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Systems synchronisation in Max-Plus algebra: a controlled invariance perspective

In memoriam Édouard Wagneur

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Abstract: We study the concept of systems synchronisation in Max-Plus algebra. We show that the problem of synchronising Max-Plus linear systems is twofold. It can be stated in terms of controlled invariance, and of coreachability. The controlled invariance is the property of a set for which a suitable control exists, that maintains the trajectory in the set. We recall some properties of controlled invariance, in particular the existence of a maximal controlled invariant set included into any given set. Beyond the determination of the admissible controlled invariant set, we define and study a problem of coreachability. The controlled invariant set is the target and we study the existence of a control law that brings the trajectory to the target from the initial state within a specific number of steps. We illustrate the study with an example from the literature.

Keywords: Systems synchronisation, controlled invariance, coreachability.

1. INTRODUCTION

The concept of (A, B) -invariance, or controlled invariance, was introduced independently by Basile & Marro (1969), and by Wonham & Morse (1970). It is the basic stone of the so-called geometric approach, for the control design of linear systems, that has given throughout the 70's the solution of many classical control problems, among which the disturbance decoupling problem, the regulator problem, and various observer design problems, see Wonham (1974); Basile & Marro (1992) for a complete account of these results. Later on, it has been instrumental in the 80's for the development of the robust control theory and the standard approach, see e.g. Trentelman *et al.* (2001). The geometric approach is an elegant framework, that permits both the verification of the existence of a control law solution of the problems, and the effective computation of solutions, when they exist.

The concept of controlled invariance was shown to be useful in the framework of systems over a ring or a semiring. Various classes of distributed systems can indeed be seen as linear systems over an operator ring or semiring (see for instance Kamen (1978); McEneaney (2006)). The concept of controlled invariant module was introduced for the sake of generalizing the classical results to such classes of systems. Hautus (1982), and Conte & Perdon (1995), are seminal references for systems over a ring.

Systems over a semiring, and particularly Max-Plus linear systems, have been thoroughly studied in the seminal

reference Baccelli *et al.* (1992). They are useful to model timed discrete-event processes subject to synchronization constraints, especially in production management, logistic systems, transportation networks. Many control problems from these areas have been formulated and solved within this framework, see for instance the complete survey from Hardouin *et al.* (2019), or Komenda *et al.* (2018). In the case of systems over a semiring, Amari *et al.* (2012) have used the notion of invariance in a problem of control design to solve a constraint satisfaction problem. Di Loreto *et al.* (2010) has addressed observation problems, and a steady-state control problem is considered by Gonçalves *et al.* (2016).

The basic results on controlled invariance for systems over a semiring are due to Katz (2007). He stated that it exists a maximal controlled invariant module in every given module in the max algebra and many other semirings. He pointed out that the computation of this maximal set is an open problem in general, and identified a number of important particular cases where the problem is solvable. Sufficient conditions are often met in practice, that allow this computation, but in turn, there are many different problems that pose the same question, and are open in general, for instance the computation of conditioned invariant modules Di Loreto *et al.* (2010), that is needed in observation problems Hardouin *et al.* (2017), or of the reachability module of a linear system (see Gaubert & Katz (2004)).

In the present paper, we define another notion that is closed to the concept of controlled invariance, that is the coreachability. We show that the synchronization of two Max-Plus linear systems comes down to a pair of subproblems. The first one is the problem of computing the maximal controlled invariant set included into a module that represent the synchronization constraint, and the second problem consists in computing the coreachable set of this maximal controlled invariant set, from which can be reached a trajectory satisfying the constraints.

The paper is organized as follows. In Section 2, we introduce the basic concepts, namely the controlled invariance, the coreachability, and elementary notions on Max-Plus modules. We treat the synchronization problem in Section 3, and illustrate the results in Section 4 using 2 examples.

2. BASIC CONCEPTS

2.1 Notation

Roman letters like a, b, \dots , denote numbers or vectors. Roman capitals A, B, \dots denote matrices, and calligraphic capitals $\mathcal{A}, \mathcal{B}, \dots$ denote sets. For two matrices A and B having the same number of rows, (A, B) denotes the matrix obtained by the concatenation of the columns of A and B . The i th component of a vector v is denoted v_i , and the entry of a matrix M that is situated in row i and column j is denoted M_{ij} . In the sequel, \mathbb{R}_{\max} denotes the max-plus semiring, that is the set $\mathbb{R} \cup \{-\infty\}$, endowed with the operations max, that is denoted \oplus , and plus, the usual addition, denoted \otimes . Both operations are extended to vectors and matrices of compatible dimensions, in a usual way. We sometimes omit the product sign \otimes , writing for instance $Ax = A \otimes x$. The max-plus semiring completed by the element $+\infty$ is denoted $\overline{\mathbb{R}}_{\max}$. Both are idempotent semirings, say the max and the plus operations are internal laws, associative. They have $-\infty$ and 0 as neutral elements, respectively. The operation plus distributes over the operation max, and in addition, the operation max is idempotent, *i.e.* for every elements x in \mathbb{R}_{\max} or $\overline{\mathbb{R}}_{\max}$, we have $x \oplus x = x$. We denote I_n the $n \times n$ matrix in which diagonal elements are equal to 0, and the other entries are equal to $-\infty$. We also denote ϵ the neutral element for operation \oplus , say $\epsilon = -\infty$, and we use the same notation ϵ to denote any matrix with all components equal to $-\infty$.

