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Chunxiao Yang * **Ina Taralova** ** **Jean Jacques Loiseau** ***

Laboratoire des Sciences du Numérique de Nantes, LS2N
UMR CNRS 6004, Ecole Centrale de Nantes
Nantes, France

* *(e-mail: chunxiao.yang@ls2n.fr)*
** *(e-mail: ina.taralova@ls2n.fr)*
*** *(e-mail: jean-jacques.loiseau@ls2n.fr)*

Abstract: In this paper, we investigate the influence of different numerical solutions methods on the chaoticity of fractional chaotic systems. In particular, fractional Chen system is discussed and approximated from both analytical and numerical points of view applying different calculation methods. The impacts of the chosen methods to the chaotic behavior of the fractional system are compared and analyzed.

Keywords: Fractional chaotic systems, bifurcation, singularity, fractional ABM corrector predictor approach, non-uniform grid, Lyapunov exponent

1. INTRODUCTION

The history of fractional calculus can be dated back to the late 17th century. In 1695, the fractional derivative made its first entrance to the domain of mathematics in a letter written to Guillaume de l'Hôpital by Gottfried Wilhelm Leibniz(Katugampola (2014)). The foundation of the fractional calculus was laid by Liouville in a paper from 1832(Liouville (1832)). Ever since then, the matter has existed solely in the domain of mathematics for centuries, during which period many characterizations of fractional integrals and derivatives have been discussed by numerous contributors(Valério et al. (2014)).

Due to the memory effect and many properties possessed by fractional calculus, researchers have found it suitable for modeling many real-life systems and have started exploring its application in science and engineering since the 1980s. Up till this days, the fractional differential equations and fractional systems have been employed in diverse disciplines like physics(Uchaikin (2013)), biology(Arfan et al. (2021)), and economics(Tarasov and Tarasova (2018)), cryptography(Yang et al. (2020)), and etc,. It has also been used to model composite materials behaviour(Krasnobrizha et al. (2016)).

However, the implementation of fractional systems remains relatively underdeveloped. One of the reasons contributing to the difficulty of using fractional systems is that there exist different characterizations for fractional derivatives. With different numerical solution approximation approaches, the system behaves differently from one method to another, which adds intricacy to the application of fractional systems as well. In addition, depending on the specific engineering problem, different requirements(analysis or control) of fractional systems behavior may be applied. Therefore, it is of great importance to understand the impacts of the different characterizations and calculation methods on the behavior of the fractional systems before applying them in specific applications. The problem is even more essential for the cases where complex, chaotic fractional systems are employed. Whether the aim is to have an excellent approximation of the original fractional system or a system with better chaotic properties(robust chaos) will lead to the adoption of different numerical approaches.

Concerning this topic, some original work on fractional nonlinear chaotic systems has been discussed in papers Yang et al. (2021a) and Yang et al. (2021b). However, a more detailed analysis needs to be conducted with insightful perceptions about how the numerical calculation methods influence the chaoticity of the systems regarding the different (control) parameters, the fractional derivatives orders, etc.

In this paper, following our previous work, we continuously oriented our research direction to the chaotic behavior changes brought upon the fractional systems by calculating the numerical solutions based on different numerical methods and initial conditions. The ranges of parameters where the systems behave chaotically with "robust chaos" are found and discussed applying different calculation methods. Bear in mind that the robust chaos is defined by the absence of periodic windows and coexisting attractors in some neighborhoods in the parameter space of a dynamical system(Zeraoulia and Sprott (2011)).

