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Predictive feedback control for inventory systems subject to delays and constraints

Berna Bou Farraa, Rosa Abbou, and Jean Jacques Loiseau

Abstract—In this paper, we discuss the inventory regulation problem of a logistic system composed of a production unit and inventory unit. The production system is characterized by a constant lead time, and the storage unit presents fixed losses due to perishable products. In this study, the system is subject to finite production resources, limited storage capacities, and the customer demands are uncertain in a given range. We address a control strategy to the input time delay system that is subject to positive and saturation constraints and to bounded disturbances. We apply a linear prediction feedback control strategy, in order to stabilize the closed-loop system and to fulfill the system constraints, so that necessary and sufficient conditions for the existence of an admissible control law are formulated.

Index Terms—Constraints verification, input-delayed system, inventory control, predictive feedback structure.

I. INTRODUCTION

The inventory regulation problem is a field of research that explores the links between the control theory domain and the production management domain. In this context, we develop a control policy to manage the complex phenomena of these dynamic systems, in order to provide stable inventory regulation level and high customer demands satisfaction. Such problems can be treated using different frameworks, in particular the analytic models associated to control theory approaches. Simon [21] was the first to study the logistic system by a servomechanism approach using Laplace Transform. Through the years, different studies were based on differential equations and feedback structures, to model and control the inventory dynamics in production systems [9], [13] and [24].

In production systems, time delays are directly related to the time required for the production and delivery process, and are often the main cause of the system instability. In the past years, several studies have been done introducing the notions of Input-Output stability for convolution systems, as well as the stability by state prediction for delayed systems, as studied in [7], [22], [23] and [25]. Riddalls and its co-authors in [20] were the first to take into account the delay exactly in logistic systems control. A second specification related to inventory systems concerns the perishable products, that generates inventory losses and have significant effects on products availability and customer service. Several works in the literature, as [3], [5], [10] and [12], have modeled the inventory items by decay processes.

Moreover, additional characteristics are considered in few works of the literature. The first feature is the positivity of the inventory level and the production order that are physical quantities. The second feature is resumed by the capacities limitation of the inventory storage and the production finite resources. These characteristics are modeled by positive and saturation constraints that should be taken into consideration when controlling the production system. The constraints verification methods are based on the bounds calculus for Input-Output systems, which are derived in [16] from the seminal results exposed in [8], and on the invariance principle formulated for finite dimensional systems, see [4] and [5].

In the continuity of these works, we discuss in this paper the inventory regulation problem for a class of input-delayed system subject to positive and saturation constraints, a constant lead time and a fixed loss factor. The contribution of this work provides an inventory control approach using a predictive feedback control structure. The resolution methodology we have adopted in our work is part of the structural approach initiated by Karcianas in [14]. This global approach consists of separating the physical system sizing from the control system design, so that both the existence and the feasibility of a control structure are guaranteed, in conformity with the system constraints and specifications. Indeed, our first contribution is to present the Input-Output bounds calculus for the convolution system, in order to prove the controlled system stability and the constraints verification. The second contribution is to describe the necessary and sufficient conditions for the existence and the implementation of a constrained control law, in terms of a specified control parameterization.

The remainder of this paper is organized as follows. The problem statement is described in Section II and some preliminaries are briefly summarized. The prediction feedback control structure is introduced in Section III, and the Input-Output reachable bounds are determined in Section IV. The main results concerning the implementation of an admissible control law are formulated in Section V. Comparative analysis and simulation examples are provided in Sections VI and VII respectively, in order to analyze and verify the theoretical results. The conclusion is drawn in Section VIII.

II. PROBLEM STATEMENT

A. Model description

In this study, we consider a logistic system composed of a production unit and an inventory unit. The system is defined by the following basic variables: the production order $u(t)$,

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the inventory level $y(t)$, and the consumer demand $d(t)$. As the demand $d(t)$ varies continuously over the time, then we consider that the inventory level $y(t)$ and the production orders $u(t)$ evolve continuously throughout the time. Thus, the dynamic of the inventory level is described by the following first order differential input-delay system of the form:

$$\dot{y}(t) = -\sigma y(t) + u(t - \theta) - d(t) \quad , \text{ for } t \geq \theta, \quad (1)$$

where $u(t) = \varphi(t)$ for $-\theta \leq t < 0$. The function of the initial conditions $\varphi(t)$ corresponds to the initial production flow over a past time horizon of the system, before the implementation of the control law $u(t)$ for $0 \leq t \leq \theta$. Using this function, we define the initial work in progress, noted w , as follows:

$$w = \int_0^\theta e^{-\sigma(\theta-\tau)} \varphi(\tau) d\tau. \quad (2)$$

The initial work in progress w includes all the final products being produced at $t = \theta$, that are ready to feed the inventory level at the following moments.

The production order $u(t)$ and the consumer demand $d(t)$ are homogeneous to flows in items per day. The inventory level $y(t)$ is a cumulative flow function of time t , taking values in number of items. In terms of control, $u(t)$ is the control input, $d(t)$ is an external perturbation, and $y(t)$ is the output to be controlled.

One remarks that the control is delayed, which comes from the fact that the production of goods or services takes time presented by θ . We consider that the delay θ is a positive constant characteristic of the considered production system, representing the estimated process time on the work stations in industrial workshops. Moreover, it is important to remark that the inventory level is naturally decreasing because of the storage losses $\sigma y(t)$. Such systems are modeled in this study by a fixed loss factor σ verifying $0 \leq \sigma < 1$, in order to best consider perishable products having limited consumption periods. Indeed, the lead time θ and the loss factor σ are supposed to be known exactly in advance in order to model deterministic characteristics in basic manufacturing processes.

