpha' + 1} \int_{\mathbb{D}} \frac{\overline{g'(s)}\, h(s)\, (1 - |s|^2)}{s\, (1 - z\overline{s})^{2+\alpha}}\, dA_{\alpha'}(s)\,.$$

On the other hand, using the duality between Bergman and Bloch spaces [23, Th. 5.3]:

$$\begin{aligned} \langle V_g^*(h), k_z \rangle_{\mathcal{B},\mathcal{A}^1_\alpha} &= \int_{\mathbb{D}} V_g^*(h)(s)\, \overline{k_z(s)}\, dA_\alpha(s) \\ &= \int_{\mathbb{D}} \frac{V_g^*(h)(s)}{(1 - \overline{s}z)^{2+\alpha}}\, dA_\alpha(s) \\ &= V_g^*(h)(z)\,. \end{aligned}$$

Hence,

$$V_g^*(h)(z) = \frac{1}{\alpha' + 1} \int_{\mathbb{D}} \frac{\overline{g'(s)}\, h(s)\, (1 - |s|^2)}{s\, (1 - z\overline{s})^{2+\alpha}}\, dA_{\alpha'}(s)\,. \tag{3.14}$$

We consider now, for any $w \in \mathbb{D}$, the function

$$F_w(z) = \frac{z\, (1 - |w|^2)^{\alpha'+2}}{(1 - \overline{w}z)^{\alpha'+3}} \in \mathcal{B}_0 \cap H^\infty\,.$$

Since $F_w(0) = 0$, and $F_w \in H^\infty$, we can apply (3.14) with $h = F_w$ and we get

$$\Big(V_g^*(F_w)\Big)'(z) = \frac{\alpha + 2}{\alpha' + 1} \int_{\mathbb{D}} \frac{\overline{s}\, \overline{g'(s)}\, (1 - |s|^2)\, (1 - |w|^2)^{\alpha'+2}}{(1 - \overline{w}s)^{\alpha'+3}\, (1 - z\overline{s})^{\alpha+3}}\, dA_{\alpha'}(s)\,.$$

Let us lower estimate the norm of $V_g^*(F_w)$ in the Bloch space $\mathcal{B}$:

$$\begin{aligned} \|V_g^*(F_w)\|_{\mathcal{B}} &= \sup_{z\in\mathbb{D}} (1 - |z|^2) \left|\big(V_g^*(F_w)\big)'(z)\right| \\ &\geq \frac{\alpha + 2}{\alpha' + 1} \left| \int_{\mathbb{D}} \frac{s\, g'(s)\, (1 - |s|^2)\, (1 - |w|^2)^{\alpha'+3}}{(1 - \overline{w}s)^{(\alpha-\alpha')}\, |1 - \overline{w}s|^{2(\alpha'+3)}}\, dA_{\alpha'}(s) \right|, \end{aligned} \tag{3.15}$$

using that the supremum over $z \in \mathbb{D}$ is larger than the value at point w.
We recall that for any analytic function G on $\mathbb{D}$, we have ([8, p.7]):

$$G(w) = (1 - |w|^2)^{3+\alpha'} \int_{\mathbb{D}} \frac{G(s)\, (1 - |s|^2)}{|1 - s\overline{w}|^{2(3+\alpha')}}\, dA_{\alpha'}(s)\,.$$

Applying this formula to the function $s \mapsto G_w(s) = \dfrac{s\, g'(s)}{(1 - \overline{w}s)^{(\alpha-\alpha')}}$ we get

$$\|V_g^*(F_w)\|_{\mathcal{B}} \geq \frac{\alpha + 2}{\alpha' + 1}\, |w|\, |g'(w)|\, (1 - |w|^2)^{\alpha'-\alpha}\,. \tag{3.16}$$

Now, we return to the fact that we know that V_g^* is 2-summing on $\mathcal{B}$. Its restriction to $\mathcal{B}_0$, whose dual space is $\mathcal{A}^1_{\alpha'}$, is 2-summing as well and $\pi_2(V^*_{g_{|\mathcal{B}_0}}) \leq \pi_2(V_g^*)$.

Using [19, Prop. 33 (b) p. 225] to the $\mathcal{B}_0$-valued random variable: $w \in \mathbb{D} \mapsto F_w \in \mathcal{B}_0$, we get

$$\int_{\mathbb{D}} \|V_g^*(F_w)\|_{\mathcal{B}}^2 \, dA(w) \leq \pi_2(V_g^*) \sup_{h \in B_{\mathcal{A}^1_{\alpha'}}} \int_{\mathbb{D}} |\langle h, F_w \rangle|^2 \, dA(w) \, . \tag{3.17}$$

On one hand, from the reproducing kernel formula, we get:

$$\begin{aligned} \langle h, F_w \rangle_{\mathcal{A}^1_{\alpha'}, \mathcal{B}_0} &= \int_{\mathbb{D}} \frac{\overline{u}\, h(u)(1-|w^2|)^{\alpha'+2}}{(1-w\overline{u})^{\alpha'+3}} \, dA_{\alpha'}(u) \\ &= (1-|w|^2)^{\alpha'+2} \int_{\mathbb{D}} \frac{\overline{u}\, h(u)}{(1-w\overline{u})^{\alpha'+3}} \, dA_{\alpha'}(u) = \frac{1}{\alpha'+2} \, (1-|w|^2)^{\alpha'+2} \, h'(w). \end{aligned}$$

From the last equality, we also see that

$$|h'(w)| \leq (\alpha'+2) \frac{\|h\|_{\mathcal{A}^1_{\alpha'}}}{(1-|w|^2)^{\alpha'+3}} \cdot$$