In the following, we make an intense use of set operations. The collection of all subsets of a given set \mathcal{S} is denoted $\mathcal{P}(\mathcal{S})$. A matrix $M \in \mathbb{R}_{\max}^{n \times m}$ being given, we denote by $\text{Im } M$ its image.

$$\text{Im } M = \{x \in \mathbb{R}_{\max}^n \mid \exists v \in \mathbb{R}_{\max}^m, x = Mv\}.$$

The image by M of any set $\mathcal{S} \subset \mathbb{R}_{\max}^m$ is denoted $M\mathcal{S}$, and the preimage by M of any set $\mathcal{T} \subset \mathbb{R}_{\max}^n$ is denoted $M^{-1}\mathcal{T}$. The difference of two sets $\mathcal{S}, \mathcal{S}' \subset \mathbb{R}_{\max}^m$, denoted $\mathcal{S} \ominus \mathcal{S}'$ is defined as $\{x'' \in \mathbb{R}_{\max}^m \mid \exists x \in \mathcal{S}, x' \in \mathcal{S}', x'' \oplus x' = x\}$.

2.2 Controlled invariance

We consider in this part a controlled system of the form

$$x(k+1) = Ax(k) \oplus Bu(k+1) \text{ for } k \in \mathbb{N}, \quad (1)$$

where $A \in \mathbb{R}_{\max}^{n \times n}$, $B \in \mathbb{R}_{\max}^{n \times m}$, n and m are positive integers, and for $k \in \mathbb{N}$, $x(k) \in \mathbb{R}_{\max}^n$ and $u(k) \in \mathbb{R}_{\max}^m$.

The variable x is called the state of the system, and u is called its input. The vector $x(0)$ is called the initial value of (1).

Such a system is called a Max-Plus linear system. This class of models is of frequent use in production management, communication or transportation networks, and economical systems (see for instance Komenda *et al.* (2018)). An important and actual tendency is to make use of a geometric point of view, and the basic concept of this approach is that of controlled invariance.

Definition 2.1. A set \mathcal{S} is said to be controlled invariant if, for every vector $x_0 \in \mathcal{S}$, there exists a control u such that the solution of system (1), initialized at $x(0) := x_0$ remains entirely in \mathcal{S} , *i.e.* $\forall k \in \mathbb{N}, x(k) \in \mathcal{S}$.

The properties of controlled invariant sets are still under investigation, however there are a number of properties that are well known (the case of the submodules of \mathbb{R}_{\max}^n , or over various other semirings, has been particularly studied, see for instance Katz (2007); Di Loreto *et al.* (2010)).

Prop 2.1. A set $\mathcal{S} \subset \mathbb{R}_{\max}^n$ is controlled invariant if and only if the following inclusion holds true: $A\mathcal{S} \subset \mathcal{S} \ominus \text{Im } B$. Equivalently, \mathcal{S} is controlled invariant if and only if the following inclusion holds true: $\mathcal{S} \subset A^{-1}(\mathcal{S} \ominus \text{Im } B)$.

Proof By definition, the set \mathcal{S} is controlled invariant if each of its elements x satisfies $Ax \oplus Bu \in \mathcal{S}$ for some vector $u \in \mathbb{R}_{\max}^m$. The result follows since Bu takes all the values in $\text{Im } B$ when u varies, and by definition of the operation \ominus .

Further, we state the property, that is well-known for systems over a field.

Prop 2.2. The union of any family of controlled invariant subsets in $\mathcal{P}(\mathbb{R}_{\max}^n)$ is also a controlled invariant set. As a consequence, every subset of \mathbb{R}_{\max}^n , say \mathcal{S} , contains a unique maximal controlled invariant subset, which contains every other controlled invariant subset of \mathcal{S} . This set is equal to the union of all the controlled invariant sets included in \mathcal{S} . This maximal controlled invariant subset is denoted $\mathcal{V}_{\mathcal{S}}^*(A, B)$.

Proof Any vector in the union, say x , lies into one of the sets of the set family that is considered. Since the sets are assumed to be controlled invariant, there exists a control u such that $Ax \oplus Bu$ lies into the same set, which in turn is an element of the union of all the sets in this family. This shows the result.

As for systems over a field, one can try to compute the maximal controlled invariant subset of a given set, using the following procedure:

$$\mathcal{V}_0 = \mathcal{S}, \quad \mathcal{V}_{i+1} = \mathcal{S} \cap A^{-1}(\mathcal{V}_i \ominus \text{Im } B), \text{ for } i \in \mathbb{N}. \quad (2)$$

This set recurrence defines a sequence of subsets $\{\mathcal{V}_i\}$, that has the following properties.