B. Control methodology

The system (1) being given, the control objective consists of calculating the production order $u(t)$ on line, so that the inventory level $y(t)$ is well regulated and the consumer demands $d(t)$ are fully satisfied. The controller should also meet the positive and saturation constraints on the system variables. From a production management point of view, the production unit $u(t)$ and the inventory level $y(t)$ are non-negative quantities having limited production resources and finite storage capacities. Mathematically, it comes down to constraints on the system variables that are interpreted as specifications to the control system. Indeed, the minimum and the maximum production rates of the system are denoted u_{min} and u_{max} respectively. The minimum and the maximum inventory capacities are denoted y_{min} and y_{max} respectively. We denote d_{min} and d_{max} respectively, the minimum and the maximum demand rates. The controller should be designed such that, for all $t \geq 0$:

$$0 \leq y_{min} < y(t) < y_{max}, \quad (3)$$

$$0 \leq u_{min} < u(t) < u_{max}, \quad (4)$$

$$0 \leq d_{min} < d(t) < d_{max}. \quad (5)$$

The problem comes down to finding a new stabilizing control strategy so that the input and output constraints remain always checked, in the presence of an arbitrarily bounded disturbance in the range defined by (5). In other terms, the control methodology consists of determining the necessary and sufficient conditions for the existence and the implementation of an admissible control law $u(t)$. Hence, the admissible property of the proposed control structure is formulated in the following definition.

Definition 1: A control law is said to be admissible for the delayed system (1) if, for any initial condition $y(0) \in [y_{min}, y_{max}]$, there exists an initial function $\varphi(\tau)$ for $\tau \in [0, \theta]$, such that system output constraint (3) and the control input constraint (4) remain fulfilled at any instant $t \geq 0$, and this for any unknown disturbance $d(t)$ verifying (5).

Before getting to the description of the control structure and its analysis, we introduce in the next section some elementary backgrounds for system stability and constraints verification.

C. About Input-Output bounds for convolution systems

We consider in our work an important family of causal Input-Output systems, which belong to the set of convolution kernels denoted by \mathcal{A} , defined as follows:

$$h(t) = \begin{cases} h_a(t) + \sum_{i=0}^{\infty} h_i \cdot \delta(t - t_i) & , \text{ for } t \geq 0, \\ 0 & , \text{ for } t < 0. \end{cases} \quad (6)$$

An element of \mathcal{A} of the form (6) is said to be non-negative if $h_a(t) \geq 0$, for $t \geq 0$, and $h_i \geq 0$, for $i \in \mathbb{N}$. In addition, the set \mathcal{A} forms a commutative Banach algebra with a norm verifying

$$\|h\|_{\mathcal{A}} = \|h_a\|_{L_1} + \sum_{i=0}^{\infty} |h_i| < \infty. \quad (7)$$

Such systems are known to belong to the class of Wiener Algebra, as in [19], which covers finite dimensional systems and infinite dimensional ones as well, such as time delayed systems. Indeed, every system with kernel defined in the class \mathcal{A} is said to be Bounded Input Bounded Output BIBO-stable, according to the following definition.

Definition 2: An Input-Output system of the form $y(t) = (h * u)(t) = \int_0^t h(t - \tau) u(\tau) d\tau$, is BIBO-stable if there exists positive real numbers u_m and y_m such as, any input verifying $|u(t)| \leq u_m$ produces a system output $|y(t)| \leq y_m$, at any time $t \geq 0$.

Therefore, the following theorem establishes the exact reachable bounds of the system output, when its input is constrained and evolves in a closed interval.

Theorem 1: Assume that h is a non-negative element of the class \mathcal{A} verifying (7), and let m_u and M_u be two non-negative real numbers. For any bounded input $u(t)$ satisfying

$m_u \leq u(t) \leq M_u$, for $t \geq 0$, the output of the system $y(t)$ expressed by $y(t) = (h * u)(t)$ is bounded for $t \geq 0$, as follows:

$$\int_0^t h(\tau) d\tau \cdot m_u \leq y(t) \leq \int_0^t h(\tau) d\tau \cdot M_u. \quad (8)$$

Moreover, we obtain the following exact bounds when t tends to infinity:

$$\|h\|_{\mathcal{A}} \cdot m_u \leq y(t) \leq \|h\|_{\mathcal{A}} \cdot M_u. \quad (9)$$

Proof: We deduce the inequalities (8) from the convolution system $y(t) = (h * u)(t)$, and the positivity of h and u . One further obtains (9) when t tends to infinity, and one checks that the infimal and supramal bounds are respectively defined by $\|h(t)\|_{\mathcal{A}} \cdot m_u = \inf_{m_u \leq u(t) \leq M_u} \{y(t)\}$ and $\|h(t)\|_{\mathcal{A}} \cdot M_u = \sup_{m_u \leq u(t) \leq M_u} \{y(t)\}$, using a constant control equal to m_u and M_u respectively. ■

This characterization allows to monitor the system output variation following the control input specification. These properties are deduced from those developed in [17] for constraints verification in multivariable systems. They are powerful tools for Input-Output reachable bounds identification and constraints meeting for delayed systems, as presented in the sequel.

III. PREDICTIVE CONTROL STRUCTURE

Several studies have been done for delayed systems introducing notions of BIBO stability as well as stabilization by state prediction principles, making possible the synthesis of stabilizing control laws for dynamic systems as logistic systems. Indeed, we propose a control structure defined by a linear predictive feedback, that stabilizes the input-delay system (1) and compensates the delay effects on the system dynamics, as follows:

$$u(t) = K(z_0 - z(t)). \quad (10)$$

The control parameter K is a static gain used to adjust the production rate, and z_0 is the reference value for the predicted storage level. This type of linear control allows the regulation of the inventory level $y(t)$, where the production order $u(t)$ presents a continuous incoming flow of products. In addition, the exact prediction $z(t)$ of the future inventory level $y(t)$ is carried out over a time horizon from t to $t + \theta$, as expressed in the following:

$$z(t) = \begin{cases} e^{-\sigma\theta} y(t) + \int_{t-\theta}^t e^{-\sigma(t-\tau)} u(\tau) d\tau & , \text{ for } t \geq \theta, \\ e^{-\sigma t} y(t) + \int_0^t e^{-\sigma(t-\tau)} u(\tau) d\tau \\ \quad + \int_t^\theta e^{-\sigma(t-\tau)} \varphi(\tau) d\tau & , \text{ for } 0 \leq t \leq \theta. \end{cases} \quad (11)$$