It follows that:

$$\begin{aligned} \sup_{h \in B_{\mathcal{A}^1_{\alpha'}}} \int_{\mathbb{D}} |\langle h, F_w \rangle|^2 \, dA(w) &\approx \sup_{h \in B_{\mathcal{A}^1_{\alpha'}}} \int_{\mathbb{D}} (1-|w|^2)^{2(\alpha'+2)} \, |h'(w)|^2 \, dA(w) \\ &\lesssim \sup_{h \in B_{\mathcal{A}^1_{\alpha'}}} \int_{\mathbb{D}} (1-|w|^2)^{2(\alpha'+2)} \, |h'(w)| \, \frac{\|h\|_{\mathcal{A}^1_{\alpha'}}}{(1-|w|^2)^{\alpha'+3}} \, dA(w) \\ &\lesssim \sup_{h \in B_{\mathcal{A}^1_{\alpha'}}} \|h\|_{\mathcal{A}^1_{\alpha'}} \int_{\mathbb{D}} (1-|w|^2) \, |h'(w)| \, dA_{\alpha'}(w) \\ &\approx \sup_{h \in B_{\mathcal{A}^1_{\alpha'}}} \|h\|^2_{\mathcal{A}^1_{\alpha'}} = 1 \, . \end{aligned} \tag{3.18}$$

On the other hand, using [7, Lemma 3.3], we get:

$$\begin{aligned} \int_{\mathbb{D}} \|V_g^*(F_w)\|_{\mathcal{B}}^2 \, dA(w) &\geq \left(\frac{\alpha+2}{\alpha'+1} \right)^2 \int_{\mathbb{D}} |w|^2 \, |g'(w)|^2 (1-|w|^2)^{2(\alpha'-\alpha)} \, dA(w) \\ &\approx \int_{\mathbb{D}} |g'(w)|^2 (1-|w|^2)^{2(\alpha'-\alpha)} \, dA(w) \, . \end{aligned} \tag{3.19}$$

Hence, combining (3.17), (3.18) and (3.19), it follows that:

$$\int_{\mathbb{D}} |g'(w)|^2 (1-|w|^2)^{2(\alpha'-\alpha)} \, dA(w) \lesssim 1 \, . \tag{3.20}$$

□

We get for free the characterizations for Volterra operators on weighted Bergman spaces $\mathcal{A}^p_\alpha$, taking $\alpha' = \alpha$.

Corollary 3.7. *Let $-1 < \alpha < \infty$ and g be an analytic function on $\mathbb{D}$. Then we have the following statements:*

(1) Let $1 \le p \le 2$ and $r \ge 1$. Then $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^p_\alpha$ is r-summing if and only if g is in the Dirichlet space $\mathcal{D}^2 = B_2$.
Moreover,

$$\pi_2(V_g) \approx \left(\int_{\mathbb{D}} |g'(z)|^2 dA(z) \right)^{1/2}.$$

(2) Let $p \ge 2$ and $1 \le r \le p'$. Then $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^p_\alpha$ is r-summing if and only if $g \in B_{p'}$.

Moreover,

$$\pi_r(V_g) \approx \left(\int_{\mathbb{D}} |g'(z)|^{p'} \, (1-|z|^2)^{p'-2} \, dA(z) \right)^{1/p'}.$$

(3) Let $p \ge 2$ and $p' \le r \le p$. Then V_g is r-summing if and only if $g \in B_r$.

Moreover,

$$\pi_r(V_g) \approx \left(\int_{\mathbb{D}} |g'(z)|^{r} \, (1-|z|^2)^{r-2} \, dA(z) \right)^{1/r}.$$

(4) Let $p \ge 2$ and $r \ge p$. Then V_g is r-summing if and only if $g \in B_p$.

Moreover,

$$\pi_r(V_g) \approx \left(\int_{\mathbb{D}} |g'(z)|^{p} \, (1-|z|^2)^{p-2} \, dA(z) \right)^{1/p}.$$

Remark 3.8. As a particular case ($\alpha = 0$), we get the characterizations of absolutely summing Volterra operators on (unweighted) classical Bergman spaces $\mathcal{A}^p$ for $p \ge 1$. It turns out that the preceding characterizations do not depend on $\alpha > -1$.

3.3. **Nuclear Volterra operators.**

Lemma 3.9. *Let h be an analytic function on $\mathbb{D}$. If $\displaystyle\int_{\mathbb{D}} \frac{|h(w)|}{1-|w|^2} \, dA(w)$ is finite, then $h = 0$.*

Proof. Let H^1 be the classical Hardy space and m the normalized Lebesgue measure on the torus.
Using polar coordinates:

$$\begin{aligned}
\int_{\mathbb{D}} \frac{|h(w)|}{1-|w|^2} \, dA(w) &= \int_0^1 \frac{2r}{1-r^2} \left(\int_{\mathbb{T}} |h(r\xi)| \, dm(\xi) \right) dr \\
&= \int_0^1 \frac{2r}{1-r^2} \, \|h_r\|_{H^1} \, dr \\
&\ge \|h_{\frac{1}{2}}\|_{H^1} \int_{1/2}^1 \frac{2r}{1-r^2} \, dr \,,
\end{aligned}$$

since $r \in (0,1) \to \|h_r\|_{H^1}$ is non-decreasing.
But,

$$\int_{1/2}^{1} \frac{2r}{1-r^2}\, dr = +\infty\,,$$

so $\|h_{\frac{1}{2}}\|_{H^1} = 0$ and then $h_{\frac{1}{2}} = 0$ on $\mathbb{D}$. Hence $h = 0$ as well on $\mathbb{D}$ since h is analytic on $\mathbb{D}$. □

Theorem 3.10. ***Characterization of nuclear Volterra operators.***
Let $p \geq 1$, $-1 < \alpha < \infty$ and g be a symbol. Then, we have $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^p_\alpha$ is a nuclear operator if and only if $g' = 0$.
In other words, there is no non trivial nuclear operator on $\mathcal{A}^p_\alpha$.

