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# Identifying the reach from high-dimensional point cloud data with connections to $r$ -convexity

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## Abstract

Convexity can be generalized to the two weaker notions of reach and  $r$ -convexity; both describe the regularity of a set’s boundary. In this article, these two notions are shown to be equivalent for closed subsets of  $\mathbb{R}^d$  with  $C^1$  smooth,  $(d - 1)$ -dimensional boundary. In the general case, for closed subsets of  $\mathbb{R}^d$ , we detail a new characterization of the reach in terms of the distance-to-set function applied to midpoints of pairs of points in the set. For compact subsets of  $\mathbb{R}^d$ , we provide methods of approximating the reach and  $r$ -convexity based on high-dimensional point cloud data. These methods are intuitive and highly tractable, and produce upper bounds that converge to the respective quantities as the density of the point cloud is increased. Simulation studies suggest that the rates at which the approximation methods converge correspond to those established theoretically.

## 1 Introduction

A number of concepts from convex geometry generalize from convex sets to much larger classes of sets. A classic example from Federer (1959) is the extension of kinematic formulas (in particular, Steiner’s formula) for convex sets to sets with *positive reach* (see Definition 3). Another example is the notion of the convex hull of a set, which can be weakened to the  $r$ -convex hull, for  $r > 0$ . This weak notion of a convex hull, instead of being expressed as the intersection of half-spaces, is expressed in terms of intersections of the compliments of open balls of radius  $r$ . The resulting intersection is said to be  *$r$ -convex* (see Definition 2), which differs subtly from the notion of reach, and these differences have been studied in Colesanti and Manselli (2010) and Cuevas et al. (2012) for example. These two notions exemplify geometric measures on sets that describe the degree to which the set is convex; each measure belonging to the closed interval  $[0, \infty]$  and carrying pertinent information.

In this paper, we bring to light the importance of  $r$ -convexity by showing its equivalence to the reach for a large class of sets. Nonetheless, the mathematical properties of  $r$ -convex sets have been widely studied (Perkal, 1956; Serra, 1984; Walther, 1999; Colesanti and Manselli, 2010; Cuevas et al., 2012) and applications of  $r$ -convexity for image smoothing have been suggested in Walther (1999) and Cuevas et al. (2007) as well as for set estimation in Mani-Levitska (1993); Cuevas et al. (2007); Pateiro López (2008); Cuevas (2009); Cuevas et al. (2012); and Aaron et al. (2022). In Cuevas and Rodríguez-Casal (2004), the authors use a rolling-type condition (weaker than  $r$ -convexity) to improve the rate of convergence of their set boundary estimator. Rodríguez Casal and Saavedra-Nieves (2016) introduce a statistical estimator for the largest  $r$  such that a set is  $r$ -convex using the test of uniformity proposed in Berrendero et al. (2012).

The reach is a popular measure of convexity largely due to its link with the Steiner formula; for the larger the reach of a set, the larger the interval on which the volume of a dilation of the set is polynomial in the dilation radius (Federer, 1959, Theorem 5.6). In the field of geometric data analysis, the reach constitutes a commonly used measure of regularity of a set’s boundary. The theoretical estimation of the reach has been of particular interest in recent literature (Chazal and Lieutier, 2008; Aamari et al., 2019, 2022; Berenfeld et al., 2022). In Cholaquidis

et al. (2022), the authors introduce a complete and tractable method for estimating the reach from point cloud data involving the computation of graph distances in a spatial network defined over the point cloud. A number of other authors have taken an interest in the mathematical properties of the reach. Poliquin et al. (2000) identifies the sets of reach  $r$  with the  $r$ -proximally smooth sets introduced in Clarke and Wolenski (1995), implying that the reach can be characterized by the gradients of the distance-to-set function. Colesanti and Manselli (2010) makes a number of insightful connections between the reach and other geometrical properties of sets. Attali and Lieutier (2015) studies Vietoris–Rips complexes via the reach, proving that the reach of a set can only increase if intersected by sufficiently small balls. An alternative characterization of the reach, involving pairwise geodesic distances, is provided in Boissonnat et al. (2019). The positivity of the reach implies a number of desirable properties, and so positive reach is a common hypothesis to guarantee convergence rates of statistical estimators of geometric quantities (Thäle, 2008; Cuevas, 2009; Rataj and Zähle, 2019; Biermé et al., 2019; Cotsakis et al., 2022).