Indeed, the prediction dynamics are determined by the following equation:

$$\dot{z}(t) = -\sigma z(t) + u(t) - e^{-\sigma\theta} d(t). \quad (12)$$

The obtained equation is well known by the model reduction or the Artstein's reduction [2]. The state prediction structure was first studied by [18] and used after by [15], [1] and [5]. The basic idea of such a basic prediction principle is to anticipate and fully compensate the system delay θ , by generating a control law that directly uses the corresponding free-delay system (12). In the frequency domain, the system output and the control law are described respectively by $(s + \sigma)\hat{y}(s) = e^{-s\theta}\hat{u}(s) - \hat{d}(s)$ and $\hat{u}(s) = \hat{C}(s)(Kz_0 - Ke^{-\sigma\theta}\hat{y}(s))$, where $\hat{C}(s) = \frac{1}{1 + K(\frac{1 - e^{-(s+\sigma)\theta}}{s+\sigma})}$. Indeed, the control loop structure is performed in Figure 1.

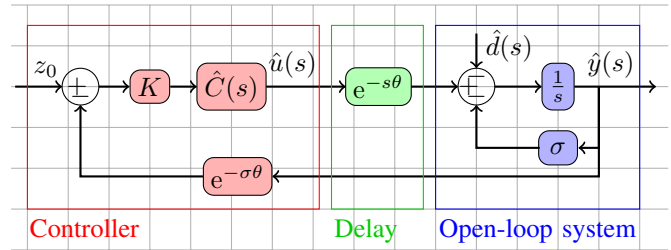


Fig. 1. Closed-loop system diagram

The characteristic equation of the closed loop system is equal to the following determinant:

$$\frac{1}{\hat{g}(s)} = s + \sigma + K = \det \begin{pmatrix} s + \sigma & -e^{-s\theta} \\ Ke^{-s\theta} & (1 + K(\frac{1 - e^{-(s+\sigma)\theta}}{s+\sigma})) \end{pmatrix}.$$

We notice that the transfer $\hat{g}(s)$ depends explicitly on the control parameter K and the loss factor σ . In the time domain, it corresponds to the Kernel $g(t) = e^{-(\sigma+K)t}$, whose norm is equal to $\|g\|_{\mathcal{A}} = \frac{1}{\sigma+K}$. Indeed, The closed-loop transfer $\hat{g}(s)$ is BIBO-stable for positive values of the control parameter K and non negative values of the loss factor σ such that $K \geq -\sigma$. The predictor linear-feedback control law being defined by (10)-(11) and illustrated in Figure 1, we move on to study in the next section the dynamics of the closed-loop system. The challenge is to determine the time evolution of the control input $u(t)$ and system output $y(t)$ based on the bounds identification, in order to check the system constraints on these variable, (4) and (3), respectively.

IV. INPUT-OUTPUT REACHABLE BOUNDS

As explained previously, the proposed methodology consists in calculating the input and the output reachable bounds, and then deducing the conditions under which these bounds would respect the affected system constraints (4) and (3) respectively. In order to use the bounds computation as introduced in Section II-C, we first describe the system dynamics in an Input-Output representation. Indeed, the temporal evolution of the control input $u(t)$ and the system output $y(t)$ are described in the following proposition.

Proposition 1: The system (1) controlled by a linear predictive feedback of the form (10)-(11), has an Input-Output

representation as follows:

$$u(t) = \begin{cases} K \frac{\sigma}{\sigma+K} z_0 + K e^{-\sigma\theta} (e^{-(\sigma+K)} * d)(t) \\ -K(y(0) + w - \frac{K}{\sigma+K} z_0) e^{-(\sigma+K)t} \end{cases}, \text{ for } t \geq 0,$$

$$y(t) = \begin{cases} T_1(t)y(0) - (T_2 * d)(t) + \phi(t) \\ + K(T_3 * (z_0 - e^{-\sigma t} w))(t) \end{cases}, \text{ for } t \geq \theta,$$

$$y(t) = \begin{cases} e^{-\sigma t} y(0) - \int_0^t e^{-\sigma(t-\tau)} d(\tau) d\tau \\ + \int_0^t e^{-\sigma(t-\tau)} \varphi(\tau) d\tau \end{cases}, \text{ for } 0 \leq t \leq \theta,$$

where the functions $T_1(t)$, $T_2(t)$, $T_3(t)$ and $\phi(t)$ are defined as follows:

$$T_1(t) = \begin{cases} e^{-\sigma t} & , \text{ for } t \in [0, \theta], \\ e^{-\sigma t} - e^{-\sigma(t-\theta)} + e^{-(\sigma+K)(t-\theta)} & , \text{ for } t \geq \theta, \end{cases}$$

$$T_2(t) = \begin{cases} e^{-\sigma t} & , \text{ for } t \in [0, \theta], \\ e^{-\sigma\theta} e^{-(\sigma+K)(t-\theta)} & , \text{ for } t \geq \theta, \end{cases}$$

$$T_3(t) = \begin{cases} 0 & , \text{ for } t \in [0, \theta], \\ e^{-(\sigma+K)(t-\theta)} & , \text{ for } t > \theta, \end{cases}$$

$$\phi(t) = \begin{cases} (e^{-\sigma t} * \varphi)(t) = \int_0^t e^{-\sigma(t-\tau)} \varphi(\tau) d\tau & , \text{ for } t \in [0, \theta], \\ w & , \text{ for } t \geq \theta. \end{cases}$$