Let us mention as an immediate consequence that, when $\alpha \geq \alpha'$, $V_g : \mathcal{A}^p_\alpha \to \mathcal{A}^p_{\alpha'}$ is nuclear if and only if $g' = 0$. Indeed, it suffices to point out that, for such indexes, the formal identity from $\mathcal{A}^p_{\alpha'}$ to $\mathcal{A}^p_\alpha$ is bounded.

Proof. Let $p > 1$:
We assume that V_g is nuclear. Then, by definition there exist two sequences $(f_n)_n \in \mathcal{A}^p_\alpha$ and $(\phi_n)_n \in (\mathcal{A}^p_\alpha)^*$ such that $\sum_n \|f_n\|\,\|\phi_n\| < \infty$ and

$$V_g = \sum_{n=1}^{\infty} \phi_n \otimes f_n \tag{3.21}$$

where $\phi_n \otimes f_n : \mathcal{A}^p_\alpha \to \mathcal{A}^p_\alpha$ is defined as the rank one operator $f \mapsto \phi_n(f)\, f_n$.
Since $(\mathcal{A}^p_\alpha)^* \simeq \mathcal{A}^{p'}_\alpha$ with equivalence of norms (cf. [8, p.19]), there exists $h_n \in B_{\mathcal{A}^{p'}_\alpha}$ such that

$$\phi_n(f) = \langle f, h_n\rangle = \int_{\mathbb{D}} f(z)\,\overline{h_n(z)}\, dA_\alpha(z)$$

for every $f \in \mathcal{A}^p_\alpha$.
Let $f \in \mathcal{A}^p_\alpha$. Since the pointwise derivation functional is continuous on $\mathcal{A}^p_\alpha$, from (3.21) we get for every $w \in \mathbb{D}$

$$g'(w)f(w) = \sum_{n=1}^{\infty} \Big(\int_{\mathbb{D}} \overline{h_n(z)}\, f(z)\, dA_\alpha(z)\Big).f_n'(w)\,.$$

Let us test the following function, which belongs to $\mathcal{A}^p_\alpha$:

$$z \in \mathbb{D} \longmapsto \frac{1}{(1-\overline{w}z)^{\alpha+2}} \qquad \text{where } w \in \mathbb{D}\,.$$

We get

$$\frac{g'(w)}{(1-|w|^2)^{\alpha+2}} = \sum_{n=1}^{\infty} \Big(\overline{\int_{\mathbb{D}} \frac{h_n(z)}{(1-\overline{z}w)^{\alpha+2}} dA_\alpha(z)}\Big).f_n'(w)\,.$$

But, since $h_n \in \mathcal{A}^{p'}_\alpha$, thanks to the reproducing kernel formula, we have for every $w \in \mathbb{D}$

$$\int_{\mathbb{D}} \frac{h_n(z)}{(1-\overline{z}w)^{\alpha+2}} dA_\alpha(z) = h_n(w)$$

Hence, integrating over $\mathbb{D}$ and using Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{D}} \frac{|g'(w)|}{(1-|w|^2)^{\alpha+1}} dA_\alpha(w) &\leq \sum_{n=1}^{\infty} \int_{\mathbb{D}} |h_n(w)|\,|f_n'(w)\,(1-|w|^2)|\,dA_\alpha(w) \\ &\leq \sum_{n=1}^{\infty} \|h_n\|_{\mathcal{A}^{p'}_\alpha}\,\|f_n\|_{\mathcal{A}^p_\alpha} < \infty.\end{aligned} \tag{3.22}$$

By Lemma 3.9, we get $g' = 0$.

This proof can be adapted to the case $p = 1$, but we shall give in the next result another proof which extends for Volterra operators between two different Bergman spaces. □

Theorem 3.11. *Characterization of nuclear Volterra operators for $p = 1$.*

Let $-1 < \alpha, \alpha' < \infty$ and g be a symbol. Then, we have $V_g : \mathcal{A}^1_\alpha \to \mathcal{A}^1_{\alpha'}$ is a nuclear operator if and only if

$$\int_{\mathbb{D}} \frac{|g'(z)|}{(1-|z|^2)^{1+\alpha-\alpha'}}\, dA(z) < \infty\,. \tag{3.23}$$

In particular, when $\alpha \geq \alpha'$, V_g is nuclear if and only if $g' = 0$.

Proof. If $V_g : \mathcal{A}^1_\alpha \to \mathcal{A}^1_{\alpha'}$ is a nuclear operator with $\alpha,\ \alpha' > -1$, then the operator

$$\begin{aligned}\widetilde{V_g} : \mathcal{A}^1_\alpha &\longrightarrow L^1(\mathbb{D}, (1-|z|^2)\,dA_{\alpha'}) \\ f &\longmapsto fg'\end{aligned}$$

is nuclear. Indeed, let us consider this composition

$$\mathcal{A}^1_\alpha \xrightarrow{V_g} \mathcal{A}^1_{\alpha'} \xrightarrow{D} L^1(\mathbb{D}, (1-|z|^2)\,dA_{\alpha'})\,,$$

where D is the derivation operator defined by $D(f) = f'$ for $f \in \mathcal{A}^1_{\alpha'}$, which is bounded since $\|D(f)\|_{L^1\left(\mathbb{D},(1-|z|^2)\,dA_{\alpha'}\right)} \lesssim \|f\|_{\mathcal{A}^1_{\alpha'}}$. Hence $\widetilde{V_g} = D \circ V_g$ is nuclear.