2. PRELIMINARIES

2.1 Fractional derivatives

During the long existence of fractional derivatives in the domain of applied mathematics, numerous characterisations(the term 'definitions' is also used) have been brought up by different mathematicians. However, for the applied problems, the most well-accepted and widely applied characterisation is the fractional derivatives in the Caputo sense. (Since it is the one whose initial conditions have clear interpretation in applied problem according to Petráš (2011)). The fractional derivative of this type can be expressed as follows(Caputo (1967)),

$$D_*^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \tag{1}$$

where α is the non-integer order; n is the ceiling of α; $t \geq 0$; and $\Gamma(.)$ in the equation stands for the Euler Gamma function given by:

$$\Gamma(\alpha) = \int_0^{\infty} \frac{t^{\alpha-1}}{e^t} dt. \tag{2}$$

2.2 Fractional chaotic system

Fractional order system. In the field of dynamic systems and control theory, fractional order systems are the dynamic systems that can be modeled by differential equations with non-integer order derivatives(Monje et al. (2010)). The equations for the fractional order dynamic system in Caputo sense can be expressed as given below,

$$\begin{aligned} D_t^{\alpha_i} \mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)) = f_i(x_1(t), x_2(t), ..., x_n(t), t) \\ x_i(0) = c_i, i = 1, 2, ..., n. \end{aligned} \tag{3}$$

In equation (3), α_i denotes the fractional derivative order for i-th differential equation consisting of the system; $x_i(0)$ stands for the initial condition; and f_i is in general a nonlinear function($i = 1, 2, ...n$).

Fractional chaotic system. The fractional chaotic systems are the fractional dynamic systems which exhibit chaotic behaviour. It has been found that the chaotic behaviour of many integer-order derivative chaotic systems can be preserved after extending them to fractional orders(Petráš (2011)). Hence, in our following work, the fractional chaotic systems adopted are obtained through extension from classical integer order derivative chaotic functions.

2.3 The stability of the fractional chaotic system

The equilibrium points $\mathbf{x}^*$ of system (3) can be obtained by solving the following equation,

$$\mathbf{f}(\mathbf{x}(t), t) = 0 \tag{4}$$

If a commensurate system with $\alpha_i = \alpha, i = 1, 2, ..., n$ is considered, then, according to the stability theorem defined in Tavazoei and Haeri (2007), the equilibrium points are locally asymptotically stable if the eigenvalue of the Jacobian matrix of the system (3) satisfies the following equation evaluated at equilibria.

$$|\arg(\mathrm{eig}(\mathbf{J}))| = |\arg(\lambda_i)| > \alpha\frac{\pi}{2}, i = 1, 2, ..., n \tag{5}$$

where $\mathbf{J}$ denotes the Jacobian matrix of (3), α is the commensurate fractional order, $\lambda_i(i = 1, 2, ...n)$ are its eigenvalues.

2.4 Lyapunov exponent

Lyapunov exponent measures the average rate of separation of close orbits of a dynamic system and has been widely used to indicate the presence of chaos. This rate can be mathematically expressed by the following equations,

$$|\delta\mathbf{X}(t)| \approx e^{\lambda t} |\delta\mathbf{X}(t)| \tag{6}$$

The λ in equation (6) is the Lyapunov exponent(LE) and $\delta(t)$ denotes the initial separation vector between two trajectories.

For an integer order derivative dynamic system with evolution equation of the form $\dot{x}_i(t) = f_i(x_1(t), x_2(t), ...x_n(t))$, $i = 1, 2, ..., n$, the Lyapunov exponents can be obtained from calculating the eigenvalues of matrix $\mathbf{\Lambda}$, given by following equations,

$$\begin{aligned} \mathbf{J}_{ij}(t) = \left.\frac{df_i(x)}{dx_j}\right|_{x(t)} \\ \dot{\mathbf{Y}} = \mathbf{J}\mathbf{Y}, \mathbf{Y_{ij}}(0) = \delta_{ij} \\ \mathbf{\Lambda} = \lim_{t\to\infty} \frac{1}{2t} \log\left(\mathbf{Y}(t)\mathbf{Y}^T(t)\right) \end{aligned} . \tag{7}$$

In the above equations, $\mathbf{J}$ is the Jacobian matrix of $\mathbf{f}$, which defines the evolution of the tangent vectors, given by the matrix $\mathbf{Y}$; δ_{ij} is the initial condition for $\mathbf{Y}$ denoting the small change at point $\mathbf{x(0)}$.

