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# Particle approximation of the doubly parabolic Keller-Segel equation in the plane

Nicolas Fournier\* and Milica Tomašević†

## Abstract

In this work, we study a stochastic system of  $N$  particles associated with the parabolic-parabolic Keller-Segel system. This particle system is singular and non Markovian in that its drift term depends on the past of the particles. When the sensitivity parameter is sufficiently small, we show that this particle system indeed exists for any  $N \geq 2$ , we show tightness in  $N$  of its empirical measure, and that any weak limit point of this empirical measure, as  $N \rightarrow \infty$ , solves some nonlinear martingale problem, which in particular implies that its family of time-marginals solves the parabolic-parabolic Keller-Segel system in some weak sense. The main argument of the proof consists of a *Markovianization* of the interaction kernel: We show that, in some loose sense, the two-by-two path-dependant interaction can be controlled by a two-by-two Coulomb interaction, as in the parabolic-elliptic case.

*Keywords and phrases:* Stochastic particle systems; Singular interaction; Non-Markovian processes; Mean-field limit; Keller-Segel equation.

*MSC 2020 classification:* 60K35, 60H30, 35K57.

## 1 Introduction and main results

In this work, we study a stochastic particle approximation of the parabolic-parabolic Keller-Segel equation for chemotaxis in the plane. This equation, with unknown  $(\rho, c)$ , writes

$$\begin{cases} \partial_t \rho_t(x) = \Delta \rho_t(x) - \chi \nabla \cdot (\rho_t(x) \nabla c_t(x)), & t > 0, \quad x \in \mathbb{R}^2, \\ \theta \partial_t c_t(x) = \Delta c_t(x) - \lambda c_t(x) + \rho_t(x), & t > 0, \quad x \in \mathbb{R}^2, \end{cases} \quad (1)$$

with  $\rho_0$  and  $c_0$  given. Here,  $\rho_t \geq 0$  represents the distribution at time  $t \geq 0$  of a cell population. These cells are attracted by a chemical substance that they emit (the so-called chemo-attractant) and whose concentration at time  $t \geq 0$  is given by  $c_t \geq 0$ . The parameters  $\chi > 0$ ,  $\theta > 0$  and  $\lambda \geq 0$  respectively stand for the sensitivity of cells to the chemo-attractant, the ratio between the diffusion time scales of bacteria and chemo-attractant, and the death rate of the chemo-attractant. All along this work, we will suppose the system is rescaled so that the total mass of the cell population (which is preserved in time) is equal to 1. We refer to the original works of Keller-Segel [18, 19, 20] for the initial motivation and some biological explanations and to the paper of Horstmann [15] for a thorough review.

An interesting feature of the Keller-Segel system is that its solutions may blow-up in finite time, although the total mass is preserved: Some point cluster may emerge due to the attraction between cells (through the chemo-attractant). Namely, it is well known that for any reasonable initial

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condition, the parabolic-elliptic version of the system, which corresponds to the case  $\theta = 0$ , explodes in finite time if  $\chi > 8\pi$ , while the global well-posedness holds when  $\chi \leq 8\pi$ . This was first rigorously established by Blanchet-Dolbeault-Perthame [3] ( $\chi < 8\pi$ ) and Biler-Karch-Laurençot-Nadzieja [2] ( $\chi \leq 8\pi$ , radial case). In the parabolic-parabolic case where  $\theta > 0$ , the global well-posedness still holds for  $\chi < 8\pi$  and any reasonable condition, see Calvez-Corrias [6]. However, in the special case of  $c_0 = 0$  and for any  $\chi > 0$ , the global well-posedness holds true when  $\theta > 0$  is large enough, see Biler-Guerra-Karch [1], and this was extended to a more general class of initial concentrations  $c_0$  (with a smallness condition depending on  $\theta$ ) by Corrias-Escobedo-Matos [9]. Concerning explosion, the situation is still largely open, let us mention that radial solutions on a disk in  $\mathbb{R}^2$  blowing-up for  $\chi > 8\pi$  have been exhibited by Herrero-Velasquez [14]. In addition, a criterion for explosion of radial solutions has been obtained by Mizoguchi [21]: The conditions are that  $\chi > 8\pi$  and that some energy of the initial condition  $(\rho_0, c_0)$  is large enough.

Our goal is to derive the system (1) as a mean-field limit of an interacting particle system. To this end, we adopt the decoupling strategy proposed by Talay-Tomašević [27] in order to obtain a stochastic particle description of the system. Namely, observe that the concentration of chemo-attractant can be made explicit in terms of  $c_0$  and of the density of bacteria: Using the Duhamel formula, we have

$$c_t(x) = b_t^{c_0, \theta, \lambda}(x) + \int_0^t (K_{t-s}^{\theta, \lambda} * \rho_s)(x) ds,$$

where we denoted, for  $(t, x) \in (0, \infty) \times \mathbb{R}^2$ ,

$$g_t^\theta(x) := \frac{\theta}{4\pi t} e^{-\frac{\theta}{4t}|x|^2}, \quad K_t^{\theta, \lambda}(x) := \frac{1}{\theta} e^{-\frac{\lambda}{\theta}t} g_t^\theta(x), \quad b_t^{c_0, \theta, \lambda}(x) := e^{-\frac{\lambda}{\theta}t} (g_t^\theta * c_0)(x). \quad (2)$$

Then, the cell density in (1) is interpreted as a Fokker-Planck equation of a non-linear stochastic process where in the place of  $\nabla c$ , the gradient of the above formulation is plugged in. Hence, the corresponding non-linear S.D.E. reads

$$X_t = X_0 + \sqrt{2}W_t + \chi \int_0^t \nabla b_s^{c_0, \theta, \lambda}(X_s) ds + \chi \int_0^t \int_0^s (\nabla K_{s-u}^{\theta, \lambda} * \rho_u)(X_s) dud s, \quad \rho_s = \text{Law}(X_s), \quad (3)$$

where the  $\rho_0$ -distributed random variable  $X_0$  and the  $2D$ -Brownian motion  $(W_t)_{t \geq 0}$  are independent. Intuitively, it represents the motion of a typical cell in an infinite cloud of cells undergoing the dynamics of (1). Then, for  $N \geq 2$ , the following particle system is its microscopic counterpart:

$$X_t^{i, N} = X_0^{i, N} + \sqrt{2}W_t^i + \chi \int_0^t \nabla b_s^{c_0, \theta, \lambda}(s, X_s^{i, N}) ds + \frac{\chi}{N-1} \sum_{j \neq i} \int_0^t \int_0^s \nabla K_{s-u}^{\theta, \lambda}(X_s^{i, N} - X_u^{j, N}) dud s,$$

where the initial condition  $(X_0^{i, N})_{i=1, \dots, N}$  is independent of the i.i.d. family  $((W_t^i)_{t \geq 0})_{i=1, \dots, N}$  of  $2D$ -Brownian motions. Noting that

$$\nabla K_t^{\theta, \lambda}(x) = -\frac{\theta}{8\pi t^2} e^{-\frac{\lambda}{\theta}t} e^{-\frac{\theta}{4t}|x|^2} x,$$

we point out here that it is not at all clear that this system is well-defined. Indeed, each particle interacts with the other particles by means of a singular functional of their trajectories.

Let us explain quickly what we mean by *singular*: Assume that particles do encounter, as is the case in the parabolic-elliptic case, see [12]. If  $X_t^{1, N} = X_t^{2, N}$  at some time  $t > 0$ , we may expect that  $|X_t^{1, N} - X_s^{2, N}| = |X_t^{1, N} - X_t^{2, N}| + |X_t^{2, N} - X_s^{2, N}| \simeq (t-s)^\alpha$ , for some  $\alpha \in (0, 1)$ . Hence the corresponding interaction (in the drift of  $X^{1, N}$ ) is of order,

$$\int_0^t |\nabla K_{t-s}^{\theta, \lambda}(X_t^{1, N} - X_s^{2, N})| ds \simeq \int_0^t \frac{(t-s)^\alpha}{(t-s)^2} e^{-\frac{\theta}{4(t-s)}(t-s)^{2\alpha}} e^{-\frac{\lambda}{\theta}(t-s)} ds.$$

This quantity diverges if and only if  $\alpha \geq 1/2$ . Since we precisely expect the paths of the particles to be slightly more irregular than Hölder( $\frac{1}{2}$ )-continuous (as the Brownian motion), it is not clear whether the drift is well-defined or not, and we are really around the critical exponent.

Our objective is to show that such a particle system exists and to prove the convergence of its empirical measure, as  $N \rightarrow \infty$  and up to extraction of a subsequence, towards a solution to (1), under an explicit (though complicated) smallness condition on the parameter  $\chi$ . To the best of our knowledge, this is the first time the doubly parabolic Keller-Segel system on the plane is derived as a mean field limit of a non-smoothed interacting stochastic particle system.

Let us start with defining our notion of solution to the system (1). We endow the set  $\mathcal{P}(\mathbb{R}^2)$  of probability measures on  $\mathbb{R}^2$  with the weak convergence topology (i.e. with continuous and bounded test functions). We denote by  $C_b^k(\mathbb{R}^2)$  the set of  $C^k$  functions on  $\mathbb{R}^2$  with bounded derivatives of order 0 to  $k$ .

**Definition 1.** Fix  $\rho_0 \in \mathcal{P}(\mathbb{R}^2)$  and some nonnegative  $c_0 \in L^p(\mathbb{R}^2)$  for some  $p > 2$ . A couple  $(\rho_t, c_t)_{t \geq 0}$  is a weak solution to (1) if  $(\rho_t)_{t \geq 0} \in C([0, \infty), \mathcal{P}(\mathbb{R}^2))$ , if for all  $t \geq 0$ ,

$$\int_0^t \int_{\mathbb{R}^2} \int_0^s \int_{\mathbb{R}^2} (K_{s-u}^{\theta, \lambda}(x-y) + |\nabla K_{s-u}^{\theta, \lambda}(x-y)|) \rho_u(dy) du \rho_s(dx) ds < \infty, \quad (4)$$

if for  $\rho_t(dx)dt$ -almost every  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$ ,

$$c_t(x) = b_t^{c_0, \theta, \lambda}(x) + \int_0^t (K_{t-s}^{\theta, \lambda} * \rho_s)(x) ds, \quad (5)$$

and if for all  $\varphi \in C_b^2(\mathbb{R}^2)$  and all  $t > 0$ ,

$$\int_{\mathbb{R}^2} \varphi(x) \rho_t(dx) = \int_{\mathbb{R}^2} \varphi(x) \rho_0(dx) + \int_0^t \int_{\mathbb{R}^2} \Delta \varphi(x) \rho_s(dx) ds + \chi \int_0^t \int_{\mathbb{R}^2} \nabla \varphi(x) \cdot \nabla c_s(x) \rho_s(dx) ds. \quad (6)$$

The fact that  $c_0 \in L^p(\mathbb{R}^2)$  implies that  $b_t^{c_0, \theta, \lambda}$  and  $\nabla b_t^{c_0, \theta, \lambda}$  are continuous on  $\mathbb{R}^d$  for each  $t > 0$  and that there is a constant  $A = A(\theta, p)$  such that for all  $t > 0$ , with  $p' = \frac{p}{p-1}$ ,

$$\begin{cases} \sup_{x \in \mathbb{R}^2} |b_t^{c_0, \theta, \lambda}(x)| \leq \|g_t^\theta\|_{L^{p'}} \|c_0\|_{L^p} \leq \frac{A}{t^{\frac{1}{p}}} \|c_0\|_{L^p}, \\ \sup_{x \in \mathbb{R}^2} |\nabla b_t^{c_0, \theta, \lambda}(x)| \leq \|\nabla g_t^\theta\|_{L^{p'}} \|c_0\|_{L^p} \leq \frac{A}{t^{\frac{1}{2} + \frac{1}{p}}} \|c_0\|_{L^p}. \end{cases} \quad (7)$$

Since  $p > 2$ , (7) and (4) imply that  $\int_0^t (c_s(x) + |\nabla c_s(x)|) \rho_s(dx) ds < \infty$  for all  $t > 0$  and that  $\nabla c_t(x) = \nabla b_t^{c_0, \theta, \lambda}(x) + \int_0^t (\nabla K_{t-s}^{\theta, \lambda} * \rho_s)(x) ds$  for  $\rho_t(dx)dt$ -almost every  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$ . Thus everything makes sense in (6).

We now introduce the martingale problem characterizing the law of the nonlinear S.D.E. (3).

**Definition 2.** Fix some  $\rho_0 \in \mathcal{P}(\mathbb{R}^2)$  and some nonnegative  $c_0 \in L^p(\mathbb{R}^2)$  for some  $p > 2$ . Consider the canonical space  $C([0, \infty), \mathbb{R}^2)$  equipped with its canonical process  $(w_t)_{t \geq 0}$  and its canonical filtration. Let  $\mathbb{Q}$  be a probability measure on this canonical space and denote  $(\mathbb{Q}_t)_{t \geq 0}$  its family of one-dimensional time marginals. We say that  $\mathbb{Q}$  solves the non-linear martingale problem  $(\mathcal{MP})$  with initial law  $\rho_0$  if  $\mathbb{Q}_0 = \rho_0$ , if

$$\int_0^t \int_{\mathbb{R}^2} \int_0^s \int_{\mathbb{R}^2} (K_{s-u}^{\theta, \lambda}(x-y) + |\nabla K_{s-u}^{\theta, \lambda}(x-y)|) \mathbb{Q}_u(dy) du \mathbb{Q}_s(dx) ds < \infty, \quad (8)$$

and if for any  $\varphi \in C_c^2(\mathbb{R}^2)$ , the process

$$M_t^\varphi := \varphi(w_t) - f(w_0) - \int_0^t \left[ \Delta \varphi(w_u) + \chi \nabla \varphi(w_u) \cdot \left( \nabla b_s^{c_0, \theta, \lambda}(w_s) + \int_0^u (\nabla K_{u-r}^{\theta, \lambda} * \mathbb{Q}_r)(w_u) dr \right) \right] du \quad (9)$$

is a  $\mathbb{Q}$ -martingale.

For  $\mathbb{Q}$  a solution to  $(\mathcal{MP})$ , setting  $c_t = b_t^{c_0, \theta, \lambda} + \int_0^t (K_{t-s}^{\theta, \lambda} * \mathbb{Q}_s) ds$ , it holds that  $(\mathbb{Q}_t, c_t)_{t \geq 0}$  is a weak solution to (1). Finally, we consider the following notion of solution to our particle system.

**Definition 3.** Fix  $N \geq 2$  and some nonnegative  $c_0 \in L^p(\mathbb{R}^2)$  for some  $p > 2$ . Consider some i.i.d. family  $(W_t^i)_{t \geq 0, i=1, \dots, N}$  of 2D-Brownian motion, as well as some exchangeable family  $(X_0^i)_{i=1, \dots, N}$  of  $\mathbb{R}^2$ -valued random variables, independent of the family of Brownian motions. A family of continuous  $\mathbb{R}^2$ -valued processes  $(X_t^{i, N})_{t \geq 0, i=1, \dots, N}$  is said to be a  $N$ -Keller-Segel particle system if a.s., for all  $t \geq 0$ , all  $i \neq j$ ,

$$\int_0^t \int_0^s |\nabla K_{s-u}^{\theta, \lambda}(X_s^{i, N} - X_u^{j, N})| dud s < \infty \quad (10)$$

and if a.s., for all  $t \geq 0$ , all  $i = 1, \dots, N$ ,

$$X_t^{i, N} = X_0^{i, N} + \sqrt{2}W_t^i + \chi \int_0^t \nabla b_s^{c_0, \theta, \lambda}(s, X_s^{i, N}) ds + \frac{\chi}{N-1} \sum_{j \neq i} \int_0^t \int_0^s \nabla K_{s-u}^{\theta, \lambda}(X_s^{i, N} - X_u^{j, N}) dud s.$$

Everything makes sense in this last expression by (7) and (10). As already mentioned, it is not at all clear that the above system has a solution and even less that it converges as  $N \rightarrow \infty$ , due to the singular nature of the path-dependent interaction of the particles. We are ready to present our main result. It gathers several statements that we make throughout the paper.