Proof: The assertions of this theorem are obtained using the Laplace transformation and its properties. Thus, the control input and the system output dynamics defined respectively by (10) and (1), are expressed based on the prediction dynamics given by (12) with $z(0) = y(0) + w$, as follows:

$$\hat{u}(s) = K \hat{g}(s) \left(\frac{s+\sigma}{s} z_0 - (y(0) + w) + e^{-\sigma\theta} \hat{d}(s) \right),$$

$$\hat{y}(s) = \hat{T}_1(s)y(0) + K \hat{T}_3(s) \left(\frac{z_0}{s} - \frac{w}{s+\sigma} \right) - \hat{T}_2(s) \hat{d}(s) + \hat{\phi}(s),$$

where

$$\hat{T}_1(s) = \left(1 + \frac{K(1 - e^{-s\theta})}{s + \sigma} \right) \hat{g}(s),$$

$$\hat{T}_2(s) = \left(1 + \frac{K(1 - e^{-(s+\sigma)\theta})}{s + \sigma} \right) \hat{g}(s),$$

$$\hat{T}_3(s) = e^{-s\theta} \hat{g}(s),$$

$$\hat{\phi}(s) = \frac{\hat{\varphi}(s)}{s + \sigma}.$$

Finally, the system input and the system output are formulated taking into consideration the functions $T_1(t)$, $T_2(t)$, $T_3(t)$ and $\phi(t)$ introduced in Proposition 1 for $0 \leq t \leq \theta$ and $t \geq \theta$. ■

Based on the input and output dynamics formulated in Proposition 1, we define the exact reachable bounds of the system variables $u(t)$ and $y(t)$, in the following Proposition.

Proposition 2: Given the system (1), controlled by a linear predictive feedback of the form (10)-(11) and a disturbance $d(t)$ verifying (5), the exact reachable bounds of the control input $u(t)$ and the system output $y(t)$ are defined by:

$$u(t) \in [u_1(t), u_2(t)], \quad \text{and} \quad y(t) \in [y_1(t), y_2(t)],$$

where

$$u_1(t) = \begin{cases} \frac{K\sigma}{\sigma+K} z_0 - K(y(0) + w - \frac{K}{\sigma+K} z_0) e^{-(\sigma+K)t} \\ + \frac{K}{\sigma+K} e^{-\sigma\theta} d_{min}(1 - e^{-(\sigma+K)t}) \end{cases}, \text{ for } t \geq 0,$$

$$u_2(t) = \begin{cases} \frac{K\sigma}{\sigma+K} z_0 - K(y(0) + w - \frac{K}{\sigma+K} z_0) e^{-(\sigma+K)t} \\ + \frac{K}{\sigma+K} e^{-\sigma\theta} d_{max}(1 - e^{-(\sigma+K)t}) \end{cases}, \text{ for } t \geq 0,$$

$$y_1(t) = \begin{cases} (e^{-\sigma t} - e^{-\sigma(t-\theta)})y(0) + w - \frac{1-e^{-\sigma\theta}}{\sigma} d_{max} \\ + \frac{1}{K+\sigma} (Kz_0 - Ke^{-\sigma t} w - e^{-\sigma\theta} d_{max}) \\ + (y(0) - \frac{K}{K+\sigma} z_0 + \frac{K}{K+\sigma} e^{-\sigma t} w \\ - \frac{e^{-\sigma\theta}}{K+\sigma} d_{max}) e^{-(\sigma+K)(t-\theta)} \end{cases}, \text{ for } t \geq \theta,$$

$$y_1(t) = \begin{cases} e^{-\sigma t} y(0) + \int_0^t e^{-\sigma(t-\tau)} \varphi(\tau) d\tau \\ - \frac{1-e^{-\sigma t}}{\sigma} d_{max} \end{cases}, \text{ for } 0 \leq t \leq \theta,$$

$$y_2(t) = \begin{cases} (e^{-\sigma t} - e^{-\sigma(t-\theta)})y(0) + w - \frac{1-e^{-\sigma\theta}}{\sigma} d_{min} \\ + \frac{1}{K+\sigma} (Kz_0 - Ke^{-\sigma t} w - e^{-\sigma\theta} d_{min}) \\ + (y(0) - \frac{K}{K+\sigma} z_0 + \frac{K}{K+\sigma} e^{-\sigma t} w \\ - \frac{e^{-\sigma\theta}}{K+\sigma} d_{min}) e^{-(\sigma+K)(t-\theta)} \end{cases}, \text{ for } t \geq \theta,$$

$$y_2(t) = \begin{cases} e^{-\sigma t} y(0) + \int_0^t e^{-\sigma(t-\tau)} \varphi(\tau) d\tau \\ - \frac{1-e^{-\sigma t}}{\sigma} d_{min} \end{cases}, \text{ for } 0 \leq t \leq \theta.$$

Proof: We apply the bounds calculation as introduced in Theorem 1, in order to formulate the bounds reached by the control input $u(t)$ and the system output $y(t)$ defined in Proposition 1. First, the variable $u(t)$ is expressed as a sum of the function $K \frac{\sigma}{\sigma+K} z_0 - K(y(0) + w - \frac{K}{K+\sigma} z_0) g(t)$ and the convolution function $K e^{-\sigma\theta} (g * d)(t)$. Since the disturbance $d(t)$ verifies (5), the convolution term is bounded as follows:

$$d_{min} \int_0^t g(t-\tau) d\tau \leq (g * d)(t) \leq d_{max} \int_0^t g(t-\tau) d\tau.$$

the input bounds are then formulated having $g(t) = e^{-(\sigma+K)t}$. Similarly for $y(t)$, the bounds of the convolution function $-(T_2 * d)(t)$ are determined for $d(t)$ verifying (5), by

$$-d_{max} \int_0^t T_2(t-\tau) d\tau \leq (T_2 * d)(t) \leq -d_{min} \int_0^t T_2(t-\tau) d\tau, \quad (13)$$