3. ANALYTICAL ANALYSIS OF FRACTIONAL CHAOTIC CHEN AND LU SYSTEMS

3.1 System equations

Hereafter we focus on the analysis of one fractional chaotic system, fractional Chen system with commensurate order α_c (smaller than 1) for all its differential equations, respectively. The system equation is given in equation (8).

$$\begin{cases} D^{\alpha_c}x_1(t) = a_c(x_2(t) - x_1(t)) \\ D^{\alpha_c}x_2(t) = (c_c - a_c)x_1(t) - x_1(t)x_3(t) + c_c x_2(t) \\ D^{\alpha_c}x_3(t) = x_1(t)x_2(t) - b_c x_3(t) \end{cases} \tag{8}$$

In the above equation, (a_c, b_c, c_c) are the parameters of the system.

To analyze the behavior of the fractional chaotic system, we first evaluate the fixed points of the fractional systems from the analytical point of view.

3.2 Equilibrium of the systems

Knowing that equilibrium points of the classical derivative order chaotic systems and their stability can be preserved with fractional derivative orders, the equilibria of fractional Chen system can be obtained using equation (4) by solving the following equations (9).

$$\begin{cases} a_c(x_2^* - x_1^*) = 0 \\ (c_c - a_c)x_1^* - x_1^*x_3^* + c_c x_2^*) = 0 \\ x_1^*x_2^* - b_c x_3^* = 0 \end{cases} \tag{9}$$

In Table.1, we give the 3 equilibrium points of the fractional Chen system. The parameters of the systems used for the calculation are $(a_c, b_c, c_c) = (35, 3, 28)$.

3.3 Singularity and the stability of the equilirbria

The singularity of the equilibrium point can be obtained through the conventional method, by calculating the eigenvalues of the system at the equilibria as given below,

$$\det(\lambda_c \mathbf{I} - \mathbf{J}_c) = \begin{bmatrix} \lambda_c + a_c & -a_c & 0 \\ -c_l + a_l + x_3^* & \lambda_c - c_c & x_1^* \\ -x_2^* & -x_1^* & \lambda_c + b_c \end{bmatrix} = 0 \quad (10)$$

where $\mathbf{J}_c$ in equation(10) represents the Jacobian matrix of the system equation for Chen system at the equilibrium points $E_c = (x_1^*, x_2^*, x_3^*)$; λ_c and stands for the eigenvalues of the system. Since both systems are of dimension 3, each equilibrium has 3 eigenvalues. The singularities of the 3-dimensional system's equilibrium can be determined through the following criteria:

(1) **Node** : all eigenvalues are real and have the same sign.
(2) **Saddle** : all eigenvalues are real and at least one of them is positive, at least and one is negative.
(3) **Focus-node** : one real eigenvalue and two conjugate complex eigenvalues with same signs for their real parts.
(4) **Saddle-Focus** : one real eigenvalue with the sign opposite to the sign of the real part of a pair of complex-conjugate eigenvalues.

With the same parameters for section 3.2, the singularities of the fractional Chen and Lu systems' corresponding equilibrium points are given in Table.1.

Analytically, according to the theorem proposed in paper Tarasov and Tarasova (2018) and equation (5), we also calculate the boundary fractional derivative orders for the equilibria of the system to be asymptotically stable. The stability results are also given in Table.1. The numerical GL method introduced in the book Petráš (2011) has been discussed in our paper Yang et al. (2021a), which confirms the analytical stability analysis.

4. NUMERICAL SIMULATION RESULTS

This section mainly analyzes and compares the numerical simulation results of the fractional Chen system calculated using two different numerical approaches. The first(M1) is the classical fractional ABM corrector predictor calculation method, which is performed on a uniform grid. The explicit formula deduction can be found in Diethelm et al. (2002). The second method(M2) is based on the classical ABM method but with a non-uniform grid that has variable steps generated by a chaotic skew tent map. A more detailed explanation can be found in our previous paper Yang et al. (2021a). For the confirmation of the stability results(last column) in Table.1, we have used the fractional derivative characterisation, and numerical calculation method introduced in paper Petráš (2011).

