



HAL
open science

On the Representation of Darwiche and Pearl's Epistemic States for Iterated Belief Revision

Nicolas Schwind, Sébastien Konieczny, Ramón Pino Pérez

► **To cite this version:**

Nicolas Schwind, Sébastien Konieczny, Ramón Pino Pérez. On the Representation of Darwiche and Pearl's Epistemic States for Iterated Belief Revision. 19th International Conference on Principles of Knowledge Representation and Reasoning KR-2022, Jul 2022, Haifa, Israel. pp.320-330, 10.24963/kr.2022/32 . hal-03855327

HAL Id: hal-03855327

<https://hal.science/hal-03855327>

Submitted on 16 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the Representation of Darwiche and Pearl's Epistemic States for Iterated Belief Revision

Nicolas Schwind¹, Sébastien Konieczny², Ramón Pino Pérez²

¹National Institute of Advanced Industrial Science and Technology, Tokyo, Japan

²CRIL - CNRS, Université d'Artois, Lens, France

nicolas-schwind@aist.go.jp, {konieczny,pinoperez}@cril.fr

Abstract

The seminal characterization of iterated belief revision was proposed by Darwiche and Pearl, which uses an abstract notion of epistemic states. In this work we look for a canonical representation of these epistemic states. Total preorders are not expressive enough to be used as such a canonical representation. Actually, we show that some operators can even not be represented on a countable epistemic space. Nonetheless, under a very reasonable assumption on the epistemic space, we show that OCFs (Ordinal Conditional Functions) can be considered as a canonical representation.

1 Introduction

Belief revision is an important issue in many domains. In particular, if one wants to conceive a truly autonomous agent, it has to be able to revise incorrect information on its model of its environment.

The standard model for belief revision as been proposed by Alchourron, Gärdenfors and Makinson (Alchourrón, Gärdenfors, and Makinson 1985; Gärdenfors 1988; Fermé and Hansson 2011), and is called the AGM theory. This theory perfectly gives account of a one step revision, as attested by several representation theorems (Alchourrón, Gärdenfors, and Makinson 1985; Alchourrón and Makinson 1985; Katsuno and Mendelzon 1991; Grove 1988; Hansson 1994; del Cerro and Herzig 1996), showing that the AGM rationality postulates correspond exactly to the most sensible way to build revision operators. The close formal links between belief revision and non-monotonic inference relations (Makinson and Gärdenfors 1989; Gärdenfors 1990) on one side and possibilistic logic (Dubois and Prade 1991) on the other side, also prove that this theory adequately captures this fundamental process.

Nonetheless, one weakness of the AGM theory is that it provides no constraint on sequences of revision, allowing several inadequate behaviors. This may be seen as a real problem, since any autonomous agent is typically expected to perform a large number of revisions during its activity.

To fill the gap, many papers aimed to identify the correct model for *iterated* belief revision, which was a very active topic during the nineties (Nayak 1994; Nayak et al. 1994; Nayak et al. 1996; Rott 2009). What is considered now as the most convincing approach for iterated belief revision was proposed by Darwiche and Pearl (1997), who in-

roduced four additional postulates about successive revisions. Some extensions of this work have also been investigated (Booth and Meyer 2006; Jin and Thielscher 2007; Konieczny and Pino Pérez 2008; Konieczny, Medina Grespán, and Pino Pérez 2010).

While the AGM framework uses simple logical theories to represent an agent's epistemic state, the Darwiche and Pearl's framework requires more complex objects (Darwiche and Pearl 1994; Freund and Lehmann 1994; Lehmann 1995; Friedman and Halpern 1999). Indeed, if one needs not only to represent the current beliefs of an agent but also some information on the relative plausibility of currently disbelieved information, then this additional information has to be represented somewhere to guide the iterated revision process. Yet epistemic states, in Darwiche and Pearl's framework, are very abstract objects: we know very little about them, besides the fact that each epistemic state is associated with a logical formula that represents the current beliefs of an agent. This can be seen both as a strength and a weakness of the approach. One can see it as a weakness, because as opposed to the standard AGM case, the Darwiche and Pearl's representation theorem is not definitional: one can characterize the revised beliefs of the agent (the formula associated with the epistemic state), but not the whole epistemic state. One can also see it as a strength, because it is general enough to encompass all possible ways to practically encode the epistemic states.

The Darwiche and Pearl's representation theorem is an extension of the Katsuno and Mendelzon's one for AGM belief revision (Katsuno and Mendelzon 1991). It states that each epistemic state can be associated with a total preorder on propositional worlds, such that the new beliefs after revision of this epistemic state is obtained by selecting the most plausible models of the new piece of information w.r.t. this total preorder. It may then be tempting to identify the epistemic states and these total preorders on worlds: not only the representation theorem itself seems to suggest total preorders on worlds as a canonical representation, but they are also a very intuitive way to represent the relative plausibility of each formula. In fact, some works explicitly choose this representation for epistemic states (Booth and Meyer 2011; Ramachandran, Nayak, and Orgun 2012; Booth and Chandler 2016; Booth and Chandler 2019), and this is enough for a lot of applications.

But one has to be aware not to make a step further, by considering these total preorders to be *the* canonical representation of Darwiche and Pearl’s epistemic states, and to restrict them to this case, as it is done in some works¹. For instance, Aravanis *et. al* (2019) showed an example of a revision operator that can not be represented as functional transitions between total preorders, would those total preorders be the ones given by the Darwiche and Pearl’s representation theorem. In this paper, we show without any further assumption that total preorders are not the canonical representation of Darwiche and Pearl’s epistemic states via an example using ordinal conditional functions (OCFs), a well-known representation of epistemic states for belief change. Then, we investigate the extent to which a canonical representation can be found for these epistemic states. To do so, based on the work initiated in (Aravanis, Peppas, and Williams 2019; Schwind and Konieczny 2020), we make precise the condition under which a revision operator defined on a given epistemic space can be “instantiated” into another epistemic space. Would such a target epistemic space exist for every Darwiche and Pearl’s revision operator, that epistemic space could be considered as an appropriate canonical representation of Darwiche and Pearl’s epistemic states.

We start from the important remark that whatever the chosen representation, every epistemic state can be viewed as a black box associating each *finite* sequence of formulae with a formula representing the beliefs of the agent after the successive revision of the epistemic state by each formula from the sequence. Based on that observation, two epistemic states are *strongly equivalent* according to a revision operator if they cannot be distinguished from each other by any such successive revision steps, which means that these epistemic states have the same behavior for that revision operator. So the question now is whether one can find a canonical representation of the whole *quotient set* of epistemic states under this strong equivalence relation. We show that in general this is not possible. The number of possible (quotient set of) epistemic states is just too large. Nevertheless, we show that under the very natural assumption that every epistemic state is reachable from an initial, “empty”, epistemic state, through a finite succession of revisions, OCFs are a possible candidate of canonical representation of epistemic states. This suggests that to encode all such possible epistemic states, what is missing in the total preorder representation is the additional numerical information OCFs accordingly provide.

2 Preliminaries

We consider a propositional language \mathcal{L}_P built up from a finite set of propositional variables P and the usual connectives. The set of consistent formulae is denoted by \mathcal{L}_P^* . \perp (resp. \top) is the Boolean constant always false (resp. true). An interpretation (or world) is a mapping from P to $\{0, 1\}$. The set of all worlds on \mathcal{L}_P is denoted by Ω . \models denotes logical entailment, \equiv logical equivalence, and $[\varphi]$ denotes the set of models of the formula φ .

¹See (Meng, Kou, and Li 2015) for instance.

In iterated belief change, it is standard to assume that the current set of beliefs of an agent is represented by an *epistemic state*. An epistemic state allows one to represent the current beliefs of the agent and some conditional information guiding the revision process. In all generality, an epistemic state can be any object Ψ from which the set of beliefs of the agent can be extracted through a mapping Bel , so that $Bel(\Psi)$ is a propositional formula from \mathcal{L}_P . Formally:

Definition 1 (Epistemic Space). *An epistemic space \mathcal{E} is a tuple $\langle E, Bel \rangle$, where E is a set and Bel is a mapping $Bel : E \rightarrow \mathcal{L}_P^*$.*

We illustrate this concept with some basic epistemic spaces:

Example 1. *Let us define the TPO-based epistemic space² The TPO-based epistemic space is the epistemic space $\mathcal{E}_{tpo} = \langle U_{tpo}, B_{tpo} \rangle$ where:*

- U_{tpo} is the set of all total preorders over the set of all worlds from Ω ;
- B_{tpo} is the mapping associating each total preorder \preceq from U_{tpo} with a formula $\psi \in \mathcal{L}_P^*$ such that $[\psi] = \min(\top, \preceq)$.

The second example of epistemic space is built with the ordinal conditional functions (OCFs) (Spohn 1988; Williams 1995). An OCF κ is a function associating each world with a non-negative integer³ such that there is a world ω such that $\kappa(\omega) = 0$.

Example 2. *The OCF-based epistemic space is the epistemic space $\mathcal{E}_{ocf} = \langle U_{ocf}, B_{ocf} \rangle$ where:*

- U_{ocf} is the set of all OCFs over Ω ;
- B_{ocf} is the mapping associating each OCF κ from U_{ocf} with a formula ψ such that $[\psi] = \{\omega \mid \kappa(\omega) = 0\}$.

Given an epistemic space $\mathcal{E} = \langle E, Bel \rangle$, a belief revision operator \circ on \mathcal{E} associates every epistemic state Ψ from E and every consistent formula μ with a new epistemic state from E , denoted by $\Psi \circ \mu$, i.e., \circ is a mapping $\circ : E \times \mathcal{L}_P^* \rightarrow E$. In the rest of the paper, when a revision operator \circ will be referred to without the epistemic space on which it is defined, we will implicitly assume that \circ is defined on some epistemic space denoted by $\mathcal{E} = \langle E, Bel \rangle$.

Let us recall the set of postulates which are expected for such operators to have a good iterative behavior (Darwiche and Pearl 1997):

Definition 2 (DP operator (Darwiche and Pearl 1997)). *A revision operator \circ is a DP operator if the following properties are satisfied, for each epistemic state Ψ and all formulae μ, μ' :*

- (R*1) $Bel(\Psi \circ \mu) \models \mu$;
- (R*2) If $Bel(\Psi) \wedge \mu \not\models \perp$, then $Bel(\Psi \circ \mu) \equiv Bel(\Psi) \wedge \mu$;
- (R*3) If $\mu \not\models \perp$, then $Bel(\Psi \circ \mu) \not\models \perp$;

²TPO stands for total preorder.

