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NEW ENERGY FUNCTIONALS FOR THE INCOMPRESSIBLE HALL-MHD SYSTEM

JIN TAN

ABSTRACT. We show the existence and uniqueness of solutions to the three-dimensional incompressible Hall-magnetohydrodynamic (Hall-MHD) system with initial data in critical Sobolev spaces. Our result works for general physical parameters, thus improves the partial result obtained very recently by Danchin and the author in [13] and fully answers a problem proposed by Chae and Lee in the Remark 2 of [6]. Considering the so-called $2\frac{1}{2}$ D flows for the Hall-MHD system (that is 3D flows independent of the vertical variable), we show that under the sole assumption that the initial magnetic field is small in the critical Sobolev space leads to a global well-posedness statement.

1. INTRODUCTION

In this paper, we consider the following three-dimensional incompressible resistive and viscous Hall-MHD system in the whole space:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P - \nu \Delta u = (\nabla \times B) \times B, \\ \operatorname{div} u = 0, \\ \partial_t B - \nabla \times (u \times B - h(\nabla \times B) \times B) = \mu \Delta B, \\ (u, B)|_{t=0} = (u_0, B_0), \end{cases} \quad (1.1)$$

where the unknowns are u , B and P , that represent the velocity field, the magnetic field and the scalar pressure, respectively. The parameters ν and μ are the fluid viscosity and the magnetic resistivity, while the positive number h measures the magnitude of the Hall effect compared to the typical length scale of the fluid. For compatibility with (1.1)₂, we assume that $\operatorname{div} u_0 = 0$ and, for physical consistency, since a magnetic field is a curl, we suppose that $\operatorname{div} B_0 = 0$, a property that is propagated by (1.1)₃.

The Hall-MHD system (1.1) is useful when modelling the magnetic reconnection phenomenon, which cannot be explained by the classical MHD system (that corresponding to $h = 0$) where the Hall electric field $E_H := hJ \times B$ (here the current J is defined by $J := \nabla \times B$) is neglected. The study of the Hall-MHD system has been initiated by Lighthill in [23]. Owing to its importance in the theory of space plasma, like e.g. star formation, solar flares or geo-dynamo, it has received lots of attention from physicists (see e.g. [2, 16, 20, 27, 30]).

The mathematical study of the Hall-MHD system has been considered only rather recently. We mention that, the results [1, 5, 14, 10] are focused on the

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weak solutions, while [6, 4, 33, 34, 9] are focused on the strong solutions with regular enough (sub-critical) initial data. In [31, 32, 11, 35], the long-time behaviour and optimal space-time decay rates of strong solutions are obtained. Recently, in [12, 13], the authors proved global (for small initial data) and local (for any initial data) well-posedness in critical Besov spaces. For Hall-MHD system without resistivity, it was showed in [7] that there exists finite time singularity formation within axisymmetry, and several strong ill-posedness results are presented in [21].

For any couple (w, z) of vector-fields on \mathbb{R}^3 , thanks to the equality

$$(\nabla \times w | z) = (w | \nabla \times z), \quad (1.2)$$

where $(\cdot | \cdot)$ denotes the scalar product in $L^2(\mathbb{R}^3; \mathbb{R}^3)$, Hall-MHD system (1.1) actually shares a same energy inequality as to the classical MHD system (see e.g. [26, 19]): for all $t \geq 0$,

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2) + \nu \|\nabla u(t)\|_{L^2}^2 + \mu \|\nabla B(t)\|_{L^2}^2 \leq 0. \quad (1.3)$$

Indeed, (1.3) plays a key role in the proofs of Leray-Hopf type weak solutions of Hall-MHD system in [1, 5]. However, we emphasis that due to the additional *Hall term*: $\nabla \times ((\nabla \times B) \times B)$, the magnetic equation (1.1)₃ becomes *quasi-linear*, and the only known natural a priori bound (1.3) is *super-critical* more than the classical MHD system, or the Navier-Stokes equations. Moreover, as mentioned in [12, 34], we shall see that the difficulties not only comes from Hall term but also from the coupling between u and B through the term $\nabla \times (u \times B)$ in (1.1)₃. Somehow, it seems difficult to get exactly the same results as for the generalized Navier-Stokes equations that presented in e.g. [3, Chap. 5].

In [12, 13], Danchin and the author pointed out that the Hall-MHD system (1.1) better behaves if $\nu = \mu$ (which is not physically motivated) since, although being still quasi-linear, the Hall term disappears in the energy estimate involving the so-called velocity of electron $v := u - hJ$. Indeed, considering v as an additional unknown, the following extended formulation of the Hall-MHD system is obtained:

$$\begin{cases} \partial_t u - \nu \Delta u = B \cdot \nabla B - u \cdot \nabla u - \nabla \pi, \\ \partial_t B - \nu \Delta B = \nabla \times (v \times B), \\ \partial_t v - \nu \Delta v = B \cdot \nabla B - u \cdot \nabla u - \nabla \times ((\nabla \times v) \times \text{curl}^{-1}(u - v)) \\ \quad + \nabla \times (v \times u) + 2\nabla \times (v \cdot \nabla \text{curl}^{-1}(u - v)) - \nabla \pi, \\ \text{div } u = \text{div } B = \text{div } v = 0, \end{cases} \quad (1.4)$$

in which the -1 -th order homogeneous Fourier multiplier (the so-called Biot-Savart operator) curl^{-1} is defined on the Fourier side by

$$\mathcal{F}(\text{curl}^{-1} z)(\xi) := \frac{i\xi \times \mathcal{F}(z)(\xi)}{|\xi|^2}.$$

That redundant equation (1.4)₃ is still quasi-linear but, owing to (1.2), the most non-linear term cancels out when performing an energy estimate, since

$$(\nabla \times ((\nabla \times v) \times B) | v) = 0.$$

Let us now briefly explain what we mean by critical regularity for the Hall-MHD system (1.1). In fact, because Hall term breaks the natural scaling of the classical MHD system (the reader may refer [25] for results in critical spaces), there does not exist a genuine scaling invariance for the Hall-MHD system (1.1). However, it

can be seen that some scaling invariance does exist for System (1.4) and the scaling invariance of (u, B, v) is the same as for the velocity in the Navier-Stokes equations:

$$(u, B, v)(t, x) \rightsquigarrow (\lambda u, \lambda B, \lambda v)(\lambda^2 t, \lambda x) \quad \text{and} \quad (u_0, B_0, v_0)(x) \rightsquigarrow (\lambda u_0, \lambda B_0, \lambda v_0)(\lambda x).$$

This observation motivates the use of criticality in [12, 13] and for more details on the definition of criticality for Hall-MHD system, one may read [28].

1.1. Main results. Our first main goal is to prove Fujita-Kato type result for the 3D Hall-MHD system (1.1) in the spirit of the celebrated work [15] (see also [8]) for the Navier-Stokes equations. In our context, this amounts to proving that System (1.1) supplemented with initial data (u_0, B_0) such that (u_0, B_0) is small enough in the critical homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times (\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$ admits a unique global solution.

Theorem 1.1. *Let $(u_0, B_0) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$, and $\nabla \times B_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. There exists a constant $c_1 > 0$ depending only on ν, μ, h such that, if*

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \|B_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \|\nabla \times B_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < c_1,$$

then there exists a unique global solution

$$(u, B) \in \mathcal{C}(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$$

to the Cauchy problem (1.1), such that $B \in L^\infty(\mathbb{R}_+; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^{\frac{5}{2}}(\mathbb{R}^3))$.

Adopting the low-high frequencies decomposition method as in [3] and [18] for the Navier-Stokes equations, we have the following consequence which states that global solutions, even if *large and with infinite energy*, enjoying a suitable integrability property have to decay to zero at infinity. In contrast with [13], the assumption that $\nu = \mu$ is not required here, thanks to an improved stability estimate in L^2 (see Proposition 1.5) and similar arguments as in [13].