**Theorem 4.** Let  $\chi > 0$ ,  $\lambda \geq 0$  and  $\theta > 0$ . Consider  $\rho_0 \in \mathcal{P}(\mathbb{R}^2)$  and some nonnegative  $c_0 \in L^p(\mathbb{R}^2)$ , for some  $p > 2$ . Consider, for each  $N \geq 2$ , some exchangeable initial condition  $(X_0^{i, N})_{i=1, \dots, N}$ . Suppose that  $\chi < \chi_{\theta, p}^*$ , where  $\chi_{\theta, p}^* > 0$  is defined in (47). Then, we have the following results.

(i) For each  $N \geq 2$ , there exists an exchangeable  $N$ -Keller-Segel particle system.

(ii) We set  $\mu^N := \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i, N})_{t \geq 0}}$ , which a.s. belongs to  $\mathcal{P}(C([0, \infty), \mathbb{R}^2))$  and, for each  $t \geq 0$ ,  $\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i, N}}$ , which a.s. belongs to  $\mathcal{P}(\mathbb{R}^2)$ . If  $\mu_0^N$  converges in probability, as  $N \rightarrow \infty$ , to  $\rho_0$ , then the family  $(\mu^N)_{N \geq 2}$  is tight in  $\mathcal{P}(C([0, \infty), \mathbb{R}^2))$  and any (possibly random) limit point  $\mu$  of  $(\mu^N)_{N \geq 2}$  a.s. solves  $(\mathcal{MP})$  with initial law  $\rho_0$ .

The following statement is an immediate consequence of the above result.

**Corollary 5.** With the assumptions and notations of Theorem 4, denoting by  $(\mu_t)_{t \geq 0}$  a (possibly random) limit point of  $((\mu_t^N)_{t \geq 0})_{N \geq 2}$  and defining  $c_t = b_t^{c_0, \theta, \lambda} + \int_0^t K_{t-s}^{\theta, \lambda} * \mu_s ds$ , the couple  $(\mu_t, c_t)_{t \geq 0}$  is a.s. a solution to (1) with initial condition  $(\rho_0, c_0)$  in the sense of Definition 1.

Let us also mention that we have some weak regularity estimates.

**Remark 6.** Adopt the assumptions and notations of Theorem 4. There exists  $\gamma \in (\frac{3}{2}, 2)$ , depending on  $\chi$ ,  $\theta$  and  $p$ , such that, with  $\beta = 2(\gamma - 1) \in (1, 2)$ ,

$$\sup_{N \geq 2} \mathbb{E} \left[ \int_0^t \frac{ds}{|X_s^{1, N} - X_s^{2, N}|^\beta} \right] + \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x - y|^\beta} \mu_s(dy) \mu_s(dx) ds \right] < \infty.$$

Moreover, we also have (40)-(41)-(42) and (45).

**About the threshold.** As shown in Remark 15, it for example holds true that (i)  $\liminf_{\theta \rightarrow 0} \chi_{\theta, p}^* \geq 3.28$  for any  $p > 2$ , (ii)  $\chi_{1, p}^* \geq 1.39$  as soon as  $p > 3.3$ , and (iii)  $\liminf_{\theta \rightarrow \infty} \sqrt{\theta} \chi_{\theta, p}^* \geq 1.65$  as soon as  $p > 3.5$ . Moreover, we do not obtain better values of  $\chi_{\theta, p}^*$  for larger values of  $p$ .

In view of the global well-posedness/explosion results concerning (1) mentioned above, we would of course prefer to have  $\chi_{\theta, p}^* = 8\pi$  for any  $\theta > 0$ , at least for large values of  $p$  (or even with stronger

regularity conditions on  $c_0$ ). Our threshold is smaller, and this is due to the fact that it does not seem easy to make use, at the level of particles, of the *macroscopic* quantities exploited in [6]. Our proof is based on a tedious *a priori* estimate that relies on some moment computations and a suitable functional inequality.

According to [1, 9], we could even hope for a very large threshold  $\chi_{\theta, c_0}^*$  when  $\theta$  is very large and  $c_0$  is very small. We do not see how to modify our argument in this direction, because our method leads to a threshold independent of  $c_0$  (at least if one assumes that  $p = \infty$ ). Observe that for  $c_0$  given and non identically null, the threshold  $\chi_{\theta, c_0}^*$  in [9] also tends to 0 as  $\theta \rightarrow \infty$ .

**References.** Particle approximations of singular P.D.E.s has been the subject of many papers. The closely related 2D Navier-Stokes and parabolic-elliptic Keller-Segel equations can be approximated by singularly interacting particle systems (singularity is of the order  $1/r$ , where  $r$  is the pairwise particle distance) that are Markovian (not depending on the past of the particles). As already mentioned, we believe that the parabolic-parabolic equation also leads to a spatial singularity of order  $1/r$ . Something in this sense appears in the paragraph *Strategy* below. Moreover, observe that, very roughly, if  $X_t^{1,N} = X_s^{2,N} + R$ , for some vector  $R$ , during the time interval  $[t-1, t]$ , then the corresponding interaction (in the drift of  $X^{1,N}$ ) looks like, e.g. when  $\lambda = 0$ ,

$$\int_{t-1}^t \nabla K_{t-s}^{\theta, \lambda}(X_t^{1,N} - X_s^{2,N}) ds = -\frac{\theta R}{8\pi} \int_{t-1}^t \frac{1}{(t-s)^2} e^{-\frac{\theta}{4(t-s)}|R|^2} ds = -\frac{R}{2\pi|R|^2} e^{-\frac{\theta|R|^2}{4}} \sim -\frac{R}{2\pi|R|^2}$$

as  $|R| \rightarrow 0$ . Of course, this is a caricatured situation.

The convergence as  $N \rightarrow \infty$  of an interacting particle system to the solution of the 2D Navier-Stokes equation has been established by Osada [23, 24] (convergence along a subsequence for a large enough viscosity), Fournier-Hauray-Mischler [10] (convergence of the whole sequence for an arbitrary positive viscosity) and finally by Jabin-Wang [16] (quantitative convergence for an arbitrary positive viscosity). The 2D parabolic-elliptic Keller-Segel equation has the same order of singularity as the 2D Navier-Stokes equation, but the interaction is *attractive* instead of being *rotating*, and solutions do explode when the sensitivity parameter  $\chi$  is greater than  $8\pi$ . Hence the situation is more delicate. The convergence of the associated particle system along a subsequence has been shown by Fournier-Jourdain [11] when  $\chi < 2\pi$ , and the quantitative convergence has been established by Bresch-Jabin-Wang [4] when  $\chi < 8\pi$ , and the critical case  $\chi = 8\pi$  is treated in the work of Tardy [28]. Let us also mention that Cattiaux-Pédèches [7] have proved the uniqueness in law of the particle system in the subcritical case  $\chi < 8\pi$ , which is far from obvious, and that a detailed study of the collisions arising near the instant of explosion has been achieved, in the supercritical case, by Fournier-Tardy in [12]. Finally, Olivera-Richard-Tomašević [22] are able to prove the quantitative convergence, in the supercritical case  $\chi > 8\pi$ , until the explosion time, of a smoothed particle system, namely when the interaction in  $1/r$  is replaced by an interaction in  $1/(r + \varepsilon_N)$ , with  $\varepsilon_N = N^{-1/2+\eta}$  for some  $\eta > 0$ .

Concerning the parabolic-parabolic Keller-Segel equation, it seems that there are very few results about interacting particle systems. Jabir-Talay-Tomašević [17] have considered the same problem as ours in dimension 1. There the well-posedness and the propagation of chaos were proved using some suitable Girsanov transforms and without any constraints on the parameters of the model (as expected). Our situation here is somewhat more singular, as (i) in the 2D setting, solutions to the limit P.D.E. may explode in finite time for  $\chi$  large and (ii) we expect that, as in the parabolic-elliptic case, particles do collide, even in the subcritical case, see [11]. Hence, it is not possible here to use Girsanov transforms as we do not expect the law of our system to be absolutely continuous w.r.t. the Wiener measure. A more computational way to see this *increase* of the singularity with the

dimension is to note that the kernel  $\nabla K$  satisfies, in dimension  $d$ , for all  $p \geq 1$ ,

$$\|\nabla K\|_{L^p(\mathbb{R}^d)} = \frac{C_{p,d}}{t^{\frac{d}{2}(1-\frac{1}{p})+\frac{1}{2}}}.$$

In particular,  $\nabla K$  belongs to  $L^1((0, T); L^2(\mathbb{R}^d))$  only for  $d = 1$  and this was crucial in [17] for justifying the Girsanov transform.

In any dimension  $d \geq 1$ , Stevens [25] studies a physically more satisfying particle system, with two populations (a population of cells and a population of chemo-attractant particles) in moderate interaction. The author proves the convergence in probability of the Kantorovich-Rubinstein distance between the empirical measure of the particle system and a solution to a generalized parabolic-parabolic Keller-Segel equation, including the supercritical case. The convergence is shown on the time interval where the limit P.D.E. has a solution belonging to  $C_b^{1,3}([0, T] \times \mathbb{R}^d, \mathbb{R}) \cap C^0([0, T], L^2(\mathbb{R}^d))$ . The moderate cutoff procedure, that we do not make precise here, decreases polynomially to 0 as  $N \rightarrow \infty$ .

Let us mention that Chen-Wang-Yang [8] also prove the convergence of a smoothed version of our particle system, where  $K_s^{\theta, \lambda}$  is replaced by  $K_{\varepsilon_N + s}^{\theta, \lambda}$  with  $\varepsilon_N = c(\log N)^{-\frac{1}{2}}$  and, of course, they actually can work in any dimension. Such a result is obtained in two steps: first, a classical propagation of chaos with rate  $e^{C/\varepsilon^2} N^{-1/2}$  using the Sznitman coupling approach [26] towards the smoothed limit equation, and then convergence of the smoothed equation with explicit rate. In the same vein, in any dimension, Budhiraja-Fan [5] study a modified version of the doubly parabolic model, where the source term  $\rho$  in the concentration equation is replaced by  $\rho * g_1$ . On the particle level,  $K_s^{\theta, \lambda}$  is replaced by  $K_{1+s}^{\theta, \lambda}$  and the authors prove the trajectorial propagation of chaos, as in [26], and the uniform convergence of the associated Euler scheme. The object of the present paper is rather to let first  $\varepsilon \rightarrow 0$  and then  $N \rightarrow \infty$ .

We also notice here that, in the two dimensional case, under a smallness condition on  $\chi$ , Tomašević [29] constructs the solution to a martingale problem related to (3). The initial condition is supposed to be a probability density function in [29], but the marginal laws of the solution to the martingale problem have some densities belonging to some mixed  $L_t^q - L_x^p$  spaces. Even with more regularity on the initial condition, our method in the present paper does not allow us to find the martingale formulation of [29] as a limit of our particle system as  $N \rightarrow \infty$ . The main reason is that the only information we recover for the limit (along a subsequence) of the empirical measure of the particle system is that its marginal laws satisfy (8). We think it is very difficult to show, only using the latter, that some initial regularity propagates in time and that two martingale problems are equivalent.

As a conclusion, this work seems to be the first one deriving the 2D parabolic-parabolic Keller-Segel equation from a *non-smoothed* particle system. We are quite satisfied to show the existence of the particle system and its convergence along a subsequence even though we have a small threshold. However, at least for  $\theta$  small or of order 1 this threshold is non-ridiculously small. It should not come as a surprise that we only have convergence along a subsequence: Our notion of solution to (1) is so weak that uniqueness seems very difficult to establish even with a smooth initial condition.

Finally, observe that we have no assumption but exchangeability on the initial condition of the particle system: We can even start with all the particles at the same place. In the same spirit, we obtain a global existence result for (1) derived from Theorem 4 that is slightly different than the previous ones, as we only assume that  $\rho_0 \in \mathcal{P}(\mathbb{R}^2)$  (which is already the case in [1]) and that  $c_0 \in L^{2+\varepsilon}(\mathbb{R}^2)$ , while all the previously cited papers work with  $c_0 \in H^1(\mathbb{R}^2)$ .

**Strategy.** The main point is to show that, when  $\chi$  is small enough, something a bit stronger than (10) holds true, uniformly in  $N \geq 1$ . To this end, we will introduce a  $\varepsilon$ -smoothed particle system, for which we will show that, considering e.g. particles 1 and 2 and setting  $D_s^{1,2,N,\varepsilon} := \int_0^s \nabla K_{s-u}^{\theta,\lambda} (X_s^{1,N,\varepsilon} - X_u^{2,N,\varepsilon}) du$

$$\sup_{N \geq 2} \mathbb{E} \left[ \int_0^t |D_s^{1,2,N,\varepsilon}|^{2(\gamma-1)} ds \right] < \infty \quad \text{for all } t > 0. \quad (11)$$

This estimate, which corresponds to (21) in Proposition 8, will be shown uniformly in  $\varepsilon$ . This will allow us to pass to the limit as  $\varepsilon \rightarrow 0$  in order to prove the existence of the non-smoothed particle system. This system satisfies, when setting  $D_s^{1,2,N} := \int_0^s \nabla K_{s-u}^{\theta,\lambda} (X_s^{1,N} - X_u^{2,N}) du$ ,

$$\sup_{N \geq 2} \mathbb{E} \left[ \int_0^t |D_s^{1,2,N}|^{2(\gamma-1)} ds \right] < \infty \quad \text{for all } t > 0. \quad (12)$$

see (42) in the proof, and then to pass to the limit as  $N \rightarrow \infty$ .

Let us show why (12) *a priori* holds. The difficulty when proving (12) is to show that particles are not too close to each other. Indeed,  $|D_s^{1,2,N}|$  may explode only if  $X_s^{1,N} = X_s^{2,N}$ , because  $(u, x) = (s, 0)$  is the only problematic point of  $\nabla K_{s-u}^{\theta,\lambda}(x)$ . Otherwise, the integral is well defined. We show in Proposition 11 that

$$\mathbb{E} \left[ \int_0^t |D_s^{1,2,N}|^{2(\gamma-1)} ds \right] \leq C \mathbb{E} \left[ \int_0^t |X_s^{1,N} - X_s^{2,N}|^{-2(\gamma-1)} ds \right]. \quad (13)$$

The proof of this estimate is difficult to summarize and relies on a slightly special application of the Itô formula shown in Lemma 10, used with a well-chosen function, on a bound of  $|\nabla K^{\theta,\lambda}|$  obtained in Remark 9, and on a key functional inequality proved in Lemma 7.

Observe that (13) reveals, in some loose sense, that the drift term  $D_s^{1,2,N}$  is controlled by  $|X_s^{1,N} - X_s^{2,N}|^{-1}$ , which is precisely the singularity of the drift in the parabolic-elliptic case. This is what we call a *Markovianization*: we bound the *path dependent* interaction by a *current time dependent* one.

Once this Markovianization is performed in Proposition 11, we may conclude the proof of Proposition 8 (including (12)) by applying the strategy of [11], that has been refined in [13, 28]: Applying the Itô formula, and using exchangeability, we find (assuming that  $c_0 = 0$  for simplicity)

$$\begin{aligned} \mathbb{E}[|X_t^{1,N} - X_t^{2,N}|^{4-2\gamma}] &= \mathbb{E}[|X_0^{1,N} - X_0^{2,N}|^{4-2\gamma}] + (4-2\gamma)^2 \int_0^t \mathbb{E}[|X_s^{1,N} - X_s^{2,N}|^{2-2\gamma}] ds \\ &\quad + \frac{4-2\gamma}{N-1} \chi \sum_{j=2}^N \int_0^t \mathbb{E}[|X_s^{1,N} - X_s^{2,N}|^{2-2\gamma} (X_s^{1,N} - X_s^{2,N}) \cdot D_s^{1,j,N}] ds. \end{aligned}$$

Using exchangeability again, (13) and the Hölder inequality, one may control the last term by  $C\chi \mathbb{E}[\int_0^t |X_s^{1,N} - X_s^{2,N}|^{-2(\gamma-1)} ds]$ . All this shows that

$$\mathbb{E}[|X_t^{1,N} - X_t^{2,N}|^{4-2\gamma}] \geq \mathbb{E}[|X_0^{1,N} - X_0^{2,N}|^{4-2\gamma}] + ((4-2\gamma)^2 - C\chi) \mathbb{E} \left[ \int_0^t |X_s^{1,N} - X_s^{2,N}|^{-2(\gamma-1)} ds \right].$$

Since now particles are subjected to attraction, there is no reason why they should be far from 0. Since  $4-2\gamma > 0$ , we expect that  $\mathbb{E}[|X_t^{1,N} - X_t^{2,N}|^{4-2\gamma}]$  should be easily controlled, uniformly in  $N \geq 2$ , by some constant  $A_t$  (actually, we use as in [13] a slightly more clever function than  $|\cdot|^{4-2\gamma}$  and this last argument is useless), and we end with

$$((4-2\gamma)^2 - C\chi) \mathbb{E} \left[ \int_0^t |X_s^{1,N} - X_s^{2,N}|^{-2(\gamma-1)} ds \right] \leq A_t.$$

Combined with (13), this gives us (12) provided that  $\chi < (4-2\gamma)/C$ .