Prop 2.3. The sequence of subsets $\{\mathcal{V}_i\}$ is decreasing. The intersection of all its terms, denoted $\mathcal{V}_{\omega} := \bigcap_{i \in \mathbb{N}} \mathcal{V}_i$, satisfies $\mathcal{V}_{\mathcal{S}}^*(A, B) \subset \mathcal{V}_{\omega}$.

In addition, this inclusion is an equality if there exists an integer k such that $\mathcal{V}_k = \mathcal{V}_{k+1}$.

Proof By the definition (2) of the sequence, we notice that $\mathcal{V}_1 \subset \mathcal{V}_0$. Further, we remark that the operator defining

the sequence iteration is isotone. As a consequence, the inclusion $\mathcal{V}_{i+1} \subset \mathcal{V}_i$ is implied by $\mathcal{V}_i \subset \mathcal{V}_{i-1}$. This permits to realize, by induction, that the sequence is decreasing. Further, we show that every step of the sequence is a set that contains the maximal controlled invariant set included into \mathcal{S} . This is the case of the initial step, since $\mathcal{V}_S^*(A, B) \subset \mathcal{S}$, by definition. In addition, $\mathcal{V}_S^*(A, B)$ is included into its image by the operator that defines the sequence (2). Since the operator is isotone, this permits to show that $\mathcal{V}_S^*(A, B)$ is a subset of \mathcal{V}_i , for $i \in \mathbb{N}$, which proves the second statement. Finally, if for some integer i , the equality $\mathcal{V}_i = \mathcal{V}_{i+1}$ is true, then the sequence stabilizes, and we have $\mathcal{V}_k = \mathcal{V}_i$ for every $k \geq i$, and therefore $\mathcal{V}_\omega = \mathcal{V}_i$. Further, we have $\mathcal{V}_\omega = \mathcal{S} \cap A^{-1}(\mathcal{V}_\omega \ominus \text{Im } B)$. Thus, we deduce using Proposition 2.1 that \mathcal{V}_ω is a controlled invariant set, which is included in \mathcal{S} , and therefore, it is included into the maximal controlled invariant set $\mathcal{V}_S^*(A, B)$. This reverse inclusion ends the proof.

2.3 Coreachability

Another basic concept that is useful in many control problems is that of coreachability. This concept is of frequent use in the framework of languages and automata. We introduce it in the context of max-plus linear systems.

Definition 2.2. Being given an integer $k \in \mathbb{N}$, a state x_0 is said to be k -coreachable w.r.t. a set $\mathcal{S} \in \mathcal{P}(\mathbb{R}_{\max}^n)$, and system (1), if there exists a control sequence u such that the state of system (1) is driven to $x(k) \in \mathcal{S}$.

The state x_0 is said to be coreachable w.r.t. a set $\mathcal{S} \in \mathcal{P}(\mathbb{R}_{\max}^n)$, and system (1), if there exists a control sequence u and an integer k such that the state of system (1) is driven to $x(k) \in \mathcal{S}$.

The set of coreachable states w.r.t. \mathcal{S} is denoted $\text{Cor}(\mathcal{S})$, and the set of k -coreachable states $\text{Cor}_k(\mathcal{S})$.

The system (1) being given, together with the set \mathcal{S} , we define the sequence of sets \mathcal{C}_i by the following recursion:

$$\mathcal{C}_0 = \mathcal{S}, \mathcal{C}_{i+1} = A^{-1}(\mathcal{C}_i \ominus \text{Im } B). \quad (3)$$

This sequence gives a characterization of the k -coreachable sets of \mathcal{S} .

Prop 2.4. With the previous notations, we have $\mathcal{C}_k = \text{Cor}_k(\mathcal{S})$, for $k \in \mathbb{N}$. As a consequence, we have $\text{Cor}(\mathcal{S}) = \bigcup_{i \in \mathbb{N}} \mathcal{C}_i$.

Proof Indeed, we can see that \mathcal{S} is the 0-coreachable set of \mathcal{S} . Further, let us define the set operator ϕ by $\phi(\mathcal{X}) = A^{-1}(\mathcal{X} \ominus \text{Im } B)$, for every $\mathcal{X} \in \mathcal{P}(\mathbb{R}_{\max}^n)$. By Definition 2.2, if a vector $x_0 \in \mathbb{R}_{\max}^n$ is 1-coreachable, then there exists two vectors $x \in \mathcal{S}$, and $u \in \mathbb{R}_{\max}^m$, such that $Ax_0 \oplus Bu = x$. This is equivalent to $x_0 \in \phi(\mathcal{S})$. More generally, $\phi(\mathcal{X})$ is equal to the 1-reachable set of \mathcal{X} , for every set $\mathcal{X} \in \mathcal{P}(\mathbb{R}_{\max}^n)$. The first conclusion comes from the fact that, indeed, the $(k+1)$ -coreachable set of \mathcal{S} is nothing but the 1-reachable set of the k reachable set of \mathcal{S} . The final statement is deduced from this characterization, and from Definition 2.2.