with $\int_0^t T_2(t-\tau) d\tau = \frac{1-e^{-\sigma t}}{\sigma}$ for $t \in [0, \theta]$, and by $T_2(t) = \frac{1-e^{-\sigma\theta}}{\sigma} + e^{-\sigma\theta} \left(\frac{1-e^{-(\sigma+K)(t-\theta)}}{\sigma+K} \right)$ for $t \geq \theta$. Moreover, the term $K(T_3 * (z_0 - e^{-\sigma t} w))(t)$ is defined by $K(T_3 * (z_0 - e^{-\sigma t} w))(t) = K(z_0 - e^{-\sigma t} w) \int_0^t e^{-(\sigma+K)(\tau-\theta)} d\tau = K(z_0 - e^{-\sigma t} w) \left(\frac{1-e^{-(\sigma+K)(t-\theta)}}{\sigma+K} \right)$. Finally, the output bounds are obtained by replacing the temporal domain functions $T_1(t)$ and $\phi(t)$ by their expressions defined in Proposition 1 for $t \geq \theta$, which completes the demonstration. ■

These input and output reachable bounds are time-dependent functions that tend asymptotically to finite values. This latter

confirms the BIBO-stability of the controlled system as mentioned in Theorem 1. Moreover, the output reachable bounds are presented by two distinct intervals, the initialization phase for $t \in [0, \theta]$ and the steady state for $t \geq \theta$ where the control parameters K and z_0 are implemented. Indeed, we study in the following section the system constraints verification, in order to determine the necessary and sufficient conditions for the existence of an admissible control law.

V. IMPLEMENTATION OF THE CONTROL LAW

The implementation of the predictive control law is divided into two parts. The first one concerns the admissibility of the control law (10)-(11) with respect to the system constraints verification. While the second axis is based on the choice of the control parameters K and z_0 , for which the existence of such an admissible control law is guaranteed. These two concepts are developed in the following two sections.

A. Admissible conditions of the control law

To determine the necessary and sufficient admissible conditions, we formulate in the following theorem the conditions for which the input and output exact reachable bounds expressed in Proposition 2 belong to the intervals defined by the constraints (4) and (3), respectively.

Theorem 2: Given the system of the form (1) with admissible initial conditions and a disturbance $d(t)$ verifying (5), then the control law (10)-(11) allows to check the system constraints (3) and (4), if and only if the following conditions are satisfied:

$$u_{min} \leq \frac{K}{K+\sigma}(\sigma z_0 + e^{-\sigma\theta} d_{min}), \quad (14)$$

$$\frac{K}{K+\sigma}(\sigma z_0 + e^{-\sigma\theta} d_{max}) \leq u_{max}, \quad (15)$$

$$y_{min} + \frac{1-e^{-\sigma\theta}}{\sigma} d_{max} + \frac{e^{-\sigma\theta}}{K+\sigma} d_{max} \leq \frac{K}{K+\sigma} z_0, \quad (16)$$

$$\frac{K}{K+\sigma} z_0 \leq y_{max} + \frac{1-e^{-\sigma\theta}}{\sigma} d_{min} + \frac{e^{-\sigma\theta}}{K+\sigma} d_{min}. \quad (17)$$

Proof: The constraints (4) and (3) are verified if the intervals $[u_1(t), u_2(t)]$ and $[y_1(t), y_2(t)]$ given by Proposition 2 are included in the intervals $[u_{min}, u_{max}]$ and $[y_{min}, y_{max}]$ respectively. We first identify the exact reachable bounds for $u(t)$ and $y(t)$ when t tends to infinity, using the bounds characterization for exponential functions. Indeed, we notice that the interval $[\frac{K}{K+\sigma}(\sigma z_0 + e^{-\sigma\theta} d_{min}), \frac{K}{K+\sigma}(\sigma z_0 + e^{-\sigma\theta} d_{max})]$ is included in the interval $[u_{min}, u_{max}]$ only if the conditions (14) and (15) are verified.

Similarly for $y(t)$, the interval $[\frac{1}{K+\sigma}(K z_0 - e^{-\sigma\theta} d_{max}) + w - \frac{1-e^{-\sigma\theta}}{\sigma} d_{max}, \frac{1}{K+\sigma}(K z_0 - e^{-\sigma\theta} d_{min}) + w - \frac{1-e^{-\sigma\theta}}{\sigma} d_{min}]$ is included in the interval $[y_{min}, y_{max}]$, if the conditions (16) and (17) are checked. This shows the sufficiency of these conditions. Their necessity comes from the fact that the bounds expressed in Proposition 2 are exact and reachable for both the control input $u(t)$ and the system output $y(t)$. ■

In the obtained results, we have effectively considered that the control parameters are fixed in advance. In this direction, we will study in the following section the co-design methodology of the controlled system, in order to identify the best control law parameterization in addition to the constraints verification.

B. Parameterization of the admissible control law

The conditions formulated in Theorem 2 are expressed in terms of different parameters classified into three categories, the intrinsic parameters θ and σ , the control parameters K and z_0 , and the parameters y_{min} , y_{max} , u_{min} , u_{max} , d_{min} and d_{max} related to the system constraints. Indeed, we respond to the problem of existence and parameterization of the controlled system in the following theorem, by formulating the conditions that are independent of the control parameters.