From [4, Cor. 5.24], it follows that $\widetilde{V_g}$ is 1-integral. Therefore, from [4, Th. 5.19], it follows that $\widetilde{V_g}$ is order bounded: there exists some $h \in L^1(\mathbb{D}, (1-|z|^2)\,dA_{\alpha'})$ such that for every $f \in B_{\mathcal{A}^1_\alpha}$ we have $|fg'| \leq h$ a.e on $\mathbb{D}$. Since $\mathcal{A}^1_\alpha$ is separable, we get that

$$\int_{\mathbb{D}} \sup_{\substack{f \in \mathcal{A}^1_\alpha \\ \|f\| \leq 1}} |f(z)|\,|g'(z)|\,(1-|z|^2)\,dA_{\alpha'}(z) < \infty\,.$$

Since the point evaluation at point z is a bounded linear functional on $\mathcal{A}^1_\alpha$ with norm equal to $\|\delta_z\|_{(\mathcal{A}^1_\alpha)^*} = \frac{1}{(1-|z|^2)^{2+\alpha}}$, we get

$$\int_{\mathbb{D}} \frac{|g'(z)|}{(1-|z|^2)^{1+\alpha-\alpha'}}\, dA(z) < \infty\,.$$

In other words, the condition (3.23) is satisfied.

Now, assume that (3.23) is satisfied, we want to prove that $V_g : \mathcal{A}^1_\alpha \to \mathcal{A}^1_{\alpha'}$ is nuclear. Let $f \in \mathcal{A}^1_\alpha$, hence from [23, Prop. 4.23] we have for every $w \in \mathbb{D}$

$$\begin{aligned} f(w) &= \int_{\mathbb{D}} \frac{f(u)}{(1-\bar{u}w)^{\alpha+2}}\, dA_\alpha(u) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+2)}{n!\,\Gamma(\alpha+2)} \Big(\int_{\mathbb{D}} f(u)\, \bar{u}^n\, dA_\alpha(u) \Big) w^n \end{aligned}$$

where Γ is the classical Euler function (see [23, p.78]).
This suggests to define the following functional on $\mathcal{A}^1_\alpha$:

$$e^*_n(f) = \frac{\Gamma(n+\alpha+2)}{n!\,\Gamma(\alpha+2)} \int_{\mathbb{D}} f(u)\, \bar{u}^n\, dA_\alpha(u) \qquad \text{where } f \in \mathcal{A}^1_\alpha\,.$$

In other words, $e^*_n(f)$ is the n^{th} Taylor coefficient of f.
Let us compute the norm of e^*_n in the dual space of $\mathcal{A}^1_\alpha$:

$$\begin{aligned} \|e^*_n\|_{(\mathcal{A}^1_\alpha)^*} &= \sup_{\|f\|_{\mathcal{A}^1_\alpha} \le 1} |e^*_n(f)| \le \frac{\Gamma(n+\alpha+2)}{n!\,\Gamma(\alpha+2)} \sup_{\|f\|_{\mathcal{A}^1_\alpha} \le 1} \int_{\mathbb{D}} |f(u)|\, |u|^n\, dA_\alpha(u) \\ &\le \frac{\Gamma(n+\alpha+2)}{n!\,\Gamma(\alpha+2)} \sup_{\|f\|_{\mathcal{A}^1_\alpha} \le 1} \int_{\mathbb{D}} |f(u)|\, dA_\alpha(u) = \frac{\Gamma(n+\alpha+2)}{n!\,\Gamma(\alpha+2)}. \end{aligned}$$

Furthermore, for every $f \in \mathcal{A}^1_\alpha$ and every $z \in \mathbb{D}$

$$\begin{aligned} V_g(f)(z) &= \int_0^z f(w)\, g'(w) dw \\ &= \sum_{n=0}^{\infty} e^*_n(f) \Big(\int_0^z g'(w)\, w^n\, dw \Big) \\ &= \sum_{n=0}^{\infty} e^*_n(f).\, h_n(z) \end{aligned}$$

where $h_n := V_g(z^n)$.

And using [8, Prop.1.11], we get

$$\begin{aligned} \|h_n\|_{\mathcal{A}^1_{\alpha'}} &\approx \int_{\mathbb{D}} |h_n'(z)|\, (1-|z|^2)\, dA_{\alpha'}(z) \\ &\approx \int_{\mathbb{D}} |z|^n\, |g'(z)|\, (1-|z|^2)^{\alpha'+1}\, dA(z)\,. \end{aligned}$$

Finally, we have

$$\sum_{n=0}^{\infty} \|e^*_n\|_{(\mathcal{A}^1_\alpha)^*}\, .\, \|h_n\|_{\mathcal{A}^1_{\alpha'}} \lesssim \int_{\mathbb{D}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+2)}{n!\,\Gamma(\alpha+2)}\, |z|^n\, |g'(z)|\, (1-|z|^2)^{\alpha'+1}\, dA(z) \tag{3.24}$$

$$\approx \int_{\mathbb{D}} \frac{|g'(z)|}{(1-|z|^2)^{1+\alpha-\alpha'}}\, dA(z) < \infty. \tag{3.25}$$

We conclude that the sequences $\{e_n^*\} \subset (\mathcal{A}^1_\alpha)^*$ and $\{h_n\} \subset \mathcal{A}^1_{\alpha'}$ satisfy $\sum \|e_n^*\|_{(\mathcal{A}^1_\alpha)^*} \|h_n\|_{\mathcal{A}^1_{\alpha'}} < \infty$, so that the operator $T = \sum_{n=0}^{\infty} e_n^* \otimes h_n$ is a nuclear operator from $\mathcal{A}^1_\alpha$ to $\mathcal{A}^1_{\alpha'}$.