Notice that, in general, (3) does not permit to easily calculate the coreachable state of \mathcal{S} . Indeed, general sets are difficult to parameterize, the computation of each iteration may be difficult, and the sequence of set does

not stabilize or converge to any set. There is an important particular case, when the set \mathcal{S} is controlled invariant. In this case indeed, the sequence is increasing, so that the union of its elements can be interpreted as a limit, in the sense of the following statement.

Theorem 2.1. Assume that \mathcal{S} is a controlled invariant set of system (1). Then, the sequence \mathcal{C}_i is increasing. Its limit, defined by $\mathcal{C}_\omega = \bigcup_{i \in \mathbb{N}} \mathcal{C}_i$, is equal to $\text{Cor}(\mathcal{S})$, and it satisfies $\mathcal{C}_\omega = A^{-1}(\mathcal{C}_\omega \ominus \text{Im } B)$, and:

$$\mathcal{C}_\omega = \inf \{ \mathcal{X} \in \mathcal{P}(\mathbb{R}_{\max}^n) \mid A^{-1}(\mathcal{X} \ominus \text{Im } B) \subset \mathcal{X} \}.$$

Proof Define the set operator ϕ by $\phi(\mathcal{X}) = A^{-1}(\mathcal{X} \ominus \text{Im } B)$, for every $\mathcal{X} \in \mathcal{P}(\mathbb{R}_{\max}^n)$. This operator defines the recurrence (3), and is also a characterization of controlled invariance. Therefore, if \mathcal{S} is controlled invariant, then $\mathcal{S} \subset \phi(\mathcal{S})$. Hence, in this situation, we have the inclusion $\mathcal{C}_0 \subset \mathcal{C}_1$. Further, since ϕ is isotone, one checks by induction that the inclusion $\mathcal{C}_i \subset \mathcal{C}_{i+1}$ is actually true for $i \in \mathbb{N}$, so that the sequence \mathcal{C}_i is increasing. We know from Proposition 2.4 that the union of all the elements of the sequence (3) is equal to the coreachable set of \mathcal{S} . We now show that $\mathcal{C}_\omega \subset \phi(\mathcal{C}_\omega)$. This ultimately comes from the fact that \mathcal{S} is controlled invariant. This first implies that $\mathcal{C}_0 \subset \mathcal{C}_1$, which in turns implies that $\bigcup_{i \in \mathbb{N}} \mathcal{C}_i = \bigcup_{i \geq 1} \mathcal{C}_i$. Then, we observe that $\bigcup_{i \geq 1} \mathcal{C}_i = \bigcup_{i \in \mathbb{N}} \phi(\mathcal{C}_i)$, and the conclusion follows from the fact that the latter union is included into $\phi(\mathcal{C}_\omega)$, like each of its terms $\phi(\mathcal{C}_i)$. Reversely, by definition of \mathcal{C}_ω , for every vector x into \mathcal{C}_ω , there exists an integer k such that $x \in \mathcal{C}_k$. This implies $\phi(\{x\}) \subset \mathcal{C}_{k+1}$, so that $\phi(\{x\}) \subset \mathcal{C}_\omega$, since \mathcal{C}_ω is the union of all the sets \mathcal{C}_i . This ends the proof.

2.4 Max-plus modules

In practice, equations (2) and (3) are useful if we are able to compute efficiently the successive images by the operator ϕ . This is in particular the case when the sets \mathcal{V}_i and \mathcal{C}_i are finitely generated modules. In this case, there are algorithms that permit to compute the image of B , the difference between any finitely generated module and this image, and its preimage by A , and all this sets are finitely generated modules. Another useful property is that the iterations stabilize. It is not always met, and, as a consequence, it is not certain that the limit is finitely generated (there are examples in the literature, which were inspired from Wagneur (1996), of sequences that tend to a limit that is not finitely generated). See also Conte & Perdon (1995), for a discussion in the case of systems over a ring, and Katz (2007) for Max-Plus linear systems.

We just recall the basic definitions and results.

By definition, a submodule of \mathbb{R}_{\max}^n is said to be of finite type, or finitely generated, if there exists an integer q , and a matrix $M \in \mathbb{R}_{\max}^{n \times q}$ so that $\mathcal{M} = \text{Im } M$. There are two different ways to represent modules of finite type. This was shown in Butkovic & Hegedüs (1984). They stated that the family of finitely generated submodules of \mathbb{R}_{\max}^n coincides with the family of cones of finite type, which are sets of the form

$$\text{Cone}(C, D) = \{x \in \mathbb{R}_{\max}^n \mid Cx = Dx\},$$

where C and D are $p \times n$ matrices, for some integer p .

Theorem 2.2. Being given a max-plus module $\mathcal{M} \subset \mathbb{R}_{\max}^n$, the following assertions are equivalent.

- (i) There exists an integer q and a matrix $M \in \mathbb{R}_{\max}^{n \times q}$ such that $\mathcal{M} = \text{Im } M$.
- (ii) There exists an integer p and matrices $C, D \in \mathbb{R}_{\max}^{p \times n}$ such that $\mathcal{M} = \text{Cone}(C, D)$.