Corollary 1.2. *Assume that $(u_0, B_0) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ and $\nabla \times B_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Suppose in addition that the Hall-MHD system supplemented with initial data (u_0, B_0) admits a global solution (u, B) such that*

$$(u, B, \nabla \times B) \in L_{loc}^\infty(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L_{loc}^4(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3)).$$

Then, (u, B) also belongs to $L_{loc}^2(\mathbb{R}_+; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$, and satisfies

$$\lim_{t \rightarrow +\infty} (\|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|B(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\nabla \times B(t)\|_{\dot{H}^{\frac{3}{2}}}^2) = 0. \quad (1.5)$$

In particular, all the solutions constructed in Theorem 1.1 satisfy (1.5).

In the last, we consider the $2\frac{1}{2}$ D flows for the Hall-MHD system (1.1) (that is 3D flows depending only on two space variables), as proposed by Chae and Lee in [6]. This issue is well-known for the incompressible Navier-Stokes equations (see e.g. the book by Bertozzi and Majda [24]). In our case, the corresponding system reads:

$$\begin{cases} \partial_t u + u \cdot \widetilde{\nabla} u + \widetilde{\nabla} \pi = B \cdot \widetilde{\nabla} B + \nu \widetilde{\Delta} u, \\ \widetilde{\operatorname{div}} u = 0, \\ \partial_t B - \widetilde{\nabla} \times (u \times B) + h \widetilde{\nabla} \times (j \times B) = \mu \widetilde{\Delta} B, \\ (u, B)|_{t=0} = (u_0, B_0), \end{cases} \quad (1.6)$$

where the unknowns u and B are functions from $\mathbb{R}_+ \times \mathbb{R}^2$ to \mathbb{R}^3 , $\widetilde{\nabla} := (\partial_1, \partial_2, 0)$, $\widetilde{\operatorname{div}} := \widetilde{\nabla} \cdot$, $\widetilde{\Delta} := \partial_1^2 + \partial_2^2$ and

$$j := \widetilde{\nabla} \times B = \begin{pmatrix} \partial_2 B^3 \\ -\partial_1 B^3 \\ \partial_1 B^2 - \partial_2 B^1 \end{pmatrix}.$$

After a small modification of the proof of [5] allows to establish that for any initial data (u_0, B_0) in $L^2(\mathbb{R}^2; \mathbb{R}^3)$ with $\widetilde{\operatorname{div}} u_0 = \widetilde{\operatorname{div}} B_0 = 0$, there exists a global-in-time Leray-Hopf weak solution $(u, B) \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^2))$ of System (1.6) that satisfies:

$$\frac{1}{2} \frac{d}{dt} \left(\|u(t)\|_{L^2(\mathbb{R}^2)}^2 + \|B(t)\|_{L^2(\mathbb{R}^2)}^2 \right) + \nu \|\widetilde{\nabla} u(t)\|_{L^2(\mathbb{R}^2)}^2 + \mu \|\widetilde{\nabla} B(t)\|_{L^2(\mathbb{R}^2)}^2 \leq 0. \quad (1.7)$$

For $2\frac{1}{2}$ D flows, we obtain a global existence and uniqueness result for System (1.6) by assuming only that $\|B_0\|_{H^1}$ is small enough and $u_0 \in H^1(\mathbb{R}^2)$. In contrast with [13], the assumption that $\nu = \mu$ is not required. It reads:

Theorem 1.3. *Let (u_0, B_0) be divergence free vector-fields with u_0 in $H^1(\mathbb{R}^2; \mathbb{R}^3)$ and B_0 in $H^1(\mathbb{R}^2; \mathbb{R}^3)$. There exists a constant $c_2 > 0$ depending only on the H^1 norm of u_0 , and on ν, μ, h such that if*

$$\|B_0\|_{H^1(\mathbb{R}^2)} \leq c_2,$$

then there exists a unique global solution (u, B) to System (1.6), with

$$(u, B) \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^2)), \quad (\widetilde{\nabla} u, \widetilde{\nabla} B) \in L^2(\mathbb{R}_+; H^1(\mathbb{R}^2)).$$

Moreover, the following energy balance is satisfied for all $t \geq t_0 \geq 0$:

$$\begin{aligned} \|u(t)\|_{L^2(\mathbb{R}^2)}^2 + \|B(t)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_{t_0}^t (\nu \|\widetilde{\nabla} u(\tau)\|_{L^2(\mathbb{R}^2)}^2 + \mu \|\widetilde{\nabla} B(\tau)\|_{L^2(\mathbb{R}^2)}^2) d\tau \\ = \|u(t_0)\|_{L^2(\mathbb{R}^2)}^2 + \|B(t_0)\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \quad (1.8)$$

Remark 1. At the same time, the critical Sobolev regularity of the $2\frac{1}{2}$ D model corresponds to u_0 in $L^2(\mathbb{R}^2)$ and to B_0 in $H^1(\mathbb{R}^2)$, if only u_0 is in $L^2(\mathbb{R}^2)$ and that the initial data are small, then it will be possible to prove a global existence and uniqueness result of solutions with critical regularity, in the spirit of the 3D one.

1.2. New energy functionals. Due to the scaling of the magnetic field for the Hall-MHD system, it seems reasonable to consider the current function $J = \nabla \times B$ as an auxiliary unknown. Applying the operator curl to both sides of (1.1)₃, one has

$$\partial_t J - \operatorname{curl} \operatorname{curl} (u \times B) + h \operatorname{curl} \operatorname{curl} (J \times B) = \mu \Delta J. \quad (1.9)$$

Using the following equality for divergence-free vector-fields

$$\operatorname{curl} (w \times z) = (\operatorname{curl} w) \times z + (\operatorname{curl} z) \times w - 2w \cdot \nabla z + \nabla (w \cdot z), \quad (1.10)$$

we rewrite (1.9) to

$$\begin{aligned} \partial_t J - \mu \Delta J = \operatorname{curl} (J \times u - 2u \cdot \nabla B) + 2h \operatorname{curl} (J \cdot \nabla B) \\ + \operatorname{curl} ((\operatorname{curl} u) \times B) - h \operatorname{curl} ((\operatorname{curl} J) \times B). \end{aligned} \quad (1.11)$$

Keeping in mind that as B is divergence free, one has the following equivalence of norms for any $s \in \mathbb{R}$:

$$\|\nabla B\|_{\dot{H}^s} \sim \|J\|_{\dot{H}^s} \quad \text{with } J := \nabla \times B. \quad (1.12)$$

It can be seen that the terms in the first line of (1.11) are of semi-linear type which are fine, while the rest terms in the second line are troublemakers, since the failure that $\dot{H}^{\frac{d}{2}}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^d)$ in the case $d = 2, 3$. Thus one really sees that the difficulties of proving existence (and uniqueness) of solutions to the Hall-MHD system in critical Sobolev spaces are come from not only the Hall term but also the coupling $\nabla \times (u \times B)$ in (1.1)₃. The ideas presented in [12, 13] in order to cancel these two trouble terms only work in the case that $\nu = \mu$.

In order to prove Theorem 1.1, our idea to handle that two troublemakers is to use the Hall term to cancel the term $\text{curl}((\text{curl } u) \times B)$ by performing a renormalized energy estimate for (1.11), and meanwhile we take advantage of the equality (1.2) to write a cancellation to deal with the Hall term. Indeed, this idea is in the spirit of the hypocoercivity method proposed by Villani [29]: construct a Lyapunov functional by adding carefully chosen lower-order terms to the natural Lyapunov functional.

We have the following a priori estimates.