Moreover, for every $f \in \mathcal{A}^1_\alpha$ and every $z \in \mathbb{D}$ we have $V_g(f)(z) = T(f)(z)$.
Hence, $V_g = T$ on $\mathcal{A}^1_\alpha$. Therefore, V_g is nuclear. □

Remark 3.12. When $\alpha' > \alpha$, we can give easily examples of non-trivial symbols such that $V_g : \mathcal{A}^1_\alpha \longrightarrow \mathcal{A}^1_{\alpha'}$ is a nuclear operator: it suffices to choose g in the Bloch space $\mathcal{B}^\eta$ for some $\eta < \alpha' - \alpha$. In particular, for any polynomial g, $V_g : \mathcal{A}^1_\alpha \longrightarrow \mathcal{A}^1_{\alpha'}$ is a nuclear operator.

4. The case of Bloch spaces

In this section, we focus on Bloch spaces.
We shall need the following lemma which is well known from the specialists and a proof can be found in [6].

Lemma 4.1. *Let $T : \ell^\infty \longrightarrow Y$ be an operator. Then T is an absolutely summing operator if and only if T is a nuclear operator.*

Our first result is:

Theorem 4.2. *Let g be an analytic function on $\mathbb{D}$ and $\eta, \beta > 0$. The following assertions are equivalent:*

(a) $V_g : \mathcal{B}^\eta \longrightarrow \mathcal{B}^\beta$ is nuclear.

(b) $V_g : \mathcal{B}^\eta \longrightarrow \mathcal{B}^\beta$ is absolutely summing.

(c) $Q = \displaystyle\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta\,|g'(z)|}{|1-\overline{w}z|^{1+\eta}}\, dA(w) < +\infty.$

Remark 4.3. We can easily see that $Q \gtrsim \|g\|_{\mathcal{B}^\beta}$. Indeed, we have $\dfrac{1}{|1-\overline{w}z|^{1+\eta}} \geq \dfrac{1}{2^{1+\eta}}$.

Proof. (a) is equivalent to (b) from Lemma 4.1 and the fact that $\mathcal{B}^\eta$ is isomorphic to ℓ^∞ (see [11, Th.1.1]).
Assume now that $V_g : \mathcal{B}^\eta \to \mathcal{B}^\beta$ is absolutely summing, then by restriction $V_{g|_{\mathcal{B}^\eta_0}} : \mathcal{B}^\eta_0 \to \mathcal{B}^\beta$ is also absolutely summing (see [4, p.37]).

Now, for $w \in \mathbb{D}$, we consider $f_w(z) = \dfrac{1}{(1-\overline{w}z)^{1+\eta}}$ which belongs to $\mathcal{B}^\eta_0 \cap H^\infty$.
Using [19, Prop. 33 (b)] to the $\mathcal{B}^\eta_0$-valued random variable: $w \in \mathbb{D} \mapsto f_w \in \mathcal{B}^\eta_0$, and the fact that the dual space of $\mathcal{B}^\eta_0$ is the space $\mathcal{A}^1$, we get

$$\int_{\mathbb{D}} \|V_g(f_w)\|_{\mathcal{B}^\beta}\, dA(w) \leq \pi_1(V_g) \sup_{h\in B_{\mathcal{A}^1}} \int_{\mathbb{D}} |\langle h, f_w\rangle|\, dA(w)\,. \tag{4.1}$$

Thanks to the reproducing kernel formula, we have

$$\begin{aligned}\langle h, f_w\rangle_{\mathcal{A}^1,\mathcal{B}_0^\eta} &= \int_{\mathbb{D}} h(z)\overline{f_w(z)}(1-|z|^2)^{\eta-1}dA(z)\\ &= \int_{\mathbb{D}} \frac{h(z)}{(1-\bar{z}w)^{1+\eta}}(1-|z|^2)^{\eta-1}dA(z)\\ &= \frac{1}{\eta}\int_{\mathbb{D}} \frac{h(z)}{(1-\bar{z}w)^{2+\eta-1}}dA_{\eta-1}(z)\\ &= \frac{1}{\eta}h(w)\end{aligned}$$

Therefore, (4.1) becomes

$$\begin{aligned}\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |g'(z)|}{|1-\overline{w}z|^{1+\eta}}\, dA(w) &\le \frac{\pi_1(V_g)}{\eta} \sup_{h\in B_{\mathcal{A}^1}} \int_{\mathbb{D}} |h(w)|\, dA(w)\\ &\le \frac{\pi_1(V_g)}{\eta} \sup_{h\in B_{\mathcal{A}^1}} \|h\|_{\mathcal{A}^1}\\ &\le \frac{1}{\eta}\pi_1(V_g)\cdot\end{aligned}$$

It remains to prove that (c) implies (b). Suppose that Q is finite, we want to show that V_g is absolutely summing. We consider $f_1, \dots, f_N \in \mathcal{B}^\eta$. Then,

$$\sum_{j=1}^N \|V_g(f_j)\|_{\mathcal{B}^\beta} = \sum_{j=1}^N \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta\, |g'(z)|\, |f_j(z)|$$

We fix an arbitrary $c > 1$. For each $j \in \{1, \dots, N\}$, there exists some $z_j \in \mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta\, |g'(z)|\, |f_j(z)| \le c(1-|z_j|^2)^\beta\, |g'(z_j)|\, |f_j(z_j)|$$

Hence, we have

$$\begin{aligned}\sum_{j=1}^N \|V_g(f_j)\|_{\mathcal{B}^\beta} &\le c\sum_{j=1}^N (1-|z_j|^2)^\beta\, |g'(z_j)|\, |f_j(z_j)|\\ &= cT\end{aligned}$$

where $T = \sum_{j=1}^{N}(1-|z_j|^2)^\beta\,|g'(z_j)|\,|f_j(z_j)|$.