Theorem 3: Given the system (1) with admissible initial conditions, there exists a linear predictive control law of the form (10)-(11) for which the system is BIBO-stable and the constraints (3) and (4) are fulfilled, for any disturbance $d(t) \in [d_{min}, d_{max}]$, if and only if the following conditions are satisfied for $t \geq \theta$:

$$u_{min} \leq \sigma y_{max} + d_{min}, \quad (18)$$

$$\sigma y_{min} + d_{max} \leq u_{max}, \quad (19)$$

and

$$y_{min} + \frac{1-e^{-\sigma\theta}}{\sigma} d_{max} < y_{max} + \frac{1-e^{-\sigma\theta}}{\sigma} d_{min}. \quad (20)$$

Moreover, if these conditions are true, an admissible control law is then implemented by choosing the reference level z_0 verifying the conditions (14), (15), (16) and (17), and a positive static gain K verifying the following condition:

$$K \geq \frac{d_{max} - d_{min} - \sigma(y_{max} - y_{min})}{y_{max} - y_{min} - \frac{1-e^{-\sigma\theta}}{\sigma}(d_{max} - d_{min})}. \quad (21)$$

In addition, if the following inequality is true,

$$e^{-\sigma\theta}(d_{max} - d_{min}) > u_{max} - u_{min}, \quad (22)$$

then, the static gain K is bounded by a lower bound defined in (21) and an upper bound defined for $\sigma \neq 0$ as follows:

$$K \leq \frac{\sigma(u_{max} - u_{min})}{e^{-\sigma\theta}(d_{max} - d_{min}) - (u_{max} - u_{min})}. \quad (23)$$

Proof: The results of this theorem are derived from the admissibility conditions of defined in Theorem 2 and expressed in terms of the control parameters K and z_0 . Therefore, the proof is divided into two parts. The first one is the procedure of elimination of reference value z_0 , while the second part is dedicated for the elimination and the choice of the static gain K . First, the conditions (14), (15), (16) and (17) depend on z_0 , and are reformulated as follows:

$$\begin{aligned} \frac{K+\sigma}{K\sigma} u_{min} - \frac{e^{-\sigma\theta}}{\sigma} d_{min} &\leq z_0 \leq \frac{K+\sigma}{K\sigma} u_{max} - \frac{e^{-\sigma\theta}}{\sigma} d_{max}, \\ \frac{K+\sigma}{K} y_{min} + \left(\frac{1-e^{-\sigma\theta}}{\sigma} \frac{K+\sigma}{K} + \frac{e^{-\sigma\theta}}{K} \right) d_{max} &\leq z_0, \\ z_0 &\leq \frac{K+\sigma}{K} y_{max} + \left(\frac{1-e^{-\sigma\theta}}{\sigma} \frac{K+\sigma}{K} + \frac{e^{-\sigma\theta}}{K} \right) d_{min}. \end{aligned}$$

The reference value z_0 being isolated, it can be eliminated assuming that $K > 0$ and $\sigma > 0$ that are the closed loop stability conditions. Hence, an equivalent set of conditions that are independent of z_0 is generated. Indeed, we obtain the conditions (18) and (19), and the two following inequalities:

$$Ke^{-\sigma\theta}(d_{max} - d_{min}) \leq (K + \sigma)(u_{max} - u_{min}), \quad (24)$$

$$((1 - e^{-\sigma\theta}) + \frac{\sigma e^{-\sigma\theta}}{K + \sigma})(d_{max} - d_{min}) \leq \sigma(y_{max} - y_{min}). \quad (25)$$

Then, we move on to the elimination procedure of the static gain K from the conditions (24) and (25). The condition (25) is reformulated by $e^{-\sigma\theta}(d_{max} - d_{min}) \leq (K + \sigma)(y_{max} - y_{min} - \frac{1 - e^{-\sigma\theta}}{\sigma}(d_{max} - d_{min}))$. Hence, the condition (20) is deduced from this latter assuming that $d_{max} > d_{min}$.

In addition, the choice of the static gain K is linked to the conditions (24) and (25). On one hand, the condition (25) is defined for a choice of K satisfying the inequality (21). On the other hand, if the inequality (22) is satisfied, the choice of the static gain is therefore limited by a maximum bound for $\sigma \neq 0$, as defined by the condition (23). As consequence, the bounds (21) and (23) resume the interval of variation of the control parameters K , which completes the proof. ■

We have determined the necessary and sufficient conditions for the existence of an admissible linear predictor feedback (10)-(11). These inequalities occur during the logistic system sizing. Knowing the system parameters, which are the specifications u_{min} , u_{max} , y_{min} , y_{max} , d_{min} and d_{max} , the lead time θ and the loss factor σ , the co-designer can test first the existence of an admissible control law, by testing the validity of the conditions (18), (19) and (20). If they are valid, an admissible control law is then implemented based on the parameterization of the control parameters K and z_0 , as formulated by the conditions (14), (15), (16), (17), (21) and (23). On the contrary, if the conditions are not satisfied, the designer of the control system is certain of not being able to control the inventory system using the proposed linear predictive feedback. This result occurs during the conception of the control system, and after ensuring the admissibility of the control law, as initiated by Karcnias in [14].

C. Study of the initialization phase

In this section, we complete the obtained results concerning the implementation of an admissible control law for $t \geq \theta$, by the study of the initialization phase for the system (1), in terms of the initial system output $y(0)$ and the initial work in progress w . Indeed, based on the Input-Output bounds formulated in Proposition 2 for $0 \leq t \leq \theta$, we characterize the set of initial admissible conditions in the following theorem.

Theorem 4: Given the system (1) subject to a disturbance $d(t)$ verifying (5), the constraints (3) and (4) are checked for $0 \leq t \leq \theta$, if and only if the following inequalities hold true:

$$z_0 - \frac{u_{max}}{K} \leq y(0) + w \leq z_0 - \frac{u_{min}}{K}, \quad (26)$$

$$y_{min} + \frac{1 - e^{-\sigma\theta}}{\sigma} d_{max} \leq e^{-\sigma\theta} y(0) + w \leq y_{max} + \frac{1 - e^{-\sigma\theta}}{\sigma} d_{min}. \quad (27)$$