**Key functional inequality.** The following functional inequality, that we will prove in Appendix A plays a central role in our main computation.

**Lemma 7.** *Let  $b > a > 0$  and  $t > 0$ . For any measurable function  $f : [0, t] \rightarrow \mathbb{R}_+$ , we have*

$$\int_0^t \frac{1}{(s + f(s))^{1+a}} ds \leq \kappa(a, b) \left( \int_0^t \frac{1}{(s + f(s))^{1+b}} ds \right)^{\frac{a}{b}}, \quad \text{where } \kappa(a, b) = \frac{a+1}{a} \left[ \frac{b}{b+1} \right]^{\frac{a}{b}}.$$

The constant  $\kappa(a, b)$  is optimal (for any value of  $t > 0$ ), as one can show by choosing  $f(s) = (\varepsilon - s)_+$  and by letting  $\varepsilon \rightarrow 0$ .

**Plan of the paper.** In Section 2, we start from a regularized particle system and we present the main computation of the paper. Once this is done, Sections 3-4-5 contain respectively a tightness result (Lemma 12), the existence result for the particle system (Proposition 13) and that any limit point satisfies  $(\mathcal{MP})$  (Theorem 14). In Section 6 we discuss our smallness condition on the chemotactic sensitivity  $\chi$  and we make it more explicit. Finally, in Appendix A we prove Lemma 7.

## 2 Main computation

For  $c_0 \in L^p(\mathbb{R}^2)$  with  $p > 2$ , for  $\varepsilon \in (0, 1]$  and  $N \geq 2$ , we introduce a smoothed version of the interaction kernel

$$H_t^{\theta, \lambda, \varepsilon}(x) := \frac{t^2}{(t + \varepsilon)^2} \nabla K_t^{\theta, \lambda}(x) = -\frac{\theta}{8\pi(t + \varepsilon)^2} e^{-\frac{\lambda}{\theta} t} e^{-\frac{\theta}{4t}|x|^2} x, \quad (14)$$

as well as the smoothed version of the Keller-Segel particle system: For all  $i = 1, \dots, N$ ,

$$X_t^{i, N, \varepsilon} = X_0^{i, N} + \sqrt{2} W_t^i + \chi \int_0^t \nabla b_{s+\varepsilon}^{c_0, \theta, \lambda}(X_s^{i, N, \varepsilon}) ds + \frac{\chi}{N-1} \sum_{j \neq i} \int_0^t \int_0^s H_{s-u}^{\theta, \lambda, \varepsilon}(X_s^{i, N, \varepsilon} - X_u^{j, N, \varepsilon}) du ds. \quad (15)$$

This system has a pathwise unique solution, as  $\nabla b_{t+\varepsilon}^{c_0, \theta, \lambda}$  and  $H_t^{\theta, \lambda, \varepsilon}$  are globally Lipschitz continuous, uniformly in  $t \geq 0$ . If the initial condition  $(X_0^{i, N})_{i=1, \dots, N}$  is exchangeable, then the family  $((X_t^{i, N, \varepsilon})_{t \geq 0})_{i=1, \dots, N}$  is also exchangeable by uniqueness in law.

The constants  $\kappa(a, b)$ , for  $b > a > 0$ , are defined in Lemma 7 and for  $\alpha, \beta, \theta > 0$  and  $\gamma > 3/2$ , we introduce

$$C_0(\beta) := \sup_{u \geq 0} \sqrt{u} (1 + \beta u)^{3/2} e^{-u}, \quad C_1(\alpha, \gamma) := (\gamma - 1)(1 - 4\alpha(\gamma - 1)), \quad (16)$$

$$C_2(\theta, \alpha, \gamma) := \frac{\sqrt{\alpha\theta}(\gamma - 1)}{2\pi} C_0\left(\frac{4\alpha}{\theta}\right) \kappa\left(\frac{1}{2}, \gamma - 1\right) \kappa\left(\gamma - \frac{3}{2}, \gamma - 1\right). \quad (17)$$

The goal of this section is to prove the following estimates, from which our main theorem will be more or less classically deduced, see e.g. Osada [24] and then [10, 11].

**Proposition 8.** *Assume that for each  $N \geq 2$ , the family  $(X_0^{i, N})_{i=1, \dots, N}$  is exchangeable and that  $c_0 \in L^p(\mathbb{R}^2)$  for some  $p > 2$ . Let  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$  and  $\alpha > 0$  such that  $C_1(\alpha, \gamma) > 0$ . Assume that  $\chi > 0$  and  $\theta > 0$  are such that*

$$C_1(\alpha, \gamma) > \chi C_2(\theta, \alpha, \gamma) \quad \text{and} \quad (4 - 2\gamma) - \chi \frac{\sqrt{\theta} C_0\left(\frac{4\alpha}{\theta}\right) \kappa\left(\frac{1}{2}, \gamma - 1\right)}{4\pi \sqrt{\alpha} [C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma)]^{\frac{1}{2(\gamma-1)}}} > 0.$$

Then for all  $t > 0$ ,

$$\sup_{\varepsilon \in (0,1], N \geq 2} \mathbb{E} \left[ \int_0^t \frac{1}{|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^{2(\gamma-1)}} ds \right] < \infty, \quad (18)$$

$$\sup_{\varepsilon \in (0,1], N \geq 2} \mathbb{E} \left[ \int_0^t \int_0^s \frac{1}{(s-u + |X_s^{1,N,\varepsilon} - X_u^{2,N,\varepsilon}|^2)^\gamma} dud s \right] < \infty, \quad (19)$$

$$\sup_{\varepsilon \in (0,1], N \geq 2} \mathbb{E} \left[ \int_0^t \int_0^s |\nabla K_{s-u}^{\theta,\lambda}(X_s^{1,N,\varepsilon} - X_u^{2,N,\varepsilon})|^{\frac{2\gamma}{3}} dud s \right] < \infty, \quad (20)$$

$$\sup_{\varepsilon \in (0,1], N \geq 2} \mathbb{E} \left[ \int_0^t \left( \int_0^s |\nabla K_{s-u}^{\theta,\lambda}(X_s^{1,N,\varepsilon} - X_u^{2,N,\varepsilon})| du \right)^{2(\gamma-1)} ds \right] < \infty. \quad (21)$$

It is important to notice that the exponents  $2(\gamma - 1)$  and  $\frac{2\gamma}{3}$  are both greater than 1. For some comments about the interest of these estimates and the strategy to prove them, we refer to the paragraph *Strategy* in Section 1. Let us first make the following observation.

**Remark 9.** For any  $\varepsilon \in (0, 1]$ , any  $\alpha > 0$ , any  $t > 0$ , any  $x \in \mathbb{R}^2$ , we have

$$|\nabla K_t^{\theta,\lambda}(x)| \leq \frac{\sqrt{\theta} C_0(\frac{4\alpha}{\theta})}{4\pi(t + \alpha|x|^2)^{\frac{3}{2}}} \quad \text{and} \quad |H_t^{\theta,\lambda,\varepsilon}(x)| \leq \frac{\sqrt{\theta} C_0(\frac{4\alpha}{\theta})}{4\pi(t + \varepsilon + \alpha|x|^2)^{\frac{3}{2}}}.$$

*Proof.* Let us for example study the case of  $H_t^{\theta,\lambda,\varepsilon}(x)$ . We write

$$|H_t^{\theta,\lambda,\varepsilon}(x)| = \frac{\theta}{8\pi(t + \varepsilon)^2} e^{-\frac{\lambda}{\theta}t} e^{-\frac{\theta}{4t}|x|^2} |x| \leq \frac{1}{(t + \varepsilon)^{3/2}} \frac{\theta}{8\pi} \frac{|x|}{(t + \varepsilon)^{1/2}} e^{-\frac{\theta}{4(t+\varepsilon)}|x|^2}.$$

Setting  $u = \frac{\theta}{4(t+\varepsilon)}|x|^2$ , this rewrites

$$|H_t^{\theta,\lambda,\varepsilon}(x)| \leq \frac{1}{(t + \varepsilon)^{3/2}} \frac{\sqrt{\theta}}{4\pi} \sqrt{u} e^{-u} \leq \frac{1}{(t + \varepsilon)^{3/2}} \frac{\sqrt{\theta}}{4\pi} \frac{C_0(\frac{4\alpha}{\theta})}{(1 + \frac{4\alpha}{\theta}u)^{3/2}}$$

by definition (16) of  $C_0$ . The result is proved, since  $(t + \varepsilon)(1 + \frac{4\alpha}{\theta}u) = t + \varepsilon + \alpha|x|^2$ .  $\square$

During the whole section we drop the superscript  $N, \varepsilon$ , i.e. we write  $X_t^i = X_t^{i,N,\varepsilon}$ , but we keep in mind that all the estimates have to be uniform in these parameters. In addition, we define

$$R_{t,s}^{i,j} := X_t^i - X_s^j \quad \text{and} \quad D_t^{i,j} := \int_0^t H_{t-s}^{\theta,\lambda,\varepsilon}(R_{t,s}^{i,j}) ds.$$

We start with the following Itô formula.

**Lemma 10.** Let  $F : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$  be of class  $C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^2)$ . For all  $t > 0$ ,

$$\begin{aligned} \mathbb{E} \left[ \int_0^t F(t-s, R_{t,s}^{1,2}) ds \right] &= \mathbb{E} \left[ \int_0^t F(0, R_{s,s}^{1,2}) ds \right] + \mathbb{E} \left[ \int_0^t \int_0^u (\partial_t F + \Delta F)(u-s, R_{u,s}^{1,2}) ds du \right] \\ &\quad + \chi \mathbb{E} \left[ \int_0^t \left( \int_0^u \nabla F(u-s, R_{u,s}^{1,2}) ds \right) \cdot \nabla b_{u+\varepsilon}^{c_0,\theta,\lambda}(X_u^1) du \right] \\ &\quad + \frac{\chi}{N-1} \sum_{j=2}^N \mathbb{E} \left[ \int_0^t \left( \int_0^u \nabla F(u-s, R_{u,s}^{1,2}) ds \right) \cdot D_u^{1,j} du \right]. \end{aligned}$$

*Proof.* Recalling (15), we have, for  $t > s > 0$ ,

$$R_{t,s}^{1,2} = R_{s,s}^{1,2} + X_t^1 - X_s^1 = R_{s,s}^{1,2} + \sqrt{2}(W_t^1 - W_s^1) + \chi \int_s^t \nabla b_{u+\varepsilon}^{c_0, \theta, \lambda}(X_u^1) du + \frac{\chi}{N-1} \sum_{j=2}^N \int_s^t D_u^{1,j} du.$$

Applying the Itô formula on  $[s, t]$ , we find

$$\begin{aligned} \mathbb{E}[F(t-s, R_{t,s}^{1,2})] &= \mathbb{E}[F(0, R_{s,s}^{1,2})] + \mathbb{E} \left[ \int_s^t (\partial_t F + \Delta F)(u-s, R_{u,s}^{1,2}) du \right] \\ &+ \chi \mathbb{E} \left[ \int_s^t \nabla F(u-s, R_{u,s}^{1,2}) \cdot \nabla b_{u+\varepsilon}^{c_0, \theta, \lambda}(X_u^1) du \right] + \frac{\chi}{N-1} \sum_{j=2}^N \mathbb{E} \left[ \int_s^t \nabla F(u-s, R_{u,s}^{1,2}) \cdot D_u^{1,j} du \right]. \end{aligned}$$

Integrating the formula in  $s$  on  $[0, t]$  and applying the Fubini theorem completes the proof.  $\square$

The next result shows that, in some loose sense, we can reduce to the parabolic-elliptic case. Gathering the estimates below, we see that  $\mathbb{E}[\int_0^t |D_u^{1,2}|^{2(\gamma-1)} du] \leq C \mathbb{E}[\int_0^t |R_{u,u}^{1,2}|^{2(1-\gamma)}] + C$ . Hence, roughly, we control the drift  $|D_u^{1,2}|$  by  $|R_{u,u}^{1,2}|^{-1}$ , which does not depend on the past of the particles and has the homogeneity of the drift of the parabolic-elliptic particle system.

**Proposition 11.** *Assume that  $(X_0^{i,N})_{i=1,\dots,N}$  is exchangeable and that  $c_0 \in L^p(\mathbb{R}^2)$  for some  $p > 2$ . Consider  $\chi > 0$ ,  $\theta > 0$ ,  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$  and  $\alpha > 0$  such that  $C_1(\alpha, \gamma) > 0$  and  $\chi C_2(\theta, \alpha, \gamma) < C_1(\alpha, \gamma)$ . Then for all  $\varepsilon \in (0, 1]$ , all  $N \geq 1$ , all  $t > 0$  and all  $j = 2, \dots, N$ ,*

$$|D_t^{1,j}| \leq \frac{\sqrt{\theta} C_0(\frac{4\alpha}{\theta}) \kappa(\frac{1}{2}, \gamma-1)}{4\pi} (S_t^{1,j})^{\frac{1}{2(\gamma-1)}} \quad \text{where} \quad S_t^{1,j} := \int_0^t \frac{1}{(t-s + \alpha |R_{t,s}^{1,j}|^2)^\gamma} ds, \quad (22)$$

and for any  $\eta > 0$ , there is a constant  $A_\eta = A_\eta(c_0, p, \chi, \theta, \gamma, \alpha)$  such that

$$\mathbb{E} \left[ \int_0^t S_u^{1,j} du \right] \leq \frac{(1+\eta)\alpha^{1-\gamma}}{C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma)} \mathbb{E} \left[ \int_0^t \frac{1}{|R_{u,u}^{1,j}|^{2(\gamma-1)}} du \right] + A_\eta t^{r_p} \quad (23)$$

for all  $t > 0$  and all  $j = 1, \dots, N$ , where  $r_p = 1 - (\gamma-1)(1 + \frac{2}{p}) > 0$  (because  $\gamma < \frac{2p+2}{p+2}$ ).

Observe that for any  $\chi > 0$ , any  $\theta > 0$  and any  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$ , the two conditions  $C_1(\alpha, \gamma) > 0$  and  $\chi C_2(\theta, \alpha, \gamma) < C_1(\alpha, \gamma)$  are satisfied for  $\alpha > 0$  small enough, because  $\lim_{\alpha \rightarrow 0} C_2(\theta, \alpha, \gamma) = 0$  and  $\lim_{\alpha \rightarrow 0} C_1(\alpha, \gamma) = \gamma - 1$ . The real restrictions will come later.

*Proof.* It of course suffices to treat the case where  $j = 2$ .