We assume now that $f_j(0) = 0$. Then, we have using [22, Cor. 4]

$$\begin{aligned}
T &\lesssim \sum_{j=1}^{N}(1-|z_j|^2)^\beta\,|g'(z_j)| \int_{\mathbb{D}} \frac{|f_j'(w)|(1-|w|^2)^\eta}{|w|.|1-\overline{w}z_j|^{1+\eta}}\,dA(w)\\
&\lesssim \sum_{j=1}^{N}(1-|z_j|^2)^\beta\,|g'(z_j)| \int_{\mathbb{D}} \frac{|f_j'(w)|(1-|w|^2)^\eta}{|1-\overline{w}z_j|^{1+\eta}}\,dA(w) \qquad \text{thanks to [6, Lemma 3.4]}\\
&\lesssim \int_{\mathbb{D}} \sum_{j=1}^{N}(1-|z_j|^2)^\beta\,|g'(z_j)| \frac{|f_j'(w)|(1-|w|^2)^\eta}{|1-\overline{w}z_j|^{1+\eta}}\,dA(w)\\
&\lesssim \int_{\mathbb{D}} \sum_{j=1}^{N}|f_j'(w)|\,(1-|w|^2)^\eta \left(\sup_j \frac{(1-|z_j|^2)^\beta|g'(z_j)|}{|1-\overline{w}z_j|^{1+\eta}}\right) dA(w)\\
&\lesssim \left(\sup_{w\in\mathbb{D}} \sum_{j=1}^{N}|f_j'(w)|\,(1-|w|^2)^\eta\right) \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta|g'(z)|}{|1-\overline{w}z|^{1+\eta}}\,dA(w)\\
&\lesssim Q\left(\sup_{w\in\mathbb{D}} \sum_{j=1}^{N}|f_j'(w)|\,(1-|w|^2)^\eta\right).
\end{aligned} \tag{4.2}$$

Let us remark that for every $w \in \mathbb{D}$, we have

$$\sum_{j=1}^{N}|f_j'(w)|(1-|w|^2)^\eta = \sup_{|\epsilon_j|=1}\Big|\sum_{j=1}^{N}\epsilon_j f_j'(w)(1-|w|^2)^\eta\Big|. \tag{4.3}$$

Hence,

$$\begin{aligned}
\sum_{j=1}^{N}\|V_g(f_j)\|_{\mathcal{B}^\beta} &\lesssim c\,Q\left(\sup_{|\epsilon_j|=1}\sup_{w\in\mathbb{D}}\Big|\sum_{j=1}^{N}\epsilon_j f_j'(w)(1-|w|^2)^\eta\Big|\right)\\
&\lesssim c\,Q\left(\sup_{|\epsilon_j|=1}\Big\|\sum_{j=1}^{N}\epsilon_j f_j\Big\|_{\mathcal{B}^\eta}\right).
\end{aligned}$$

Therefore, V_g is absolutely summing.

We now remove the restriction $f_j(0) = 0$. Take the function $\widetilde{f_j} = f_j - f_j(0)$ which belongs also to $\mathcal{B}^\eta$. Since $\widetilde{f_j}(0) = 0$, we have from the first case

$$\sum_{j=1}^{N}\|V_g(\widetilde{f_j})\|_{\mathcal{B}^\beta} \lesssim c\,Q\left(\sup_{|\epsilon_j|=1}\Big\|\sum_{j=1}^{N}\epsilon_j\widetilde{f_j}\Big\|_{\mathcal{B}^\eta}\right).$$

Now,

$$\sum_{j=1}^{N} \|V_g(f_j)\|_{\mathcal{B}^\beta} = \sum_{j=1}^{N} \|V_g(\widetilde{f_j} + f_j(0))\|_{\mathcal{B}^\beta}$$
$$\leq \sum_{j=1}^{N} \|V_g(\widetilde{f_j})\|_{\mathcal{B}^\beta} + \sum_{j=1}^{N} |f_j(0)|\, \|V_g(1)\|_{\mathcal{B}^\beta}$$

But, from the remark 4.3 we have

$$\|V_g(1)\|_{\mathcal{B}^\beta} = \|g - g(0)\|_{\mathcal{B}^\beta} \leq \|g\|_{\mathcal{B}^\beta} \lesssim Q < \infty$$

and from

$$\sum_{j=1}^{N} |f_j(0)| = \sup_{|\epsilon_j|=1} \Big| \sum_{j=1}^{N} \epsilon_j f_j(0) \Big|$$

we get

$$\sum_{j=1}^{N} \|V_g(f_j)\|_{\mathcal{B}^\beta} \lesssim c\, Q \sup_{|\epsilon_j|=1} \Big\| \sum_{j=1}^{N} \epsilon_j \widetilde{f_j} \Big\|_{\mathcal{B}^\eta} + Q \sup_{|\epsilon_j|=1} \Big| \sum_{j=1}^{N} \epsilon_j f_j(0) \Big|$$
$$\leq c\, Q \sup_{|\epsilon_j|=1} \Big\| \sum_{j=1}^{N} \epsilon_j f_j \Big\|_{\mathcal{B}^\eta} + Q \sup_{|\epsilon_j|=1} \Big\| \sum_{j=1}^{N} \epsilon_j f_j \Big\|_{\mathcal{B}^\eta}$$
$$\lesssim Q \sup_{|\epsilon_j|=1} \Big\| \sum_{j=1}^{N} \epsilon_j f_j \Big\|_{\mathcal{B}^\eta}$$

Therefore, V_g is absolutely summing. □

Corollary 4.4. *V_g is nuclear on $\mathcal{B}^\eta$ if and only if $g' = 0$.*
In other words, there is no non trivial nuclear operator on $\mathcal{B}^\eta$.