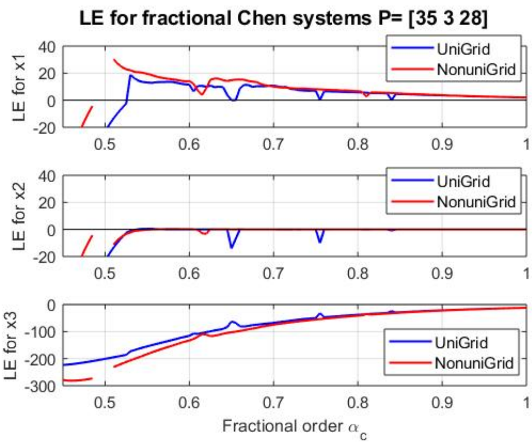


Fig. 1. Lyapunov exponent of fractional Chen system over fractional derivative order (α_c)

4.1 Lyapunov exponent results over fractional derivative orders

The Lyapunov exponent(LE), as mentioned in section 2.4, measures the separation of two closely initialized orbits of the dynamic systems. It is considered that for a dynamic system, a LE value greater than 0 indicates the presence of chaotic behavior. In addition, the greater LE value one system has, the higher level of disorder it possesses. For our following Lyapunov exponent analysis, the LEs of the fractional systems are calculated the same way as given in Danca and Kuznetsov (2018).

We first calculated the LE values for the Chen system over different fractional derivative orders. The ranges of fractional orders are from 0.45 to 1 and the system parameters are set to (35, 3, 28). Fig.1 shows the LE simulation results for the three state components(x_1, x_2, x_3) of the systems. The fractional derivative orders are evaluated with the step of 0.005. For each fractional order, we randomly choose the initial states other than the equilibria of the system and iterate 300000 times. The red curves stand for the results employing the non-uniform grid calculation method(M2), and the blue curves are for the classical calculation method(M1).

The first thing to be noticed is that, for both systems, the x_1 component of their state vectors possess LE greater than 0. This means that the systems are chaotic under the parameters given within some specific fractional derivative ranges. In terms of the different calculation methods adopted, we observe that the red and blue curves behave differently for the system's LE figure. To begin with, the red one for the x_1 component crosses the horizontal axis(where LE equals 0) at a smaller fractional derivative order prior to the blue one. With the red curves representing the results for the non-uniform grid calculation method(M2), and blue curves for the classical ABM method(M1), we can conclude that the fractional order's range for the system to be chaotic is greater applying the former method than the latter one. In addition, it is observable that the curves for the non-uniform grid are relatively smoother with fewer drop-down values along the different orders than the uniform grid curves. This

Table 1. Fractional systems equilibria and their stability

System	Equilibrium f(x)=0	Eigenvalue			Singularity	Fractional order for analytically stable system(eq.5)
		λ_1	λ_2	λ_3		
Fractional Chen	(0,0,0)	-30.8359	23.8359	-3	Saddle	$\alpha_c < 0$
	(-7.9373,-7.9373,21)	-18.4280	4.2140+14.8846i	4.2140-14.8846i	Saddle Focus	$\alpha_c < 0.8244$
	(7.9373,7.9373,21)	-18.4280	4.2140+14.8846i	4.2140-14.8846i	Saddle Focus	$\alpha_c < 0.8244$

shows that more robust chaos is acquired through the non-uniform grid calculation method.

Another point worth mentioning is that one can observe a gap around fractional derivative order $\alpha_c = 0.5$ for the non-uniform grid curve in Fig.4.1. In other words, the LEs calculated around this fractional order could not be computed. This means the robustness of the chaos in this range for the fractional Chen system is questionable and not guaranteed.

The reason that causes these discontinuous LE curves is inconclusive but most likely linked to the floating-point arithmetic that MATLAB performed. Looking into the numerical solutions of the system with these fractional orders, we noticed that after numerous iterations, the system solutions expand dramatically(even to the magnitude of 10^{300} sometimes), which renders the MATLAB incapable of identifying and computing furthermore. The initial conditions adopted for the system may also contribute to this discontinuity and will be illustrated in our following bifurcation and phase portrait analysis.