This result has been often commented in the literature. The algorithms to pass from a representation to the other one have been first introduced by Butkovic & Hegedüs (1984). They were interpreted in terms of duality by Gaubert & Katz (2007), and they have been refined and generalized by Allamigeon *et al.* (2010).

3. FORCED SYNCHRONIZATION OF MAX-PLUS LINEAR SYSTEMS

3.1 Problem definition

Consider two systems, say (Σ_1) , and (Σ_2) defined as follows, for $k \in \mathbb{N}$:

$$\begin{aligned} (\Sigma_1) \quad & \begin{cases} x(k+1) = A \otimes x(k) \oplus B \otimes u(k+1), \\ y(k) = C \otimes x(k), \end{cases} \\ (\Sigma_2) \quad & \begin{cases} w(k+1) = D \otimes w(k), \\ v(k) = E \otimes w(k). \end{cases} \end{aligned}$$

The systems (Σ_1) and (Σ_2) are said to have synchrone outputs if their output solutions coincide: $y(k) = v(k)$, for $k \geq 0$. When no input is applied (say $u(k) = \epsilon$, for $k \geq 0$), we say that they synchronize in finite time if there exist an integer k_0 such that $y(k) = v(k)$, for $k \geq k_0$. The interest for these notions comes from the definition of convergence in the framework of timed discrete-event systems.

The problem that is addressed here is that of the forced synchronization, which is formulated as follows.

Problem 3.1. (Forced synchronization) We say that the system is synchronizable, by control, if there exists a control law u that forces the synchronisation of (Σ_1) and (Σ_2) in finite time, other said there exists an integer k_0 , such that $\forall k \geq k_0$, the outputs of both systems (Σ_1) and (Σ_2) are equal: $y(k) = v(k)$.

Such a control u is said to be admissible for the synchronization of (Σ_1) and (Σ_2) . If the conditions arise for a given k_0 , then the system is said to be k_0 -synchronizable

There are many variants of this problem. We introduce here two of them.

Problem 3.2. (Forced subsynchronization) We say that the system is subsynchronizable, by control, if there exists a control law u and an integer k_0 , such that $\forall k \geq k_0$, the output of (Σ_1) is less than or equal to that of (Σ_2) : $y(k) \leq v(k)$.

If the conditions arise for a given k_0 , then the system is said to be k_0 -subsynchronizable

This last problem is very useful in practice. When the exact synchronization is not possible, one seeks for the subsynchronizability. When the max-plus systems represent the evolution of the dates of timed discrete event systems (see for instance Hardouin *et al.* (2019)), the system (Σ_2) represents the dynamics or the trajectory that must be tracked, which can be interpreted as the specification of deadlines for the occurrence of events. The inequality in Problem 3.2 means that the outputs of the controlled

system are produced before the deadline. In manufacturing problems, one often seeks for just-in-time strategies, where the difference between the prescribed dates and the ones that are actually realized is minimized.

Remark 3.1. (Forced state synchronization) A particular case that is often considered in the literature is the synchronization of states. We say that the system is state synchronizable, by control, if there exists a control law u and an integer k_0 , such that $\forall k \geq k_0$, the states of both systems (Σ_1) and (Σ_2) are equal: $x(k) = w(k)$. If the conditions arise for a given k_0 , then the system is said to be k_0 -state synchronizable. This last problem is often implicitly considered in the literature, in the sense that many an author propose control design methods that are illustrated by examples where the trajectories are synchronized. To handle it, one simply chooses $C = I_n$, and $E = I_n$.

3.2 Twofold conditions of synchronization

We show here that synchronizability is reduced to two sub problems. We should identify a region that is both controlled invariant and coreachable from the initial conditions. This is the region of vectors $x(k_0)$ that appear in Problem 3.1.

The first step toward this formulation consists in defining the extended system (Σ_e) , composed of (Σ_1) and (Σ_2) :

$$(\Sigma_e) \quad x_e(k+1) = A_e \otimes x_e(k) \oplus B_e \otimes u(k+1),$$

where $x_e^T = (x^T, w^T)$, and :

$$A_e = \begin{pmatrix} A & \epsilon \\ \epsilon & D \end{pmatrix}, B_e = \begin{pmatrix} B \\ \epsilon \end{pmatrix}.$$

The second step is to show that the condition for forced synchronization, in Problem 3.1, can be stated in terms of properties of $x_e(k_0)$. This is the following fact.

Fact 3.1. The system is synchronizable if and only if there exists a control u , and an integer k_0 , such that both following conditions hold true:

- (i) $x_e(k_0)$ is the initial condition of a trajectory that remains in $\text{Cone}(C_e, D_e)$, where the matrices C_e and D_e are defined as:

$$C_e = (C \ \epsilon), \ D_e = (\epsilon \ E).$$

- (ii) $x_e(k_0)$ is reachable from $x_e(0)$.

Proof The first item comes from the definition of Problem 3.1, and from the fact that the system (Σ_e) is time invariant, as are Σ_1 and Σ_2 . The second item directly comes from the statement of Problem 3.1.