The novelties in this paper are listed as follows. Firstly, we establish that for the compact subsets of  $\mathbb{R}^d$  with  $C^1$  smooth,  $(d - 1)$ -dimensional boundary, the notions of  $r$ -convexity and reach are equivalent (see Theorem 1). A conjecture mentioned in Perkal (1956) is proven false by constructing a set that does not have positive reach, but is  $r$ -convex for all  $r \in \mathbb{R}^+$ . In the same spirit as Boissonnat et al. (2019), we introduce a characterization of the reach in terms of the distance-to-set function applied to pairwise midpoints (see Theorem 2). This gives rise to a tractable method of approximating the reach of a compact set  $A \subset \mathbb{R}^d$  based on a point cloud that is known to be contained, and relatively dense, in  $A$ . We show that this method produces an upper bound for the reach of  $A$ , and that the upper bound converges to the reach (with a known rate) if the point cloud is taken to be arbitrarily dense in  $A$  (see Theorem 4). In addition, we propose a natural and computationally efficient method for bounding the largest  $r$  such that a compact set  $A \subset \mathbb{R}^d$  is  $r$ -convex based on point cloud data. Like the bound for the reach, the bound on the  $r$ -convexity converges to the correct limit as the density of the point cloud increases (see Theorem 3). Existing methods for estimating the reach and  $r$ -convexity from point cloud data suffer from complexity and do not obtain the rate of convergence that we have established for the bound on the reach.

The organization of the paper is as follows. Section 2 introduces the notation and set operations that we use throughout the document, as well as the formal definitions of reach and  $r$ -convexity. Section 3 is split into two subsections. In Section 3.1, we provide the link between reach and  $r$ -convexity and the counterexample to the conjecture in Perkal (1956); in Section 3.2, we provide an alternative formulation of the reach. Section 4 is dedicated to the tractable methods of approximating the reach and  $r$ -convexity from point cloud data. The methods developed in Section 4 are tested on sets with known reach and  $r$ -convexity in Section 4.3, and empirical rates of convergence are provided. The main proofs can be found just after their corresponding claims, however, those in Section 3.1 are postponed to Section 5, in which some additional auxiliary results are stated and proved.

## 2 Definitions and notation

The sets that we study in this paper are subsets of  $\mathbb{R}^d$ , which is endowed with the Euclidean metric  $\|\cdot\|$ . For a set  $S \subseteq \mathbb{R}^d$ , let  $\partial S$  denote its boundary, and let  $\text{cl}(S) := S \cup \partial S$  denote its closure. Denote the closed ball with radius  $r \in \mathbb{R}^+$  centered at  $s \in \mathbb{R}^d$  by  $B(s, r) := \{t \in \mathbb{R}^d : \|t - s\| \leq r\}$ . For  $t \in \mathbb{R}^d$ , denote the distance between  $t$  and a non-empty set  $S$  by  $\delta_S(t) := \inf\{\|t - s\| : s \in S\}$ . Intervals in  $\mathbb{R}$  with open (resp. closed) endpoints are denoted with round (resp. square) brackets.

**Definition 1** (Operations on subsets of  $\mathbb{R}^d$ ). *We recall the Minkowski addition of two sets  $A, B \subseteq \mathbb{R}^d$ ,*

$$A \oplus B := \{x + y : x \in A, y \in B\}.$$

*The Minkowski difference is given by*

$$A \ominus B := \bigcap_{y \in B} (A \oplus \{y\}) = (A^c \oplus B)^c,$$

*where  $A^c$  denotes the compliment of  $A$  in  $\mathbb{R}^d$ . For  $r \in \mathbb{R}$ , let*

$$A_r := \begin{cases} A \oplus B(0, r), & \text{for } r \geq 0, \\ A \ominus B(0, -r), & \text{for } r < 0, \end{cases}$$

denote either the dilation or erosion of a set  $A$ , depending on the sign of  $r$ . Finally, define  $A_{\bullet r} := (A_r)_{-r}$  to be the closing of  $A$  by  $B(0, r)$  if  $r \geq 0$  and the opening of  $A$  by  $B(0, -r)$  if  $r < 0$ .

The Hausdorff distance between two closed sets  $A, B \subseteq \mathbb{R}^d$  is denoted  $d_H(A, B) := \inf\{r \in \mathbb{R}^+ : A \subseteq B_r, B \subseteq A_r\}$ .

**Lemma 1.** *Let  $r, s > 0$ , and let  $A \subseteq \mathbb{R}^d$ . The following identities hold:*

- (a)  $(A_r)^c = (A^c)_{-r}$ ,
- (b)  $A \subseteq A_{\bullet r}$ ,
- (c)  $(A_r)_s = A_{r+s}$ ,
- (d)  $(A_r)_{-s} \supseteq A_{r-s}$ ,
- (e)  $(A_{-r})_s \subseteq A_{s-r}$ .