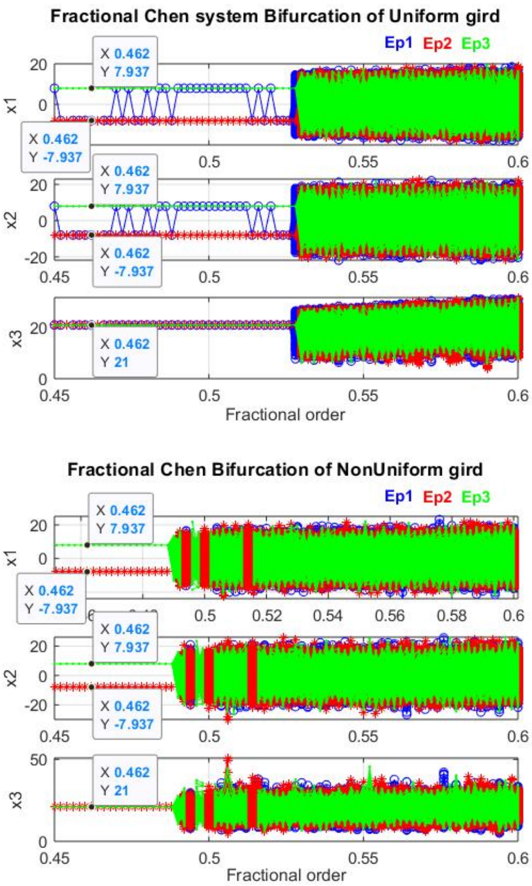
4.2 Bifurcation diagrams and phase space analysis over fractional orders

To acquire further information on the impact of the numerical calculation methods on the systems' chaoticity and justify the LE results in section 4.1, we also analyze the numerical solutions of the system from the observation of bifurcation diagrams and phase space.

The window of fractional order $\alpha_c \in [0.45, 6]$ for the fractional Chen system is chosen and evaluated with a step of 0.002. For each evaluated fractional order, we calculated its numerical solutions with 100000 iterations applying both methods starting from the random values close to each equilibrium. The parameters used for simulation are the same as those adopted in the previous sections.

We plotted in Fig.2 the bifurcation diagrams(last 500 iterations) for the 3 state components of the system applying both methods. For each sub-figure, the blue color represents the phase portrait obtained with initial condition close to first equilibrium(namely Ep1 $= (0, 0, 0)$, for fractional Chen system with $(a_c, b_c, c_c) = (35, 3, 28)$); the red color stands for the states acquired starting from the second equilibrium of the system(Ep2 $= (-7.937, -7.937, 21)$ for the given Chen system); and the green for those of the third equilibrium(Ep3 $= (7.937, 7.937, 21)$).

It can be observed from Fig.2(a) that for the fractional Chen system starting close to Ep1(blue circles), before the system exhibits the behavior of chaos, the last 500 states acquired applying uniform grid method altered between the two other equilibria, Ep2 and Ep3. In comparison, the system solutions(represented by red '*'), starting from the initial conditions close to Ep2, stay at the equilibrium



(a) Chen Bifurcation

Fig. 2. Bifurcation diagram of fractional Chen system over fractional order with initial conditions generated randomly close to the equilibria

Ep2($(-7.937, -7.937, 21)$) after the transient. The same phenomenon occurs for the Chen system with initial conditions close to Ep3. The results are in accordance with the analytical analysis where Ep1 is a saddle point, which is unstable and repulses the trajectory starting from initial conditions close to Ep1.

As mentioned in the previous section, the different choices of initial conditions contribute to the different appearance of chaotic behavior of the systems. This phenomenon can be clearly observed from Fig.2(a) applying both methods. For the bifurcation diagram of uniform grid method M1, the Chen system with an initial condition close to the equilibrium point $(0, 0, 0)$ starts to act chaotically before the systems starting from the initial conditions in the vicinity of the two other equilibria at derivative order of 0.528. The phase portrait given in Fig.3(a) confirms this finding. Whereas applying the non-uniform grid method

M2, for the fractional Chen system, we can observe areas where there are either no blue color or green color data at fractional order smaller than 0.52. This means that there exist fractional orders for which the system does not have MATLAB-computable solutions if it starts from initial conditions close to Ep1 or Ep3. Hence, we can conclude that non-robust chaos appears within these regions. We also give here some phase space figures in Fig.3(b) to justify the observations of bifurcation diagrams.