Proposition 1.4. *Define*

$$\begin{aligned} \mathcal{L}(t) &:= \frac{1}{2} \left(\alpha \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|B(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|J(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \right) - \frac{1}{h} \left(\Lambda^{\frac{1}{2}} u(t) \mid \Lambda^{\frac{1}{2}} J(t) \right), \\ \mathcal{D}(t) &:= \alpha \nu \|u(t)\|_{\dot{H}^{\frac{3}{2}}}^2 + \mu \|B(t)\|_{\dot{H}^{\frac{3}{2}}}^2 + \mu \|J(t)\|_{\dot{H}^{\frac{3}{2}}}^2, \quad \forall t \geq 0, \end{aligned}$$

and

$$\alpha := \left(\frac{\mu}{\nu} + \frac{\nu}{\mu} \right) \frac{8}{h^2}, \quad \beta := \frac{(1 + \alpha)(1 + h^2)}{\min\{1, h\} \min\{\frac{\nu}{h^2}, \mu\}}.$$

Let (u, B) be a smooth solution of the 3D Hall-MHD system (1.1) on the time interval $[0, T_0]$. There exists a universal constant C such that on $[0, T_0]$, we have

$$\frac{d}{dt} \mathcal{L}(t) + \frac{1}{4} \mathcal{D}(t) \leq C \beta \sqrt{\mathcal{L}(t)} \mathcal{D}(t). \quad (1.13)$$

About the fractional derivative operators, the reader may check Definition 2.1. Using the fact that $\frac{\mu}{\nu} + \frac{\nu}{\mu} \geq 2$ for $\nu, \mu > 0$, and Hölder's inequality and Young's inequality, one has

$$\begin{aligned} \mathcal{L}(t) &\geq \frac{1}{2} \left(\alpha \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|B(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|J(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \right) - \frac{1}{h^2} \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2 - \frac{1}{4} \|J(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \\ &\geq \frac{\min\{1, h^2\}}{4h^2} \|(u, B, J)(t)\|_{\dot{H}^{\frac{1}{2}}}^2. \end{aligned} \quad (1.14)$$

We know that thanks to (1.14) and Proposition 1.4 we do obtain desired estimates in order to prove the existence part of Theorem 1.1.

Let us now simply explain our idea. Indeed, when taking L^2 scalar product with $J - \frac{1}{h} u$ (but not J) for (1.11), we find from (1.2) and basic vector identity that

$$\begin{aligned} \left(\text{curl}((\text{curl } u) \times B) \mid J - \frac{1}{h} u \right) &= (\text{curl}((\text{curl } u) \times B) \mid J) \\ &= ((\text{curl } u) \times B \mid \text{curl } J) \end{aligned}$$

$$= (B \times (\operatorname{curl} J) \mid \operatorname{curl} u), \quad (1.15)$$

$$\begin{aligned} -h \left(\operatorname{curl}((\operatorname{curl} J) \times B) \mid J - \frac{1}{h} u \right) &= (\operatorname{curl}((\operatorname{curl} J) \times B) \mid u) \\ &= ((\operatorname{curl} J) \times B \mid \operatorname{curl} u) \\ &= -(B \times (\operatorname{curl} J) \mid \operatorname{curl} u). \end{aligned} \quad (1.16)$$

Obviously, the sum of the above two scalar products cancels out and when performing similar (renormalized) energy estimate of the higher-order $\Lambda^{\frac{1}{2}}$, the above products still cancel each other, up to some lower-order commutator terms. Next, to couple with the cross term $(\partial_t \Lambda^{\frac{1}{2}} J \mid -\frac{1}{h} \Lambda^{\frac{1}{2}} u)$, we take L^2 scalar product with $\Lambda^{\frac{1}{2}}(\alpha u - \frac{1}{h} J)$ for the resulting equation of applying $\Lambda^{\frac{1}{2}}$ to (1.4)₁ and since the non-linear terms in (1.4)₁ are of lower-order, we can choose some appropriate α and use Young's inequality to handle the another cross term $(\nabla \Lambda^{\frac{1}{2}} u \mid \nabla \Lambda^{\frac{1}{2}} J)$.

About the uniqueness part of Theorem 1.1, we emphasis that a direct estimate in the space L^2 maybe not enough, since the obtained L^2 stability estimate in [13] strongly rely on the condition $\nu = \mu$. We consider slightly regular space and obtain the following stability estimates:

Proposition 1.5. *Let (u_1, B_1) and (u_2, B_2) be two solutions of the Hall-MHD system on $[0, T] \times \mathbb{R}^3$, supplemented with the divergence-free initial data $(u_{i,0}, B_{i,0})$ and such that, for $i = 1, 2$,*

$$(u_i, B_i, \nabla \times B_i) \in L^\infty(0, T; \dot{H}^{\frac{1}{2}}) \quad \text{and} \quad (u_i, B_i, \nabla \times B_i) \in L^4(0, T; \dot{H}^1).$$

Then there exists a constant M depends only on ν, μ, h such that for all $t \in [0, T]$,

$$\begin{aligned} &\|(u_1 - u_2)(t)\|_{H^{\frac{1}{2}}}^2 + \|(B_1 - B_2)(t)\|_{H^1}^2 \\ &\quad + \int_0^t \left(\nu \|\nabla(u_1 - u_2)\|_{H^{\frac{1}{2}}}^2 + \mu \|\nabla(B_1 - B_2)\|_{H^1}^2 \right) d\tau \\ &\leq \left(\|u_{1,0} - u_{2,0}\|_{H^{\frac{1}{2}}}^2 + \|B_{1,0} - B_{2,0}\|_{H^1}^2 \right) \exp \left(M \int_0^t V(\tau) d\tau \right) \end{aligned}$$

with $V(\tau) := \|(u_1, u_2, B_1, B_2, \nabla \times B_1, \nabla \times B_2)(\tau)\|_{\dot{H}^1}^4$, for all $\tau \in [0, t]$.

We remark that after localization of the 3D Hall-MHD system (1.1) by means of the Littlewood-Paley spectral cut-off operators (one may read [3] for its definition), the idea illustrated previously may valid in critical Besov spaces of the type $\dot{B}_{2,r}^{\frac{1}{2}}$ (for any $r \in [1, \infty]$), which does not need the assumption $\nu = \mu$ sated in Theorem 2.3 of [12]. The details are left to the interested reader.

For $2^{\frac{1}{2}}$ D flows, the proof of Theorem 1.3 is essentially relies on the following proposition.

Proposition 1.6. *Define $\omega := \tilde{\nabla} \times u$ and*

$$\begin{aligned} \tilde{\mathcal{L}}(t) &:= \frac{1}{2} \left(\tilde{\alpha} \|u(t)\|_{L^2}^2 + \tilde{\alpha} \|B(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 - \frac{1}{h} (\omega(t) \mid B(t)) \right), \\ \tilde{\mathcal{D}}(t) &:= \tilde{\alpha} \nu \|\tilde{\nabla} u(t)\|_{L^2}^2 + \tilde{\alpha} \mu \|\tilde{\nabla} B(t)\|_{L^2}^2 + \nu \|\tilde{\nabla} \omega(t)\|_{L^2}^2, \quad \forall t \geq 0, \end{aligned}$$

and

$$\tilde{\alpha} := \left(\frac{\mu}{\nu} + \frac{\nu}{\mu} \right) \frac{8}{h^2}, \quad \tilde{\beta} := \left(\frac{1}{\tilde{\alpha}^2 \mu^2 \nu^3 h^4} + \frac{1}{\nu^5} \right) \frac{h^2}{\min\{1, h^2\}}.$$

Let (u, B) be a smooth solution of the $2\frac{1}{2}D$ Hall-MHD system (1.6) on the time interval $[0, T_1]$. Then (1.7) is satisfied and there exists a universal constant C such that on $[0, T_1]$, we have

$$\tilde{\mathcal{L}}(t) + \int_0^t \tilde{\mathcal{D}}(\tau) d\tau \leq \left(\tilde{\mathcal{L}}(0) + \frac{\|(u_0, B_0)\|_{L^2}^4}{\nu^{\frac{3}{2}} \mu^{\frac{1}{2}}} \right) \exp(C \tilde{\beta} \|(u_0, B_0)\|_{L^2}^4). \quad (1.17)$$

Due to the coupling between magnetic field and fluid velocity, it seems hard to impose smallness condition only on initial magnetic field. As found in [13], if one consider $\omega = \tilde{\nabla} \times u$ as an auxiliary unknown then, indeed, an a priori L^2 estimate can be obtained for ω *without any smallness assumption on the initial data*, under the assumption that $\nu = \mu$. Here, we use again the idea of adding some harmless terms into the natural energy functional, in order to cancel out the troublemakers when performing L^2 estimate of ω . More details on the proof of Proposition 1.6 will be presented in Section 4.

To prove the uniqueness part of Theorem 1.3, we apply a weak-strong uniqueness theorem from [13] which valid for all positive parameters ν, μ and h .

Finally, we mention that some cross products between the magnetic (current) field and the fluid velocity (vorticity) field are involved in several conservation laws that have been discussed in [14].