³Note that in the original definition (Spohn 1988) OCFs are defined on ordinals. But here, as in mosts cases, the integer restriction is enough. Note however than using the full power of ordinals can prove useful in some cases (Konieczny 2009).

- (R*4)** If $\mu \equiv \mu'$, then $Bel(\Psi \circ \mu) \equiv Bel(\Psi \circ \mu')$;
(R*5) $Bel(\Psi \circ \mu) \wedge \mu' \models Bel(\Psi \circ (\mu \wedge \mu'))$;
(R*6) If $Bel(\Psi \circ \mu) \wedge \mu' \not\models \perp$,
then $Bel(\Psi \circ (\mu \wedge \mu')) \models Bel(\Psi \circ \mu) \wedge \mu'$.
(C1) If $\alpha \models \mu$, then $Bel((\Psi \circ \mu) \circ \alpha) \equiv Bel(\Psi \circ \alpha)$
(C2) If $\alpha \models \neg\mu$, then $Bel((\Psi \circ \mu) \circ \alpha) \equiv Bel(\Psi \circ \alpha)$
(C3) If $Bel(\Psi \circ \alpha) \models \mu$, then $Bel((\Psi \circ \mu) \circ \alpha) \models \mu$
(C4) If $Bel(\Psi \circ \alpha) \not\models \neg\mu$, then $Bel((\Psi \circ \mu) \circ \alpha) \not\models \neg\mu$

The postulates **(R*1-R*6)** are a direct adaptation of the standard KM postulates to epistemic states. The remaining four postulates, **(C1-C4)**, add constraints w.r.t. iteration.

Darwiche and Pearl also provided a characterization of DP operators in terms of total preorders over worlds:

Definition 3 (Faithful assignment). *Given an epistemic space $\mathcal{E} = \langle E, Bel \rangle$, a mapping $\Psi \mapsto \preceq_\Psi$ associating each epistemic state $\Psi \in E$ with a total preorder⁴ over worlds \preceq_Ψ is a faithful assignment (on \mathcal{E}) if and only if for all worlds $\omega, \omega' \in \Omega$:*

1. If $\omega \models Bel(\Psi)$ and $\omega' \models Bel(\Psi)$, then $\omega \simeq_\Psi \omega'$
2. If $\omega \models Bel(\Psi)$ and $\omega' \not\models Bel(\Psi)$, then $\omega \prec_\Psi \omega'$

Theorem 1 ((Darwiche and Pearl 1997)). *An operator \circ is a DP operator if and only if there exists a faithful assignment $\Psi \mapsto \preceq_\Psi$ that satisfies the following properties:*

- CR1. If $\omega \models \mu$ and $\omega' \models \mu$, then $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi \circ \mu} \omega'$
- CR2. If $\omega \not\models \mu$ and $\omega' \not\models \mu$, then $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi \circ \mu} \omega'$
- CR3. If $\omega \models \mu$ and $\omega' \not\models \mu$, then $\omega \prec_\Psi \omega' \Rightarrow \omega \prec_{\Psi \circ \mu} \omega'$
- CR4. If $\omega \models \mu$ and $\omega' \not\models \mu$, then $\omega \preceq_\Psi \omega' \Rightarrow \omega \preceq_{\Psi \circ \mu} \omega'$,

and such that for each epistemic state Ψ and each formula μ , $[Bel(\Psi \circ \mu)] = \min([\mu], \preceq_\Psi)$.

When a faithful assignment exists for \circ , we will call such a faithful assignment a *DP assignment corresponding to \circ* . As a matter of fact, when such assignment exists it is unique⁵.

Conditions (CR1-CR4) above correspond to the iteration postulates **(C1-C4)**. They impose constraints on the total preorder $\preceq_{\Psi \circ \mu}$: CR1 and CR2, the ‘‘rigidity’’ conditions, say that the order between models of μ is preserved and the order between models of $\neg\mu$ is also preserved. CR3 and CR4 say that there is no worsening between the models of μ and the models of $\neg\mu$.

Theorem 1 has important applications. One of them is that, for some epistemic spaces, it allows to define some DP operators in a constructive way, for which the verification that they satisfy indeed the DP postulates is done almost trivially. Let us illustrate this through the following examples.

Example 3. *We consider the Boutilier’s natural revision operator \circ_B defined over the epistemic space \mathcal{E}_{tpo} . This operator associates each total preorder $\Psi \in \mathcal{U}_{tpo}$ and each formula μ with a total preorder $\Psi \circ_B \mu$ that satisfies $\min(\Psi \circ \mu) = \min([\mu], \Psi)$ and the following condition:*

⁴For each preorder \preceq , \simeq denotes the corresponding indifference relation, and \prec the corresponding strict ordering.

⁵That is due to the fact that if an assignment satisfies $[Bel(\Psi \circ \mu)] = \min([\mu], \preceq_\Psi)$, then $\omega \preceq_\Psi \omega'$ iff $\omega \in [\Psi \circ \alpha_{\omega, \omega'}]$ (where $[\alpha_{\omega, \omega'}] = \{\omega, \omega'\}$), from which the unicity follows.

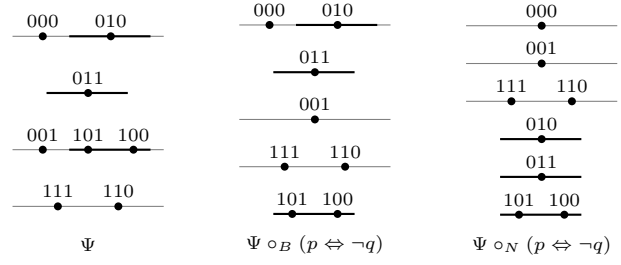


Figure 1: The Boutilier and Nayak operators.

B. If $\omega, \omega' \notin \min([\mu], \Psi)$, then $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi \circ_B \mu} \omega'$, where \preceq_Ψ denotes Ψ and $\preceq_{\Psi \circ_B \mu}$ denotes $\Psi \circ_B \mu$.

That is, Boutilier’s revision operator on \mathcal{E}_{tpo} consists in selecting the set of all models of μ that are minimal according to an input preorder, and defining this set as the first level of the revised preorder while leaving the rest of the preorder unchanged. This is a DP operator: one of the most easy ways to see that is to note that the assignment $\Psi \mapsto \Psi$ is a faithful assignment which satisfies the conditions (CR1-CR4), and that $\min(\Psi \circ \mu) = \min([\mu], \Psi)$. Then by Theorem 1, \circ_B is a DP operator.

Example 4. *Another example is the Nayak’s lexicographic operator \circ_N defined also in the epistemic space \mathcal{E}_{tpo} . It is defined by $\min(\Psi \circ \mu) = \min([\mu], \Psi)$, conditions (CR1-CR4) and:*

- N. If $\omega \models \mu$ and $\omega' \not\models \mu$, then $\omega \prec_{\Psi \circ_N \mu} \omega'$

Nayak’s revision moves all models of μ below all models of $\neg\mu$, and keeps the relationships between worlds of μ (resp. of $\neg\mu$) unchanged. Using the same argument as in the previous example, one can see that this operator is also a DP operator.

Let us illustrate how the behaviors of Boutilier and Nayak revision operators depart from each other.

Example 5. *Let $P = \{p, q, r\}$. Figure 1 depicts a total preorder Ψ over worlds⁶, and the revised total preorders $\Psi \circ_B (p \Leftrightarrow \neg q)$ and $\Psi \circ_N (p \Leftrightarrow \neg q)$. We have that $B_{tpo}(\Psi) \equiv p \wedge q$, and $B_{tpo}(\Psi \circ_B (p \Leftrightarrow \neg q)) \equiv B_{tpo}(\Psi \circ_N (p \Leftrightarrow \neg q)) \equiv p \wedge \neg q$. Then it is easy to see from the figure that:*

- $B_{tpo}(\Psi \circ_B (p \Leftrightarrow \neg q)) \circ_B q \equiv p \wedge q$
 - $B_{tpo}(\Psi \circ_N (p \Leftrightarrow \neg q)) \circ_N q \equiv \neg p \wedge q \wedge r$,
- i.e., $B_{tpo}(\Psi \circ_B (p \Leftrightarrow \neg q)) \circ_B q \not\equiv B_{tpo}(\Psi \circ_N (p \Leftrightarrow \neg q)) \circ_N q$.

The next example is an operator on the epistemic space of OCFs, that is, it is defined on \mathcal{E}_{ocf} .

Example 6. *Given an OCF κ and a formula α , we define $\kappa(\alpha) = \min\{n : \exists \omega \in [\alpha], \kappa(\omega) = n\}$.*

Consider an OCF κ , a formula μ , and an integer x such that $x \geq 1$. The (μ, x) -conditionalization (Spohn 1988) of κ

⁶A world ω is at the same or at a lower level than a world ω' iff $\omega \preceq_\Psi \omega'$. So minimal (i.e., most plausible) worlds are at the lowest levels.

is the OCF κ' defined for each world ω by:

$$\kappa'(\omega) = \begin{cases} \kappa(\omega) - \kappa(\mu) & \text{if } \omega \models \mu \\ \kappa(\omega) - \kappa(\neg\mu) + x & \text{if } \omega \not\models \mu \end{cases}$$

The operator \circ_C on \mathcal{E}_{ocf} is defined for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$ by $\kappa \circ_C \mu = \kappa'$, where κ' is the (μ, x) -conditionalization of κ .

Conditionalization was the first change operator proposed on OCFs (Spohn 1988) (see (Williams 1995) for a generalization and other operators on OCFs). The conditionalization of an OCF performs a (DP) revision under some conditions (if $x > 0$ and $\kappa(\mu) > 0$), but in the other cases we can obtain a contraction or a re-ordering. Please see Definition 7 in Section 4 for an example of a pure DP revision on OCFs.

The most commonly used epistemic spaces are \mathcal{E}_{tpo} and \mathcal{E}_{ocf} . Their popularity can be explained by their simplicity and by the fact that the representation theorems for iterated change operators as (Darwiche and Pearl 1997; Booth and Meyer 2006; Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010; Medina Grespan and Pino Pérez 2013) allow to define operators as transitions between states (cf. Example 3, 4 and 6. The choice of such epistemic spaces is for instance used in (Booth and Meyer 2011; Booth and Chandler 2019; Spohn 1988).