However, despite the problem of this non-robust chaos, the fractional Chen system acquired using the non-uniform grid calculation method still has a greater chaotic range in terms of fractional derivative orders compared to that calculated applying the uniform grid method. The former starts to show robust chaos after order 0.514, while the latter starts at fractional derivative order 0.53.

4.3 Analysis in terms of parameter impacts

To better understand the impacts of the chosen numerical calculation methods on the system's chaoticity in terms of the range of system parameters, we also evaluated the fractional Chen system behavior over different parameter values by applying classical and non-uniform grid calculation methods. Some LE and phase portrait results over different parameters for the fractional Chen system's chaotic vector component x_1 are given in Fig.4(a).

In the figures, the red color stands for the result of the Chen system calculated by the non-uniform grid method M2, while the blue curves represent the result for the uniform grid method M1. For the first and third parameters a_c and c_c, we give here the LEs of the fractional derivative order 0.65 and 0.85, respectively. The ranges of the parameters are $a_c \in [25, 50]$ and $c_c \in [15, 35]$ with a step of 0.25. For the second parameter b_c, the parameter is evaluated every 0.05 from $b_c = 0.05$ to 5 with fractional derivative order 0.75. It is to be mentioned the originally the parameters are set as $(a_c, b_c, c_c) = (36, 3, 28)$. Only the parameter under investigation is varied for the evaluation of each parameter, leaving the other two unchanged.

It can be easily observed from the first subfigure of Fig.4(a) that applying the non-uniform gird method, the range of the parameter a_c for LE>0 is greater compared to that of the uniform grid at fractional derivative order 0.65. This means the chaotic range is larger with the Chen system calculated using the non-uniform grid method. The observation is confirmed by the phase portraits given in Fig.4(b). The last 1000 states among the total 100000 iterations for $a_c = 46.5$ are plotted. The states calculated by the non-uniform calculation method show the shape of the attractor. Whereas the states of uniform grid method stay at a fixed point which is one of the equilibrium $(-5.539, -5.539, 9.5)$ with system parameters $(a_c, b_c, c_c) = (46.5, 3, 28)$. From the second sub-figure in Fig.4(a), it can be seen that with fractional order equal to 0.75, even though the region where LE>0 is relatively similar in size, the LE curve obtained by the non-uniform grid method has relatively greater values and fewer dropdowns. The latter indicates that chaos with a greater chaotic window is obtained by applying the non-uniform grid calculation method for the given order and parameters. The phase portrait in Fig.4(c) also confirms the finding by showing that the states calculated by the uniform grid follow a more regular and predictable trajectory, whereas the behavior of the non-uniform grid states appears to be much more irregular.

5. CONCLUSION

In this paper, we show that the chaoticity of fractional chaotic systems can be affected by the employed numerical calculation methods. We first discussed the chaoticity and stability of the fractional Chen system from the conventional analytical point of view through the equilibrium point analysis. Then we calculate the solutions of the systems numerically by applying two different calculation methods. The impacts of the different approximation methods adopted on the system's chaotic behavior are illustrated and analyzed. The results show that different calculation methods, as well as the initial conditions employed, lead to varied chaotic behavior for the fractional chaotic systems.

From the aspect of fractional derivative orders, even though the chaos exhibited by the Chen system is not robust for some specific fractional derivative orders, the conclusion of a greater chaotic range possessed by the system calculated using the non-uniform grid method compared to the classical ABM method still holds. In terms of the fractional parameters, systems calculated by the non-uniform grid acquired relatively enhanced chaotic properties in terms of greater chaotic parameter ranges and more robust chaos compared to the classical method.

For the enlarged chaotic ranges for fractional derivative orders and parameters, we also give here one explanation related to the memory effect, which are taken into consideration when calculating the solutions numerically. When all the previous states are taken into consideration, like in the long memory GL method, the stability of the numerically obtained results will be the same as that of the analytically acquired. With respect to this assumption, further work is to be conducted.