*Proof of Lemma 1.* Fix  $r, s > 0$ . The identity in (a) follows directly from Definition 1. Item (b) is proved by contradiction. Let  $a \in A$  and suppose  $a \in (A_{\bullet r})^c = (A_r)^c \oplus B(0, r)$ . Then there is a  $p \in (A_r)^c$  such that  $a \in B(p, r) \Leftrightarrow \|p - a\| \leq r \Leftrightarrow p \in B(a, r) \Leftrightarrow p \in A_r$ ; thus, a contradiction. To prove (c), remark that  $B(0, r) \oplus B(0, s) = B(0, r + s)$  and that Minkowski addition is associative. To prove (d), consider the case where  $r \geq s$ , then  $(A_r)_{-s} = (A_{r-s+s})_{-s} = ((A_{r-s})_s)_{-s} = (A_{r-s})_{\bullet s} \supseteq A_{r-s}$ . For the case  $r < s$ , write  $(A_r)_{-s} = (A_r)_{r-s-r} = ((A_r)_{-r})_{r-s} = (A_{\bullet r})_{r-s} \supseteq A_{r-s}$ . To show (e), consider the compliments of the sets in (d) and apply (a) repeatedly.  $\square$

**Weak notions of convexity** Here, we precise the weak notions of convexity that we study in detail.

**Definition 2.** *A set  $A \subseteq \mathbb{R}^d$  is said to be  $r$ -convex for  $r \in \mathbb{R}^+$  if it is closed and  $A_{\bullet s} = A$  for all  $s \in (0, r)$  (see, e.g., Perkal (1956)). Define the quantity  $\text{rconv}(A) := \sup\{r \in \mathbb{R} : A_{\bullet r} = A\}$ .*

A useful interpretation of the above definition is that a set  $A$  is  $r$ -convex if it can be expressed as the compliment of a union of open balls of radius  $r$ .

**Definition 3.** *Recall from Federer (1959) that the reach of a set  $A \subseteq \mathbb{R}^d$  is given by*

$$\text{reach}(A) := \sup\{r \in \mathbb{R}^+ : \forall y \in A_r \exists! x \in A \text{ nearest to } y\}.$$

If  $\text{reach}(A) > 0$ , then  $A$  is said to have positive reach.

It is shown in Proposition 1 of Cuevas et al. (2012) that for a compact set  $A \subset \mathbb{R}^d$  with  $\text{reach}(A) \geq r > 0$ , it holds that  $A$  is  $r$ -convex. Thus, the reach of a set provides a stronger notion of convexity than  $r$ -convexity. However, both notions are *generalizations* of convexity, since a set  $A$  has  $\text{reach}(A) = \infty$  if and only if it is convex and closed (Federer, 1959, p. 433).

## 3 Equivalent notions of reach

### 3.1 Connections to $r$ -convexity

The weak notions of convexity introduced in Section 2 (see Definitions 2 and 3) are closely related. It is the goal of this section to highlight the connections between  $r$ -convexity and the reach, as well as identify conditions for which the two notions are equivalent. Theorem 1 provides the principal insight that we leverage for many of our results.

**Theorem 1.** *Let  $A$  be closed in  $\mathbb{R}^d$ . It holds that*

$$\text{reach}(A) \leq \text{rconv}(A). \tag{1}$$

Moreover, if  $\partial A$  is a  $C^1$  smooth  $(d - 1)$ -dimensional manifold, then

$$\text{reach}(A) = \text{rconv}(A). \tag{2}$$

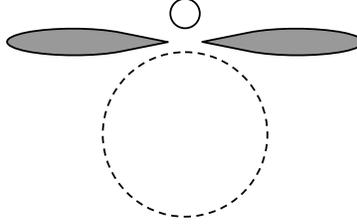


Figure 1: The closed set  $A$  (in grey) has  $\text{reach}(A)$  equal to the radius of the small circle, but it is  $r$ -convex for all positive  $r$  less than the radius of the larger dashed circle.

Equation (1) is proven in Cuevas et al. (2012, Proposition 1) for compact  $A$ . Nonetheless, we reprove the statement for closed  $A$  in the proof of Theorem 1, which we postpone to Section 5. Figure 1 depicts a closed set  $A$  that obeys (1) but not (2).

The following corollary provides the link between the  $r$ -convexity and the reach of sets in Serra's regular model (Serra, 1984, p. 144).