*Step 1.* We first prove (22). By Remark 9, it holds that

$$|D_t^{1,2}| \leq \frac{\sqrt{\theta} C_0(\frac{4\alpha}{\theta})}{4\pi} \int_0^t \frac{1}{(t-s + \varepsilon + \alpha |R_{t,s}^{1,2}|^2)^{\frac{3}{2}}} ds = \frac{\sqrt{\theta} C_0(\frac{4\alpha}{\theta})}{4\pi} \int_0^t \frac{1}{(s + \varepsilon + \alpha |R_{t,t-s}^{1,2}|^2)^{\frac{3}{2}}} ds.$$

Applying Lemma 7 with  $a = \frac{1}{2}$ ,  $b = \gamma - 1$  and  $f(s) = \alpha |R_{t,t-s}^{1,2}|^2$ , it comes

$$\begin{aligned} |D_t^{1,2}| &\leq \frac{\sqrt{\theta} C_0(\frac{4\alpha}{\theta}) \kappa(\frac{1}{2}, \gamma-1)}{4\pi} \left( \int_0^t \frac{1}{(s + \varepsilon + \alpha |R_{t,t-s}^{1,2}|^2)^\gamma} ds \right)^{\frac{1}{2(\gamma-1)}} \\ &= \frac{\sqrt{\theta} C_0(\frac{4\alpha}{\theta}) \kappa(\frac{1}{2}, \gamma-1)}{4\pi} (\bar{S}_t^{1,2,\varepsilon})^{\frac{1}{2(\gamma-1)}}, \end{aligned} \quad (24)$$

where for any  $\delta > 0$ , we have set

$$\bar{S}_t^{1,j,\delta} := \int_0^t \frac{1}{(t-s+\delta+\alpha|R_{t,s}^{1,j}|^2)^\gamma} ds \leq S_t^{1,j}. \quad (25)$$

*Step 2.* Let  $F(t, x) = -(t + \alpha|x|^2)^{1-\gamma}$ . We have

$$\nabla F(t, x) = 2\alpha(\gamma - 1)(t + \alpha|x|^2)^{-\gamma}x$$

and

$$\begin{aligned} (\partial_t F + \Delta F)(t, x) &= (\gamma - 1)(t + \alpha|x|^2)^{-\gamma-1}[(1 + 4\alpha)(t + \alpha|x|^2) - 4\alpha^2\gamma|x|^2] \\ &\geq (\gamma - 1)(1 + 4\alpha - 4\alpha\gamma)(t + \alpha|x|^2)^{-\gamma} \\ &= C_1(\alpha, \gamma)(t + \alpha|x|^2)^{-\gamma}. \end{aligned}$$

*Step 3.* We now prove (23). We apply Lemma 10 with the function  $F$  introduced in Step 2, or rather with the smooth function  $F(\delta + t, x)$ , for some  $\delta > 0$  that we will tend to 0. At first reed, one can take  $\delta = 0$ : The computations are then slightly informal since  $F$  is not smooth, but this makes disappear some terms that are actually not very important. We find

$$I_t^{\delta,1} = I_t^{\delta,2} + I_t^{\delta,3} + I_t^{\delta,4} + \frac{1}{N-1} \sum_{j=2}^N A_t^{\delta,1,j}, \quad (26)$$

where

$$I_t^{\delta,1} := \mathbb{E} \left[ \int_0^t F(\delta + t - s, R_{t,s}^{1,2}) ds \right] \leq 0, \quad (27)$$

$$I_t^{\delta,2} := \mathbb{E} \left[ \int_0^t F(\delta, R_{s,s}^{1,2}) ds \right] = -\mathbb{E} \left[ \int_0^t (\delta + \alpha|R_{s,s}^{1,2}|^2)^{1-\gamma} ds \right], \quad (28)$$

$$I_t^{\delta,3} := \mathbb{E} \left[ \int_0^t \int_0^u (\partial_t F + \Delta F)(\delta + u - s, R_{u,s}^{1,2}) ds du \right],$$

$$I_t^{\delta,4} := \chi \mathbb{E} \left[ \int_0^t \left( \int_0^u \nabla F(\delta + u - s, R_{u,s}^{1,2}) ds \right) \cdot \nabla b_{u+\varepsilon}^{c_0, \theta, \lambda}(X_u^1) du \right],$$

$$A_t^{\delta,1,j} := \chi \mathbb{E} \left[ \int_0^t \left( \int_0^u \nabla F(\delta + u - s, R_{u,s}^{1,2}) ds \right) \cdot D_u^{1,j} du \right].$$

Using Step 2, we find

$$I_t^{\delta,3} \geq C_1(\alpha, \gamma) \mathbb{E} \left[ \int_0^t \int_0^u (\delta + u - s + \alpha|R_{u,s}^{1,2}|^2)^{-\gamma} ds du \right] = C_1(\alpha, \gamma) \mathbb{E} \left[ \int_0^t \bar{S}_u^{1,2,\delta} du \right], \quad (29)$$

recall (25). Next, we write

$$A_t^{\delta,1,j} \geq -\chi \mathbb{E} \left[ \int_0^t |D_u^{1,j}| T_u du \right], \quad (30)$$

with  $T_u = \int_0^u |\nabla F(\delta + u - s, R_{u,s}^{1,2})| ds = \int_0^u |\nabla F(\delta + s, R_{u,u-s}^{1,2})| ds$ . By Step 2 again,

$$T_u = 2\alpha(\gamma - 1) \int_0^u \frac{|R_{u,u-s}^{1,2}| ds}{(s + \delta + \alpha|R_{u,u-s}^{1,2}|^2)^\gamma} \leq 2\sqrt{\alpha}(\gamma - 1) \int_0^u \frac{ds}{(s + \delta + \alpha|R_{u,u-s}^{1,2}|^2)^{\gamma-1/2}}.$$

Apply Lemma 7 with  $a = \gamma - 3/2$ ,  $b = \gamma - 1$  and  $f(s) = \delta + \alpha |R_{u,u-s}^{1,j}|^2$ . It comes

$$\begin{aligned} T_u &\leq 2\sqrt{\alpha}(\gamma - 1)\kappa\left(\gamma - \frac{3}{2}, \gamma - 1\right) \left(\int_0^u \frac{ds}{(s + \delta + \alpha |R_{u,u-s}^{1,2}|^2)^\gamma}\right)^{\frac{\gamma-3/2}{\gamma-1}} \\ &= 2\sqrt{\alpha}(\gamma - 1)\kappa\left(\gamma - \frac{3}{2}, \gamma - 1\right) \left(\bar{S}_u^{1,2,\delta}\right)^{\frac{\gamma-3/2}{\gamma-1}}. \end{aligned} \quad (31)$$

This last inequality, plugged together with (24) in (30), gives us

$$\begin{aligned} A_t^{\delta,1,j} &\geq -2\chi\sqrt{\alpha}(\gamma - 1) \frac{\sqrt{\theta}C_0\left(\frac{4\alpha}{\theta}\right)\kappa\left(\frac{1}{2}, \gamma - 1\right)\kappa\left(\gamma - \frac{3}{2}, \gamma - 1\right)}{4\pi} \mathbb{E}\left[\int_0^t \left(\bar{S}_u^{1,2,\delta}\right)^{\frac{\gamma-3/2}{\gamma-1}} \left(\bar{S}_u^{1,j,\varepsilon}\right)^{\frac{1}{2(\gamma-1)}} du\right] \\ &= -\chi C_2(\theta, \alpha, \gamma) \mathbb{E}\left[\int_0^t \left(\bar{S}_u^{1,2,\delta}\right)^{\frac{\gamma-3/2}{\gamma-1}} \left(\bar{S}_u^{1,j,\varepsilon}\right)^{\frac{1}{2(\gamma-1)}} du\right] \\ &\geq -\chi C_2(\theta, \alpha, \gamma) \mathbb{E}\left[\int_0^t \left(\bar{S}_u^{1,2,\delta}\right)^{\frac{\gamma-3/2}{\gamma-1}} \left(\bar{S}_u^{1,j,\delta}\right)^{\frac{1}{2(\gamma-1)}} du\right] \end{aligned}$$

if  $\delta \in (0, \varepsilon]$ . By the Hölder inequality (both for  $\mathbb{E}$  and  $\int_0^t$ ) with  $p = \frac{\gamma-1}{\gamma-3/2}$  and  $p' = 2(\gamma - 1)$ ,

$$\begin{aligned} A_t^{\delta,1,j} &\geq -\chi C_2(\theta, \alpha, \gamma) \left(\mathbb{E}\left[\int_0^t \bar{S}_u^{1,2,\delta} du\right]\right)^{\frac{\gamma-3/2}{\gamma-1}} \left(\mathbb{E}\left[\int_0^t \bar{S}_u^{1,j,\delta} du\right]\right)^{\frac{1}{2(\gamma-1)}} \\ &= -\chi C_2(\theta, \alpha, \gamma) \mathbb{E}\left[\int_0^t \bar{S}_u^{1,2,\delta} du\right] \end{aligned} \quad (32)$$

by exchangeability. Finally, recalling (7) and (31), for some constant  $A = A(c_0, p, \chi, \theta, \gamma, \alpha)$  that may change from line to line,

$$I_t^{\delta,4} \geq -A\mathbb{E}\left[\int_0^t T_u \frac{du}{u^{\frac{1}{2}+\frac{1}{p}}}\right] \geq -A\mathbb{E}\left[\int_0^t \left(\bar{S}_u^{1,2,\delta}\right)^{\frac{\gamma-3/2}{\gamma-1}} \frac{du}{u^{\frac{1}{2}+\frac{1}{p}}}\right].$$

Using the Young inequality with  $p = \frac{\gamma-1}{\gamma-3/2}$  and  $p' = 2(\gamma - 1)$ , we find that for any  $\zeta > 0$ , there is a constant  $A_\zeta = A_\zeta(c_0, p, \chi, \theta, \gamma, \alpha)$  such that

$$I_t^{\delta,4} \geq -\zeta\mathbb{E}\left[\int_0^t \bar{S}_u^{1,2,\delta} du\right] - A_\zeta \int_0^t \frac{du}{u^{(\gamma-1)(1+\frac{2}{p})}} du \geq -\zeta\mathbb{E}\left[\int_0^t \bar{S}_u^{1,2,\delta} du\right] - A_\zeta t^{r_p}, \quad (33)$$

recall that  $r_p = 1 - (\gamma - 1)(1 + \frac{2}{p}) > 0$ .

Plugging (27), (28), (29), (32), (33) into (26) we obtain, for any  $\delta \in (0, \varepsilon]$ , any  $\zeta > 0$ ,

$$\begin{aligned} (C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma) - \zeta) \mathbb{E}\left[\int_0^t \bar{S}_u^{1,2,\delta} du\right] &\leq \mathbb{E}\left[\int_0^t (\delta + \alpha |R_{s,s}^{1,2}|^2)^{1-\gamma} ds\right] + A_\zeta t^{r_p} \\ &\leq \alpha^{1-\gamma} \mathbb{E}\left[\int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds\right] + A_\zeta t^{r_p}. \end{aligned}$$

Letting  $\delta \rightarrow 0$ , we find that if  $\zeta \in (0, C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma))$ ,

$$\mathbb{E}\left[\int_0^t S_u^{1,2} du\right] \leq \frac{\alpha^{1-\gamma}}{C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma) - \zeta} \mathbb{E}\left[\int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds\right] + A_\zeta t^{r_p}.$$

The conclusion immediately follows.  $\square$

We are now ready to conclude this section. We use here some ideas of [13], which is natural since in Proposition 11, we loosely showed that the singularity here is of the same order as in the parabolic-elliptic case. In particular, we borrow the functions  $\phi$  and  $\psi$  below.

*Proof of Proposition 8.* We set  $\nu = 4 - 2\gamma \in (0, 1)$  and divide the proof in several steps.

*Step 1.* We introduce the function  $\phi(r) := (1 + r^{\frac{\nu}{2}})^{-1} r^{\frac{\nu}{2}}$  on  $\mathbb{R}_+$  and set  $\psi(x, y) = \phi(|x - y|^2)$  for  $x, y \in \mathbb{R}^2$ . As in [13, Proof of Proposition 5], it holds that

$$\nabla_x \psi(x, y) = \nu \frac{|x - y|^{\nu-2}}{(1 + |x - y|^\nu)^2} (x - y) \quad \text{and} \quad \Delta_x \psi(x, y) = \nu^2 \frac{|x - y|^{\nu-2}}{(1 + |x - y|^\nu)^2} \left(1 - 2 \frac{|x - y|^\nu}{1 + |x - y|^\nu}\right).$$

For any  $\eta > 0$ , there exists a constant  $L_\eta = L_\eta(\gamma) > 0$  (recall that  $\nu = 4 - 2\gamma$ ) such that

$$\Delta_x \psi(x, y) \geq (\nu^2 - \eta) |x - y|^{\nu-2} - L_\eta$$

for all  $x, y \in \mathbb{R}^2$ . To check this claim, it suffices to prove that the function

$$f_\eta(r) = \nu^2 \frac{r^{\nu-2}}{(1 + r^\nu)^2} \left(1 - 2 \frac{r^\nu}{1 + r^\nu}\right) - (\nu^2 - \eta) r^{\nu-2}$$

is bounded from below (possibly by a negative constant) on  $(0, \infty)$ . This follows from the facts that  $f_\eta$  is continuous on  $(0, \infty)$  and that  $\lim_{r \rightarrow 0} f_\eta(r) = +\infty$  and  $\lim_{r \rightarrow +\infty} f_\eta(r) = 0$ .

*Step 2.* Applying the Itô formula (as in the previous proof, one should first consider a smooth approximation of  $\psi$ , but we will not repeat this here), we obtain

$$\begin{aligned} \mathbb{E}[\psi(X_t^1, X_t^2)] &= \mathbb{E}[\psi(X_0^1, X_0^2)] + \mathbb{E}\left[\int_0^t [\Delta_x \psi(X_s^1, X_s^2) + \Delta_y \psi(X_s^1, X_s^2)] ds\right] \\ &+ \mathbb{E}\left[\int_0^t [\nabla_x \psi(X_s^1, X_s^2) \cdot \nabla b_{s+\varepsilon}^{c_0, \theta, \lambda}(X_s^1) + \nabla_y \psi(X_s^1, X_s^2) \cdot \nabla b_{s+\varepsilon}^{c_0, \theta, \lambda}(X_s^2)] ds\right] \\ &+ \frac{\chi}{N-1} \sum_{j \neq 1} \mathbb{E}\left[\int_0^t \nabla_x \psi(X_s^1, X_s^2) \cdot D_s^{1,j} ds\right] + \frac{\chi}{N-1} \sum_{j \neq 2} \mathbb{E}\left[\int_0^t \nabla_y \psi(X_s^1, X_s^2) \cdot D_s^{2,j} ds\right]. \end{aligned}$$

By symmetry of  $\psi$  and exchangeability of the particle system, we have

$$J_t^1 = J_0^1 + 2J_t^2 + 2J_t^3 + \frac{2}{N-1} \sum_{j=2}^N B_t^{1,j}, \quad (34)$$

where

$$J_t^1 := \mathbb{E}[\psi(X_t^1, X_t^2)] \leq 1, \quad J_0^1 := \mathbb{E}[\psi(X_0^1, X_0^2)] \geq 0, \quad (35)$$

and where

$$\begin{aligned} J_t^2 &:= \mathbb{E}\left[\int_0^t \Delta_x \psi(X_s^1, X_s^2) ds\right], \\ J_t^3 &:= \chi \mathbb{E}\left[\int_0^t \nabla_x \psi(X_s^1, X_s^2) \cdot \nabla b_{s+\varepsilon}^{c_0, \theta, \lambda}(X_s^1) ds\right], \\ B_t^{1,j} &:= \chi \mathbb{E}\left[\int_0^t \nabla_x \psi(X_s^1, X_s^2) \cdot D_s^{1,j} ds\right]. \end{aligned}$$

By Step 1, it holds that for any  $\eta > 0$ ,

$$J_t^2 \geq (\nu^2 - \eta) \mathbb{E}\left[\int_0^t |R_{s,s}^{1,2}|^{\nu-2} ds\right] - L_\eta t = ((4 - 2\gamma)^2 - \eta) \mathbb{E}\left[\int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds\right] - L_\eta t. \quad (36)$$