The paper will unfold in the following way: In the next section, a few definitions and technical results will recalled. Section 3 is devoted to the proof of Theorem 1.1. In the last section, we prove Theorem 1.3.

We end this introductory part by presenting a few notations. As usual, we denote by C harmless positive constants that may change from line to line, and $A \lesssim B$ means that $A \leq CB$. For X a Banach space, $p \in [1, \infty]$ and $T > 0$, the notation $L^p(0, T; X)$ or $L_T^p(X)$ designates the set of measurable functions $f : [0, T] \rightarrow X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0, T)$, endowed with the norm $\|\cdot\|_{L_T^p(X)} := \|\|\cdot\|_X\|_{L^p(0, T)}$. We agree that $\mathcal{C}([0, T]; X)$ denotes the set of continuous functions from $[0, T]$ to X . We will keep the same notations for multi-component functions. Throughout the paper, the commutator of two elements, f and g , is the element $[f, g] := fg - gf$.

2. PRELIMINARY

For the reader's convenience, we here recall a few results that will be repeatedly used in the paper (more details may be found in e.g. [3, Chap. 1]). Let us first recall the definitions of Sobolev spaces and fractional derivation operators.

Definition 2.1. *Let s be in \mathbb{R} . The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ (also denoted by \dot{H}^s) is the set of tempered distributions u on \mathbb{R}^d , with Fourier transform in $L_{loc}^1(\mathbb{R}^d)$, satisfying*

$$\|u\|_{\dot{H}^s} := \|\Lambda^s u\|_{L^2} < \infty,$$

where Λ^s stands for the fractional derivative operator defined in terms of the Fourier transform by

$$\mathcal{F}(\Lambda^s u)(\xi) := |\xi|^s \mathcal{F}u(\xi), \quad \xi \in \mathbb{R}^d.$$

The nonhomogeneous Sobolev space $H^s(\mathbb{R}^d)$ (also denoted by H^s) is the set of tempered distributions u on \mathbb{R}^d , with Fourier transform in $L^1_{loc}(\mathbb{R}^d)$, satisfying

$$\|u\|_{H^s} := \|\langle D \rangle^s u\|_{L^2} < \infty \quad \text{with} \quad \mathcal{F}(\langle D \rangle^s u)(\xi) := (1 + |\xi|^2)^{s/2} \mathcal{F}u(\xi).$$

We have the following proposition.

Proposition 2.2. *Let $s_0 \leq s \leq s_1$. Then, $\dot{H}^{s_0} \cap \dot{H}^{s_1}$ is included in \dot{H}^s , and we have for all θ in $[0, 1]$,*

$$\|u\|_{\dot{H}^s} \leq \|u\|_{\dot{H}^{s_0}}^{1-\theta} \|u\|_{\dot{H}^{s_1}}^\theta \quad \text{with} \quad s = (1-\theta)s_0 + \theta s_1. \quad (2.1)$$

We will often use the following Sobolev embedding for $0 \leq s < d/2$:

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2s}}(\mathbb{R}^d), \quad (2.2)$$

and sometimes the following Gagliardo-Nirenberg inequality:

$$\|u\|_{L^\infty(\mathbb{R}^3)} \lesssim \|u\|_{\dot{H}^s(\mathbb{R}^3)}^{1-\theta} \|u\|_{\dot{H}^{s'}(\mathbb{R}^3)}^\theta, \quad s < \frac{3}{2} < s', \quad \theta = \frac{\frac{3}{2} - s}{s' - s}. \quad (2.3)$$

The following inequalities (see e.g. [22], Lemma 2.10) are needed.

Lemma 2.3. *Let $s > 0$ and $1 < p, p_1, p_2, p_3, p_4 < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. There exists a constant $C > 0$ such that*

$$\|\Lambda^s(uv) - u\Lambda^s v\|_{L^p} \leq C(\|\nabla u\|_{L^{p_1}} \|\Lambda^{s-1} v\|_{L^{p_2}} + \|\Lambda^s u\|_{L^{p_3}} \|v\|_{L^{p_4}})$$

and

$$\|\Lambda^s(uv)\|_{L^p} \leq C(\|\Lambda^s u\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{p_3}} \|\Lambda^s v\|_{L^{p_4}}).$$

Finally, we recall a bootstrap argument from [13].

Lemma 2.4. *Let X, D, W be three nonnegative measurable functions on $[0, T]$ such that X is also differentiable. Assume that there exist two nonnegative real numbers C and α such that*

$$\frac{d}{dt} X^2 + D^2 \leq CWX^2 + CX^\alpha D^2. \quad (2.4)$$

If, in addition,

$$2CX^\alpha(0) \exp\left(\frac{C\alpha}{2} \int_0^T W dt\right) < 1, \quad (2.5)$$

then, for any $t \in [0, T]$, one has

$$X^2(t) + \frac{1}{2} \int_0^t D^2 d\tau \leq X^2(0) \exp\left(C \int_0^t W d\tau\right). \quad (2.6)$$

3. GLOBAL FUJITA-KATO TYPE SOLUTIONS FOR THE 3D FLOWS

This section is devoted to prove Theorem 1.1, it includes the existence part and the uniqueness part.

3.1. Existence. First and foremost, we prove Proposition 1.4.

Proof of Proposition 1.4. Applying the fractional derivative operator $\Lambda^{\frac{1}{2}}$ to both sides of equations (1.4)₁, (1.1)₃ and (1.11), taking the L^2 scalar product with $\Lambda^{\frac{1}{2}}u$, $\Lambda^{\frac{1}{2}}B$, $\Lambda^{\frac{1}{2}}(J - \frac{1}{h}u)$, respectively, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu \|u\|_{\dot{H}^{\frac{3}{2}}}^2 &= -(u \cdot \nabla u | \Lambda u) + (B \cdot \nabla B | \Lambda u) =: A_1 + A_2, \\ \frac{1}{2} \frac{d}{dt} \|B\|_{\dot{H}^{\frac{1}{2}}}^2 + \mu \|B\|_{\dot{H}^{\frac{3}{2}}}^2 &= (\nabla \times ((u - hJ) \times B) | \Lambda B) =: A_3 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|J\|_{\dot{H}^{\frac{1}{2}}}^2 + \mu \|J\|_{\dot{H}^{\frac{3}{2}}}^2 - \frac{1}{h} (\Lambda^{\frac{1}{2}} \partial_t J | \Lambda^{\frac{1}{2}} u) - \frac{\mu}{h} (\nabla \Lambda^{\frac{1}{2}} J | \nabla \Lambda^{\frac{1}{2}} u) \\ = A_4 + A_5 + A_6 + A_7, \end{aligned}$$

where

$$\begin{aligned} A_4 &:= \left(\Lambda^{\frac{1}{2}} (J \times u - 2u \cdot \nabla B) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} \left(J - \frac{1}{h} u \right) \right), \\ A_5 &:= 2h \left(\Lambda^{\frac{1}{2}} (J \cdot \nabla B) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} \left(J - \frac{1}{h} u \right) \right), \\ A_6 &:= \left(\Lambda^{\frac{1}{2}} ((\operatorname{curl} u) \times B) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} \left(J - \frac{1}{h} u \right) \right), \\ A_7 &:= -h \left(\Lambda^{\frac{1}{2}} ((\operatorname{curl} J) \times B) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} \left(J - \frac{1}{h} u \right) \right). \end{aligned}$$