Using the concepts introduced in Section 2.2, one can rephrase the above facts as necessary and sufficient conditions for synchronizability, stated in terms of controlled invariance and coreachability. It ultimately appears that synchronizability is a property of the initial condition $x_e(0)$. We formulate here the conditions both for synchronizability, and k_0 -synchronizability when a finite horizon is prescribed.

Theorem 3.1. Both following statements hold true.

- (i) The systems (Σ_1) and (Σ_2) are synchronizable if and only if the initial condition of (Σ_e) , that is $x_e(0)$, lies in the coreachable set of the supremal controlled invariant module included into $\text{Cone}(C_e, D_e)$:

$$x_e(0) \in \mathcal{C}or(\mathcal{V}_{\text{Cone}(C_e, D_e)}^*) .$$

(ii) Be given an integer $k_0 \in \mathbb{N}$, the systems (Σ_1) and (Σ_2) are k_0 -synchronizable if and only if the initial condition of (Σ_e) , that is $x_e(0)$, lies in the k_0 -coreachable set of the supremal controlled invariant module included into $\text{Cone}(C_e, D_e)$:

$$x_e(0) \in \mathcal{C}or_{k_0}(\mathcal{V}_{\text{Cone}(C_e, D_e)}^*) .$$

Proof According to section 2.2, the first condition of Fact 3.1 means that $x(k_0)$ lies in the maximal controlled invariant module included into $\text{Cone}(C_e, D_e)$, say $\mathcal{V}_{\text{Cone}(C_e, D_e)}^*$. The results mentioned in Section 2.3 show in turn that $x(k_0)$ must be reachable from the initial condition, which leads to the second statement of the theorem. If the horizon k_0 is not prescribed, then there exists an integer k_0 such that the trajectory reaches $\mathcal{V}_{\text{Cone}(C_e, D_e)}^*$ at time k_0 if and only if this set is reachable from the initial condition. This ends the proof.

We can state the solution of problems 3.2. The proof directly follows from that of Theorem 3.1.

Corollary 3.1. We have the following solution of problems 3.2.

(iii) Problem 3.2 has a solution if and only if

$$x_e(0) \in \mathcal{C}or(\mathcal{V}_{\text{Cone}(C'_e, D_e)}^*) ,$$

with $C'_e = (CE)$.

(iv) Problem 3.2 has a solution for a given k_0 if and only if

$$x_e(0) \in \mathcal{C}or_{k_0}(\mathcal{V}_{\text{Cone}(C'_e, D_e)}^*) .$$

In the following, we shall illustrate Theorem 3.1 with a simple example, and Corollary 3.1 on an example taken from the literature.

4. EXAMPLES

We have implemented the algorithms (2) (3) on ScicosLab, using in particular the Max-Plus toolbox and a personal implementation of the algorithms developed by Allami-geon *et al.* (2010) to calculate cone or image representations of Max-Plus modules.

Example 4.1. In order to illustrate Theorem 3.1, we consider two systems. (Σ_1) is a system to be controlled and (Σ_2) is a reference system. We wish to find integer \hat{k} from which the output trajectory of (Σ_1) will synchronise with that of (Σ_2) , starting with

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , w_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} .$$

The systems are defined as follows:

$$\begin{aligned} (\Sigma_1) \quad & \begin{cases} x(k+1) = A_1 \otimes x(k) \oplus B_1 \otimes u(k+1) , \\ y(k) = C \otimes x(k) , \end{cases} \\ (\Sigma_2) \quad & \begin{cases} w(k+1) = A_2 \otimes w(k) , \\ v(k) = E \otimes w(k) , \end{cases} \end{aligned}$$

with

$$A_1 = \begin{pmatrix} 1 & 0 \\ \epsilon & 0 \end{pmatrix} , B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , A_2 = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} .$$

Here C and E are the 2×2 identity matrices. We first build the extended system (Σ_e) and seek for the supremal controlled invariant module included into

$$\text{Cone} \left(\begin{pmatrix} 0 & \epsilon & \epsilon \\ \epsilon & 0 & \epsilon \end{pmatrix} , \begin{pmatrix} \epsilon & \epsilon & 0 \\ \epsilon & \epsilon & 0 \end{pmatrix} \right) .$$

This module is generated by the columns of a matrix M , that is defined below. Here, starting from (x_0, w_0) , no synchronisation is possible before $k = 2$. Indeed, $x_e(0)$ lies in \mathcal{C}_2 , the second step of the algorithm (3) initialized by $\mathcal{S} = \mathcal{V}_{\text{Cone}(C_e, D_e)}^*$. The set \mathcal{C}_2 is generated from the columns of matrix N , defined as:

$$M = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \\ 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} , N = \begin{pmatrix} \epsilon & \epsilon & \epsilon & 0 & -1 & \epsilon \\ \epsilon & \epsilon & 0 & \epsilon & \epsilon & 0 \\ 0 & \epsilon & \epsilon & -1 & \epsilon & -2 \\ \epsilon & 0 & 0 & \epsilon & 0 & \epsilon \end{pmatrix} .$$

Example 4.2. In their article Necoara *et al.* (2006) propose a design method, called Receding Horizon Control method. They give an example of model tracking, where the aim is to design a control law so that the state vector of the system follows a reference signal r_{sys} given by

$$r_{sys}(k) = (17 \ 15 \ 1 \ 10)^T + (4.5 \ 4.5 \ 4.5 \ 4.5)^T * k ,$$

with initial state $r_0 = (17 \ 15 \ 1 \ 10)^T$.