REFERENCES

Arfan, M., Shah, K., Ullah, A., Shutaywi, M., Kumam, P., and Shah, Z. (2021). On fractional order model of tumor dynamics with drug interventions under nonlocal fractional derivative. *Results in Physics*, 21, 103783.

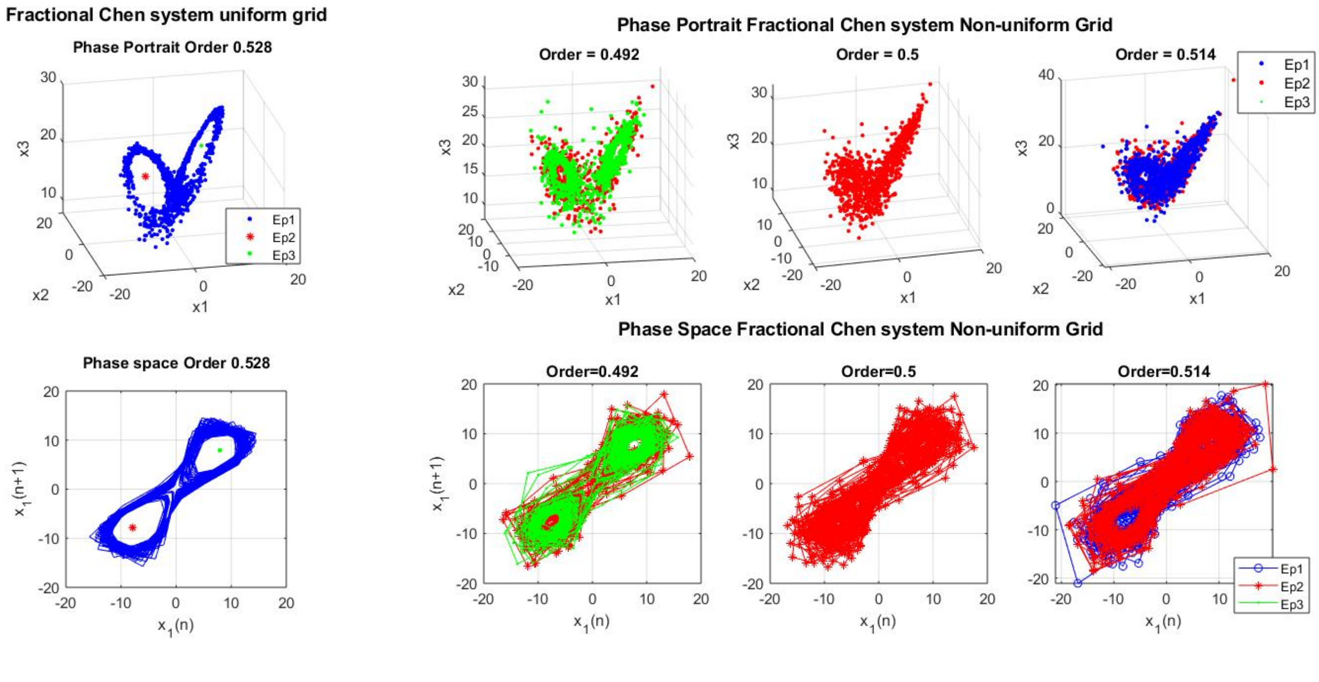
Caputo, M. (1967). Linear Models of Dissipation whose Q is almost Frequency Independent—II. *Geophysical Journal International*, 13(5), 529–539.

Danca, M.F. and Kuznetsov, N.V. (2018). Matlab code for lyapunov exponents of fractional-order systems. *Int. J. Bifurc. Chaos*, 28, 1850067:1–1850067:14.

Diethelm, K., Ford, N.J., and Freed, A.D. (2002). A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics*, 29, 3–22.