Proof. Assume that V_g is nuclear on $\mathcal{B}^\eta$, it follows from the preceding theorem that

$$\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\eta\, |g'(z)|}{|1-\overline{w}z|^{1+\eta}}\, dA(w) \geq \int_{\mathbb{D}} \frac{|g'(w)|}{(1-|w|^2)} dA(w)$$

using that the supremum over $z \in \mathbb{D}$ is larger than the value at point w.

From Lemma 3.9, we get that $g' = 0$. □

Our second result is:

Theorem 4.5. *Let $r \geq 1$ and $\eta, \beta > 0$. Let g be an analytic function on $\mathbb{D}$ and we consider the following assertions:*

(a) $V_g : \mathcal{B}^\eta \longrightarrow \mathcal{B}^\beta$ is r-summing.

(b) $V_g : \mathcal{B}^\eta_0 \longrightarrow \mathcal{B}^\beta$ is r-summing.

(c) $K = \displaystyle\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(r-1)}\,(1-|z|^2)^{\beta r}\,|g'(z)|^r}{|1-\overline{w}z|^{(1+\eta)r}}\,dA(w) < +\infty.$

Then, we have $(a) \Rightarrow (b) \Rightarrow (c)$.
If, in addition $\eta > 1$*, then (a), (b) and (c) are equivalent. Moreover,*

$$\pi_r(V_g) \approx K^{1/r}\,.$$

Proof. (a) $\Rightarrow$ (b) is obvious.
(b) implies (c): we assume that $V_g : \mathcal{B}_0^\eta \longrightarrow \mathcal{B}^\beta$ is r-summing.
Let us consider, for every $w \in \mathbb{D}$, $f_w(z) = \dfrac{(1-|w|^2)^{2/r'}}{(1-\overline{w}z)^{1+\eta}}$ which belongs to $\mathcal{B}_0^\eta \cap H^\infty$.
Again, using [19, Prop. 33 (b)] to the $\mathcal{B}_0^\eta$-valued random variable: $w \in \mathbb{D} \mapsto f_w \in \mathcal{B}_0^\eta$, and the fact that the dual space of $\mathcal{B}_0^\eta$ is the space $\mathcal{A}^1$, we get

$$\int_{\mathbb{D}} \|V_g(f_w)\|_{\mathcal{B}^\beta}^r\,dA(w) \le \pi_r^r(V_g) \sup_{h\in B_{\mathcal{A}^1}} \int_{\mathbb{D}} |\langle h, f_w\rangle|^r\,dA(w)\,. \tag{4.4}$$

Thanks to the reproducing kernel formula, we have

$$\begin{aligned}
\langle h, f_w\rangle_{\mathcal{A}^1,\mathcal{B}_0^\eta} &= \int_{\mathbb{D}} h(z)\,\overline{f_w(z)}(1-|z|^2)^{\eta-1}\,dA(z)\\
&= (1-|w|^2)^{2/r'} \int_{\mathbb{D}} \frac{h(z)}{(1-\overline{z}w)^{1+\eta}}(1-|z|^2)^{\eta-1}\,dA(z)\\
&= \frac{1}{\eta}(1-|w|^2)^{2/r'} \int_{\mathbb{D}} \frac{h(z)}{(1-\overline{z}w)^{2+\eta-1}}\,dA_{\eta-1}(z)\\
&= \frac{1}{\eta}(1-|w|^2)^{2/r'} h(w)\,.
\end{aligned}$$

Using the fact that the point evaluation δ_z is bounded on $\mathcal{A}^1$, with norm equal to $\|\delta_z\|_{(\mathcal{A}^1)^*} = \dfrac{1}{(1-|z|^2)^2}$, we get from (4.4)

$$\begin{aligned}
K = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(r-1)}\,(1-|z|^2)^{\beta r}\,|g'(z)|^r}{|1-\overline{w}z|^{(1+\eta)r}}\,dA(w) &\le \frac{\pi_r^r(V_g)}{\eta^r} \sup_{h\in B_{\mathcal{A}^1}} \int_{\mathbb{D}} |h(w)|^r\,(1-|w|^2)^{2(r-1)}\,dA(w)\\
&\le \frac{\pi_r^r(V_g)}{\eta^r} \sup_{h\in B_{\mathcal{A}^1}} \int_{\mathbb{D}} |h(w)| \frac{\|h\|_{\mathcal{A}^1}^{r-1}}{(1-|w|^2)^{2(r-1)}}\,(1-|w|^2)^{2(r-1)}\,dA(w)\\
&\le \frac{\pi_r^r(V_g)}{\eta^r} \sup_{h\in B_{\mathcal{A}^1}} \|h\|_{\mathcal{A}^1}^r\\
&\le \frac{\pi_r^r(V_g)}{\eta^r}\,.
\end{aligned}$$