Before concluding this section, let us introduce a few notations that will be used in some subsequent proofs. Given a set E , $card(E)$ denotes the cardinality of E . Given any total preorder $\Psi \in U_{tpo}$ and any world ω , $rank(\Psi, \omega)$ denotes the rank of ω in Ψ . More precisely, $rank(\Psi, \omega) = k - 1$ when there is a chain $(\omega_1, \dots, \omega_k)$ of size k such that $\omega = \omega_k$ and for each i, j such that $i < j$, $\omega_i \prec \omega_j$ (\prec denotes the strict ordering corresponding to Ψ), and such that no such chain of size k' with $k < k'$ exists. Then, given a non-negative integer i , $lvl(\Psi, i)$ denotes the set of models at the i^{th} level in Ψ , i.e., $lvl(\Psi, i) = \{\omega \mid rank(\Psi, \omega) = i\}$, and $max(\Psi) = \max\{i \mid i \in \mathbb{N}, lvl(\Psi, i) \neq \emptyset\}$. Lastly, given an OCF $\kappa \in U_{ocf}$, $tpo(\kappa)$ denotes the total preorder associated with κ , i.e., $tpo(\kappa)$ is defined as the total preorder \preceq where for all worlds $\omega, \omega', \omega \preceq \omega'$ if and only if $\kappa(\omega) \leq \kappa(\omega')$.

3 Instantiability of Revision Operators

Each revision operator is defined on a given epistemic space, but can be equivalently defined on a different epistemic space. Intuitively, given two epistemic spaces $\mathcal{E} = \langle E, Bel \rangle$ and $\mathcal{I} = \langle U, B \rangle$, we say that a revision operator \circ on \mathcal{E} is “safely \mathcal{I} -instantiable” whenever one can find a revision operator $\circ_{\mathcal{I}}$ on \mathcal{I} that has the same “behavior” as \circ , i.e., $\circ_{\mathcal{I}}$ models the same transitions between epistemic states from U as \circ between those from \mathcal{E} , modulo the beliefs that can be observed from these epistemic states in the context of belief revision. The set of all epistemic states from E is mapped onto another set U , and B is used to associate with each mapped epistemic state from U its corresponding beliefs. Formally:

Definition 4 ((Safe) translation). *Let \circ be a revision operator on $\mathcal{E} = \langle E, Bel \rangle$, and let $\mathcal{I} = \langle U, B \rangle$ be an epistemic*

space. A translation of \circ into \mathcal{I} is a pair $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$, where $f_{\mathcal{E}}$ is a mapping from E to U and $\circ_{\mathcal{I}}$ is a revision operator on \mathcal{I} such that for each $\Psi \in \mathcal{E}$, $Bel(\Psi) \equiv B(f_{\mathcal{E}}(\Psi))$, and for each $\Psi \in E$ and each $\mu \in \mathcal{L}_P^$, $f_{\mathcal{E}}(\Psi \circ \mu) = f_{\mathcal{E}}(\Psi) \circ_{\mathcal{I}} \mu$. Such a translation is said to be safe if there exists a mapping $h_{\mathcal{I}} : U \rightarrow E$ such that $(h_{\mathcal{I}}, \circ)$ is a translation of $\circ_{\mathcal{I}}$ into \mathcal{E} .*

When a such a (safe) translation exists we say that \circ is (safely) \mathcal{I} -instantiable.

Note that the notion of translation is close to the notions of simulation in (Aravanis, Peppas, and Williams 2019) and of instantiation in (Schwind and Konieczny 2020). We intend now to explain why safe translations are of particular interest. Let $\mathcal{I} = (U, B)$, $\mathcal{I}' = (U', B')$ be two epistemic spaces, let $\circ_{\mathcal{I}}$ be a revision operator on \mathcal{I} and $\circ_{\mathcal{I}'}$ be a revision operator on \mathcal{I}' . Given $\Psi \in U$ and $\Psi' \in U'$, we say that Ψ is *strongly equivalent*⁷ to Ψ' w.r.t. $(\circ_{\mathcal{I}}, \circ_{\mathcal{I}'})$, denoted by $\Psi \equiv_{(\circ_{\mathcal{I}}, \circ_{\mathcal{I}'})} \Psi'$, if $B(\Psi) \equiv B'(\Psi')$ and for each finite sequence of formulae $\sigma = (\mu_1, \dots, \mu_k)$ with $k \geq 1$, $B(\Psi \circ_{\mathcal{I}} \sigma) \equiv B'(\Psi' \circ_{\mathcal{I}'} \sigma)$.

In the particular case where $\mathcal{I} = \mathcal{I}'$ and $\circ_{\mathcal{I}} = \circ_{\mathcal{I}'} = \circ$, we simply say that Ψ and Ψ' are strongly equivalent w.r.t. \circ and simplify the notation $\Psi \equiv_{(\circ_{\mathcal{I}}, \circ_{\mathcal{I}'})} \Psi'$ into $\Psi \equiv_{\circ} \Psi'$. Then given a revision operator \circ on $\mathcal{I} = (U, B)$, we denote by $U_{\equiv_{\circ}}$ the the quotient of U by the equivalence relation \equiv_{\circ} . That is, if $(\Psi)_{\equiv_{\circ}}$ denotes the equivalence class⁸ of Ψ , then $U_{\equiv_{\circ}} = \{(\Psi)_{\equiv_{\circ}} \mid \Psi \in U\}$.

The notion of strong equivalence between epistemic states is very important. Indeed, each epistemic state, whatever the epistemic space to which it belongs, can actually be seen as a black box that associates each finite sequence of formulae with a formula representing the beliefs of the agent. And even if epistemic states may contain more complex information, this additional information does not have any further impact in the iterated revision process. That is, if two epistemic states have the same behavior for any Darwiche and Pearl’s iterated revision, then they can not be distinguished from one another in this context, and in that case they belong to the same equivalence class.

Let us now use the notion of strong equivalence between epistemic states to define the notion of equivalence between two operators defined on two different epistemic spaces:

Definition 5. *Let $\mathcal{I} = (U, B)$ and $\mathcal{I}' = (U', B')$ be two epistemic spaces, $\circ_{\mathcal{I}}$ be a revision operator on \mathcal{I} , and $\circ_{\mathcal{I}'}$ be a revision operator on \mathcal{I}' . We say that $\circ_{\mathcal{I}}$ and $\circ_{\mathcal{I}'}$ are equivalent if there is a one-to-one and onto correspondence g from $U_{\equiv_{\circ_{\mathcal{I}}}}$ to $U'_{\equiv_{\circ_{\mathcal{I}'}}}$ such that for each $V \in U_{\equiv_{\circ_{\mathcal{I}}}}$ and for all $\Psi \in V$, $\Psi' \in g(V)$, we have that $\Psi \equiv_{(\circ_{\mathcal{I}}, \circ_{\mathcal{I}'})} \Psi'$.*

Let us state an important lemma:

Lemma 1. *Let \circ be a revision operator, $\mathcal{I} = (U, B)$ be an epistemic space, and $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$ be a translation of \circ into \mathcal{I} . Then for each $\Psi \in E$, we have that $\Psi \equiv_{(\circ, \circ_{\mathcal{I}})} f_{\mathcal{E}}(\Psi)$.*

⁷In standard DP papers, two epistemic states are said to be equivalent iff $B(\Psi) \equiv B(\Psi')$. This is why we call this notion strong equivalence. Note that a similar distinction was made in (Konieczny and Pino Pérez 2000).

⁸ $(\Psi)_{\equiv_{\circ}} = \{\Psi' \in U \mid \Psi' \equiv_{\circ} \Psi\}$

Proof. Let $\Psi \in E$. First, by definition of $f_{\mathcal{E}}$ we get that $Bel(\Psi) \equiv B(f_{\mathcal{E}}(\Psi))$. Second, for each sequence of formulae $\sigma = (\mu_1, \dots, \mu_k)$, $k \geq 1$, we know that $Bel(\Psi \circ \sigma) \equiv B(f_{\mathcal{E}}(\Psi \circ \sigma))$. Since $f_{\mathcal{E}}(\Psi \circ \mu) = f_{\mathcal{E}}(\Psi) \circ_{\mathcal{I}} \mu$ for any formula μ , it is easy to see, by induction, that $f_{\mathcal{E}}(\Psi \circ \sigma) = f_{\mathcal{E}}(\Psi) \circ_{\mathcal{I}} \sigma$, thus $Bel(\Psi \circ \sigma) \equiv B(f_{\mathcal{E}}(\Psi) \circ_{\mathcal{I}} \sigma)$. This shows that $\Psi \equiv_{(\circ, \circ_{\mathcal{I}})} f_{\mathcal{E}}(\Psi)$. \square

Taking advantage of Lemma 1, we are ready to make clear why the notion of safe translation is of particular interest:

Proposition 1. *Let \circ be a revision operator and $\mathcal{I} = (U, B)$ be an epistemic space. If $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$ is a safe translation of \circ into \mathcal{I} , then \circ and $\circ_{\mathcal{I}}$ are equivalent.*

Proof. Let \circ be a revision operator on $\mathcal{E} = (E, Bel)$, $\mathcal{I} = (U, B)$ be an epistemic space, and $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$ be a safe translation of \circ into \mathcal{I} . For each $(\Psi)_{\equiv_{\circ}} \in E_{\equiv_{\circ}}$, let g be the mapping $(\Psi)_{\equiv_{\circ}} \mapsto (f_{\mathcal{E}}(\Psi))_{\equiv_{\circ_{\mathcal{I}}}}$. From Lemma 1, it is easy to see that g is well defined and defines an injective mapping from $E_{\equiv_{\circ}}$ to $U_{\equiv_{\circ_{\mathcal{I}}}}$, and we directly get for each $(\Psi)_{\equiv_{\circ}} \in E_{\equiv_{\circ}}$ and for all $\Psi_1 \in (\Psi)_{\equiv_{\circ}}$, $\Psi_2 \in (f_{\mathcal{E}}(\Psi))_{\equiv_{\circ_{\mathcal{I}}}}$, that $\Psi_1 \equiv_{(\circ, \circ_{\mathcal{I}})} \Psi_2$. Since $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$ is safe, there is a mapping $h_{\mathcal{I}} : U \rightarrow E$ such that $(h_{\mathcal{I}}, \circ)$ is a translation of $\circ_{\mathcal{I}}$ into \mathcal{E} . Then, similar to g , let us define the function g' , mapping each $(\Psi')_{\equiv_{\circ_{\mathcal{I}}}} \in U_{\equiv_{\circ_{\mathcal{I}}}}$ into $(h_{\mathcal{I}}(\Psi'))_{\equiv_{\circ}}$. From Lemma 1 again, g' defines an injective mapping from $U_{\equiv_{\circ_{\mathcal{I}}}}$ to $E_{\equiv_{\circ}}$, and it is easy to verify that for each $(\Psi)_{\equiv_{\circ}} \in E_{\equiv_{\circ}}$ and for each $(\Psi')_{\equiv_{\circ_{\mathcal{I}}}} \in U_{\equiv_{\circ_{\mathcal{I}}}}$ we have $g'(g((\Psi)_{\equiv_{\circ}})) = (\Psi)_{\equiv_{\circ}}$ and $g(g'((\Psi')_{\equiv_{\circ_{\mathcal{I}}}})) = (\Psi')_{\equiv_{\circ_{\mathcal{I}}}}$. Thus, g is a one-to-one and onto correspondence from $E_{\equiv_{\circ}}$ to $U_{\equiv_{\circ_{\mathcal{I}}}}$. Therefore, g satisfies the requirements of Definition 5, which means that \circ and $\circ_{\mathcal{I}}$ are equivalent. \square