Next, recalling (7) and using that  $|\nabla_x \psi(x, y)| \leq \nu |x - y|^{\nu-1} = (4 - 2\gamma) |x - y|^{3-2\gamma}$ ,

$$|J_t^3| \leq L' \mathbb{E} \left[ \int_0^t |R_{s,s}^{1,2}|^{3-2\gamma} \frac{ds}{s^{\frac{1}{2} + \frac{1}{p}}} \right]$$

for some constant  $L' = L'(c_0, p, \chi, \theta, \gamma, \alpha)$ . By the Young inequality with  $p = \frac{2(\gamma-1)}{2\gamma-3}$  and  $p' = 2(\gamma-1)$ , we see that for all  $\zeta > 0$ , there is a constant  $L'_\zeta = L'_\zeta(c_0, p, \chi, \theta, \gamma, \alpha)$  such that

$$|J_t^3| \leq \zeta \mathbb{E} \left[ \int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds \right] + L'_\zeta \int_0^t \frac{ds}{s^{(\gamma-1)(1+\frac{2}{p})}} = \zeta \mathbb{E} \left[ \int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds \right] + L'_\zeta t^{r_p}, \quad (37)$$

recall that  $r_p = 1 - (\gamma-1)(1+\frac{2}{p}) > 0$ . Finally, by (22) and since  $|\nabla_x \psi(x, y)| \leq (4-2\gamma)|x-y|^{3-2\gamma}$ ,

$$|B_t^{1,j}| \leq \chi(4-2\gamma) \frac{\sqrt{\theta} C_0(\frac{4\alpha}{\theta}) \kappa(\frac{1}{2}, \gamma-1)}{4\pi} \mathbb{E} \left[ \int_0^t (S_s^{1,j})^{\frac{1}{2(\gamma-1)}} |R_{s,s}^{1,2}|^{3-2\gamma} ds \right].$$

By the Hölder inequality (both for  $\mathbb{E}$  and  $\int_0^t$ ) with  $p = 2(\gamma-1)$  and  $p' = \frac{2(\gamma-1)}{2\gamma-3}$ ,

$$\begin{aligned} |B_t^{1,j}| &\leq \chi \frac{(4-2\gamma)\sqrt{\theta} C_0(\frac{4\alpha}{\theta}) \kappa(\frac{1}{2}, \gamma-1)}{4\pi} \left( \mathbb{E} \left[ \int_0^t S_s^{1,j} ds \right] \right)^{\frac{1}{2(\gamma-1)}} \left( \mathbb{E} \left[ \int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds \right] \right)^{\frac{2\gamma-3}{2(\gamma-1)}} \\ &\leq \chi \frac{(4-2\gamma)\sqrt{\theta} C_0(\frac{4\alpha}{\theta}) \kappa(\frac{1}{2}, \gamma-1)}{4\pi} \left( \frac{(1+\eta)\alpha^{1-\gamma}}{C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma)} \mathbb{E} \left[ \int_0^t |R_{u,u}^{1,2}|^{2(1-\gamma)} du \right] + A_\eta t^{r_p} \right)^{\frac{1}{2(\gamma-1)}} \\ &\quad \times \left( \mathbb{E} \left[ \int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds \right] \right)^{\frac{2\gamma-3}{2(\gamma-1)}} \end{aligned}$$

for any  $\eta > 0$ , by (23) (with  $A_\eta = A_\eta(c_0, p, \chi, \theta, \gamma, \alpha)$ ). Since  $\frac{1}{2(\gamma-1)} < 1$ , allowing  $A_\eta$  to change from line to line,

$$\begin{aligned} |B_t^{1,j}| &\leq \chi(1+\eta)^{\frac{1}{2(\gamma-1)}} \frac{(4-2\gamma)\sqrt{\theta} C_0(\frac{4\alpha}{\theta}) \kappa(\frac{1}{2}, \gamma-1)}{4\pi \sqrt{\alpha} [C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma)]^{\frac{1}{2(\gamma-1)}}} \mathbb{E} \left[ \int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds \right] \\ &\quad + A_\eta t^{\frac{r_p}{2(\gamma-1)}} \left( \mathbb{E} \left[ \int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds \right] \right)^{\frac{2\gamma-3}{2(\gamma-1)}}. \end{aligned}$$

We easily deduce, again by the Young inequality with  $p = 2(\gamma-1)$  and  $p' = \frac{2(\gamma-1)}{2\gamma-3}$ , that for all  $\zeta > 0$ , there is a constant  $L''_\zeta = L''_\zeta(c_0, \chi, \theta, \gamma, \alpha)$  such that

$$|B_t^{1,j}| \leq \left( \chi \frac{(4-2\gamma)\sqrt{\theta} C_0(\frac{4\alpha}{\theta}) \kappa(\frac{1}{2}, \gamma-1)}{4\pi \sqrt{\alpha} [C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma)]^{\frac{1}{2(\gamma-1)}}} + \zeta \right) \mathbb{E} \left[ \int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds \right] + L''_\zeta t^{r_p}. \quad (38)$$

Plugging (35), (36), (37) and (38) in (34), we have proved that for all  $\eta > 0$ , all  $\zeta > 0$ ,

$$\begin{aligned} 1 &\geq \left( 2(4-2\gamma)^2 - 2\chi \frac{(4-2\gamma)\sqrt{\theta} C_0(\frac{4\alpha}{\theta}) \kappa(\frac{1}{2}, \gamma-1)}{4\pi \sqrt{\alpha} [C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma)]^{\frac{1}{2(\gamma-1)}}} - 2\eta - 4\zeta \right) \mathbb{E} \left[ \int_0^t |R_{s,s}^{1,2}|^{2(1-\gamma)} ds \right] \\ &\quad - 2L_\eta t - 2(L'_\zeta + L''_\zeta) t^{r_p}. \end{aligned}$$

By assumption, it is possible to find  $\eta > 0$  and  $\zeta > 0$  small enough so that the constant in front of the first term of the right hand side is positive. Since this constant does not depend on  $N$  nor on  $\varepsilon$ , which is also the case of  $L_\eta$ ,  $L'_\zeta$  and  $L''_\zeta$ , this ends the proof of (18).

Next, recalling that  $S_t^{1,j}$  was defined in the statement of Proposition 11, we have

$$\mathbb{E} \left[ \int_0^t \int_0^s \frac{1}{(s-u+|X_s^1 - X_u^2|)^\gamma} du ds \right] \leq \mathbb{E} \left[ \int_0^t \int_0^s \frac{1}{(s-u+\alpha|X_s^1 - X_u^2|)^\gamma} du ds \right] = \mathbb{E} \left[ \int_0^t S_s^{1,2} ds \right],$$

because  $\alpha \in (0, 1)$  (since  $C_1(\alpha, \gamma) > 0$  implies that  $\alpha < \frac{1}{4(\gamma-1)} < \frac{1}{2}$ ). Hence (19) directly follows from (23) (with e.g.  $\eta = 1$ ) and (18).

By Remark 9 with  $\alpha = 1$ , (20) immediately follows from (19) and

$$\begin{aligned} & \mathbb{E} \left[ \int_0^t \left| \int_0^s \nabla K_{s-u}^{\theta, \lambda}(X_s^1 - X_u^2) du \right|^{2(\gamma-1)} ds \right] \\ & \leq \left[ \frac{\sqrt{\theta} C_0(\frac{4}{\theta})}{4\pi} \right]^{2(\gamma-1)} \mathbb{E} \left[ \int_0^t \left( \int_0^s \frac{1}{(s-u+|R_{s,u}^{1,2}|)^{\frac{3}{2}}} du \right)^{2(\gamma-1)} ds \right] \\ & \leq \left[ \frac{\sqrt{\theta} C_0(\frac{4}{\theta})}{4\pi} \kappa\left(\frac{1}{2}, \gamma-1\right) \right]^{2(\gamma-1)} \mathbb{E} \left[ \int_0^t \int_0^s \frac{1}{(s-u+|R_{s,u}^{1,2}|)^\gamma} du ds \right] \end{aligned}$$

by Lemma 7 with  $a = \frac{1}{2}$  and  $b = \gamma - 1$ . Hence (21) also follows from (19).  $\square$

### 3 Tightness

Here we prove the tightness in  $N \geq 2$  and  $\varepsilon \in (0, 1]$  of the smoothed particle system. We closely follow [11, Lemma 11], although some additional moment conditions were assumed there.

**Lemma 12.** *Consider some nonnegative  $c_0 \in L^p(\mathbb{R}^2)$  for some  $p > 2$ . Let  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$ ,  $\alpha > 0$ ,  $\chi > 0$  and  $\theta > 0$  satisfy the conditions of Proposition 8. For each  $N \geq 2$ , each  $\varepsilon \in (0, 1]$ , consider the unique solution  $(X_t^{i,N,\varepsilon})_{t \in [0, \infty), i=1, \dots, N}$  to (15) with some exchangeable initial condition  $(X_0^{i,N})_{i=1, \dots, N}$ .*

(i) *For  $N \geq 2$  fixed, the family  $((X_t^{1,N,\varepsilon})_{t \geq 0}, \varepsilon \in (0, 1])$  is tight in  $C([0, \infty), \mathbb{R}^2)$ .*

(ii) *If  $(X_0^{1,N})_{N \geq 2}$  is tight in  $\mathbb{R}^2$ , then  $((X_t^{1,N,\varepsilon})_{t \geq 0}, N \geq 2, \varepsilon \in (0, 1])$  is tight in  $C([0, \infty), \mathbb{R}^2)$ .*

*Proof.* We start with (ii). The space  $C([0, \infty), \mathbb{R}^2)$  being endowed with the uniform convergence on compact time intervals, we only have to check that  $((X_t^{1,N,\varepsilon})_{t \in [0, T]}, N \geq 2, \varepsilon \in (0, 1])$  is tight in  $C([0, T], \mathbb{R}^2)$  for any  $T > 0$ . By definition,  $X_t^{1,N,\varepsilon} = X_0^{1,N} + \sqrt{2}W_t^1 + \chi G_t^{1,N,\varepsilon} + \chi \Gamma_t^{1,N,\varepsilon}$ , where

$$\begin{aligned} G_t^{1,N,\varepsilon} & := \int_0^t \nabla b_{s+\varepsilon}^{c_0, \theta, \lambda}(X_s^{1,N,\varepsilon}) ds, \\ \Gamma_t^{1,N,\varepsilon} & := \frac{1}{N-1} \sum_{j=2}^N \int_0^t D_s^{1,j,N,\varepsilon} ds, \quad \text{with } D_s^{1,j,N,\varepsilon} := \int_0^s H_{s-u}^{\theta, \lambda, \varepsilon}(X_s^{1,N,\varepsilon} - X_u^{j,N,\varepsilon}) du. \end{aligned}$$

The family  $(X_0^{1,N})_{N \geq 2}$  is tight by hypothesis and  $(W_t^1)_{t \in [0, T]}$  does not depend on  $N \geq 2$ . The family  $((G_t^{1,N,\varepsilon})_{t \in [0, T]}, N \geq 2, \varepsilon \in (0, 1])$  is tight because by (7), it a.s. takes values in the set  $\mathcal{K}$  of functions  $x : [0, T] \mapsto \mathbb{R}^2$  such that  $x(0) = 0$  and for all  $0 \leq s < t \leq T$ ,  $|x(t) - x(s)| \leq A|t - s|^{\frac{1}{2} - \frac{1}{p}}$  (for some constant  $A = A(c_0, p, T)$ ), and because  $\mathcal{K}$  is compact in  $C([0, T], \mathbb{R}^2)$  by Ascoli's theorem.

It remains to prove that the family  $((\Gamma_t^{1,N,\varepsilon})_{t \in [0, T]}, N \geq 2, \varepsilon \in (0, 1])$  is tight in  $C([0, T], \mathbb{R}^2)$ . Let  $0 \leq s < t \leq T$ . We use Hölder's inequality with  $p = \frac{2(\gamma-1)}{2\gamma-3}$  and  $p' = 2(\gamma-1)$  to get

$$\left| \Gamma_t^{1,N,\varepsilon} - \Gamma_s^{1,N,\varepsilon} \right| \leq \frac{\chi}{N-1} \sum_{j=2}^N \int_s^t |D_u^{1,j,N,\varepsilon}| du \leq |t-s|^{\frac{2\gamma-3}{2(\gamma-1)}} \frac{\chi}{N-1} \sum_{j=2}^N \left( \int_s^t |D_u^{1,j,N,\varepsilon}|^{2(\gamma-1)} du \right)^{\frac{1}{2(\gamma-1)}}.$$



Setting  $\beta = \frac{2\gamma-3}{2(\gamma-1)} > 0$  and using that  $x^{\frac{1}{2(\gamma-1)}} \leq 1+x$  (because  $2(\gamma-1) > 1$ ), we conclude that

$$\left| \Gamma_t^{1,N,\varepsilon} - \Gamma_s^{1,N,\varepsilon} \right| \leq Z_T^{N,\varepsilon} |t-s|^\beta, \quad \text{where} \quad Z_T^{N,\varepsilon} := \frac{\chi}{N-1} \sum_{j=2}^N \left[ 1 + \int_0^T |D_u^{1,j,N,\varepsilon}|^{2(\gamma-1)} du \right].$$

Since  $|H_s^{\theta,\lambda,\varepsilon}(x)| \leq |\nabla K_s^{\theta,\lambda}(x)|$ , we have by (21) and exchangeability

$$C_T := \sup_{\varepsilon \in (0,1], N \geq 2} \mathbb{E}[Z_T^{N,\varepsilon}] \leq \chi \sup_{\varepsilon \in (0,1], N \geq 2} \mathbb{E} \left[ 1 + \int_0^T \left( \int_0^u |\nabla K_{u-s}^{\theta,\lambda}(X_s^{1,N,\varepsilon} - X_u^{j,N,\varepsilon})| ds \right)^{2(\gamma-1)} du \right] < \infty.$$

Now, let  $\mathcal{K}'_M$  the set of functions  $x : [0, T] \mapsto \mathbb{R}^2$  such that  $x(0) = 0$  and for all  $0 \leq s < t \leq T$ ,  $|x(t) - x(s)| \leq M|t-s|^\beta$ . For all  $\varepsilon \in (0, 1]$ , all  $N \geq 2$  and all  $M > 0$ ,

$$\mathbb{P}((\Gamma_t^{1,N,\varepsilon})_{t \in [0,T]} \notin \mathcal{K}'_M) \leq \mathbb{P}(Z_T^{N,\varepsilon} > M) \leq \frac{C_T}{M}.$$

Since  $\mathcal{K}'_M$  is compact in  $C([0, T], \mathbb{R}^2)$  by Ascoli's theorem, the proof of (ii) is complete. The proof of (i) is the same, but we do not need the tightness of the family  $(X_0^{1,N})_{N \geq 2}$  since  $N \geq 2$  is fixed.  $\square$

## 4 Existence of the particle system

Here we show the existence of the particle system without cutoff. We follow the ideas of [11, Theorem 5] and combine them with our results from Section 2.

**Proposition 13.** *Consider some nonnegative  $c_0 \in L^p(\mathbb{R}^2)$  for some  $p > 2$ . Let  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$ ,  $\alpha > 0$ ,  $\chi > 0$  and  $\theta > 0$  satisfy the conditions of Proposition 8. Fix  $N \geq 2$  and consider some exchangeable initial condition  $(X_0^{i,N})_{i=1,\dots,N}$ . There exists a (weak) solution  $(X_t^{i,N})_{t \geq 0, i=1,\dots,N}$  to (3). Moreover, the family  $((X_t^{i,N})_{t \geq 0, i=1,\dots,N})$  is exchangeable, and for all  $t > 0$ ,*

$$\sup_{N \geq 2} \mathbb{E} \left[ \int_0^t \frac{1}{|X_s^{1,N} - X_s^{2,N}|^{2(\gamma-1)}} ds \right] < \infty, \quad (39)$$

$$\sup_{N \geq 2} \mathbb{E} \left[ \int_0^t \int_0^s \frac{1}{(s-u + |X_s^{1,N} - X_u^{2,N}|^2)^\gamma} duds \right] < \infty, \quad (40)$$

$$\sup_{N \geq 2} \mathbb{E} \left[ \int_0^t \int_0^s |\nabla K_{s-u}^{\theta,\lambda}(X_s^{1,N} - X_u^{2,N})|^{\frac{2\gamma}{3}} duds \right] < \infty, \quad (41)$$

$$\sup_{N \geq 2} \mathbb{E} \left[ \int_0^t \left( \int_0^s |\nabla K_{s-u}^{\theta,\lambda}(X_s^{1,N} - X_u^{2,N})| du \right)^{2(\gamma-1)} ds \right] < \infty. \quad (42)$$

*Proof.* The only difference w.r.t. the proof of [11, Theorem 5] lies in the last step.