It remains to see that (c) implies (a) when $\eta > 1$. We consider $f_1, \dots, f_N \in \mathcal{B}^\eta$, then

$$\sum_{j=1}^{N} \|V_g(f_j)\|_{\mathcal{B}^\beta}^r = \sum_{j=1}^{N} \Big(\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \, |g'(z)| \, |f_j(z)| \Big)^r$$

We fix an arbitrary $c > 1$. For each $j \in \{1, \dots, N\}$, there exists some $z_j \in \mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \, |g'(z)| \, |f_j(z)| \le c \, (1-|z_j|^2)^\beta \, |g'(z_j)| \, |f_j(z_j)|$$

Hence, we have

$$\sum_{j=1}^{N} \|V_g(f_j)\|_{\mathcal{B}^\beta}^r \le c^r \sum_{j=1}^{N} (1-|z_j|^2)^{\beta r} \, |g'(z_j)|^r \, |f_j(z_j)|^r \, .$$

Since $f_1, ..., f_N$ belong to B^η, then for all $j \in \{1, ..., N\}$, $f_j \in \mathcal{A}^1_\alpha$ for $\alpha \ge \eta$. Using [7, Lemma 3.1] with $\alpha = (1+\eta)r - 1 \ge \eta$, we get

$$\begin{aligned}
\sum_{j=1}^{N} (1-|z_j|^2)^{\beta r} \, |g'(z_j)|^r \, |f_j(z_j)|^r &\lesssim \sum_{j=1}^{N} (1-|z_j|^2)^{\beta r} \, |g'(z_j)|^r \int_{\mathbb{D}} \frac{|f_j(w)|^r (1-|w|^2)^{(1+\eta)r-1}}{|1-\overline{w} z_j|^{(1+\eta)r+1}} \, dA(w) \\
&\lesssim \int_{\mathbb{D}} \sum_{j=1}^{N} (1-|z_j|^2)^{\beta r} \, |g'(z_j)|^r \, \frac{|f_j(w)|^r (1-|w|^2)^{(1+\eta)r-1}}{|1-\overline{w} z_j|^{(1+\eta)r+1}} \, dA(w) \\
&\lesssim \int_{\mathbb{D}} \sum_{j=1}^{N} |f_j(w)|^r \, (1-|w|^2)^{(\eta-1)r} \, \sup_j \frac{(1-|w|^2)^{2(r-1)} (1-|z_j|^2)^{\beta r} |g'(z_j)|^r}{|1-\overline{w} z_j|^{(1+\eta)r}} \, dA(w) \\
&\lesssim \sup_{w\in\mathbb{D}} \sum_{j=1}^{N} |f_j(w)|^r \, (1-|w|^2)^{(\eta-1)r} \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(r-1)} (1-|z|^2)^{\beta r} |g'(z)|^r}{|1-\overline{w} z|^{(1+\eta)r}} \, dA(w) \\
&\lesssim K \Big(\sup_{w\in\mathbb{D}} \sum_{j=1}^{N} |f_j(w)|^r \, (1-|w|^2)^{r(\eta-1)} \Big).
\end{aligned}$$

For every $w \in \mathbb{D}$, we have:

$$\sum_{j=1}^{N} |f_j(w)|^r \, (1-|w|^2)^{(\eta-1)r} = \sup_{a\in B_{\ell^{r'}}} \Big| \sum_{j=1}^{N} a_j f_j(w) \, (1-|w|^2)^{(\eta-1)} \Big|^r .$$

Using [23, Prop. 7], since $\eta > 1$, we get

$$\sum_{j=1}^{N} |f_j(w)|^r \, (1-|w|^2)^{(\eta-1)r} \le \sup_{a\in B_{\ell^{r'}}} \Big\| \sum_{j=1}^{N} a_j f_j \Big\|_{\mathcal{B}^\eta}^r .$$

Hence,

$$\Big(\sum_{j=1}^{N} \|V_g(f_j)\|_{\mathcal{B}^\beta}^r \Big)^{1/r} \lesssim K^{1/r} \sup_{a\in B_{\ell^{r'}}} \| \sum_{j=1}^{N} a_j f_j \|_{\mathcal{B}^\eta}.$$

We obtain that $V_g : \mathcal{B}^\eta \to \mathcal{B}^\beta$ is r-summing and $\pi_r(V_g) \lesssim K^{1/r}$. □

As an immediate corollary of this result:

Corollary 4.6. *If $V_g : \mathcal{B}^\beta \longrightarrow \mathcal{B}^\beta$ is r-summing, where $\beta > 0$, and $\gamma > \max(1,\beta)$ then $V_g : \mathcal{B}^\gamma \longrightarrow \mathcal{B}^\gamma$ is r-summing.*

Remark 4.7. Let us give an example of non-trivial symbols such that $V_g : \mathcal{B}^\eta \longrightarrow \mathcal{B}^\beta$ is a nuclear operator.

If $\beta - \eta > 1$, and $g \in \mathcal{B}^{\beta-\eta-1}$, then the Volterra operator $V_g : \mathcal{B}^\eta \longrightarrow \mathcal{B}^\beta$ is a nuclear operator. Indeed, let us remark that

$$\begin{aligned} Q = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta\,|g'(z)|}{|1-\overline{w}z|^{1+\eta}}\,dA(w) &\leq \sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta-\eta-1}|g'(z)| \\ &= \|g\|_{\mathcal{B}^{\beta-\eta-1}} \end{aligned} \tag{4.5}$$

Hence, $Q < +\infty$ and the conclusion follows from Theorem 4.2.
In particular, if g is a polynomial, then $V_g : \mathcal{B}^\eta \longrightarrow \mathcal{B}^\beta$ is nuclear when $\beta > \eta + 1$.

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