By Hölder's inequality and Sobolev embedding (2.2), we get

$$\begin{aligned} |A_1| &\leq C \|u\|_{L^6} \|\nabla u\|_{L^3} \|\Lambda u\|_{L^2} \leq C \|u\|_{\dot{H}^1}^2 \|u\|_{\dot{H}^{\frac{3}{2}}}, \\ |A_2| &\leq C \|B\|_{L^6} \|\nabla B\|_{L^2} \|\Lambda u\|_{L^3} \leq C \|B\|_{\dot{H}^1}^2 \|u\|_{\dot{H}^{\frac{3}{2}}}, \\ |A_3| &\leq C (\|u - hJ\|_{\dot{H}^1} \|B\|_{L^6} + \|u - hJ\|_{L^6} \|B\|_{\dot{H}^1}) \|\Lambda B\|_{L^3} \\ &\leq C (\|u\|_{\dot{H}^1} + h \|J\|_{\dot{H}^1}) \|B\|_{\dot{H}^1} \|B\|_{\dot{H}^{\frac{3}{2}}} \end{aligned}$$

and by Lemma 2.3 and (1.12) we have

$$\begin{aligned} |A_5| &\leq Ch (\|\Lambda^{\frac{1}{2}} J\|_{L^3} \|\nabla B\|_{L^6} + \|J\|_{L^6} \|\Lambda^{\frac{1}{2}} \nabla B\|_{L^3}) \left(\|\operatorname{curl} \Lambda^{\frac{1}{2}} J\|_{L^2} + \frac{1}{h} \|\operatorname{curl} \Lambda^{\frac{1}{2}} u\|_{L^2} \right) \\ &\leq C \left(h \|J\|_{\dot{H}^1}^2 \|J\|_{\dot{H}^{\frac{3}{2}}} + \|J\|_{\dot{H}^1}^2 \|u\|_{\dot{H}^{\frac{3}{2}}} \right). \end{aligned}$$

Term A_4 may be bounded similarly as A_5 :

$$|A_4| \leq C \left(\|J\|_{\dot{H}^1} \|u\|_{\dot{H}^1} \|J\|_{\dot{H}^{\frac{3}{2}}} + \frac{1}{h} \|J\|_{\dot{H}^1} \|u\|_{\dot{H}^1} \|u\|_{\dot{H}^{\frac{3}{2}}} \right).$$

Now, we decompose A_6 and A_7 as

$$\begin{aligned} A_6 &= \left(\Lambda^{\frac{1}{2}} ((\operatorname{curl} u) \times B) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} J \right) - \frac{1}{h} \left(\Lambda^{\frac{1}{2}} ((\operatorname{curl} u) \times B) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} u \right) =: A_{61} + A_{62}, \\ A_7 &= \left(\Lambda^{\frac{1}{2}} ((\operatorname{curl} J) \times B) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} u \right) - h \left(\Lambda^{\frac{1}{2}} ((\operatorname{curl} J) \times B) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} J \right) =: A_{71} + A_{72} \end{aligned}$$

and we rewrite

$$A_{61} = - \left([\Lambda^{\frac{1}{2}}, B \times] (\operatorname{curl} u) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} J \right) - \left(B \times \Lambda^{\frac{1}{2}} (\operatorname{curl} u) \mid \operatorname{curl} \Lambda^{\frac{1}{2}} J \right)$$

$$A_{71} = - \left([\Lambda^{\frac{1}{2}}, B \times] (\operatorname{curl} J) \Big| \operatorname{curl} \Lambda^{\frac{1}{2}} u \right) - \left(B \times \Lambda^{\frac{1}{2}} (\operatorname{curl} J) \Big| \operatorname{curl} \Lambda^{\frac{1}{2}} u \right).$$

Not surprisingly, the last terms in A_{61} and A_{71} cancel out each other and thus gives

$$A_{61} + A_{71} = - \left([\Lambda^{\frac{1}{2}}, B \times] (\operatorname{curl} u) \Big| \operatorname{curl} \Lambda^{\frac{1}{2}} J \right) - \left([\Lambda^{\frac{1}{2}}, B \times] (\operatorname{curl} J) \Big| \operatorname{curl} \Lambda^{\frac{1}{2}} u \right).$$

Meanwhile, thanks to the commutator estimate in Lemma 2.3 and (1.12), one easily has

$$\begin{aligned} |A_{62}| &\leq \frac{1}{h} \| [\Lambda^{\frac{1}{2}}, B \times] (\operatorname{curl} u) \|_{L^2} \| \operatorname{curl} \Lambda^{\frac{1}{2}} u \|_{L^2} \\ &\leq \frac{C}{h} (\| \nabla B \|_{L^6} \| \Lambda^{-\frac{1}{2}} \operatorname{curl} u \|_{L^3} + \| \Lambda^{\frac{1}{2}} B \|_{L^6} \| \operatorname{curl} u \|_{L^3}) \| u \|_{\dot{H}^{\frac{3}{2}}} \\ &\leq \frac{C}{h} (\| J \|_{\dot{H}^1} \| u \|_{\dot{H}^1} + \| J \|_{\dot{H}^{\frac{1}{2}}} \| u \|_{\dot{H}^{\frac{3}{2}}}) \| u \|_{\dot{H}^{\frac{3}{2}}} \end{aligned}$$

and similarly

$$\begin{aligned} |A_{72}| &\leq h \| [\Lambda^{\frac{1}{2}}, B \times] (\operatorname{curl} J) \|_{L^2} \| \operatorname{curl} \Lambda^{\frac{1}{2}} J \|_{L^2} \\ &\leq Ch (\| J \|_{\dot{H}^1}^2 + \| J \|_{\dot{H}^{\frac{1}{2}}} \| J \|_{\dot{H}^{\frac{3}{2}}}) \| J \|_{\dot{H}^{\frac{3}{2}}}, \\ |A_{61} + A_{71}| &\leq C (\| u \|_{\dot{H}^1} \| J \|_{\dot{H}^1} \| J \|_{\dot{H}^{\frac{3}{2}}} + \| u \|_{\dot{H}^{\frac{3}{2}}} \| J \|_{\dot{H}^1}^2 + \| u \|_{\dot{H}^{\frac{3}{2}}} \| J \|_{\dot{H}^{\frac{1}{2}}} \| J \|_{\dot{H}^{\frac{3}{2}}}). \end{aligned}$$

Next, applying the fractional derivative operator $\Lambda^{\frac{1}{2}}$ to both sides of equation (1.4)₁ and then taking the L^2 scalar product with $-\frac{1}{h} \Lambda^{\frac{1}{2}} J$, we get

$$-\frac{1}{h} (\Lambda^{\frac{1}{2}} \partial_t u \Big| \Lambda^{\frac{1}{2}} J) - \frac{\nu}{h} (\nabla \Lambda^{\frac{1}{2}} u \Big| \nabla \Lambda^{\frac{1}{2}} J) = -\frac{1}{h} (B \cdot \nabla B \Big| \Lambda J) + \frac{1}{h} (u \cdot \nabla u \Big| \Lambda J) =: A_8.$$

By Young's inequality, one has

$$\begin{aligned} \left| \frac{\nu + \mu}{h} (\nabla \Lambda^{\frac{1}{2}} u \Big| \nabla \Lambda^{\frac{1}{2}} J) \right| &\leq \frac{2(\nu^2 + \mu^2)}{\mu h^2} \| u \|_{\dot{H}^{\frac{3}{2}}}^2 + \frac{\mu}{4} \| J \|_{\dot{H}^{\frac{3}{2}}}^2 \\ &\leq \frac{\alpha \nu}{4} \| u \|_{\dot{H}^{\frac{3}{2}}}^2 + \frac{\mu}{4} \| J \|_{\dot{H}^{\frac{3}{2}}}^2. \end{aligned}$$

Term A_8 may be bounded similarly as A_1 :

$$|A_8| \leq \frac{C}{h} (\| B \|_{\dot{H}^1}^2 + \| u \|_{\dot{H}^1}^2) \| J \|_{\dot{H}^{\frac{3}{2}}}.$$

Hence, recalling the definitions of $\mathcal{L}(t)$ and $\mathcal{D}(t)$ and using repeatedly the interpolation inequality (2.1) and Young's inequality, it is easy to deduce from the above inequalities that

$$\begin{aligned} &\frac{d}{dt} \mathcal{L}(t) + \mathcal{D}(t) \\ &\leq \frac{\nu + \mu}{h} (\nabla \Lambda^{\frac{1}{2}} u \Big| \nabla \Lambda^{\frac{1}{2}} J) + \alpha (|A_1| + |A_2|) + |A_3 + A_4 + \cdots + A_8| \\ &\leq \frac{\alpha \nu}{4} \| u \|_{\dot{H}^{\frac{3}{2}}}^2 + \frac{\mu}{4} \| J \|_{\dot{H}^{\frac{3}{2}}}^2 + C \frac{(1 + \alpha)(1 + h^2)}{h} \| (u, B, J) \|_{\dot{H}^{\frac{1}{2}}} \| (u, B, J) \|_{\dot{H}^{\frac{3}{2}}}^2. \quad (3.1) \end{aligned}$$