We have also the following result:

Proposition 2. *If a revision operator \circ is not \mathcal{I} -instantiable, then there is no revision operator on \mathcal{I} equivalent to \circ .*

Proof. We prove the contrapositive of the statement. Let \circ be a revision operator, $\mathcal{I} = (U, B)$ be an epistemic space, and let $\circ_{\mathcal{I}}$ be an operator on \mathcal{I} that is equivalent to \circ . So there is a bijection $g : E_{\equiv_{\circ}} \rightarrow U_{\equiv_{\circ_{\mathcal{I}}}}$ such that for each $F \in E_{\equiv_{\circ}}$, each $\Psi \in F$ and each $\Psi' \in g(F)$, $\Psi \equiv_{(\circ, \circ_{\mathcal{I}})} \Psi'$. First, let $\delta_U : U_{\equiv_{\circ_{\mathcal{I}}}} \rightarrow U$ be a mapping associating every equivalence class $F \in U_{\equiv_{\circ_{\mathcal{I}}}}$ with an arbitrary chosen epistemic state from F , in particular, if $\delta_U(V) = \Psi$, then $V = (\Psi)_{\equiv_{\circ_{\mathcal{I}}}}$. Define now the revision operator $\circ'_{\mathcal{I}}$ on \mathcal{I} for each $\Psi \in U$ and each $\mu \in \mathcal{L}_P^*$ by $\Psi \circ'_{\mathcal{I}} \mu = \delta_U((\Psi \circ_{\mathcal{I}} \mu)_{\equiv_{\circ_{\mathcal{I}}}})$. Obviously enough, $\circ'_{\mathcal{I}}$ is equivalent to $\circ_{\mathcal{I}}$, thus $\circ'_{\mathcal{I}}$ is equivalent to \circ . Now, define $f_{\mathcal{E}} : E \rightarrow U$ for each $\Psi \in E$ as $f_{\mathcal{E}}(\Psi) = \delta_U(g((\Psi)_{\equiv_{\circ}}))$. It is easy to see for each $\Psi \in E$ that $Bel(\Psi) \equiv B(f_{\mathcal{E}}(\Psi))$. And for each $\Psi \in \mathcal{E}$ and each $\mu \in \mathcal{L}_P^*$, $f_{\mathcal{E}}(\Psi) \circ'_{\mathcal{I}} \mu = \delta_U(g((\Psi)_{\equiv_{\circ}})) \circ'_{\mathcal{I}} \mu = \delta_U((\delta_U(g((\Psi)_{\equiv_{\circ}})) \circ_{\mathcal{I}} \mu)_{\equiv_{\circ_{\mathcal{I}}}}) = \delta_U(g((\Psi \circ \mu)_{\equiv_{\circ}})) = f_{\mathcal{E}}(\Psi \circ \mu)$. This shows that $(f_{\mathcal{E}}, \circ'_{\mathcal{I}})$ is a translation of \circ into \mathcal{I} , i.e., \circ is \mathcal{I} -instantiable. \square

Propositions 1 and 2 show why the notion of (safe) instantiability is a key concept: (safe) \mathcal{I} -instantiability it is a

(sufficient and) necessary condition for defining an operator equivalent to \circ in the space \mathcal{I} .

Another important consequence of Lemma 1 is the following:

Proposition 3. *Let \circ be a revision operator and $\mathcal{I} = (U, B)$ be an epistemic space. If \circ is \mathcal{I} -instantiable, then $card(E_{\equiv_{\circ}}) \leq card(U)$.*

Proof. Let \circ be a revision operator and $\mathcal{I} = (U, B)$ be an epistemic space. Assume that \circ is \mathcal{I} -instantiable, and let $(f_{\mathcal{E}}, \circ_{\mathcal{I}})$ be a translation of \circ into \mathcal{I} . So let $\Psi, \Psi' \in E$ and assume that $\Psi \not\equiv_{\circ} \Psi'$. From Lemma 1, we know that $f_{\mathcal{E}}(\Psi) \not\equiv f_{\mathcal{E}}(\Psi')$, thus $f_{\mathcal{E}}(\Psi) \neq f_{\mathcal{E}}(\Psi')$. This means that there is an injective mapping from $E_{\equiv_{\circ}}$ to U . Hence, $card(E_{\equiv_{\circ}}) \leq card(U)$. \square

This result is quite intuitive: in order to be used as an appropriate space for defining an operator equivalent to a given operator \circ , an epistemic space has to be “large enough” to encode all the equivalence classes (w.r.t. \circ) of epistemic states in the initial epistemic space. This point is important for our subsequent results.

4 TPO vs OCF

In this section, we investigate the links between the TPO and OCF epistemic spaces. We start by defining the notion of “structure preservation”, in order to work with epistemic states that are closely related to the total preorders obtained by the representation theorem. Then we show how to find for every operator on the TPO epistemic space an equivalent operator on the OCF epistemic space. Lastly, we show that a converse translation is not possible.

A very interesting feature of operators defined in the spaces \mathcal{E}_{tpo} and \mathcal{E}_{ocf} is that they are typically built using the ordered structure of the epistemic states in a natural way. For instance, the DP assignment corresponding to the operators in Examples 3 and 4 is the identity, and the one corresponding to the operator on \mathcal{E}_{ocf} introduced later in Definition 7 is “almost” the identity. An appealing property of an operator defined on \mathcal{E}_{tpo} or \mathcal{E}_{ocf} is thus to preserve the internal ordered structure of their epistemic states according to its corresponding assignment. Let us formalize this property:

Definition 6. *Let \circ be a DP operator defined on \mathcal{E}_{tpo} and $\Psi \mapsto \preceq_{\Psi}$ be the DP assignment corresponding to \circ . We say that \circ is structure preserving if for each $\Psi \in U_{tpo}$, $\Psi = \preceq_{\Psi}$. Likewise, let \circ be a DP operator defined on \mathcal{E}_{ocf} and $\kappa \mapsto \preceq_{\kappa}$ be the DP assignment corresponding to \circ . We say that \circ is structure preserving if for each $\kappa \in U_{ocf}$, $tpo(\kappa) = \preceq_{\kappa}$.*

One can wonder if there exist DP operators that are not structure preserving. The following example shows an operator defined on \mathcal{E}_{tpo} which is not structuring preserving.

Example 7. *Consider a propositional language with two propositional variables. Thus, there are four interpretations $\omega_1, \omega_2, \omega_3$ and ω_4 . Let \circ_N be the Nayak operator on \mathcal{E}_{tpo} . Consider the two following total preorders: Ψ_1 defined by $\omega_1 \sim \omega_2 \prec \omega_3 \prec \omega_4$ and Ψ_2 defined by $\omega_1 \sim \omega_2 \prec \omega_4 \prec$*

ω_3 . Clearly $\Psi_1 \neq \Psi_2$ and $B_{t_{po}}(\Psi_1) \equiv B_{t_{po}}(\Psi_2)$. Now, define a new operator \star as follows:

$$\Psi \star \mu = \begin{cases} \Psi_2 \circ_N \mu & \text{if } \Psi = \Psi_1 \\ \Psi_1 \circ_N \mu & \text{if } \Psi = \Psi_2 \\ \Psi \circ_N \mu, & \text{otherwise} \end{cases}$$

It is easy to see that \star is a DP operator and, moreover, $\Psi_1 \preceq_{\Psi_2}^* \Psi_2$. Therefore \star is not structure preserving.

However, we can show that:

Proposition 4. *An operator \circ on $\mathcal{E}_{t_{po}}$ is a DP operator if and only if there is a structure preserving DP operator \circ' on $\mathcal{E}_{t_{po}}$ that is equivalent to \circ .*

Proof. The (if) part of the proof is direct: since \circ' is equivalent to \circ and \circ' is a DP operator, \circ is a DP operator as well. Let us prove the (only if) part of the proof. Let \circ be a DP operator. Let τ denote the faithful assignment corresponding to \circ (recall that it is unique), and let us first show that τ is a one-to-one-correspondence on $U_{t_{po}}$. It is enough to prove that τ is surjective. In the following, $\tau(\Psi)$ will sometimes be denoted by \preceq_{Ψ} .

Let us start with a useful observation. Let $\Psi \in U_{t_{po}}$ and α be a formula such that $[\alpha] \subseteq \text{lvl}(\preceq_{\Psi}, \text{max}(\preceq_{\Psi}))$, i.e., α is a formula whose models lie at the last level of \preceq_{Ψ} . Then from the representation theorem (Theorem 1) and condition CR2, the total preorder $\preceq_{\Psi \circ \alpha}$ is uniquely defined by:

- if $\omega, \omega' \models \alpha$, then $\omega \preceq_{\Psi \circ \alpha} \omega'$
- if $\omega \models \alpha, \omega' \not\models \alpha$, then $\omega \prec_{\Psi \circ \alpha} \omega'$
- if $\omega, \omega' \not\models \alpha$, then $\omega \preceq_{\Psi \circ \alpha} \omega'$ iff $\omega \preceq_{\Psi} \omega'$

From this observation, we can show that every total preorder \preceq corresponds to $\tau(\Psi)$ from some $\Psi \in U_{t_{po}}$. Indeed, let \preceq be any total preorder from $U_{t_{po}}$, let $\Psi_{\top} \in U_{t_{po}}$ be such that $B_{t_{po}}(\Psi_{\top}) \equiv \top$ (note that $\preceq_{\Psi_{\top}}$ is always the “flat” total preorder defined for all worlds ω, ω' by $\omega \simeq_{\Psi_{\top}} \omega'$), and let $\sigma = (\alpha_1, \dots, \alpha_{\text{max}(\preceq)})$ be a sequence of formulae such that for each α_i , $[\alpha_i] = \text{lvl}(\preceq, \text{max}(\preceq) - i)$. It is easy to see from the observation above that $\tau(\Psi_{\top} \circ \sigma) = \preceq$. This shows that τ is surjective, so it is a one-to-one correspondence.