*Step 1.* For each  $\varepsilon \in (0, 1]$ , let  $(X_t^{i,N,\varepsilon})_{t \in [0,\infty), i=1,\dots,N}$  solve to (15). By Lemma 12-(i), we know that the family  $((X_t^{1,N,\varepsilon})_{t \geq 0}, \varepsilon \in (0, 1])$  is tight in  $C([0, \infty), \mathbb{R}^2)$ . By exchangeability, the family  $((X_t^{1,N,\varepsilon}, \dots, X_t^{N,N,\varepsilon})_{t \geq 0}, \varepsilon \in (0, 1])$  is tight in  $C([0, \infty), (\mathbb{R}^2)^N)$  and consequently, the family  $((X_t^{1,N,\varepsilon}, W_t^1), \dots, (X_t^{N,N,\varepsilon}, W_t^N))_{t \geq 0}, \varepsilon \in (0, 1])$  is tight in  $C([0, \infty), (\mathbb{R}^2 \times \mathbb{R}^2)^N)$ . Hence, there exists a decreasing sequence  $\varepsilon_k \rightarrow 0$  such that  $((X_t^{1,N,\varepsilon_k}, W_t^1), \dots, (X_t^{N,N,\varepsilon_k}, W_t^N))_{t \geq 0}$  converges in law in  $C([0, \infty), (\mathbb{R}^2 \times \mathbb{R}^2)^N)$  as  $k \rightarrow \infty$ . Applying the Skorokhod representation theorem, we can find, for each  $k \geq 1$ , a solution  $(\tilde{X}_t^{1,N,\varepsilon_k}, \dots, \tilde{X}_t^{N,N,\varepsilon_k})_{t \geq 0}$  to (15), associated to some Brownian motions  $(\tilde{W}_t^{1,N,\varepsilon_k}, \dots, \tilde{W}_t^{N,N,\varepsilon_k})_{t \geq 0}$ , in such a way that  $((\tilde{X}_t^{1,N,\varepsilon_k}, \tilde{W}_t^{1,N,\varepsilon_k}), \dots, (\tilde{X}_t^{N,N,\varepsilon_k}, \tilde{W}_t^{N,N,\varepsilon_k}))_{t \geq 0}$  a.s. goes to some limit  $((X_t^{1,N}, W_t^1), \dots, (X_t^{N,N}, W_t^N))_{t \geq 0}$ , as  $k \rightarrow \infty$ , in  $C([0, \infty), (\mathbb{R}^2 \times \mathbb{R}^2)^N)$ . Of

course,  $((X_t^{i,N})_{t \geq 0}, i = 1, \dots, N)$  is exchangeable and we deduce (39)-(42) from (18)-(21) and the Fatou Lemma. Observe that (10) follows from (41) (by exchangeability) since  $\gamma > 3/2$ .

*Step 2.* We introduce  $\mathcal{F}_t = \sigma((X_s^{i,N}, W_s^i)_{i=1, \dots, N, s \in [0, t]})$ . Of course,  $(X_t^{i,N})_{i=1, \dots, N, t \geq 0}$  is  $(\mathcal{F}_t)_{t \geq 0}$ -adapted. Exactly as in Step 2 of the proof of [11, Theorem 5], one can show that  $(W_t^i)_{i=1, \dots, N, t \geq 0}$  is a  $2N$ -dimensional  $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion.

*Step 3.* It only remains to check that for each  $i \in \{1, \dots, N\}$ , each  $t \geq 0$ ,

$$X_t^{i,N} = X_0^{i,N} + \sqrt{2}W_t^i + \chi Y_t^{i,N} + \frac{\chi}{N-1} \sum_{j \neq i} Z_t^{i,j,N},$$

where

$$Y_t^{i,N} = \int_0^t \nabla b_s^{c_0, \theta, \lambda}(X_s^{i,N}) ds \quad \text{and} \quad Z_t^{i,j,N} = \int_0^t \int_0^s \nabla K_{s-u}^{\theta, \lambda}(X_s^{i,N} - X_u^{j,N}) duds.$$

We start from  $\tilde{X}_t^{i,N, \varepsilon_k} = \tilde{X}_0^{i,N, \varepsilon_k} + \sqrt{2}\tilde{W}_t^{i,N, \varepsilon_k} + \chi Y_t^{i,N, \varepsilon_k} + \frac{\chi}{N-1} \sum_{j \neq i} Z_t^{i,j,N, \varepsilon_k}$ , where

$$Y_t^{i,j,N, \varepsilon} = \int_0^t \nabla b_{s+\varepsilon}^{c_0, \theta, \lambda}(\tilde{X}_s^{i,N, \varepsilon}) ds \quad \text{and} \quad Z_t^{i,j,N, \varepsilon} = \int_0^t \int_0^s H_{s-u}^{\theta, \lambda, \varepsilon}(\tilde{X}_s^{i,N, \varepsilon} - \tilde{X}_u^{j,N, \varepsilon}) du$$

and pass to the limit as  $k \rightarrow \infty$ , e.g. in probability. Of course,  $(\tilde{X}_t^{i,N, \varepsilon_k}, \tilde{X}_0^{i,N, \varepsilon_k}, \tilde{W}_t^{i,N, \varepsilon_k})$  a.s. tends to  $(X_t^{i,N}, X_0^{i,N}, W_t^i)$  by construction, and  $Y_t^{i,N, \varepsilon_k}$  a.s. tends to  $Y_t^{i,N}$  by dominated convergence, recalling (7) and noting that a.s.,  $b_{s+\varepsilon}^{c_0, \theta, \lambda}(\tilde{X}_s^{i,N, \varepsilon_k})$  tends to  $b_s^{c_0, \theta, \lambda}(X_s^{i,N})$  for all  $s > 0$  (because  $(s, x) \mapsto \nabla b_s^{c_0, \theta, \lambda}(x)$  is continuous on  $(0, \infty) \times \mathbb{R}^2$ ).

It remains to show that  $Z_t^{i,j,N, \varepsilon_k} \rightarrow Z_t^{i,j,N}$  in probability as  $k \rightarrow \infty$ . We fix  $\eta > 0$  and decompose

$$\begin{aligned} |Z_t^{i,j,N, \varepsilon_k} - Z_t^{i,j,N}| &\leq \int_0^t \int_0^s |H_{s-u}^{\theta, \lambda, \varepsilon_k}(\tilde{X}_s^{i,N, \varepsilon_k} - \tilde{X}_u^{j,N, \varepsilon_k}) - H_{s-u}^{\theta, \lambda, \eta}(\tilde{X}_s^{i,N, \varepsilon_k} - \tilde{X}_u^{j,N, \varepsilon_k})| duds \\ &\quad + \int_0^t \int_0^s |H_{s-u}^{\theta, \lambda, \eta}(\tilde{X}_s^{i,N, \varepsilon_k} - \tilde{X}_u^{j,N, \varepsilon_k}) - H_{s-u}^{\theta, \lambda, \eta}(X_s^{i,N} - X_u^{j,N})| duds \\ &\quad + \int_0^t \int_0^s |H_{s-u}^{\theta, \lambda, \eta}(X_s^{i,N} - \tilde{X}_u^{j,N}) - \nabla K_{s-u}^{\theta, \lambda}(X_s^{i,N} - X_u^{j,N})| duds \\ &=: I_{1,k,\eta,t}^{i,j} + I_{2,k,\eta,t}^{i,j} + I_{3,\eta,t}^{i,j}. \end{aligned}$$

First, for each  $\eta > 0$ ,  $\lim_{k \rightarrow \infty} I_{2,k,\eta,t}^{i,j} = 0$  a.s., by dominated convergence, because  $x \mapsto H_s^{\theta, \lambda, \eta}(x)$  is continuous and uniformly bounded and since  $(\tilde{X}_s^{i,N, \varepsilon_k}, \tilde{X}_u^{j,N, \varepsilon_k}) \rightarrow (X_s^{i,N}, X_u^{j,N})$  a.s.

We now check that

$$\lim_{\eta \rightarrow 0} \limsup_{k \rightarrow \infty} \mathbb{E}[I_{1,k,\eta,t}^{i,j} + I_{3,\eta,t}^{i,j}] = 0, \quad (43)$$

and this will complete the proof. Recalling that  $H_s^{\theta, \lambda, \varepsilon} = \frac{s^2}{(s+\varepsilon)^2} \nabla K_s^{\theta, \lambda, \varepsilon}$ , one verifies that, if  $k$  is large enough so that  $\varepsilon_k \in (0, \eta]$ , since  $\frac{s^2}{(s+\eta)^2} \leq \frac{s^2}{(s+\varepsilon_k)^2} \leq 1$ ,

$$I_{1,k,\eta,t}^{i,j} + I_{3,\eta,t}^{i,j} \leq \int_0^t \int_0^s \left(1 - \frac{(s-u)^2}{(s-u+\eta)^2}\right) \left(|\nabla K_{s-u}^{\theta, \lambda}(\tilde{X}_s^{i,N, \varepsilon_k} - \tilde{X}_u^{j,N, \varepsilon_k})| + |\nabla K_{s-u}^{\theta, \lambda}(X_s^{i,N} - X_u^{j,N})|\right) duds.$$

Applying Hölder's inequality with  $p = \frac{2\gamma}{2\gamma-3}$  and  $p' = \frac{2\gamma}{3}$ , we find

$$\begin{aligned} I_{1,k,\eta,t}^{i,j} + I_{3,\eta,t}^{i,j} &\leq \left( \int_0^t \int_0^s \left(1 - \frac{(s-u)^2}{(s-u+\eta)^2}\right)^{\frac{2\gamma}{2\gamma-3}} duds \right)^{\frac{2\gamma-3}{2\gamma}} \\ &\quad \times \left( \int_0^t \int_0^s \left(|\nabla K_{s-u}^{\theta, \lambda}(\tilde{X}_s^{i,N, \varepsilon_k} - \tilde{X}_u^{j,N, \varepsilon_k})| + |\nabla K_{s-u}^{\theta, \lambda}(X_s^{i,N} - X_u^{j,N})|\right)^{\frac{2\gamma}{3}} duds \right)^{\frac{3}{2\gamma}}. \end{aligned}$$

By (20) and (41) (and since  $\frac{3}{2\gamma} < 1$ ), we deduce that for some constant  $C > 0$ ,

$$\limsup_{k \rightarrow \infty} \mathbb{E}[I_{1,k,\eta,t}^{i,j} + I_{3,\eta,t}^{i,j}] \leq C \left( \int_0^t \int_0^s \left( 1 - \frac{(s-u)^2}{(s-u+\eta)^2} \right)^{\frac{2\gamma}{2\gamma-3}} \text{d}u \text{d}s \right)^{\frac{2\gamma-3}{2\gamma}},$$

which tends to 0 as  $\eta \rightarrow 0$  by dominated convergence. This proves (43).  $\square$

## 5 Convergence

We prove that the empirical measure of the particle system converges, up to extraction of a subsequence, to a solution of the martingale problem. We recall that  $\mathcal{P}(\mathbb{R}^2)$  and  $\mathcal{P}(C([0, \infty), \mathbb{R}^2))$  are endowed with their weak convergence topologies.

**Theorem 14.** *Consider some nonnegative  $c_0 \in L^p(\mathbb{R}^2)$  for some  $p > 2$ . Let  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$ ,  $\alpha > 0$ ,  $\chi > 0$  and  $\theta > 0$  satisfy the conditions of Proposition 8. Consider, for each  $N \geq 2$ , the particle system  $(X_t^{i,N})_{t \geq 0, i=1, \dots, N}$  built in Proposition 13, as well as  $\mu^N = N^{-1} \sum_1^N \delta_{(X_t^{i,N})_{t \geq 0}}$ , which a.s. belongs to  $\mathcal{P}(C([0, \infty), \mathbb{R}^2))$ . For each  $t \geq 0$ , we set  $\mu_t^N = N^{-1} \sum_1^N \delta_{X_t^{i,N}}$ , which a.s. belongs to  $\mathcal{P}(\mathbb{R}^2)$ . We assume that  $\mu_0^N$  converges in probability, as  $N \rightarrow \infty$ , to some  $\rho_0 \in \mathcal{P}(\mathbb{R}^2)$ .*

*The family  $(\mu^N, N \geq 2)$  is tight in  $\mathcal{P}(C([0, \infty), \mathbb{R}^d))$  and any (possibly random) limit point  $\mu$  of  $(\mu^N)_{N \geq 2}$  a.s. solves (MP) with initial law  $\rho_0$ . Moreover, for  $(\mu_t)_{t \geq 0}$  its family of time marginals, for all  $t \geq 0$ ,*

$$\mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} \int_0^s \int_{\mathbb{R}^2} (|K_{s-u}^{\theta, \lambda}(x-y)|^\gamma + |\nabla K_{s-u}^{\theta, \lambda}(x-y)|^{\frac{2\gamma}{3}}) \mu_u(\text{d}y) \text{d}u \mu_s(\text{d}x) \text{d}s \right] < \infty, \quad (44)$$

$$\mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|^{2(\gamma-1)}} \mu_s(\text{d}y) \mu_s(\text{d}x) \text{d}s \right] < \infty. \quad (45)$$

*Proof.* Again, we follow closely the proof of [11, Theorem 6]. During this proof, we use the shortened notation  $C = C([0, \infty), \mathbb{R}^2)$ .

*Step 1.* For each  $N \geq 2$ ,  $(X_t^{i,N})_{t \geq 0, i=1, \dots, N}$  has been built as a limit point of  $(X_t^{i,N,\varepsilon})_{t \in [0, \infty), i=1, \dots, N}$  as  $\varepsilon \rightarrow 0$ . By Lemma 12-(ii), the family  $((X_t^{1,N})_{t \geq 0}, N \geq 2)$  is thus tight in  $C$ . Since the system is exchangeable, this implies, see Sznitman [26, Proposition 2.2], that the family  $(\mu^N, N \geq 2)$  is tight in  $\mathcal{P}(C)$ . We now consider a (non relabelled) subsequence of  $\mu^N$  that converges in law to some  $\mu$  as  $N \rightarrow \infty$ . We denote by  $(\mu_t)_{t \geq 0}$  its family of time-marginals. Since  $\mu_0^N$  goes to  $\rho_0$  by assumption, we have  $\mu_0 = \rho_0$  a.s.

Moreover, since  $\mu^N$  converges in law to  $\mu$ , it also holds true that  $\mu^N \otimes \mu^N$  and  $\mu^N \odot \mu^N$  both converge in law to  $\mu \otimes \mu$  in  $\mathcal{P}(C \times C)$ , where we have set

$$\mu^N \odot \mu^N = \frac{1}{N(N-1)} \sum_{i \neq j} \delta_{((X_t^{i,N})_{t \geq 0}, (X_t^{j,N})_{t \geq 0})}.$$

Hence we deduce from the Fatou lemma that for all  $t \geq 0$ ,

$$\begin{aligned} \mathbb{E} \left[ \int_0^t \int_0^s \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\mu_u(\text{d}y) \mu_s(\text{d}x) \text{d}u \text{d}s}{(|s-u+(x-y)|^2)^\gamma} \right] &= \mathbb{E} \left[ \int_C \int_C \int_0^t \int_0^s \frac{\text{d}u \text{d}s}{(|s-u+(x_s-y_u)|^2)^\gamma} (\mu \otimes \mu)(\text{d}x, \text{d}y) \right] \\ &\leq \liminf_N \mathbb{E} \left[ \int_C \int_C \int_0^t \int_0^s \frac{\text{d}u \text{d}s}{(|s-u+(x_s-y_u)|^2)^\gamma} (\mu^N \odot \mu^N)(\text{d}x, \text{d}y) \right] \\ &= \liminf_N \frac{1}{N(N-1)} \sum_{i \neq j} \mathbb{E} \left[ \int_0^t \int_0^s \frac{\text{d}u \text{d}s}{(|s-u+(X_s^{i,N} - X_u^{j,N})|^2)^\gamma} \right], \end{aligned}$$

which is finite by (40) and exchangeability. Moreover, there is a constant  $A = A(\theta) > 0$  such that

$$K_s^{\theta,\lambda}(x) \leq \frac{1}{4\pi s} e^{-\frac{\theta}{4s}|x|^2} \leq \frac{A}{s + |x|^2} \quad \text{and} \quad |\nabla K_s^{\theta,\lambda}(x)| \leq \frac{A}{(s + |x|^2)^{3/2}}.$$

The first estimate easily follows from the fact  $z \mapsto (1+z)e^{-z}$  is bounded on  $\mathbb{R}_+$ , and the second one has been shown in Remark 9. We conclude that (44) holds true. Observe that this implies (8) because  $\gamma > 3/2$ . Similarly, (45) is deduced from (39) and the Fatou lemma.