Notice that

$$D(t) \geq \min \left\{ \frac{\nu}{h^2}, \mu \right\} \| (u, B, J)(t) \|_{\dot{H}^{\frac{3}{2}}}^2,$$

we thus complete the proof of Proposition 1.4 by substituting the above inequality and (1.14) into (3.1). \square

Now, combining Proposition 1.4 with Lemma 2.4 implies that there exists a constant $c_1 > 0$ such that, if

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|B_0\|_{\dot{H}^{\frac{1}{2}}} + \|J_0\|_{\dot{H}^{\frac{1}{2}}} < c_1 \leq \frac{1}{8C\beta \max\{\sqrt{\alpha}, 1\}},$$

then

$$\sqrt{\mathcal{L}(0)} \leq \frac{1}{8C\beta}$$

and we have for all $t \in [0, T]$

$$\mathcal{L}(t) + \int_0^t \mathcal{D}(\tau) d\tau \leq C\mathcal{L}(0). \quad (3.2)$$

We conclude from (3.2) that we obtained a priori bounds for (u, B, J) in the space $L^\infty(0, T; \dot{H}^{\frac{1}{2}}) \cap L^2(0, T; \dot{H}^{\frac{3}{2}})$ uniformly in time. In order to prove rigorously the existence part of Theorem 1.1, one may resort to the following classical procedure:

- (1) smooth out the initial data and get a sequence $(u^n, B^n)_{n \in \mathbb{N}}$ of smooth solutions to System (1.1) on the maximal time interval $[0, T^n]$;
- (2) apply (3.2) to $(u^n, B^n)_{n \in \mathbb{N}}$ so as to prove that $T^n = \infty$ and that the sequence $(u^n, B^n, J^n)_{n \in \mathbb{N}}$ with $J^n := \nabla \times B^n$ is bounded in $L^\infty(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}_+; \dot{H}^{\frac{3}{2}})$;
- (3) use compactness to prove that $(u^n, B^n)_{n \in \mathbb{N}}$ converges, up to extraction, to a solution of System (1.1) supplemented with initial data (u_0, B_0) .

Since each steps of proving existence are extremely similar to the ones that are clearly presented in [13], we omit the details here.

3.2. Uniqueness. Let us finally prove the uniqueness part of the theorem. Indeed, one only needs to prove Proposition 1.5.

Proof. In order to estimate the difference $(\delta u, \delta B) := (u_1 - u_2, B_1 - B_2)$ in the space

$$E(T) := \left(L_T^\infty(L^2 \cap \dot{H}^{\frac{1}{2}}) \cap L_T^2(\dot{H}^1 \cap \dot{H}^{\frac{3}{2}}) \right) \times \left(L_T^\infty(L^2 \cap \dot{H}^1) \cap L_T^2(\dot{H}^1 \cap \dot{H}^2) \right).$$

We first need to justify that, indeed, $(\delta u, \delta B)$ belongs to that space. One can observe that

$$\begin{cases} \partial_t \delta u - \nu \Delta \delta u = R_1, \\ \partial_t \delta B - \mu \Delta \delta B = R_2, \\ (\delta u, \delta B)|_{t=0} = (\delta u(0), \delta B(0)) \end{cases} \quad (3.3)$$

where

$$\begin{aligned} R_1 &:= \mathcal{P}(B_1 \cdot \nabla \delta B + \delta B \cdot \nabla B_2 - u_1 \cdot \nabla \delta u - \delta u \cdot \nabla u_2), \\ R_2 &:= \nabla \times (u_1 \times \delta B + \delta u \times B_2) - h \nabla \times (J_1 \times \delta B + \delta J \times B_2), \\ J_1 &:= \nabla \times B_1, \quad J_2 := \nabla \times B_2, \quad \delta J := J_1 - J_2. \end{aligned}$$

To achieve our goal, it suffices to prove that R_1 belongs to the space $L^2(0, T; \dot{H}^{-1}) \cap L^2(0, T; \dot{H}^{-\frac{1}{2}})$ and $R_2, \nabla R_2 \in L^2(0, T; \dot{H}^{-1})$. Now, since $(\delta u, \delta B, \delta J) \in L^4(0, T; \dot{H}^1)$ and using repeatedly the product laws (see e.g. Corollary 2.55 in [3]):

$$\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3), \quad \dot{H}^1(\mathbb{R}^3) \times \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3),$$

we get

$$\|R_1\|_{L^2(0, T; \dot{H}^{-1})} \lesssim \|B_1\|_{L_T^\infty(L^3)} \|\delta B\|_{L_T^2(L^6)} + \|\delta B\|_{L^2(0, T; L^6)} \|B_2\|_{L_T^\infty(L^3)}$$

$$\begin{aligned}
& + \|u_1\|_{L_T^\infty(L^3)} \|\delta u\|_{L_T^2(L^6)} + \|\delta u\|_{L_T^2(L^6)} \|u_2\|_{L_T^\infty(L^3)} \\
& \lesssim T^{\frac{1}{4}} \|(u_1, u_2, B_1, B_2)\|_{L^\infty(0, T; \dot{H}^{\frac{1}{2}})} \|(\delta u, \delta B)\|_{L^4(0, T; \dot{H}^1)}, \\
\|R_1\|_{L^2(0, T; \dot{H}^{-\frac{1}{2}})} & \lesssim \|B_1\|_{L_T^4(\dot{H}^1)} \|\nabla \delta B\|_{L_T^4(L^2)} + \|\delta B\|_{L^4(0, T; \dot{H}^1)} \|\nabla B_2\|_{L_T^4(L^2)} \\
& \quad + \|u_1\|_{L_T^4(\dot{H}^1)} \|\nabla \delta u\|_{L_T^4(L^2)} + \|\delta u\|_{L_T^4(\dot{H}^1)} \|\nabla u_2\|_{L_T^4(L^2)} \\
& \lesssim \|(u_1, u_2, B_1, B_2)\|_{L^4(0, T; \dot{H}^1)} \|(\delta u, \delta B)\|_{L^4(0, T; \dot{H}^1)}, \\
\|R_2\|_{L^2(0, T; \dot{H}^{-1})} & \lesssim \|(u_1 - h J_1) \times \delta B\|_{L^2(0, T; L^2)} + \|(\delta u - h \delta J) \times B_2\|_{L^2(0, T; L^2)} \\
& \lesssim T^{\frac{1}{4}} \|(u_1, B_2, h J_1)\|_{L^\infty(0, T; \dot{H}^{\frac{1}{2}})} \|(\delta B, \delta u, h \delta J)\|_{L^4(0, T; \dot{H}^1)}
\end{aligned}$$

and

$$\begin{aligned}
\|R_2\|_{L_T^2(L^2)} & \lesssim \|(u_1 - h J_1) \times \delta B\|_{L^2(0, T; \dot{H}^1)} + \|(\delta u - h \delta J) \times B_2\|_{L^2(0, T; \dot{H}^1)} \\
& \lesssim \|\nabla(u_1 - h J_1)\|_{L_T^4(L^2)} \|\delta B\|_{L_T^4(L^\infty)} + \|u_1 - h J_1\|_{L_T^\infty(L^3)} \|\nabla \delta B\|_{L_T^4(L^6)} \\
& \quad + \|\delta u - h \delta J\|_{L_T^4(\dot{H}^1)} \|B_2\|_{L_T^4(L^\infty)} + \|\delta u - h \delta J\|_{L_T^\infty(L^3)} \|\nabla B_2\|_{L_T^4(L^6)}.
\end{aligned}$$

Note that our assumptions ensure that B_i and ∇B_i are in $L^4(0, T; \dot{H}^1)$ and thus we do have, by the Gagliardo-Nirenberg inequality (2.3), $\delta B, B_1, B_2$ in $L^4(0, T; L^\infty)$. Moreover, thanks to Sobolev embeddings (2.2), we are sure that $R_2 \in L^2(0, T; L^2)$.