Now, let us consider the operator \circ' on $\mathcal{E}_{t_{po}}$ defined for each $\Psi \in U_{t_{po}}$ and each $\mu \in \mathcal{L}_P^*$ by $\Psi \circ' \mu = \tau(\tau^{-1}(\Psi) \circ \mu)$, and let us show that (τ, \circ') is a safe translation of \circ into $\mathcal{E}_{t_{po}}$. Let $\Psi \in U_{t_{po}}$ and $\mu \in \mathcal{L}_P^*$. The fact that $B_{t_{po}}(\Psi) \equiv B_{t_{po}}(\tau(\Psi))$ is direct from the representation theorem (cf. Theorem 1) and conditions 1 and 2 of a faithful assignment. Yet we also have that $\tau(\Psi) \circ' \mu = \tau(\tau^{-1}(\tau(\Psi)) \circ \mu) = \tau(\Psi \circ \mu)$. This shows that (τ, \circ') is a translation of \circ into $\mathcal{E}_{t_{po}}$. To show that (τ, \circ') is safe, it is enough to remark that (τ^{-1}, \circ) is a translation of \circ' into $\mathcal{E}_{t_{po}}$, since $B_{t_{po}}(\Psi) \equiv B_{t_{po}}(\tau^{-1}(\Psi))$ and $\tau^{-1}(\Psi \circ' \mu) \equiv \tau^{-1}(\tau(\tau^{-1}(\Psi) \circ \mu)) = \tau^{-1}(\Psi) \circ \mu$.

Since (τ, \circ') is a safe translation of \circ into $\mathcal{E}_{t_{po}}$, from Proposition 1 we get that \circ and \circ' are equivalent, which concludes the proof. \square

This result is very important, since it means that we can always assume to work with structure preserving operators for any DP operator on $\mathcal{E}_{t_{po}}$.

Now, let us show that every revision operator on $\mathcal{E}_{t_{po}}$ is safely \mathcal{E}_{ocf} -instantiable. Let \circ be any revision operator on $\mathcal{E}_{t_{po}}$. Let $f_{\mathcal{E}_{t_{po}}}$ be the mapping from $U_{t_{po}}$ to U_{ocf} defined for each $\Psi \in U_{t_{po}}$ by $f_{\mathcal{E}_{t_{po}}}(\Psi) = \kappa_{\Psi}$, where κ_{Ψ} is the OCF defined for each world ω by $\kappa_{\Psi}(\omega) = \text{rank}(\Psi, \omega)$. Now, let us define a revision operator $\circ_{\mathcal{E}_{ocf}}$ on \mathcal{E}_{ocf} for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$ as $\kappa \circ_{\mathcal{E}_{ocf}} \mu = f_{\mathcal{E}_{t_{po}}}(t_{po}(\kappa) \circ \mu)$. Then:

Proposition 5. *$(f_{\mathcal{E}_{t_{po}}}, \circ_{\mathcal{E}_{ocf}})$ is a safe translation of \circ into \mathcal{E}_{ocf} .*

Proof. First, it is easy to see that for each $\Psi \in B_{t_{po}}$, we have that $B_{t_{po}}(\Psi) = B_{ocf}(\kappa_{\Psi})$. Then, remark that for each $\Psi \in U_{t_{po}}$, by construction of κ_{Ψ} we get that $t_{po}(\kappa_{\Psi}) = \Psi$. Thus for each $\Psi \in U_{t_{po}}$ and each $\mu \in \mathcal{L}_P^*$ we get that $f_{\mathcal{E}_{t_{po}}}(\Psi) \circ_{\mathcal{E}_{ocf}} \mu = \kappa_{\Psi} \circ_{\mathcal{E}_{ocf}} \mu = f_{\mathcal{E}_{t_{po}}}(t_{po}(\kappa_{\Psi}) \circ \mu) = f_{\mathcal{E}_{t_{po}}}(\Psi \circ \mu)$. This shows that $(f_{\mathcal{E}_{t_{po}}}, \circ_{\mathcal{E}_{ocf}})$ is a translation of \circ into \mathcal{E}_{ocf} .

To see that this is also a safe translation, simply consider the mapping $h_{\mathcal{E}_{ocf}} : U_{ocf} \rightarrow U_{t_{po}}$ defined for each $\kappa \in U_{ocf}$ as $h_{\mathcal{E}_{ocf}}(\kappa) = t_{po}(\kappa)$. On the one hand, it is easy to see that $B_{ocf}(\kappa) \equiv B_{t_{po}}(h_{\mathcal{E}_{ocf}}(\kappa))$. On the other hand, remark that for each $\Psi \in U_{t_{po}}$, $h_{\mathcal{E}_{ocf}}(f_{\mathcal{E}_{t_{po}}}(\Psi)) = t_{po}(\kappa_{\Psi}) = \Psi$. Hence, we get for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$ that $h_{\mathcal{E}_{ocf}}(\kappa \circ_{\mathcal{E}_{ocf}} \mu) = h_{\mathcal{E}_{ocf}}(f_{\mathcal{E}_{t_{po}}}(t_{po}(\kappa) \circ \mu)) = t_{po}(\kappa) \circ \mu = h_{\mathcal{E}_{ocf}}(\kappa) \circ \mu$. This shows that $(h_{\mathcal{E}_{ocf}}, \circ)$ is a translation of $\circ_{\mathcal{E}_{ocf}}$ into $U_{t_{po}}$, which proves that $(f_{\mathcal{E}_{t_{po}}}, \circ_{\mathcal{E}_{ocf}})$ is a safe translation of \circ into \mathcal{E}_{ocf} . \square

We call the pair $(f_{\mathcal{E}_{t_{po}}}, \circ_{\mathcal{E}_{ocf}})$ the *canonical* translation of \circ into \mathcal{E}_{ocf} .

A particular consequence of Proposition 5 is that every revision operator \circ on $\mathcal{E}_{t_{po}}$ is safely \mathcal{E}_{ocf} -instantiable. Moreover, it is easy to see by construction of $\circ_{\mathcal{E}_{ocf}}$ that if \circ is a structure preserving DP operator, then $\circ_{\mathcal{E}_{ocf}}$ is also structure preserving. Hence, a direct consequence of Propositions 4 and 5 is that:

Corollary 1. *For every DP operator \circ on $\mathcal{E}_{t_{po}}$, there is an operator on \mathcal{E}_{ocf} equivalent to \circ that is structure preserving.*

So we can encode any DP revision defined on TPOs into the OCF epistemic space by a structure preserving operator. Let us now show that the converse translation is not possible, by providing an example of operator that is defined on OCFs and that is not $\mathcal{E}_{t_{po}}$ -instantiable.

Definition 7. *Let \bullet_+ be the revision operator on \mathcal{E}_{ocf} defined for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$ by $\kappa' = \kappa \bullet_+ \mu$, where for each world $\omega \in \Omega$, $\kappa'(\omega)$ is defined by:*

$$\kappa'(\omega) = \begin{cases} 0 & \text{if } \omega \models \mu \text{ and } \kappa(\omega) = \kappa(\mu) \\ \kappa(\omega) & \text{if } \omega \models \mu \text{ and } \kappa(\omega) > \kappa(\mu) \\ \kappa(\omega) + 1 & \text{if } \omega \not\models \mu \end{cases}$$

This operator can be related with the \bullet operator proposed by Darwiche and Pearl to illustrate their DP postulates (Darwiche and Pearl 1997), since \bullet and \bullet_+ deal with the countermodels of μ in a similar way. However, \bullet_+ depart from \bullet in its treatment of the models of μ that do not move (except in the case where they are the minimal ones).

First, please observe that :

Proposition 6. \bullet_+ is a DP revision operator.

Proof. Let τ be the mapping from U_{ocf} to U_{tpo} defined for each $\kappa \in U_{ocf}$ by $\tau(\kappa) = tpo(\kappa)$. The fact that \bullet_+ is a DP operator is shown as follows: one can easily observe by definition of \bullet_+ that τ satisfies conditions 1 and 2 of a faithful assignment and conditions (CR1-CR4) of Theorem 1, and for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$, $[B_{ocf}(\kappa \bullet_+ \mu)] = \min([\mu], tpo(\kappa))$. That is, τ defines a DP assignment corresponding to \bullet_+ , which from Theorem 1 shows that \bullet_+ is a DP operator. \square

Let us illustrate the behavior of this operator through the following example:

Example 8. Let $P = \{p, q\}$, and consider the two OCFs κ and π defined by:⁹

$$\begin{aligned} \kappa(00) = 0 \quad \kappa(01) = 1 \quad \kappa(10) = 1 \quad \kappa(11) = 2 \\ \pi(00) = 0 \quad \pi(01) = 1 \quad \pi(10) = 1 \quad \pi(11) = 4 \end{aligned}$$

Then $\kappa \bullet_+ p = \kappa_1$, $\pi \bullet_+ p = \pi_1$, $\pi_1 \bullet_+ p = \pi_2$, $\pi_2 \bullet_+ p = \pi_3$, as illustrated in Figure 2.¹⁰

Note that \bullet_+ is structure preserving, so its corresponding faithful assignment associates κ and π with the same total preorder \preceq . Would \bullet_+ be safely \mathcal{E}_{tpo} -instantiable, Propositions 1 and 4 together show that there would be a structure preserving operator on \mathcal{E}_{tpo} that is equivalent to \bullet_+ , and so κ and π would be translated to the same total preorder. But this example shows that this can not be the case, since $B_{ocf}((\kappa \bullet_+ p) \bullet_+ q) \not\equiv B_{ocf}((\pi \bullet_+ p) \bullet_+ q)$, and so κ and π are not strongly equivalent. Similarly, one can remark that the TPO translated from π_1 and π_2 should be exactly the same, but $B_{ocf}((\pi_1 \bullet_+ p) \bullet_+ q) \not\equiv B_{ocf}((\pi_2 \bullet_+ p) \bullet_+ q)$.

This example is enough to conclude that \bullet_+ is not safely \mathcal{E}_{tpo} -instantiable, but let us give a more general result:

Proposition 7. $U_{ocf, \equiv \bullet_+}$ is countably infinite.