*Step 2.* It only remains to prove that a.s., for any  $\varphi \in C_c^2(\mathbb{R}^2)$ , the process  $(M_t^\varphi)_{t \geq 0}$ , defined in (9), is a  $\mu$ -martingale. To this end, it suffices to show that for all  $t > s > 0$ , all continuous bounded function  $\Phi : C \rightarrow \mathbb{R}$ , we have  $\Psi(\mu) = 0$  a.s., where for  $\mathbb{Q} \in \mathcal{P}(C)$ ,

$$\begin{aligned} \Psi(\mathbb{Q}) &= \int_C \Phi((x_r)_{r \in [0,s]}) \left( \varphi(x_t) - \varphi(x_s) - \int_s^t \left[ \Delta \varphi(x_u) + \chi \nabla \varphi(x_u) \cdot \nabla b_s^{c_0, \theta, \lambda}(x_u) \right. \right. \\ &\quad \left. \left. + \chi \nabla \varphi(x_u) \cdot \int_0^u (\nabla K_{u-v}^{\theta, \lambda} * \mathbb{Q}_v)(x_u) dv \right] du \right) \mathbb{Q}(dx). \end{aligned}$$

We observe that for any  $\mathbb{Q} \in \mathcal{P}(C)$ , it holds that  $\Psi(\mathbb{Q}) = \Theta(\mathbb{Q} \otimes \mathbb{Q})$ , where for  $\Pi \in \mathcal{P}(C \times C)$ ,

$$\begin{aligned} \Theta(\Pi) &= \int_{C \times C} \Phi((x_r)_{r \in [0,s]}) \left( \varphi(x_t) - \varphi(x_s) - \int_s^t \left[ \Delta \varphi(x_u) + \chi \nabla \varphi(x_u) \cdot \nabla b_s^{c_0, \theta, \lambda}(x_u) \right. \right. \\ &\quad \left. \left. + \chi \nabla \varphi(x_u) \cdot \int_0^u \nabla K_{u-v}^{\theta, \lambda}(x_u - y_v) dv \right] du \right) \Pi(dx, dy). \end{aligned}$$

*Step 2.1.* Here we show that for some constant  $A$ , for all  $N \geq 2$ ,

$$\mathbb{E} \left[ [\Theta(\mu^N \odot \mu^N)]^2 \right] \leq \frac{A}{N}. \quad (46)$$

We have

$$\begin{aligned} \Theta(\mu^N \odot \mu^N) &= \frac{1}{N} \sum_{i=1}^N \Phi((X_r^{i,N})_{r \in [0,s]}) \left( \varphi(X_t^{i,N}) - \varphi(X_s^{i,N}) - \int_s^t \left[ \Delta \varphi(X_u^{i,N}) \right. \right. \\ &\quad \left. \left. + \chi \nabla \varphi(X_u^{i,N}) \cdot \nabla b_s^{c_0, \theta, \lambda}(X_u^{i,N}) + \frac{\chi}{N-1} \sum_{j \neq i} \int_0^u \nabla K_{u-v}^{\theta, \lambda}(X_u^{i,N} - X_v^{j,N}) dv \right] du \right) \\ &= \frac{1}{N} \sum_{i=1}^N \Phi((X_r^{i,N})_{r \in [0,s]}) (O_t^{i,N} - O_s^{i,N}), \end{aligned}$$

where

$$\begin{aligned} O_t^{i,N} &:= \varphi(X_t^{i,N}) - \int_0^t \Delta \varphi(X_s^{i,N}) ds - \chi \int_0^t \nabla \varphi(X_s^{i,N}) \cdot \nabla b_s^{c_0, \theta, \lambda}(X_s^{i,N}) ds \\ &\quad - \frac{\chi}{N-1} \sum_{j \neq i} \int_0^t \nabla \varphi(X_s^{i,N}) \cdot \left( \int_0^u \nabla K_{u-v}^{\theta, \lambda}(X_u^{i,N} - X_v^{j,N}) dv \right) ds \\ &= \varphi(X_0^{i,N}) + \sqrt{2} \int_0^t \nabla \varphi(X_s^{i,N}) \cdot dW_s^i \end{aligned}$$

by the Itô formula (starting from (3)). Then (46) follows from some easy stochastic calculus arguments, because  $\Phi$  and  $\nabla \varphi$  are bounded and since the Brownian motions  $(W_t^1)_{t \geq 0}, \dots, (W_t^N)_{t \geq 0}$  are independent.

*Step 2.2.* Next we introduce, for  $\eta \in (0, 1]$ ,  $\Theta_\eta$  defined as  $\Theta$  with  $\nabla K_s^{\theta, \lambda}(x)$  replaced by the smooth and bounded kernel  $H_s^{\theta, \lambda, \eta}(x) = \frac{s^2}{(s+\eta)^2} \nabla K_s^{\theta, \lambda}(x)$ , recall (14). Then one easily checks that the map  $\Pi \mapsto \Theta_\eta(\Pi)$  is continuous and bounded from  $\mathcal{P}(C \times C)$  to  $\mathbb{R}$ . This uses in particular (7) and that  $x \mapsto \nabla b_s^{c_0, \theta, \lambda}(x)$  is continuous for all  $s > 0$ . Since  $\mu^N$  goes in law to  $\mu$  and thus, as already seen,  $\mu^N \odot \mu^N$  goes in law to  $\mu \otimes \mu$ , we deduce that for any  $\eta \in (0, 1]$ ,

$$\mathbb{E}[|\Theta_\eta(\mu \otimes \mu)|] = \lim_N \mathbb{E}[|\Theta_\eta(\mu^N \odot \mu^N)|].$$

*Step 2.3.* We now prove that  $\lim_{\eta \rightarrow 0} \Delta_\eta = 0$ , where

$$\Delta_\eta := \mathbb{E}[|\Theta(\mu \otimes \mu) - \Theta_\eta(\mu \otimes \mu)|] + \sup_{N \geq 2} \mathbb{E}[|\Theta(\mu^N \odot \mu^N) - \Theta_\eta(\mu^N \odot \mu^N)|].$$

We proceed as in the proof of (43). Since  $\Phi$  and  $\nabla \varphi$  are bounded, we see that for some constant  $A$ , for any  $\Pi \in \mathcal{P}(\mathbb{R}^2 \times \mathbb{R}^2)$ ,

$$\begin{aligned} |\Theta(\Pi) - \Theta_\eta(\Pi)| &\leq A \int_{C \times C} \int_0^t \int_0^u |\nabla K_{u-v}^{\theta, \lambda}(x_u - y_v) - H_{u-v}^{\theta, \lambda, \eta}(x_u - y_v)| dv du \Pi(dx, dy) \\ &= A \int_{C \times C} \int_0^t \int_0^u \left(1 - \frac{(s-u)^2}{(s-u+\eta)^2}\right) |\nabla K_{u-v}^{\theta, \lambda}(x_u - y_v)| dv du \Pi(dx, dy) \\ &\leq A \varepsilon_\eta \int_{C \times C} \left(\int_0^t \int_0^s |\nabla K_{s-u}^{\theta, \lambda}(x_u - y_v)|^{\frac{2\gamma}{3}} duds\right)^{\frac{3}{2\gamma}} \Pi(dx, dy) \end{aligned}$$

by the Hölder inequality, with  $\varepsilon_\eta = \left(\int_0^t \int_0^s \left(1 - \frac{(s-u)^2}{(s-u+\eta)^2}\right)^{\frac{2\gamma}{2\gamma-3}} duds\right)^{\frac{2\gamma-3}{2\gamma}}$ . Hence

$$\begin{aligned} \Delta_\eta &\leq A \varepsilon_\eta \mathbb{E} \left[ \int_{C \times C} \left(\int_0^t \int_0^s |\nabla K_{s-u}^{\theta, \lambda}(x_u - y_v)|^{\frac{2\gamma}{3}} duds\right)^{\frac{3}{2\gamma}} (\mu \otimes \mu + \mu^N \odot \mu^N)(dx, dy) \right] \\ &= A \varepsilon_\eta \mathbb{E} \left[ \left(\int_0^t \int_{\mathbb{R}^2} \int_0^s \int_{\mathbb{R}^2} |\nabla K_{s-u}^{\theta, \lambda}(x - y)|^{\frac{2\gamma}{3}} \mu_u(dy) du \mu_s(dx) ds\right)^{\frac{3}{2\gamma}} \right] \\ &\quad + A \varepsilon_\eta \mathbb{E} \left[ \left(\int_0^t \int_0^s \left(\frac{1}{N(N-1)} \sum_{i \neq j} |\nabla K_{s-u}^{\theta, \lambda}(X_s^{i, N} - X_u^{j, N})|^{\frac{2\gamma}{3}} duds\right)^{\frac{3}{2\gamma}} \right), \right] \end{aligned}$$

Since  $\lim_{\eta \rightarrow 0} \varepsilon_\eta = 0$  by dominated convergence, the conclusion follows from (44), (41) and exchangeability (recall that  $\frac{3}{2\gamma} < 1$ ).

*Step 2.4.* Recalling that  $\Psi(\mu) = \Theta(\mu \otimes \mu)$ , we may write, for any  $\eta \in (0, 1]$ ,

$$\begin{aligned} \mathbb{E}[|\Psi(\mu)|] &\leq \mathbb{E}[|\Theta(\mu \otimes \mu) - \Theta_\eta(\mu \otimes \mu)|] + \limsup_N |\mathbb{E}[|\Theta_\eta(\mu \otimes \mu)|] - \mathbb{E}[|\Theta_\eta(\mu^N \odot \mu^N)|]| \\ &\quad + \limsup_N \mathbb{E}[|\Theta_\eta(\mu^N \odot \mu^N) - \Theta(\mu^N \odot \mu^N)|] + \limsup_N \mathbb{E}[|\Theta(\mu^N \odot \mu^N)|]. \end{aligned}$$

The last term is equal to 0 by Step 2.1, as well as the second one by Step 2.2. Hence

$$\mathbb{E}[|\Psi(\mu)|] \leq \mathbb{E}[|\Theta(\mu \otimes \mu) - \Theta_\eta(\mu \otimes \mu)|] + \limsup_N \mathbb{E}[|\Theta_\eta(\mu^N \odot \mu^N) - \Theta(\mu^N \odot \mu^N)|].$$

Step 2.3 thus implies that  $\mathbb{E}[|\Psi(\mu)|] = 0$ , which was our goal.  $\square$

## 6 Conclusion and discussion about the constants

Recall that for  $\alpha, \beta, \theta > 0$  and  $\gamma \in (3/2, 2)$ ,

$$C_0(\beta) := \sup_{u \geq 0} \sqrt{u}(1 + \beta u)^{3/2} e^{-u}, \quad C_1(\alpha, \gamma) := (\gamma - 1)(1 - 4\alpha(\gamma - 1)),$$

$$C_2(\theta, \alpha, \gamma) := \frac{\sqrt{\alpha\theta}(\gamma - 1)}{2\pi} C_0\left(\frac{4\alpha}{\theta}\right) \kappa\left(\frac{1}{2}, \gamma - 1\right) \kappa\left(\gamma - \frac{3}{2}, \gamma - 1\right).$$

Fix  $\rho_0 \in \mathcal{P}(\mathbb{R}^2)$ ,  $c_0 \in L^p(\mathbb{R}^2)$  for some  $p > 2$ ,  $\chi > 0$  and  $\theta > 0$ . By Proposition 13 and Theorem 14, the conclusions of Theorem 4 hold true provided there are  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$  and  $\alpha > 0$  such that

$$C_1(\alpha, \gamma) > 0, \quad C_1(\alpha, \gamma) > \chi C_2(\theta, \alpha, \gamma), \quad (4 - 2\gamma) - \chi \frac{\sqrt{\theta} C_0\left(\frac{4\alpha}{\theta}\right) \kappa\left(\frac{1}{2}, \gamma - 1\right)}{4\pi \sqrt{\alpha} [C_1(\alpha, \gamma) - \chi C_2(\theta, \alpha, \gamma)]^{\frac{1}{2(\gamma-1)}}} > 0.$$

The first condition implies that  $\alpha \in (0, \frac{1}{4(\gamma-1)})$  and the two other ones can be summarized as

$$\chi C_2(\theta, \alpha, \gamma) + [\chi C_3(\theta, \alpha, \gamma)]^{2(\gamma-1)} < C_1(\alpha, \gamma),$$

where we have set

$$C_3(\theta, \alpha, \gamma) := \frac{\sqrt{\theta} C_0\left(\frac{4\alpha}{\theta}\right) \kappa\left(\frac{1}{2}, \gamma - 1\right)}{4\pi \sqrt{\alpha} (4 - 2\gamma)}.$$

Hence if we set, for  $\theta > 0$ ,  $\gamma \in (\frac{3}{2}, 2)$  and  $\alpha \in (0, \frac{1}{4(\gamma-1)})$ ,

$$\chi_{\theta, \alpha, \gamma} = \sup \left\{ \chi > 0 : \chi C_2(\theta, \alpha, \gamma) + [\chi C_3(\theta, \alpha, \gamma)]^{2(\gamma-1)} < C_1(\alpha, \gamma) \right\},$$

Theorem 4 holds true with

$$\chi_{\theta, p}^* = \sup \left\{ \chi_{\theta, \alpha, \gamma} : \gamma \in \left(\frac{3}{2}, \frac{2p+2}{p+2}\right), \alpha \in \left(0, \frac{1}{4(\gamma-1)}\right) \right\}. \quad (47)$$

We now discuss the numerical values of this threshold, obtained by numerical trials. We do not really take care of  $p$ : We try to find the values of  $\gamma \in (\frac{3}{2}, 2)$  and  $\alpha \in (0, \frac{1}{4(\gamma-1)})$  maximizing  $\chi_{\theta, \alpha, \gamma}$  and then see to which values of  $p > 2$  this applies.

**Remark 15.** (i) For any  $p > 2$ ,  $\liminf_{\theta \rightarrow 0} \chi_{\theta, p}^* \geq 3.28$ .

(ii) For any  $p > 2.6$ ,  $\chi_{0.1, p}^* \geq 2.42$ .

(iii) For any  $p > 3.3$ ,  $\chi_{1, p}^* \geq 1.39$ .

(iv) For any  $p > 3.5$ ,  $\chi_{10, p}^* \geq 0.51$ .

(v) For any  $p > 3.5$ ,  $\liminf_{\theta \rightarrow \infty} \sqrt{\theta} \chi_{\theta, p}^* \geq 1.65$ .

*Proof.* We start with (i). Fix  $p > 2$ . We choose  $\gamma = \frac{3}{2} + \sqrt{\theta}$ , which belongs to  $(\frac{3}{2}, \frac{2p+2}{p+2})$  for all  $\theta > 0$  small enough, and  $\alpha = \sigma \theta$  (for some  $\sigma > 0$  to be chosen later) which belongs to  $(0, \frac{1}{4(\gamma-1)})$  for all  $\theta > 0$  small enough. It holds that  $\lim_{\theta \rightarrow 0} C_1(\alpha, \gamma) = \frac{1}{2}$ . Moreover,

$$C_2(\theta, \alpha, \gamma) = \frac{\sqrt{\sigma} \theta (\frac{1}{2} + \sqrt{\theta}) C_0(4\sigma) \kappa(\frac{1}{2}, \frac{1}{2} + \sqrt{\theta}), \kappa(\sqrt{\theta}, \frac{1}{2} + \sqrt{\theta})}{2\pi} \rightarrow 0$$

as  $\theta \rightarrow 0$ , because  $\kappa(\frac{1}{2}, \frac{1}{2} + \sqrt{\theta}) \rightarrow 1$  and  $\kappa(\sqrt{\theta}, \frac{1}{2} + \sqrt{\theta}) \sim \frac{1}{\sqrt{\theta}}$ . Finally,

$$C_3(\theta, \alpha, \gamma) = \frac{C_0(4\sigma) \kappa(\frac{1}{2}, \frac{1}{2} + \sqrt{\theta})}{\sqrt{\sigma} 4\pi (1 - 2\sqrt{\theta})} \rightarrow \frac{C_0(4\sigma)}{4\pi \sqrt{\sigma}}.$$

All in all, with these values of  $\alpha$  and  $\gamma$ , the condition  $\chi C_2(\theta, \alpha, \gamma) + [\chi C_3(\theta, \alpha, \gamma)]^{2(\gamma-1)} < C_1(\alpha, \gamma)$  asymptotically writes  $\frac{C_0(4\sigma)}{4\pi\sqrt{\sigma}}\chi < 0.5$  for  $\theta > 0$  small (since  $2(\gamma-1) \rightarrow 1$ ). The choice  $\sigma = 0.13$  seems to be a good one and we find numerically  $C_0(4\sigma) \simeq 0.6895$ , whence the condition  $\chi \leq 3.2856$ .