Now, estimating $(\delta u, \delta B)$ in the space $E(T)$ follows from Lemma 5.10 in [3] and standard energy estimates applied on (3.3). More precisely, using Hölder's inequality and (2.2), we have

$$\begin{aligned}
& \frac{1}{2} \|\delta u(t)\|_{L^2}^2 - \frac{1}{2} \|\delta u(0)\|_{L^2}^2 + \nu \int_0^t \|\delta u(\tau)\|_{\dot{H}^1}^2 d\tau \\
& \lesssim \int_0^t (\|B_1 \otimes \delta B\|_{L^2} + \|B_2 \otimes \delta B\|_{L^2} + \|u_2 \otimes \delta u\|_{L^2}) \|\nabla \delta u\|_{L^2} d\tau \\
& \lesssim \int_0^t (\|B_1\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^{\frac{1}{2}}} + \|B_2\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^{\frac{1}{2}}} + \|u_2\|_{\dot{H}^1} \|\delta u\|_{\dot{H}^{\frac{1}{2}}}) \|\delta u\|_{\dot{H}^1} d\tau
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2} \|\delta u(t)\|_{\dot{H}^{\frac{1}{2}}}^2 - \frac{1}{2} \|\delta u(0)\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu \int_0^t \|\delta u(\tau)\|_{\dot{H}^{\frac{3}{2}}}^2 d\tau \\
& \leq \int_0^t (\|B_1 \cdot \nabla \delta B\|_{L^{\frac{3}{2}}} + \|\delta B \cdot \nabla B_2\|_{L^{\frac{3}{2}}} + \|u_1 \cdot \nabla \delta u\|_{L^{\frac{3}{2}}} + \|\delta u \cdot \nabla u_2\|_{L^{\frac{3}{2}}}) \|\Lambda \delta u\|_{L^3} d\tau \\
& \lesssim \int_0^t (\|B_1\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^1} + \|\delta B\|_{\dot{H}^1} \|B_2\|_{\dot{H}^1} + \|u_1\|_{\dot{H}^1} \|\delta u\|_{\dot{H}^1} + \|\delta u\|_{\dot{H}^1} \|u_2\|_{\dot{H}^1}) \|\delta u\|_{\dot{H}^{\frac{3}{2}}} d\tau
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2} \|\delta B(t)\|_{L^2}^2 - \frac{1}{2} \|\delta B(0)\|_{L^2}^2 + \mu \int_0^t \|\delta B(\tau)\|_{\dot{H}^1}^2 d\tau \\
& \lesssim \int_0^t (\|(u_1 - h J_1) \times \delta B\|_{L^2} + \|(\delta u - h \delta J) \times B_2\|_{L^2}) \|\nabla \times \delta B\|_{L^2} d\tau \\
& \lesssim \int_0^t (\|(u_1 - h J_1)\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^{\frac{1}{2}}} + \|\delta u - h \delta J\|_{\dot{H}^{\frac{1}{2}}} \|B_2\|_{\dot{H}^1}) \|\delta B\|_{\dot{H}^1} d\tau.
\end{aligned}$$

Moreover, by Lemma 2.3 and using repeatedly (1.12) and the product law:

$$\|wz\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|w\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} \|z\|_{\dot{H}^1(\mathbb{R}^3)} + \|w\|_{\dot{H}^1(\mathbb{R}^3)} \|z\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)},$$

we write

$$\frac{1}{2} \|\Lambda \delta B(t)\|_{L^2}^2 - \frac{1}{2} \|\Lambda \delta B(0)\|_{L^2}^2 + \mu \int_0^t \|\Lambda \delta B(\tau)\|_{\dot{H}^1}^2 d\tau$$

$$\begin{aligned}
&\lesssim \int_0^t \left(\|u_1 \times \delta B + \delta u \times B_2 - h J_1 \times \delta B\|_{\dot{H}^1} \|\Lambda^2 \delta B\|_{L^2} + h \|[\Lambda, B_2 \times] \delta J\|_{L^2} \|\Lambda \delta J\|_{L^2} \right) d\tau \\
&\lesssim \int_0^t \left(\|u_1\|_{\dot{H}^{\frac{3}{2}}} \|\delta B\|_{\dot{H}^1} + \|u_1\|_{\dot{H}^1} \|\delta J\|_{\dot{H}^{\frac{1}{2}}} + h (\|J_1\|_{\dot{H}^1} \|\delta B\|_{L^\infty} + \|J_1\|_{\dot{H}^1} \|\delta J\|_{\dot{H}^{\frac{1}{2}}}) \|\delta J\|_{\dot{H}^1} \right. \\
&\quad \left. + h \|J_2\|_{\dot{H}^1} \|\delta J\|_{\dot{H}^{\frac{1}{2}}} \|\delta J\|_{\dot{H}^1} \right) d\tau.
\end{aligned}$$

At this stage, interpolation inequality (2.1) and Young's inequality imply that

$$\begin{aligned}
\|B_1\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^{\frac{1}{2}}} \|\delta u\|_{\dot{H}^1} &\leq \|B_1\|_{\dot{H}^1} \|\delta B\|_{L^2}^{\frac{1}{2}} \|\delta B\|_{\dot{H}^1}^{\frac{1}{2}} \|\delta u\|_{\dot{H}^1} \\
&\leq \frac{\nu}{20} \|\delta u\|_{\dot{H}^1}^2 + \frac{\mu}{20} \|\delta B\|_{\dot{H}^1}^2 + \frac{C}{\nu^2 \mu} \|\delta B\|_{L^2}^2 \|B_1\|_{\dot{H}^1}^4, \\
\|B_1\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^1} \|\delta u\|_{\dot{H}^{\frac{3}{2}}} &\leq \frac{\nu}{20} \|\delta u\|_{\dot{H}^{\frac{3}{2}}}^2 + \frac{C}{\nu} \|\Lambda \delta B\|_{L^2}^2 \|B_1\|_{\dot{H}^1}^2, \\
\|u_1\|_{\dot{H}^1} \|\delta u\|_{\dot{H}^1} \|\delta u\|_{\dot{H}^{\frac{3}{2}}} &\leq \|u_1\|_{\dot{H}^1} \|\delta u\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|\delta u\|_{\dot{H}^{\frac{3}{2}}}^{\frac{3}{2}} \\
&\leq \frac{C}{\nu^3} \|u_1\|_{\dot{H}^1}^4 \|\delta u\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{\nu}{20} \|\delta u\|_{\dot{H}^{\frac{3}{2}}}^2, \\
\|(u_1 - h J_1)\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^{\frac{1}{2}}} \|\delta B\|_{\dot{H}^1} &\leq \|(u_1 - h J_1)\|_{\dot{H}^1} \|\delta B\|_{L^2}^{\frac{1}{2}} \|\delta B\|_{\dot{H}^1}^{\frac{3}{2}} \\
&\leq \frac{C}{\mu^3} \|(u_1, h J_1)\|_{\dot{H}^1}^4 \|\delta B\|_{L^2}^2 + \frac{\mu}{20} \|\delta B\|_{\dot{H}^1}^2, \\
h \|J_1\|_{\dot{H}^1} \|\delta B\|_{L^\infty} \|\delta J\|_{\dot{H}^1} &\leq h \|J_1\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^1}^{\frac{1}{2}} \|\delta J\|_{\dot{H}^1}^{\frac{3}{2}} \\
&\leq \frac{Ch^4}{\mu^3} \|J_1\|_{\dot{H}^1}^4 \|\delta B\|_{\dot{H}^1}^2 + \frac{\mu}{20} \|\delta J\|_{\dot{H}^1}^2, \\
h \|J_1\|_{\dot{H}^1} \|\delta J\|_{\dot{H}^{\frac{1}{2}}} \|\delta J\|_{\dot{H}^1} &\leq h \|J_1\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^1}^{\frac{1}{2}} \|\delta J\|_{\dot{H}^1}^{\frac{3}{2}} \\
&\leq \frac{Ch^4}{\mu^3} \|J_1\|_{\dot{H}^1}^4 \|\delta B\|_{\dot{H}^1}^2 + \frac{\mu}{20} \|\delta J\|_{\dot{H}^1}^2,
\end{aligned}$$

and similar inequalities for the other terms of the right-hand sides of the above inequalities. In the end, we get for all $t \in (0, T)$,

$$\begin{aligned}
&\|\delta u(t)\|_{L^2 \cap \dot{H}^{\frac{1}{2}}}^2 + \|\delta B(t)\|_{L^2 \cap \dot{H}^1}^2 + \int_0^t \left(\nu \|\delta u(\tau)\|_{\dot{H}^1 \cap \dot{H}^{\frac{3}{2}}}^2 + \mu \|\delta B(\tau)\|_{\dot{H}^1 \cap \dot{H}^2}^2 \right) d\tau \\
&\leq (\|\delta u(0)\|_{L^2 \cap \dot{H}^{\frac{1}{2}}}^2 + \|\delta B(0)\|_{L^2 \cap \dot{H}^1}^2) + M \int_0^t V(\tau) \left(\|\delta u\|_{L^2 \cap \dot{H}^{\frac{1}{2}}}^2 + \|\delta B\|_{L^2 \cap \dot{H}^1}^2 \right) d\tau.
\end{aligned}$$

Since our assumptions ensure that V is integrable on $[0, T]$, applying Gronwall's inequality completes the proof of Proposition 1.5. \square

We have complete the proof of Theorem 1.1. \square

4. GLOBAL UNIQUE SOLVABILITY OF THE $2\frac{1}{2}$ D FLOWS WITH LARGE VELOCITY

In order to prove Theorem 1.3, let us first prove Proposition 1.6.