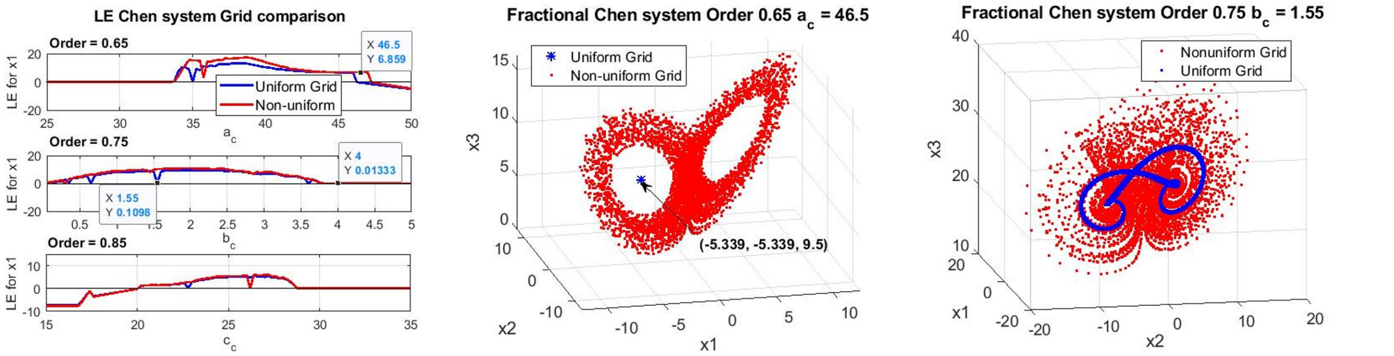
Katugampola, U.N. (2014). A new approach to generalized fractional derivatives. *arXiv: Classical Analysis and ODEs*.

Krasnobrizha, A., Rozycki, P., Gornet, L., and Cosson, P. (2016). Hysteresis behaviour modelling of woven composite using a collaborative elastoplastic damage model with fractional derivatives. *Composite Structures*, 158, 101–111.



(a) Non-uniform grid (b) Uniform grid

Fig. 3. Phase portrait and phase space for fractional Chen system obtained using different methods



(a) LE for x1 (b) Phase Portrait for order 0.65, $b_c = 46.5$ (c) Phase Portrait for order 0.75, $b_c = 1.55$

Fig. 4. LE for fractional Chen system x_1 over different parameters

Liouville, J. (1832). Mémoire sur quelques quéstions de géometrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces quéstions. *Journal de l'école Polytechnique*, 1–16.

Monje, C.A., Chen, Y.Q., Vinagre, B.M., Xue, D., and Feliu, V. (2010). *Fractional-order systems and control : fundamentals and applications*. Springer.

Petráš, I. (2011). *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*. Springer Science & Business Media.

Tarasov, V. and Tarasova, V. (2018). Macroeconomic models with long dynamic memory: Fractional calculus approach. *Applied Mathematics and Computation*, 466–486.

Tavazoei, M.S. and Haeri, M. (2007). A necessary condition for double scroll attractor existence in fractional-order systems. *Physics Letters A*, 367(1), 102–113.

Uchaikin, V.V. (2013). *Fractional derivatives for physicists and engineers*, volume 2. Springer.

Valério, D., Machado, J., and Kiryakova, V. (2014). Some pioneers of the applications of fractional calculus. *Fractional Calculus and Applied Analysis*, 17(2), 552–578.

Yang, C., Taralova, I., and Loiseau, J.J. (2021a). Fractional Chaotic system solutions and their impact on chaotic behaviour. In *The 14th CHAOS 2021 International Conference*. Athens(Turn to virtual), Greece.

Yang, C., Taralova, I., and Loiseau, J.J. (2021b). Improving chaotic features of fractional chaotic maps. *IFAC-PapersOnLine*, 54(17), 154–159. 6th IFAC Conference on Analysis and Control of Chaotic Systems CHAOS 2021.

Yang, C., Taralova, I., Loiseau, J.J., and El-Assad, S. (2020). A stream cipher based on fractional pseudo chaotic random number generator. In *2020 15th International Conference for Internet Technology and Secured Transactions (ICITST)*, 1–6.

Zeraoulia, E. and Sprott, J.C. (2011). *Robust Chaos and Its Applications*. WORLD SCIENTIFIC.