Proof. Without loss of generality, assume that \mathcal{L}_P is such that $|P| \geq 2$. Let $\omega_1, \omega_2, \omega_3$ be three distinct worlds from Ω . Since U_{ocf} is a countable set, $U_{ocf, \equiv \bullet_+}$ is a countable set. Let \mathbb{N}_* denotes the set of positive integers. To conclude the proof that $U_{ocf, \equiv \bullet_+}$ is countably infinite, since \mathbb{N}_* is infinite it is enough to build an injective mapping τ from \mathbb{N}_* to $U_{ocf, \equiv \bullet_+}$. So let us define $\tau : \mathbb{N}_* \rightarrow U_{ocf, \equiv \bullet_+}$ for each $i \in \mathbb{N}_*$ as $\tau(i) = \kappa^i$, where κ^i is any arbitrary OCF that satisfies $\kappa^i(\omega_1) = 0$, $\kappa^i(\omega_2) = 1$, and $\kappa^i(\omega_3) = i + 2$. For each $k \in \mathbb{N}_*$, let σ^k be the sequence of formulae $(\alpha_{13}, \dots, \alpha_{13})$ where $[\alpha_{13}] = \{\omega_1, \omega_3\}$ and α_{13} appears exactly k times in the sequence. Let us denote by κ_k^i the OCF $\kappa^i \bullet_+ \sigma^k$. By definition of \bullet_+ , we get for all $i, k \in \mathbb{N}_*$ that $\kappa_k^i(\omega_1) = \kappa^i(\omega_1) = 0$, $\kappa_k^i(\omega_2) = \kappa^i(\omega_2) + k = k + 1$, and $\kappa_k^i(\omega_3) = \kappa^i(\omega_3) = i + 2$. Hence,

- $\kappa_{i+1}^i(\omega_2) = \kappa_{i+1}^i(\omega_3)$
- if $k < i + 1$, then $\kappa_k^i(\omega_2) < \kappa_k^i(\omega_3)$
- otherwise, i.e., if $k > i + 1$, then $\kappa_k^i(\omega_2) > \kappa_k^i(\omega_3)$

⁹A world is denoted by an ordered pair of two symbols from $\{0, 1\}$ according to the ordering $p < q$.

¹⁰ $\kappa(\omega) = i$ iff ω appears at the level associated to the integer i .

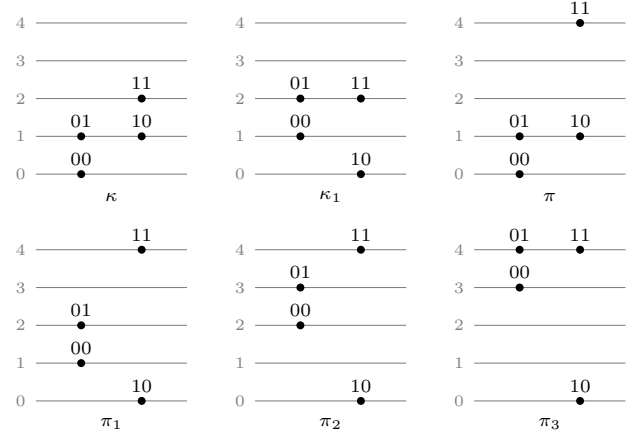


Figure 2: The DP operator \bullet_+

This shows that $B_{ocf}(\kappa_k^i \bullet_+ \alpha_{23}) \equiv \alpha_{23}$ if and only if $k = i + 1$, where $[\alpha_{23}] = \{\omega_2, \omega_3\}$. Stated otherwise, for all $i, j \in \mathbb{N}_*$, if $i \neq j$ then the sequence $\sigma = \sigma^{i+1} \cdot \alpha_{23}$ is such that $B_{ocf}(\kappa^i \bullet_+ \sigma) \not\equiv B_{ocf}(\kappa^j \bullet_+ \sigma)$. This means that if $i \neq j$ then $\kappa_i \not\equiv_{\bullet_+} \kappa_j$, which shows that τ is an injective mapping from \mathbb{N}_* to $U_{ocf, \equiv \bullet_+}$, and concludes the proof that $U_{ocf, \equiv \bullet_+}$ is countably infinite. \square

Then from Propositions 3 and 7 and from the fact that U_{tpo} is a finite set, we can conclude that:

Corollary 2. \bullet_+ is not \mathcal{E}_{tpo} -instantiable.

Proposition 7 is more interesting than the counterexample, since the fact that $U_{ocf, \equiv \bullet_+}$ is countably infinite means not only that an operator on \mathcal{E}_{tpo} that is equivalent to \bullet_+ can not be found on the same propositional language, but also that trying to use a larger language in order to gain in expressivity (i.e., artificially adding new atoms to have more TPOs) can not work either.

5 An Impossibility Result

Let us start this section by the definition of an epistemic space \mathcal{I} being C -complete where C is a class of revision operators, meaning that \mathcal{I} is rich enough to endorse any operator of a given class C (e.g., satisfying a set of postulates).

Definition 8. Given a class C of revision operators, we say that an epistemic space $\mathcal{I} = (U, B)$ is C -complete if every operator from C is safely \mathcal{I} -instantiable.

When C is the class of DP operators and \mathcal{I} is C -complete, we simply say that \mathcal{I} is DP-complete.

In this section, we intend to introduce an impossibility result, that is, no epistemic space $\mathcal{I} = (U, B)$ where U is countable is DP-complete. To this end, we intend to introduce a specific DP revision operator that is not \mathcal{I} -instantiable, for any $\mathcal{I} = (U, B)$ where U is a countable set.

Let us first define our revision operator, denoted by \bullet^* . Without loss of generality, we can assume that \mathcal{L}_P is such that $|P| \geq 2$. Let $\omega_1, \omega_2, \omega_3$ be three distinct worlds from Ω . Let $\mathbb{B}^{\mathbb{N}}$ be the set of all (countably infinite) binary sequences

of values from $\{0, 1\}$. For each $\delta \in \mathbb{B}^{\mathbb{N}}$ and for each integer $i \geq 1$, $\delta(i)$ denotes the i^{th} value of δ in the sequence, and $(\delta + 1)$ denotes the sequence from $\mathbb{B}^{\mathbb{N}}$ defined for each integer $i \geq 1$ as $(\delta + 1)(i) = \delta(i + 1)$ (e.g., if $\delta = (00101 \dots)$, then $(\delta + 1) = (0101 \dots)$). Moreover, let \circ_B and \circ_N be respectively the Boutilier revision operator and the Nayak lexicographic operator on \mathcal{E}_{tpo} (recall that both operators are DP operators). Lastly, let Φ_{123} denote the subset of total preorders from U_{tpo} that satisfy $\omega_1 < \omega_2 < \omega_3$ and for each $\omega \notin \{\omega_1, \omega_2, \omega_3\}$, $\omega_3 < \omega$.

Let us now define an epistemic space $\mathcal{E}^* = \langle E^*, Bel^* \rangle$ as follows. Let $E^* = U_{tpo} \times \mathbb{B}^{\mathbb{N}}$, and let Bel^* be defined for each $(\Psi, \delta) \in E^*$ as $Bel^*((\Psi, \delta)) = B_{tpo}(\Psi)$. We now define the revision operator \bullet^* on \mathcal{E}^* as follows, for each $(\Psi, \delta) \in E^*$:

- $(\Psi, \delta) \bullet^* \mu = (\Psi \circ_N \mu, (\delta + 1))$, if $\Psi \in \Phi_{123}$, $[\mu] = \{\omega_1, \omega_3\}$, and $\delta(1) = 1$,
- $(\Psi, \delta) \bullet^* \mu = (\Psi \circ_B \mu, (\delta + 1))$, if $\Psi \in \Phi_{123}$, $[\mu] = \{\omega_1, \omega_3\}$, and $\delta(1) = 0$,
- $(\Psi, \delta) \bullet^* \mu = (\Psi \circ_B \mu, \delta)$, in the remaining cases.

Let us first show that:

Proposition 8. \bullet^* is a DP operator.

Proof. Let τ be the mapping associating each epistemic state $(\Psi, \delta) \in E^*$ with the total preorder Ψ . On the one hand, since $Bel^*((\Psi, \delta)) = B_{tpo}(\Psi)$ for every $(\Psi, \delta) \in E^*$, by definition of $B_{tpo}(\Psi)$ we directly get that τ satisfies conditions 1 and 2 of a faithful assignment. On the other hand, by construction of \bullet^* , for every $(\Psi, \delta) \in E^*$, we have that $\tau((\Psi, \delta) \bullet^* \mu) \in \{\Psi \circ_N \mu, \Psi \circ_B \mu\}$. Yet \circ_B and \circ_N are both structure preserving, so the identity function on U_{tpo} defines a DP assignment corresponding to both \circ_B and \circ_N . Then τ satisfies conditions (CR1-CR4) as well, and for each epistemic state (Ψ, δ) and each formula μ , $[Bel^*((\Psi, \delta) \bullet^* \mu)] = \min([\mu], \Psi)$. This means that τ is the DP assignment corresponding to \bullet^* . Then, by Theorem 1, \bullet^* is a DP revision operator. \square

Our impossibility result is based on the following statement, which shows E_{\equiv, \bullet^*}^* is uncountable:

Proposition 9. $card(\mathbb{B}^{\mathbb{N}}) \leq card(E_{\equiv, \bullet^*}^*)$.

Proof. Let σ_{321} be the sequence of formulae $(\alpha_3, \alpha_2, \alpha_1)$ with $[\alpha_3] = \{\omega_3\}$, $[\alpha_2] = \{\omega_2\}$, and $[\alpha_1] = \{\omega_1\}$. Let Ψ be any total preorder from U_{tpo} , and let us note $\Psi' = \Psi \circ_B \sigma_{123}$. It is easy to verify by construction of \circ_B that $\Psi' \in \Phi_{123}$. Yet by definition of \circ_* , we know that for each epistemic state $(\Psi, \delta) \in E^*$, $(\Psi, \delta) \bullet^* \sigma_{123} = (\Psi \circ_B \sigma_{123}, \delta)$. Hence, for each epistemic state $(\Psi, \delta) \in E^*$, $(\Psi, \delta) \bullet^* \sigma_{123}$ is an epistemic state (Ψ', δ) such that $\Psi' \in \Phi_{123}$.