In this case, we consider that the system (Σ_1) is a system to be controlled and (Σ_2) is the reference system $r_{sys}(k)$.

We wish to find integer \hat{k} from which the state trajectory of (Σ_1) will subsynchronise with the state of (Σ_2) , starting with $w_0 = r_0$ and $x_0 = (20 \ 31.5 \ 42 \ 51.5)^T$. Here we have:

$$\begin{aligned} (\Sigma_1) \quad & \begin{cases} x(k+1) = A_1 \otimes x(k) \oplus B_1 \otimes u(k+1) , \\ y(k) = C \otimes x(k) , \end{cases} \\ (\Sigma_2) \quad & \begin{cases} w(k+1) = A_2 \otimes w(k) , \\ v(k) = E \otimes w(k) , \end{cases} \end{aligned}$$

with

$$A_1 = \begin{pmatrix} \epsilon & 1 & \epsilon & \epsilon \\ \epsilon & \epsilon & 2 & \epsilon \\ \epsilon & \epsilon & \epsilon & 3 \\ 4 & \epsilon & \epsilon & \epsilon \end{pmatrix} , B_1 = \begin{pmatrix} 2 \\ \epsilon \\ \epsilon \\ \epsilon \end{pmatrix} ,$$

and

$$A_2 = \begin{pmatrix} 4.5 & \epsilon & \epsilon & \epsilon \\ \epsilon & 4.5 & \epsilon & \epsilon \\ \epsilon & \epsilon & 4.5 & \epsilon \\ \epsilon & \epsilon & \epsilon & 4.5 \end{pmatrix} .$$

Here C and E are 4×4 identity matrices. In order to solve this forced subsynchronization problem (3.2), we first build the extended system (Σ_e) and seek for the supremal controlled invariant module included into

$$\text{Cone} \left(\begin{pmatrix} 0 & \epsilon & \epsilon & \epsilon & 0 & \epsilon & \epsilon \\ \epsilon & 0 & \epsilon & \epsilon & 0 & \epsilon & \epsilon \\ \epsilon & \epsilon & 0 & \epsilon & \epsilon & 0 & \epsilon \\ \epsilon & \epsilon & \epsilon & 0 & \epsilon & \epsilon & 0 \end{pmatrix} , \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon & 0 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & 0 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & 0 & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & 0 \end{pmatrix} \right) .$$

This module is not controlled invariant. Therefore we proceed as in Cárdenas *et al.* (2017) seeking for the supremal controlled invariant module $\mathcal{V}_{\text{Cone}(C'_e, D_e)}^* = \text{Im}M$, where:

$$M = \begin{pmatrix} \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & 0 & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & 0 & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & 0 \\ -7.5 & 0 & \epsilon & \epsilon & \epsilon & 0 & -3.5 & -6 \\ -4 & \epsilon & 0 & \epsilon & \epsilon & -4.5 & 0 & -2.5 \\ -1.5 & \epsilon & \epsilon & 0 & \epsilon & -2 & -5.5 & 0 \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & 0 \\ 0 & \epsilon & \epsilon & \epsilon & 0 & -0.5 & -4 & -6.5 \end{pmatrix} .$$

Here, starting from (x_0, w_0) , no synchronisation is possible before $k = 26$, indeed, $x_e(0) \in \mathcal{C}_{26}$, the 26th step of algorithms (3) initialized by $\mathcal{S} = \mathcal{V}_{\text{Cone}(C'_e, D_e)}^*$. Here, \mathcal{C}_{26} is generated from the columns of matrix N defined as:

$$N = \begin{pmatrix} \epsilon & \epsilon & 0 & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \\ 0 & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & 0 & \epsilon \\ \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & \epsilon & 0 \\ -59.5 & 0 & -56 & \epsilon & \epsilon & \epsilon & -54 & -55.5 \\ -56 & \epsilon & -52.5 & 0 & \epsilon & \epsilon & -58.5 & -52 \\ -53.5 & \epsilon & -50 & \epsilon & 0 & \epsilon & -56 & -57.5 \\ -52 & \epsilon & -56.5 & \epsilon & \epsilon & 0 & -54.5 & -56 \end{pmatrix}.$$

Problem 3.2 is solved with $k_0 = 26$, after Corollary 3.1.