Proof of Proposition 1.6. Applying the operator $\tilde{\nabla} \times$ to (1.6)₁, it is easy to find that ω satisfies

$$\partial_t \omega + \tilde{\nabla} \times (\omega \times u) = \nu \tilde{\Delta} \omega + \tilde{\nabla} \times (j \times B).$$

Taking the L^2 scalar product with $\omega + \frac{1}{h} B$ (but not ω) for above equation, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\tilde{\nabla} \omega\|_{L^2}^2 + \frac{1}{h} (\partial_t \omega | B) + \frac{\nu}{h} (\tilde{\nabla} \omega | \tilde{\nabla} B) - (j \times B | \tilde{\nabla} \times \omega) \\ = \mathcal{A}_1 + \mathcal{A}_2, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} \mathcal{A}_1 &:= - \left(\omega \times u \mid \tilde{\nabla} \times \omega \right), \\ \mathcal{A}_2 &:= - \frac{1}{h} \left(\omega \times u \mid \tilde{\nabla} \times B \right). \end{aligned}$$

Also, taking the L^2 scalar product with $\frac{1}{h} \omega$ for (1.6)₃, we get

$$\frac{1}{h} (\omega | \partial_t B) - \frac{\mu}{h} (\tilde{\nabla} \omega | \tilde{\nabla} B) + (j \times B | \tilde{\nabla} \times \omega) = \mathcal{A}_3, \quad (4.2)$$

where

$$\mathcal{A}_3 := \frac{1}{h} \left(u \times B \mid \tilde{\nabla} \times \omega \right).$$

Adding (4.2) into (4.1), and putting together with the result of energy inequality (1.7) multiplied by $\tilde{\alpha}$, we obtain that

$$\frac{d}{dt} \tilde{\mathcal{L}}(t) + \tilde{\mathcal{D}}(t) + \frac{\nu + \mu}{h} (\tilde{\nabla} \omega | \tilde{\nabla} B) \leq \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3. \quad (4.3)$$

Using Hölder's inequality, Sobolev embedding (2.2), interpolation inequality (2.1) and then Young's inequality, we write

$$\begin{aligned} |\mathcal{A}_2| &\leq \frac{1}{h} \|\omega\|_{L^4} \|u\|_{L^4} \|\tilde{\nabla} B\|_{L^2} \\ &\leq \frac{C}{\tilde{\alpha} \mu h^2} \|\omega\|_{L^2} \|\tilde{\nabla} \omega\|_{L^2} \|u\|_{L^2} \|\tilde{\nabla} u\|_{L^2} + \frac{\tilde{\alpha} \mu}{4} \|\tilde{\nabla} B\|_{L^2}^2 \\ &\leq \frac{C}{\tilde{\alpha}^2 \mu^2 \nu h^4} \|u\|_{L^2}^2 \|\tilde{\nabla} u\|_{L^2}^2 \|\omega\|_{L^2}^2 + \frac{\nu}{8} \|\tilde{\nabla} \omega\|_{L^2}^2 + \frac{\tilde{\alpha} \mu}{4} \|\tilde{\nabla} B\|_{L^2}^2. \end{aligned}$$

Very similarly, we have

$$\begin{aligned} |\mathcal{A}_1| &\leq \|\omega\|_{L^4} \|u\|_{L^4} \|\tilde{\nabla} \omega\|_{L^2} \leq \frac{C}{\nu^3} \|u\|_{L^2}^2 \|\tilde{\nabla} u\|_{L^2}^2 \|\omega\|_{L^2}^2 + \frac{\nu}{8} \|\tilde{\nabla} \omega\|_{L^2}^2, \\ |\mathcal{A}_3| &\leq \|B\|_{L^4} \|u\|_{L^4} \|\tilde{\nabla} \omega\|_{L^2} \leq \frac{C}{\nu} \|u\|_{L^2} \|B\|_{L^2} \|\tilde{\nabla} u\|_{L^2} \|\tilde{\nabla} B\|_{L^2} + \frac{\nu}{8} \|\tilde{\nabla} \omega\|_{L^2}^2. \end{aligned}$$

Meanwhile, it is easy to see that

$$\begin{aligned} \left(\frac{\nu}{h} + \frac{\mu}{h} \right) |(\tilde{\nabla} \omega | \tilde{\nabla} B)| &\leq \frac{4(\nu^2 + \mu^2)}{\nu h^2} \|\tilde{\nabla} B\|_{L^2}^2 + \frac{\nu}{8} \|\tilde{\nabla} \omega\|_{L^2}^2 \\ &\leq \frac{\tilde{\alpha} \mu}{2} \|\tilde{\nabla} B\|_{L^2}^2 + \frac{\nu}{8} \|\tilde{\nabla} \omega\|_{L^2}^2. \end{aligned}$$

Now, notice that

$$\begin{aligned} \tilde{\mathcal{L}}(t) &\geq \frac{1}{2} (\tilde{\alpha} \|u(t)\|_{L^2}^2 + \tilde{\alpha} \|B(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2) - \frac{1}{h^2} \|B(t)\|_{L^2}^2 - \frac{1}{4} \|\omega(t)\|_{L^2}^2 \\ &\geq \frac{\min\{1, h^2\}}{4h^2} \|(u, B, \omega)(t)\|_{L^2}^2. \end{aligned} \quad (4.4)$$

Combining above estimates, we can rewrite (4.3) to

$$\frac{d}{dt} \tilde{\mathcal{L}}(t) + \frac{1}{4} \tilde{\mathcal{D}}(t) \leq \left(\frac{C}{\tilde{\alpha}^2 \mu^2 \nu h^4} + \frac{C}{\nu^3} \right) \frac{h^2}{\min\{1, h^2\}} \|u\|_{L^2}^2 \|\tilde{\nabla} u\|_{L^2}^2 \tilde{\mathcal{L}}(t)$$

$$+ \frac{C}{\nu} \|u\|_{L^2} \|B\|_{L^2} \|\tilde{\nabla} u\|_{L^2} \|\tilde{\nabla} B\|_{L^2}.$$

Applying Gronwall's lemma enables us to finish the proof of Proposition 1.6. \square

Existence and uniqueness. At this stage, thanks to Proposition 1.6 and inequality (4.4), one can follow the proof of inequality (5.3) in [13] to get the \dot{H}^1 estimate of the magnetic field by assuming that B_0 is small enough in H^1 . Then, one can work out a standard procedure similar to that of sub-section 3.1 and complete the proof of the existence part of Theorem 1.3. In order to prove the uniqueness, we apply the weak-strong uniqueness Theorem 2.6 established in [13] which valid for general physical parameters.

Finally, to check that the energy balance (1.8) is fulfilled, one can use the same approximation scheme as in the proof of existence (the energy balance is clearly satisfied by approximate solution sequence (u_n, B_n)) then observe that $(u_n, B_n)_{n \in \mathbb{N}}$ is actually a Cauchy sequence in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^2))$, which may be checked by arguing as in the proof of uniqueness.

It completes the proof of Theorem 1.3. \square

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