Now, given any such epistemic state $(\Psi', \delta) \in \mathcal{E}^*$ such that $\Psi' \in \Phi_{123}$, when α_{13}, α_{23} are formulae such that $[\alpha_{13}] = \{\omega_1, \omega_3\}$ and $[\alpha_{23}] = \{\omega_2, \omega_3\}$, we have that:

- if $\delta(1) = 1$, then $(\Psi', \delta) \bullet^* \alpha_{13} \bullet^* \alpha_{23} = (\Psi' \circ_N \alpha_{13}, (\delta + 1)) \bullet^* \alpha_{23} = (\Psi' \circ_N \alpha_{13} \circ_B \alpha_{23}, (\delta + 1))$, which we denote by $(\Psi_1, (\delta + 1))$; and

- if $\delta(1) = 0$, then $(\Psi', \delta) \bullet^* \alpha_{13} \bullet^* \alpha_{23} = (\Psi' \circ_B \alpha_{13}, (\delta + 1)) \bullet^* \alpha_{23} = (\Psi' \circ_B \alpha_{13} \circ_B \alpha_{23}, (\delta + 1))$, which we denote by $(\Psi_2, (\delta + 1))$.

And it is easy to verify by construction of \circ_B and \circ_N that $B_{tpo}(\Psi_1) \equiv \alpha_3$ and $B_{tpo}(\Psi_2) \equiv \alpha_2$, thus $Bel((\Psi_1, (\delta + 1))) \equiv \alpha_3$ and $Bel((\Psi_2, (\delta + 1))) \equiv \alpha_2$.

Overall, we have shown that for each epistemic state $(\Psi, \delta) \in E^*$, when σ is the sequence of formulae $\sigma = \sigma_{123} \cdot \alpha_{13} \cdot \alpha_{23}$,

- if $\delta(1) = 1$, then $(\Psi, \delta) \bullet^* \sigma = (\Psi_1, (\delta + 1))$ and $Bel((\Psi_1, (\delta + 1))) \equiv \alpha_3$, and
- if $\delta(1) = 0$, then $(\Psi, \delta) \bullet^* \sigma = (\Psi_2, (\delta + 1))$ and $Bel((\Psi_2, (\delta + 1))) \equiv \alpha_2$.

That is, for all epistemic states $(\Psi, \delta), (\Psi, \delta') \in E^*$,

$$\delta(1) \neq \delta'(1) \Rightarrow Bel((\Psi, \delta) \bullet^* \sigma) \neq Bel((\Psi, \delta') \bullet^* \sigma) \quad (1)$$

We intend now to show that if $(\Psi, \delta), (\Psi, \delta')$ are two epistemic states from E^* such that $\delta \neq \delta'$, then $(\Psi, \delta) \not\equiv_{\bullet^*} (\Psi, \delta')$. Given a vector $\delta \in \mathbb{B}^{\mathbb{N}}$ and an integer $k \geq 1$, δ can be rewritten as the concatenation of two vectors δ_{-k} and δ_{+k} , i.e., $\delta = \delta_{-k} \cdot \delta_{+k}$, where $\delta_{-k} = (\delta(1), \dots, \delta(k-1))$ (δ_{-k} is possibly empty) and $\delta_{+k} = (\delta(k), \delta(k+1), \dots)$. Now, let $\delta \neq \delta'$, then let i be the smallest integer such that $\delta(i) \neq \delta'(i)$, formally $i = \min(\{j \in \mathbb{N} \mid \delta(j) \neq \delta'(j)\})$. We have that $\delta_{-i} = \delta'_{-i}$ and $\delta_{+i}(1) \neq \delta'_{+i}(1)$. On the one hand, by definition of \bullet^* , we get that $(\Psi, \delta) \bullet^* \sigma^{(i)} = (\Psi_i, \delta_{+i})$ and $(\Psi, \delta') \bullet^* \sigma^{(i)} = (\Psi_i, \delta'_{+i})$ for some unique total preorder Ψ_i . On the other hand, we have shown in Equation 1 above that since $\delta_{+i}(1) \neq \delta'_{+i}(1)$, $Bel(\Psi_i, \delta_{+i}) \bullet^* \sigma \neq Bel(\Psi_i, \delta'_{+i}) \bullet^* \sigma$. That is, the sequence of formulae $\sigma^{(i+1)}$ is such that $Bel((\Psi, \delta) \bullet^* \sigma^{(i+1)}) \neq Bel((\Psi, \delta') \bullet^* \sigma^{(i+1)})$. This shows that $(\Psi, \delta) \not\equiv_{\bullet^*} (\Psi, \delta')$.

We have shown that each total preorder $\Psi \in U_{tpo}$ and for all $\delta, \delta' \in \mathbb{B}^{\mathbb{N}}$, if $\delta \neq \delta'$ then $(\Psi, \delta) \not\equiv_{\bullet^*} (\Psi, \delta')$. This shows that there is an injective mapping from $\mathbb{B}^{\mathbb{N}}$ to E_{\equiv, \bullet^*}^* , from which we can conclude that $card(\mathbb{B}^{\mathbb{N}}) \leq card(E_{\equiv, \bullet^*}^*)$. \square

And so, as a consequence of Propositions 3 and 9 and from the fact that the set $\mathbb{B}^{\mathbb{N}}$ is uncountable, we get that:

Corollary 3. Let $\mathcal{I} = (U, B)$ be an epistemic space such that U is a countable set. Then \bullet^* is not \mathcal{I} -instantiable.

We are ready to state our impossibility result. Since \bullet^* is a DP operator (cf. Proposition 8), we get as a direct consequence of Corollary 3 that:

Corollary 4. There is no DP-complete epistemic space $\mathcal{I} = (U, B)$ where U is a countable set.

In particular, \mathcal{E}_{ocf} and \mathcal{E}_{tpo} are not DP-complete, and thus these epistemic spaces cannot be used to define every DP operator. This can be seen as a very negative result. But this is a simple consequence of the generality of Darwiche and Pearl's definition of epistemic states. Nevertheless, one can be more positive if one makes a very natural assumption on the operator and the epistemic space on which it is defined. This is the topic of the next section.

6 A Possibility Result

Let us now investigate a particular, yet very natural, case, where all epistemic states can be finitely generated from an initial empty epistemic state and a sequence of revisions.

Let $\mathcal{E} = \langle E, Bel \rangle$ be an epistemic space and \circ be a revision operator on \mathcal{E} . We consider the following property:

(G) There exists a unique $\Psi_{\top} \in \mathcal{E}$ such that $Bel(\Psi_{\top}) \equiv \top$, and for each $\Psi \in \mathcal{E}$, there exists a finite sequence of formulae $\sigma = (\mu_1, \dots, \mu_k)$ such that $\Psi = \Psi_{\top} \circ \sigma$

A DP operator satisfying **(G)** is said to be a DP+G operator. And when C is the class of DP operators satisfying **(G)** and \mathcal{I} is C -complete, we simply say that \mathcal{I} is DP+G-complete.

We intend to show that every DP+G operator is safely \mathcal{E}_{ocf} -instantiable. For this purpose, we intend to provide a canonical, safe translation of any DP+G operator \circ into an equivalent operator on OCFs, denoted by $\circ_{\mathcal{E}_{ocf}}$. So let \circ be any DP+G operator on $\mathcal{E} = (E, Bel)$. By condition **(G)**, every epistemic state from E can be identified by a finite sequence of formulae. Since the set of all finite sequence of formulae is a countable set, E is also countable. So one can denote by (Ψ_1, Ψ_2, \dots) the list of all epistemic states from E . For each $\Psi_i \in E$, let \preceq_{Ψ_i} be the total preorder associated with Ψ_i by the DP assignment corresponding to \circ . Now, let $f_{\mathcal{E}}$ be the mapping from E to U_{ocf} defined for each $\Psi_i \in \mathcal{E}$ by $f_{\mathcal{E}}(\Psi_{\top}) = \kappa_{\top}$, where κ_{\top} is the OCF defined for each world ω by $\kappa_{\top}(\omega) = 0$; and if $\Psi_i \neq \Psi_{\top}$, then $f_{\mathcal{E}}(\Psi_i) = \kappa_i$, where κ_i is the OCF defined for each world ω by $\kappa_i(\omega) = 0$ if ω is at the first level of \preceq_{Ψ_i} otherwise $\kappa_i(\omega) = |\Omega| + i$ if $\omega \in lvl(\preceq_{\Psi_i}, max(\preceq_{\Psi_i}))$ (i.e., if ω lies at the last level of \preceq_{Ψ_i}), otherwise $\kappa_i(\omega) = rank(\preceq_{\Psi_i}, \omega)$. By construction, $f_{\mathcal{E}}$ is injective. Let $Im(f_{\mathcal{E}})$ be the subset of OCFs from U_{ocf} that is the image of $f_{\mathcal{E}}$, i.e., $Im(f_{\mathcal{E}}) = \{f_{\mathcal{E}}(\Psi_i) \mid \Psi_i \in \mathcal{E}\}$. The fact that $f_{\mathcal{E}}$ is injective allows one to define another mapping $h_{\mathcal{E}_{ocf}} : U_{ocf} \rightarrow E$ defined for each OCF $\kappa \in U_{ocf}$ by $h_{\mathcal{E}_{ocf}}(\kappa) = f_{\mathcal{E}}^{-1}(\kappa)$ if $\kappa \in Im(f_{\mathcal{E}})$, otherwise $h_{\mathcal{E}_{ocf}}(\kappa) = \Psi_i$, where Ψ_i is any arbitrary chosen epistemic state from E such that $tpo(\kappa) = \preceq_{\Psi_i}$. (Note that such an epistemic state $\Psi_i \in E$ exists, since the DP assignment corresponding to \circ is a surjective mapping from E to U_{tpo} : similar to the proof of Proposition 4 and by condition **(G)**, we can prove that for any total preorder \preceq , there exists a finite sequence σ such that $\Psi_i = \Psi_{\top} \circ \sigma$ and $\preceq = \preceq_{\Psi_i}$.) Then we define the revision operator $\circ_{\mathcal{E}_{ocf}}$ on \mathcal{E}_{ocf} defined for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$ by $\kappa \circ_{\mathcal{E}_{ocf}} \mu = f_{\mathcal{E}}(h_{\mathcal{E}_{ocf}}(\kappa) \circ \mu)$.

Using this translation, we get that:

Proposition 10. $(f_{\mathcal{E}}, \circ_{\mathcal{E}_{ocf}})$ is a safe translation of \circ into \mathcal{E}_{ocf} .