5. CONCLUSION

We have introduced the concept of coreachable set, recalled the definition of controlled invariance, and defined a controlled synchronization problem for a pair of systems, in the context of Max-Plus linear systems. The problem consists of driving the first system in such a way that its solution coincides with that of the second system, after a finite time. We have shown that synchronization is a property of the initial conditions of the two systems. The computation of the set of admissible initial conditions is reduced to the computation of the maximal controlled invariant module, contained in a cone that represents the synchronization constraint, and of its infimal coreachable module. We also treated on the same way a problem of subsynchronization, that can appear when formalizing a just-in-time control design question. The results have been illustrated on two examples.

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REFERENCES

- X. Allamigeon, S. Gaubert, and É. Goubault (2010). The Tropical Double Description Method. In J.-Y. Marion & T. Schwentick (Eds.), *Proceedings of the 27th International Symposium on Theoretical Aspects of Computer Science (STACS 2010)*, 5, 47–58. Dagstuhl, Germany.
- S. Amari, I. Demongodin, J. J. Loiseau, and C. Martinez (2012). Max-plus control design for temporal constraints meeting in timed event graphs. *IEEE Trans. Automatic Control*, 57(2), 462–467.
- F. Baccelli, G. Cohen, G. Olsder, and J.-P. Quadrat (1992). *Synchronization and Linearity*. Wiley.
- G. Basile, and G. Marro (1969). Controlled and conditioned invariant subspaces in linear system theory. *Journal of Optimization Theory and Applications*, 3(5), 306–315.
- G. Basile, and G. Marro (1992). *Controlled and conditioned invariants in linear system theory*. Prentice Hall Englewood Cliffs.
- P. Butkovic, and G. Hegedüs (1984). An elimination method for finding all solutions of the system of linear equations over an extremal algebra. *Ekonomicko-matematicky Obzor*, 20, 203–214.
- C. Cárdenas, J.J. Loiseau, and C. Martinez (2017). Invariance par retour d'état sur le demi-anneau max-plus. In *MSR 2017, Modélisation des Systèmes Réactifs*. Marseille.
- G. Conte, and A. M. Perdon (1995). The disturbance decoupling problem for systems over ring. *SIAM J. Control & Optimization*, 33, 750–764.
- M. Di Loreto, S. Gaubert, R. D. Katz, and J. J. Loiseau (2010). Duality between invariant spaces for max-plus linear discrete event systems. *SIAM J. Control & Optimization*, 48(8) 5606–5628.
- S. Gaubert and R. Katz. Rational semimodules over the max-plus semiring and geometric approach of discrete event systems. *Kybernetika*, 40(2), 153–180.
- S. Gaubert, and R. D. Katz. The Minkowski theorem for max-plus convex sets. *Linear Algebra and its Applications*, 421, 356–369.
- V.M. Gonçalves, C.A. Maia, and L. Hardouin (2016). On the steady-state control of timed event graphs with firing date constraints. *IEEE Transactions on Automatic Control*, 61(8), 2187–2202.
- L. Hardouin, Y. Shang, C.A. Maia, and B. Cottenceau (2017). Observer-based controllers for max-plus linear systems. *IEEE Trans. Automatic Control*, 62(5), 2153–2165.
- L. Hardouin, B. Cottenceau, Y. Shang, J. Raisch (2019). Control and State Estimation for max-plus Linear Systems. *Journal on Foundations and Trends in Systems and Control*, 6(1) 1–116.
- M. L. J. Hautus (1982). Controlled invariance in systems over ring. In: D. Hinrichsen, and A. Isidori (Eds.), *Proceeding of the Joint Workshop on Feedback and Synthesis of Linear and Nonlinear Systems* Lecture Notes in Control and information Sci., 39, 107–122. Springer, New York.
- E.W. Kamen (1978). An operator theory for functional differential equations. *Journal of Differential Equations*, 27 274–297.
- R. Katz (2007). Max-plus (A, B)-invariant spaces and control of timed discrete-event systems. *IEEE Trans. Automatic Control*, 52(2) 229–241.
- J. Komenda, S. Lahaye, J.-L. Boismond, and T. van den Boom (2018). Max-plus algebra in the history of discrete event systems. *Annual Review in Control*, 45, 240–249.
- W.M. McEneaney (2006). *Max-plus methods for nonlinear control and estimation*. Springer.
- I. Necoara, T. van den Boom, B. De Schutter, and J. Hellendoorn (2006). Stabilization of max-plus-linear systems using receding horizon control: The unconstrained case. In *Proceedings of the 2nd IFAC Conference on Analysis and Design of Hybrid Systems (ADHS'06)*, 148–153. Alghero, Italy.
- H. L. Trentelman, A. A. Stoorvogel, and M. Hautus (2001). *Control Theory for Linear Systems*. New York: Springer-Verlag.
- E. Wagneur (1996). Torsion matrices in the max-algebra. In *Proceedings of the Workshop on Discrete Event Systems (WODES'96)*, 165–168. Edinburgh, United Kingdom.
- W.M. Wonham (1974). Linear multivariable control. In *Optimal control theory and its applications*, 392–424. Springer.
- W.M. Wonham, and A.S. Morse (1970). Decoupling and pole assignment in linear multivariable systems: a geometric approach. *SIAM Journal on Control*, 8(1), 1–18.