Proof: The constraints (3) and (4) are fulfilled if the intervals given in Proposition 2 verify the following inclusions: $[u_1(t), u_2(t)] \subset [u_{min}, u_{max}]$ and $[y_1(t), y_2(t)] \subset [y_{min}, y_{max}]$. Thus, the condition (26) is checked if $\lim_{t \rightarrow 0} u_1(t) \geq u_{min}$ and $\lim_{t \rightarrow 0} u_2(t) \leq u_{max}$ with $u_1(t) < u_2(t)$ for $t \geq 0$. Similarly, the condition (27) is formulated for $t \geq \theta$ based on $y_1(t) \geq y_{min}$ and $y_2(t) \leq y_{max}$, with $y_1(t) < y_2(t)$ for $t \geq \theta$. This shows the sufficiency of the stated conditions. The necessity comes from the fact that the bounds expressed in Proposition 2 are exact and reachable for $u(t)$ and $y(t)$. ■

Based on the conditions of Theorem 4, we notice that the initial values are modulated according to the predetermined choice of the static gain K and the reference level z_0 of the control law. If the initial conditions are checked, then we can choose the control parameters K and z_0 as described in Theorem 3. Indeed, the predictive control approach is efficient in both the initialization and the steady states. We end by a compared analysis with published works from the literature.

VI. COMPARATIVE ANALYSIS AND DISCUSSION

The above study concerning the control of inventory systems is compared to recent frameworks, in the following remarks.

Remark 1: The authors in [12] have applied a linear feedback based on a Smith predictor principle to study the inventory regulation problem. Indeed, the Smith predictor based controller is a successful method for dead-time compensation, but the closed-loop system stability is not necessarily guaranteed. In this direction, an enhanced linear control law was developed in [11], in order to establish smooth and non-oscillatory system responses. In this study, the control strategy is improved using the stabilization by a state prediction principle, where the closed-loop system stability is guaranteed.

Remark 2: Furthermore, we study in this paper a general class of logistic systems with a positive loss factor contrary to [1] and [16] where $\sigma = 0$ is considered. Moreover, the model description is completed in this paper where both positive and saturation constraints are considered, contrary to [16] where only saturation constraints are modeled.

Remark 3: In [5], an affine control law was developed in order to solve the inventory regulation problem (1). It is defined by the same linear predictive feedback (10)-(11), and checks the additional specifications:

$$u(t) = \begin{cases} u_1 & , \text{ for } z(t) = z_{min}, \\ u_2 & , \text{ for } z(t) = z_{max}. \end{cases} \quad (28)$$

The control values u_1 , u_2 , z_{min} , and z_{max} are chosen such that the hypotheses $u_1, u_2 \in [u_{min}, u_{max}]$ and $z(t) \in [z_{min}, z_{max}]$ are verified, and the control parameters are defined as follows:

$$K = \frac{u_1 - u_2}{z_{max} - z_{min}}, \quad \text{and} \quad z_0 = \frac{u_1 z_{max} - u_2 z_{min}}{u_1 - u_2}. \quad (29)$$

The control parameterization defined by (29) represents one choice of the general set of admissible control solutions using the linear predictive feedback (10)-(11). These results are completed in our paper, by a global parameterization of the proposed control law. Indeed, the choice of the static gain K and the reference value z_0 as defined by (29) for the affine control law, verifies the general implementation of the linear control law as described in Theorems 2 and 3.

Remark 4: In general, the control law parameterization can allow the best choice of the control parameters, in order to optimize certain cost criteria. In particular, the robustness property is one of the most important characteristics for the control system conception and for the logistic system management, where delays are known with uncertainties. In this case, an approximation of the prediction based controller is incorporated, based on a delay estimation θ_0 of the real uncertain input-delay θ , as follows:

$$z_0(t) = e^{-\sigma\theta_0}y(t) + \int_{t-\theta_0}^t e^{-\sigma(t-\tau)}u(\tau)d\tau, \text{ for } t \geq \theta_0. \quad (30)$$

The variable $z_0(t)$ does not constitute an exact prediction as expressed in (11), but an approximation of the future inventory level, due to the delay error of estimation. Moreover, the system dynamics are defined by the following delayed differential equation, that depends on the two non commensurable delays contrary to the free-delay system (12):

$$\dot{z}(t) = -\sigma z(t) + u(t) - e^{-\sigma\theta_0}d(t) + e^{-\sigma\theta_0}(u(t-\theta) - u(t-\theta_0)). \quad (31)$$

Compared to Figure 1, the controller structure is illustrated in Figure 2, where $\hat{C}_0(s) = \frac{1}{1+K(\frac{1-e^{-(s+\sigma)\theta_0}}{s+\sigma})}$.

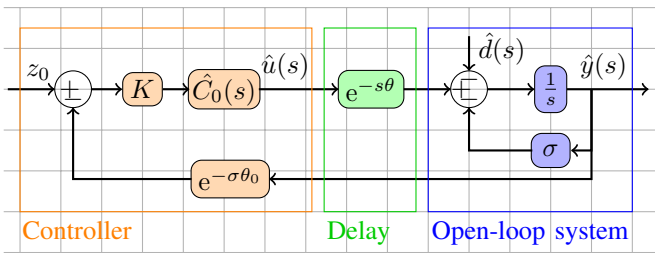


Fig. 2. Closed-loop system diagram with uncertain input-delay

The characteristic quasi-polynomial of the closed loop system depends explicitly on the control parameter K and the delays θ and θ_0 , as expressed in the following:

$$\frac{1}{\hat{g}_0(s)} = s + \sigma + K - K e^{-\sigma\theta_0}(e^{-s\theta_0} - e^{-s\theta}).$$

Indeed, necessary and sufficient conditions were provided in the recent work [6], in order to guarantee the system stability considering the approximation of the predictor-based control. We notice that this latter is related to this paper since the closed loop spectrum $\frac{1}{\hat{g}_0(s)}$ is reduced to $\frac{1}{\hat{g}(s)} = s + \sigma + K$, when the delay is known exactly without uncertainty for $\theta = \theta_0$. However, stability alone is not a sufficient goal for inventory systems which are subject to significant constraints.