For (ii), choose  $\gamma = 1.56$  and  $\alpha = 0.009$ . The result follows from a numerical computation. For any  $p > 2.6$ , it holds that  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$ .

For (iii), choose  $\gamma = 1.62$  and  $\alpha = 0.045$ . For any  $p > 3.3$ , it holds that  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$ .

For (iv), choose  $\gamma = 1.63$  and  $\alpha = 0.067$ . For any  $p > 3.5$ , it holds that  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$ .

For (v), we choose  $\gamma = 1.63$  and  $\alpha = 0.08$ . For any  $p > 3.5$ , it holds that  $\gamma \in (\frac{3}{2}, \frac{2p+2}{p+2})$ . We have  $C_1(\alpha, \gamma) \simeq 0.502$  and  $\kappa(\frac{1}{2}, \gamma - 1) \simeq 1.411$  and  $\kappa(\gamma - \frac{3}{2}, \gamma - \frac{1}{2}) < 7.751$ . Hence

$$\begin{aligned}\chi C_2(\theta, \alpha, \gamma) &= \chi\sqrt{\theta} \frac{\sqrt{\alpha}(\gamma-1)C_0(\frac{4\alpha}{\theta})\kappa(\frac{1}{2}, \gamma-1)\kappa(\gamma-\frac{3}{2}, \gamma-1)}{2\pi} \simeq 0.286 C_0\left(\frac{4\alpha}{\theta}\right)\chi\sqrt{\theta}, \\ \chi C_3(\theta, \alpha, \gamma) &= \chi\sqrt{\theta} \frac{C_0(\frac{4\alpha}{\theta})\kappa(\frac{1}{2}, \gamma-1)}{4\pi\sqrt{\alpha}(4-2\gamma)} \simeq 0.537 C_0\left(\frac{4\alpha}{\theta}\right)\chi\sqrt{\theta}.\end{aligned}$$

Since  $\lim_{\theta \rightarrow \infty} C_0(\frac{4\alpha}{\theta}) = C_0(0) \simeq 0.429$ , the condition  $\chi C_2(\theta, \alpha, \gamma) + [\chi C_3(\theta, \alpha, \gamma)]^{2(\gamma-1)} < C_1(\alpha, \gamma)$  asymptotically rewrites  $0.123 \chi\sqrt{\theta} + [0.231 \chi\sqrt{\theta}]^{1.26} < 0.502$ . This holds true if  $\chi\sqrt{\theta} < 1.65$ .  $\square$

## A Proof of Lemma 7

We fix  $b > a > 0$ , introduce  $\mathcal{G} = \{g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ : 0 \leq g(s) \leq 1/s \text{ a.e.}\}$  and, for  $g \in \mathcal{G}$ ,  $I_a(g) = \int_0^\infty g^{1+a}$  and  $I_b(g) = \int_0^\infty g^{1+b}$ . We set  $\mathcal{G}_b = \{g \in \mathcal{G} : 0 < I_b(g) < \infty\}$  and

$$\bar{\kappa}(a, b) = \sup\{[I_b(g)]^{-\frac{a}{b}} I_a(g) : g \in \mathcal{G}_b\}.$$

We will show that  $\bar{\kappa}(a, b) = \kappa(a, b)$  and this will complete the proof, because for  $f : [0, t] \rightarrow \mathbb{R}_+$ , the function  $g(s) = \frac{1}{s+f(s)} \mathbf{1}_{\{s \in [0, t]\}}$  belongs to  $\mathcal{G}$ .

*Step 1.* Here we show that  $\bar{\kappa}(a, b) < \infty$  and that there exists  $g \in \mathcal{G}_b$  realizing the supremum.

First note that for any  $g \in \mathcal{G}$ , by the Hölder inequality,

$$I_a(g) \leq \int_0^1 g^{1+a} + \int_1^\infty \frac{ds}{s^{1+a}} \leq [I_b(g)]^{\frac{1+a}{1+b}} + \frac{1}{a}.$$

Next we observe that  $\bar{\kappa}(a, b) = \sup\{I_a(g) : g \in \mathcal{G}_{b,1}\}$ , where  $\mathcal{G}_{b,1} = \{g \in \mathcal{G} : I_b(g) = 1\}$ . This easily follows from the fact that for any  $g \in \mathcal{G}_b$  and any  $\lambda > 0$ , the function  $g_\lambda(s) = \lambda g(\lambda s)$  still belongs to  $\mathcal{G}_b$ , and  $I_a(g_\lambda) = \lambda^a I_a(g)$  and  $I_b(g_\lambda) = \lambda^b I_b(g)$ .

The two above points show that  $\bar{\kappa}(a, b) \leq 1 + \frac{1}{a} < \infty$ . Now we consider a sequence  $(g_n)_{n \geq 1}$  of  $\mathcal{G}_{b,1}$  such that  $\lim_n I_a(g_n) = \bar{\kappa}(a, b)$  and we set  $h_n = g_n^{1+a}$ . Since  $I_b(g_n) = 1$ , the family  $(h_n)_{n \geq 1}$  takes values in the unit ball of  $L^p(\mathbb{R}_+)$ , where  $p = \frac{1+b}{1+a} > 1$ , so that we can find a (non relabeled) subsequence of  $(h_n)_{n \geq 1}$  converging weakly in  $L^p(\mathbb{R}_+)$  to some function  $h$ . One easily verifies that  $g := h^{\frac{1}{1+a}} \in \mathcal{G}$ , and it classically holds true that

$$I_b(g) = \|h\|_p^p \leq \liminf_n \|h_n\|_p^p = \liminf_n I_b(g_n) = 1.$$

We now show that  $I_a(g) = \bar{\kappa}(a, b)$ . The weak convergence of  $h_n$  to  $h$  implies that for every  $\varepsilon \in (0, 1)$ , one has  $\int_\varepsilon^{1/\varepsilon} g_n^{1+a} = \int_\varepsilon^{1/\varepsilon} h_n \rightarrow \int_\varepsilon^{1/\varepsilon} h = \int_\varepsilon^{1/\varepsilon} g^{1+a}$ . To conclude that  $I_a(g) = \lim_n I_a(g_n) = \bar{\kappa}(a, b)$ , it suffices to note that, by the Hölder inequality and since  $g_n \in \mathcal{G}_{b,1}$ ,

$$\limsup_{\varepsilon \rightarrow 0} \sup_{n \geq 1} \left( \int_0^\varepsilon g_n^{1+a} + \int_{1/\varepsilon}^\infty g_n^{1+a} \right) \leq \limsup_{\varepsilon \rightarrow 0} \sup_{n \geq 1} \left( [I_b(g_n)]^{\frac{1+a}{1+b}} \varepsilon^{\frac{b-a}{1+b}} + \int_{1/\varepsilon}^\infty \frac{ds}{s^{1+a}} \right) = 0.$$

All in all,  $g \in \mathcal{G}$  and  $[I_b(g)]^{-\frac{a}{b}} I_a(g) \geq \bar{\kappa}(a, b)$ , whence necessarily  $[I_b(g)]^{-\frac{a}{b}} I_a(g) = \bar{\kappa}(a, b)$ .

*Step 2.* By Step 1, there is  $g \in \mathcal{G}_b$  realizing the supremum. Here we show that there is a constant  $k > 0$ , namely  $k = [\frac{b(1+a)I_b(g)}{a(1+b)I_a(g)}]^{-\frac{1}{b-a}}$ , such that for a.e.  $s > 0$ , we have  $g(s) = \max\{k, s^{-1}\}$ .

*Step 2.1.* We first show that  $g > 0$  a.e. It suffices to show that for all  $\varepsilon \in (0, 1)$ ,  $\lambda(A_\varepsilon) = 0$ , where  $A_\varepsilon = \{s \in [0, \frac{1}{\varepsilon}] : g(s) = 0\}$  and where  $\lambda$  is the Lebesgue measure. For any  $\alpha \in (0, \varepsilon)$ , the function  $g_\alpha = g + \alpha \mathbf{1}_{A_\varepsilon}$  belongs to  $\mathcal{G}_b$ . Hence

$$\frac{I_a(g)}{[I_b(g)]^{\frac{a}{b}}} \geq \frac{I_a(g_\alpha)}{[I_b(g_\alpha)]^{\frac{a}{b}}} = \frac{I_a(g) + \lambda(A_\varepsilon)\alpha^{1+a}}{[I_b(g) + \lambda(A_\varepsilon)\alpha^{1+b}]^{\frac{a}{b}}} = \frac{I_a(g)}{[I_b(g)]^{\frac{a}{b}}} \left(1 + \lambda(A_\varepsilon) \left[ \frac{\alpha^{1+a}}{I_a(g)} - \frac{a}{b} \frac{\alpha^{1+b}}{I_b(g)} \right] + O(\alpha^{2+a+b})\right)$$

as  $\alpha \rightarrow 0$ . This implies that  $\lambda(A_\varepsilon) = 0$ , because  $\alpha^{1+a} \gg \alpha^{1+b}$  as  $\alpha \rightarrow 0$ .

*Step 2.2.* Now we show that  $g \leq k$  a.e. By Step 2.1, it suffices to prove that for all  $\varepsilon \in (0, 1)$ , we have  $g \leq k$  a.e. on  $B_\varepsilon = \{s \in [0, \frac{1}{\varepsilon}] : g(s) \geq \varepsilon\}$ . For all  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\delta \leq \mathbf{1}_{B_\varepsilon}$ , all  $\alpha \in (0, \varepsilon)$ , it holds that  $g_{\delta, \alpha} = g - \alpha\delta$  belongs to  $\mathcal{G}_b$ . The function  $\frac{\delta}{g}$  is bounded and compactly supported, which allows one to justify the following computation: Since  $g_{\delta, \alpha}^{1+a} = g^{1+a}(1 - \alpha\frac{\delta}{g})^{1+a} = g^{1+a} - \alpha(1+a)g^a\delta + O(\alpha^2)$  and  $g_{\delta, \alpha}^{1+b} = g^{1+b} - \alpha(1+b)g^b\delta + O(\alpha^2)$  as  $\alpha \rightarrow 0$ ,

$$\frac{I_a(g)}{[I_b(g)]^{\frac{a}{b}}} \geq \frac{I_a(g_{\delta, \alpha})}{[I_b(g_{\delta, \alpha})]^{\frac{a}{b}}} = \frac{I_a(g) - (1+a)\alpha \int_0^\infty g^a \delta + O(\alpha^2)}{[I_b(g) - (1+b)\alpha \int_0^\infty g^b \delta + O(\alpha^2)]^{\frac{a}{b}}}.$$

Hence

$$\frac{I_a(g)}{[I_b(g)]^{\frac{a}{b}}} \geq \frac{I_a(g)}{[I_b(g)]^{\frac{a}{b}}} \left(1 - \frac{(1+a)\alpha}{I_a(g)} \int_0^\infty g^a \delta + \frac{a(1+b)\alpha}{bI_b(g)} \int_0^\infty g^b \delta + O(\alpha^2)\right).$$

This implies that

$$\int_0^\infty g^b \delta \leq \frac{b(1+a)I_b(g)}{a(1+b)I_a(g)} \int_0^\infty g^a \delta.$$

Since this holds true for any measurable  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\delta \leq \mathbf{1}_{B_\varepsilon}$ , we conclude that  $g \leq [\frac{b(1+a)I_b(g)}{a(1+b)I_a(g)}]^{-\frac{1}{b-a}}$  a.e. on  $B_\varepsilon$ , which was our goal.

*Step 2.3.* We next show that  $g \geq k$  a.e. on  $C = \{s > 0 : g(s) < \frac{1}{s}\}$ . By Step 2.1, it suffices to show that for all  $\varepsilon \in (0, 1)$ ,  $g \geq k$  a.e. on  $C_\varepsilon = \{s \in [0, \frac{1}{\varepsilon}] : \varepsilon \leq g(s) \leq \frac{1-\varepsilon}{s}\}$ . For all  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\delta \leq \mathbf{1}_{C_\varepsilon}$ , all  $\alpha \in (0, \varepsilon^2)$ , it holds that  $g_{\delta, \alpha} = g + \alpha\delta$  belongs to  $\mathcal{G}_b$ . The function  $\frac{\delta}{g}$  is again bounded and compactly supported and, proceeding exactly as in Step 2.2, we find that

$$\frac{I_a(g)}{[I_b(g)]^{\frac{a}{b}}} \geq \frac{I_a(g)}{[I_b(g)]^{\frac{a}{b}}} \left(1 + \frac{(1+a)\alpha}{I_a(g)} \int_0^\infty g^a \delta - \frac{a(1+b)\alpha}{bI_b(g)} \int_0^\infty g^b \delta + O(\alpha^2)\right)$$

as  $\alpha \rightarrow 0$ . This implies that

$$\int_0^\infty g^b \delta \geq \frac{b(1+a)I_b(g)}{a(1+b)I_a(g)} \int_0^\infty g^a \delta.$$

Since this holds true for any measurable  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\delta \leq \mathbf{1}_{C_\varepsilon}$ , we conclude that  $g \geq [\frac{b(1+a)I_b(g)}{a(1+b)I_a(g)}]^{-\frac{1}{b-a}}$  a.e. on  $C_\varepsilon$ .

*Step 2.4.* We now conclude that  $g(s) = \min\{k, s^{-1}\}$  for a.e.  $s > 0$ . We know from Steps 2.2 and 2.3 that  $g \leq k$  a.e. and that  $g \geq k$  a.e. on  $C = \{s > 0 : g(s) < \frac{1}{s}\}$ . We thus have  $g = k$  a.e. on  $C$ . Let  $D = \mathbb{R}_+ \setminus C = \{s > 0 : g(s) = \frac{1}{s}\}$  and  $r = \text{ess inf } D$ . We claim that  $r = \frac{1}{k}$ .

We know that for a.e.  $\varepsilon \in (0, r)$ ,  $r - \varepsilon \in C$ , so that  $\frac{1}{r-\varepsilon} \geq g(r - \varepsilon) = k$ . Thus  $r \leq \frac{1}{k}$ . Now consider  $r_n \in D$  such that  $\lim_n r_n = r$ . We have  $k \geq g(r_n) = \frac{1}{r_n}$ , so that  $r \geq \frac{1}{k}$ .



We have shown that  $g = k$  a.e. on  $[0, \frac{1}{k}] \subset C$ , and it remains to verify that for a.e.  $s > \frac{1}{k}$ , we have  $g(s) = \frac{1}{s}$ . This follows from the fact that if  $g(s) < \frac{1}{s}$  for some  $s > \frac{1}{k}$ , then  $s \in C$ , so that  $g(s) = k$ , which is not possible since  $k > \frac{1}{s}$ .

*Step 3.* For any  $k > 0$ , the function  $g(s) = \max\{k, s^{-1}\}$  satisfies  $I_a(g) = k^a + \frac{1}{a}k^a$  and  $I_b(g) = k^b + \frac{1}{b}k^b$ . We deduce from Step 2 that

$$\bar{\kappa}(a, b) = \frac{k^a + \frac{1}{a}k^a}{(k^b + \frac{1}{b}k^b)^{a/b}} = \frac{1 + \frac{1}{a}}{(1 + \frac{1}{b})^{a/b}},$$

which is nothing but  $\kappa(a, b)$ . □

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