Proof. First, for each $\Psi_i \in E$, let \preceq_{Ψ_i} be the total preorder associated with Ψ_i by the DP assignment corresponding to \circ . Then we know by conditions 1 and 2 of a faithful assignment that $[Bel(\Psi_i)] = \min(\Omega, \preceq_{\Psi_i})$. And by construction of $f_{\mathcal{E}}(\Psi_i) = \kappa_i$, we have that $\kappa_i(\omega) = 0$ if and only if $rank(\preceq_{\Psi_i}, \omega) = 0$ if and only if $\omega \in \min(\Omega, \preceq_{\Psi_i})$. Hence, for each $\Psi_i \in E$, we have that $Bel(\Psi_i) = B_{ocf}(f_{\mathcal{E}}(\Psi_i))$. Second, for each $\Psi_i \in E$ and each $\mu \in \mathcal{L}_P^*$ we get that $f_{\mathcal{E}}(\Psi_i) \circ_{\mathcal{E}_{ocf}} \mu = \kappa_i \circ_{\mathcal{E}_{ocf}} \mu = f_{\mathcal{E}}(h_{\mathcal{E}_{ocf}}(\kappa_i) \circ \mu) =$

$f_{\mathcal{E}}(f_{\mathcal{E}}^{-1}(\kappa_i) \circ \mu) = f_{\mathcal{E}}(\Psi_i \circ \mu)$. This shows that $(f_{\mathcal{E}}, \circ_{\mathcal{E}_{ocf}})$ is a translation of \circ into \mathcal{E}_{ocf} .

Yet by definition of $h_{\mathcal{E}_{ocf}}$, on the one hand we get for each $\kappa \in U_{ocf}$ that $B_{ocf}(\kappa) \equiv Bel(h_{\mathcal{E}_{ocf}}(\kappa))$ (since by denoting $\Psi_i = h_{\mathcal{E}_{ocf}}(\kappa)$, we have that $tpo(\kappa) = \preceq_{\Psi_i}$). On the other hand, for each $\kappa \in U_{ocf}$ and each $\mu \in \mathcal{L}_P^*$ we get that $h_{\mathcal{E}_{ocf}}(\kappa \circ_{\mathcal{E}_{ocf}} \mu) = h_{\mathcal{E}_{ocf}}(f_{\mathcal{E}}(h_{\mathcal{E}_{ocf}}(\kappa) \circ \mu)) = f_{\mathcal{E}}^{-1}(f_{\mathcal{E}}(\Psi_i \circ \mu)) = \Psi_i \circ \mu = h_{\mathcal{E}_{ocf}}(\kappa_i) \circ \mu$. This shows that $(h_{\mathcal{E}_{ocf}}, \circ)$ is a translation of $\circ_{\mathcal{E}_{ocf}}$ into \mathcal{E} . Hence, $(f_{\mathcal{E}}, \circ_{\mathcal{E}_{ocf}})$ is a safe translation of \circ into \mathcal{E}_{ocf} . \square

We know now that every DP+G operator \circ is safely \mathcal{E}_{ocf} -instantiable. Moreover, by construction, $\circ_{\mathcal{E}_{ocf}}$ is structure preserving. Therefore:

Corollary 5. For every DP+G operator \circ , the operator $\circ_{\mathcal{E}_{ocf}}$ is equivalent to \circ and is structure preserving.

And we can conclude that:

Corollary 6. \mathcal{E}_{ocf} is DP+G-complete.

This means that every DP operator defined on any epistemic space and satisfying the property **(G)** (that simply states that there is an initial, empty, epistemic state and that each epistemic state is reachable from it through a finite sequence of revisions) can be safely translated to the OCF epistemic space. As a consequence, all these operators can be defined equivalently in the OCF epistemic space.

7 Conclusion

In this work, we focused on Darwiche and Pearl's epistemic states and iterated revision operators from the representational perspective. We defined epistemic spaces as the structures in which these epistemic states and revision operators are defined. We highlighted the fact that epistemic states can be distinguished from each other by revision operators only by their behavior under any finite sequence of successive revisions. This led us to focus on the notion of strong equivalence between epistemic states, and to define a notion of equivalence between revision operators defined on different epistemic spaces. Based on these notions, we made precise the condition under which an operator is "instantiable" into another epistemic space.

Then, we looked for a canonical representation for the whole quotient set of epistemic states under this strong equivalence relation. But we showed that in general, such a canonical representation can not exist. The number of possible (quotient set of) epistemic states is just too large.

But then, if we restrict ourselves to epistemic states finitely generated from an initial, "empty" epistemic state, that is a quite reasonable assumption, then we showed that we can find such a canonical representation in terms of OCFs (Ordinal Conditional Functions). OCFs are a well-known representation choice for belief change operators, and are really close to total preorders. But this paper shows that the numerical information they add compared to total preorders is what is missing in order to encode all possible finitely generated epistemic state spaces.

Acknowledgements

This work has benefited from the support of the AI Chair BE4musIA of the French National Research Agency (ANR-20-CHIA-0028) and of the JSPS KAKENHI Grant Number JP20K11947. The third author has also been partially funded by the program PAUSE of Collège de France.

References

- Alchourrón, C. E., and Makinson, D. 1985. On the logic of theory change: Safe contraction. *Studia Logica* 44:405–422.
- Alchourrón, C. E.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic* 50:510–530.
- Aravanis, T. I.; Peppas, P.; and Williams, M. 2019. Observations on Darwiche and Pearl’s approach for iterated belief revision. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI’19)*, 1509–1515.
- Booth, R., and Chandler, J. 2016. Extending the harper identity to iterated belief change. In *Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI’16)*, 987–993.
- Booth, R., and Chandler, J. 2019. From iterated revision to iterated contraction: Extending the harper identity. *Artificial Intelligence* 277.
- Booth, R., and Meyer, T. A. 2006. Admissible and restrained revision. *Journal of Artificial Intelligence Research* 26:127–151.
- Booth, R., and Meyer, T. A. 2011. How to revise a total preorder. *Journal of Philosophical Logic* 40(2):193–238.
- Darwiche, A., and Pearl, J. 1994. On the logic of iterated belief revision. In *Proceedings of the 5th Conference on Theoretical Aspects of Reasoning about Knowledge (TARK’94)*, 5–23.
- Darwiche, A., and Pearl, J. 1997. On the logic of iterated belief revision. *Artificial Intelligence* 89(1-2):1–29.
- del Cerro, L. F., and Herzig, A. 1996. Belief change and dependence. In *Proceedings of the 6th Conference of Theoretical Aspects of Rationality and Knowledge (TARK’96)*, 147–161.
- Dubois, D., and Prade, H. 1991. Epistemic entrenchment and possibilistic logic. *Artificial Intelligence* 50:223–239.
- Fermé, E., and Hansson, S. O. 2011. AGM 25 years: Twenty-five years of research in belief change. *Journal of Philosophical Logic* 40(2):295–331.
- Freund, M., and Lehmann, D. 1994. Belief revision and rational inference. Technical Report TR-94-16, Institute of Computer Science, The Hebrew University of Jerusalem.
- Friedman, N., and Halpern, J. Y. 1999. Belief revision: A critique. *Journal of Logic, Language and Information* 8(4):401–420.
- Gärdenfors, P. 1988. *Knowledge in flux*. MIT Press.
- Gärdenfors, P. 1990. Belief revision and nonmonotonic logic: Two sides of the same coin? In *Proceedings of the 9th European Conference on Artificial Intelligence (ECAI’90)*, 768–773.
- Grove, A. 1988. Two modellings for theory change. *Journal of Philosophical Logic* 17(157-180).
- Hansson, S. O. 1994. Kernel contraction. *The Journal of Symbolic Logic* 59(3):845–859.
- Jin, Y., and Thielscher, M. 2007. Iterated belief revision, revised. *Artificial Intelligence* 171(1):1–18.
- Katsuno, H., and Mendelzon, A. O. 1991. Propositional knowledge base revision and minimal change. *Artificial Intelligence* 52:263–294.
- Konieczny, S., and Pino Pérez, R. 2008. Improvement operators. In *Proceedings of the 11th International Conference on Principles of Knowledge Representation and Reasoning (KR’08)*, 177–187.
- Konieczny, S., and Pino Pérez, R. 2000. A framework for iterated revision. *Journal of Applied Non-Classical Logics* 10(3-4):339–367.
- Konieczny, S.; Medina Grespan, M.; and Pino Pérez, R. 2010. Taxonomy of improvement operators and the problem of minimal change. In *Proceedings of the 12th International Conference on Principles of Knowledge Representation and Reasoning (KR’10)*, 161–170.
- Konieczny, S. 2009. Using transfinite ordinal conditional functions. In *Proceedings of the 10th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU’09)*, 396–407.
- Lehmann, D. 1995. Belief revision, revised. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence (IJCAI’95)*, 1534–1540.
- Makinson, D., and Gärdenfors, P. 1989. Relations between the logic of theory change and nonmonotonic logic. In *The Logic of Theory Change*, 185–205.
- Medina Grespan, M., and Pino Pérez, R. 2013. Representation of basic improvement operators. In *Trends in Belief Revision and Argumentation Dynamics*. College Publications. 195–227.
- Meng, H.; Kou, H.; and Li, S. 2015. Belief revision with general epistemic states. In *Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI’15)*.
- Nayak, A. C.; Foo, N. Y.; Pagnucco, M.; and Sattar, A. 1994. Entrenchment kinematics 101. In *Proceedings of the 7th Australian Joint Conference on Artificial Intelligence (AI’94)*, 157–164.
- Nayak, A. C.; Foo, N. Y.; Pagnucco, M.; and Sattar, A. 1996. Changing conditional beliefs unconditionally. In *Proceedings of the 6th Conference of Theoretical Aspects of Rationality and Knowledge (TARK’96)*, 119–135.
- Nayak, A. 1994. Iterated belief change based on epistemic entrenchment. *Erkenntnis* 41:353–390.
- Ramachandran, R.; Nayak, A. C.; and Orgun, M. A. 2012. Three approaches to iterated belief contraction. *Journal of Philosophical Logic* 41(1):115–142.
- Rott, H. 2009. Shifting priorities: Simple representations for twenty-seven iterated theory change operators. In Makinson, D.; Malinowski, J.; and Wansing, H., eds., *Towards Mathematical Philosophy*. Springer. 269–296.

Schwind, N., and Konieczny, S. 2020. Non-prioritized iterated revision: Improvement via incremental belief merging. In *Proceedings of the 17th International Conference on Principles of Knowledge Representation and Reasoning (KR'20)*, 738–747.

Spohn, W. 1988. *Ordinal Conditional Functions: A Dynamic Theory of Epistemic States*. Springer Netherlands. 105–134.

Williams, M.-A. 1995. Iterated theory base change: A computational model. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence (IJCAI'95)*, 1541–1549.