In this direction, we develop in this paper, in addition to the stability conditions, the conditions for which the closed-loop system would meet the system constraints and specifications as formulated in Theorems 3 and 4. These results would be extended when the input-delay θ is known with uncertainty in order to complete the robust study initiated in [6].

These remarks concerning the system constraints verification, the inventory loss factor consideration and the production delay uncertainty are illustrated in the simulation examples.

VII. SIMULATION EXAMPLE

We use the Simulink-Matlab environment to construct a simulation model of the inventory controlled system. The system constraints are mentioned in Table I. In addition, the demand signal is given in Figure 4 reflecting the semi-annual trend and the seasonal changes over a half-year margin. Thus, the system responses are illustrated in Figures 3 and 4 for different cases: (i) the real time delay is well defined with $\theta = \theta_0 = 5$ and the loss factor is varying so that $\sigma = 0$, $\sigma = 0.07$ and $\sigma = 0.2$, (ii) the real time delay is expressed with an uncertainty such that $\theta = 6$ and $\theta_0 = 5$.

TABLE I
SYSTEM PARAMETERS AND CONSTRAINTS

$u(t)$ [item/day]		θ	$y(t)$ [item]		$d(t)$ [item/day]	
u_{min}	u_{max}		y_{min}	y_{max}	d_{min}	d_{max}
4	20	5	0	70	5	15

Based on Figure 3, we notice that the inventory level $y(t)$ does not present any overflow of $y_{max} = 70$, nor a storage failure for $y(t) < 0$. In addition, when the customer demand is saturated for $d(t) = 15$, the inventory level is decreasing to $y_{min} = 0$. This latter reduces the storage losses for $\sigma = 0.07$, and minimizes the storage costs when $\sigma = 0$. However, when the storage losses increase, for $\sigma = 0.2$, a non-negligible security storage level is provided in order to avoid any storage lack and to deliver the final products to the customers on time. Then, when there is a demand shutdown, the controlled structure makes it possible to replenish the storage level in order to reach the inventory reference.

Moreover, the production order $u(t)$ as illustrated in Figure 4 follows linearly and closely the seasonal changes in the customer demand when $\sigma = 0$. In practice when the inventory losses are considered, for $\sigma = 0.07$ and $\sigma = 0.2$, the control input provides higher production levels and lower production dynamics in order to anticipate the storage losses. Then, the initial production work in progress is predefined for each case, so that the storage dynamics provide a high service level for $t \leq \theta$ before the implementation of the control law. In addition, the system stability and the constraints verification are guaranteed for a static gain adjusted to $K = 0.75$ verifying the conditions of Theorems 3 and 4, and an inventory reference z_0 modulated based on the conditions of Theorem 2.

As introduced in Remark 4, we consider in the last case that the delay is subject to uncertainties such that the delay $\theta = 6$ becomes different from its estimation $\theta_0 = 5$. We notice that

the system behavior presents some fluctuations, that never lead to negative responses. In this case, the control approach is implemented based on an approximation of the predictor-based controller, and the control parameters K and z_0 are modulated verifying the robust stability conditions developed in [6].

admissible control law have been formulated in terms of the system specifications and the control law parameterization. These results are compared to recent works modeling the system constraints and input-delay uncertainty in particular.

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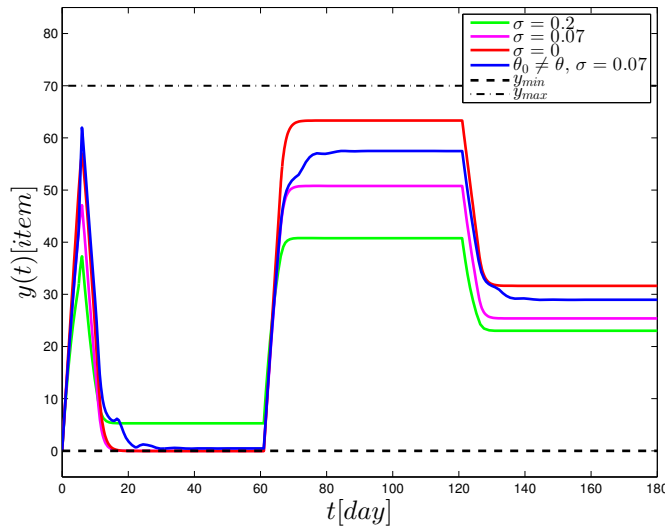


Fig. 3. System output dynamics with various specifications

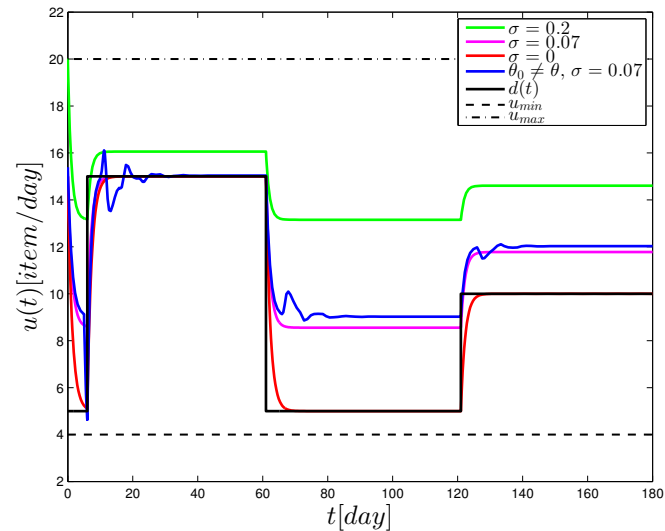


Fig. 4. Control input dynamics for rectangular disturbance

VIII. CONCLUSION

The paper deals with the inventory regulation problem of a production system subject to a constant lead time and a fixed loss factor, based on a predictive feedback control structure. Since the input-delay system is subject to positive and saturation constraints, then the Input-Output reachable bounds are identified, and the Input-Output system stability and the constraints verification are guaranteed. Indeed, the necessary and sufficient conditions for the existence of an