**Corollary 1.** *Let  $A \subset \mathbb{R}^d$  be non-empty, compact, and path-connected. If  $\text{rconv}(\text{cl}(A^c)) > 0$ , then  $\text{reach}(A) = \text{rconv}(A)$ .*

The proof of Corollary 1, which relies heavily on Theorem 1 in Walther (1999), is postponed to Section 5. Likewise, we prove the following result in Section 5.

**Corollary 2.** *Let  $A$  be a closed set in  $\mathbb{R}^d$  and let  $\epsilon \in \mathbb{R}^+$ . Then  $\text{reach}(A_\epsilon) = \text{rconv}(A_\epsilon)$ .*

Cockreham and Gao (2018) concerns the coating (*i.e.*, a small dilation) of sets with positive reach. Corollary 2 concerns precisely these sets.

**A counterexample to Borsuk's conjecture.** Here, we provide a simple counterexample to Borsuk's conjecture, that  $r$ -convex sets are locally contractible (see, *e.g.*, Perkal (1956) and Remark 2 of Cuevas et al. (2012)). Consider the following subset of  $\mathbb{R}^2$ ,

$$\Lambda := \left\{ \left( \frac{1}{n}, 0 \right) : n \in \mathbb{N}^+ \right\} \cup \{(0, 0)\}$$

is not locally contractible at  $(0, 0) \in \mathbb{R}^2$ , but is  $r$ -convex for all  $r \in \mathbb{R}^+$ , *i.e.*,  $\text{rconv}(\Lambda) = \infty$ . This is consistent with Cuevas et al. (2012, Remark 4) and Federer (1959, Remark 4.15), since  $\text{reach}(\Lambda) = 0$ . In general, any closed subset of  $\mathbb{R}^d$  contained in a  $(d - 1)$ -dimensional affine linear subspace is  $r$ -convex for all  $r > 0$ .

### 3.2 Alternative formulations of the reach in the general case

Boissonnat et al. (2019) introduces a new formulation of the reach by considering pairs of points in the set, and remarking that they must satisfy a certain condition depending on the reach. In this section, we provide an equivalent notion of the reach, inspired by Colesanti and Manselli (2010, Theorem 3.8) and the formulation in Boissonnat et al. (2019, Theorem 1). Our new formulation is expressed in terms of the following geometric objects.

**Definition 4.** *Define for  $\alpha \in \mathbb{R}^+$  and  $x \in [0, \alpha/2]$ ,*

$$g_\alpha(x) := \begin{cases} \frac{\alpha^2}{8x} + \frac{x}{2}, & x > 0, \\ \infty, & x = 0, \end{cases} \quad (3)$$

*and its inverse for  $r \geq \alpha/2$ ,*

$$g_\alpha^{-1}(r) := r - \sqrt{r^2 - \frac{\alpha^2}{4}}. \quad (4)$$

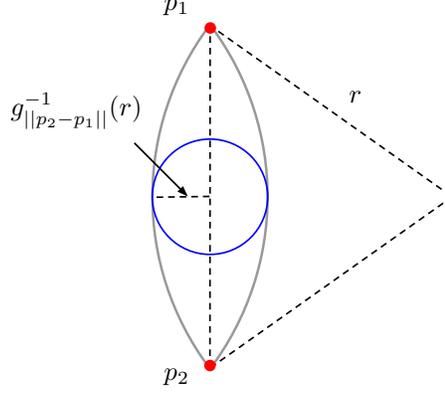


Figure 2: A geometric interpretation of the objects in Definition 4. The two grey arcs with radius  $r$  outline the set  $\mathfrak{h}(p_1, p_2, r)$ . The blue circle marks the boundary of the set  $\mathfrak{b}(p_1, p_2, r)$  in (5).

For  $p_1, p_2 \in \mathbb{R}^d$ , denote

$$\mathfrak{b}(p_1, p_2, r) := B\left(\frac{p_1 + p_2}{2}, g_{\|p_2 - p_1\|}^{-1}(r)\right). \quad (5)$$

In addition, let  $\mathfrak{h}(p_1, p_2, r)$  be the intersection of all closed balls in  $\mathbb{R}^d$  of radius  $r$  that contain both  $p_1$  and  $p_2$  (see Colesanti and Manselli (2010, Definition 3.1)).

**Theorem 2.** Let  $A$  be closed in  $\mathbb{R}^d$ . Two equivalent formulations of the reach are

$$\text{reach}(A) = \sup \{r \in \mathbb{R}^+ : \forall a_1, a_2 \in A, \|a_2 - a_1\| < 2r \Rightarrow \mathfrak{b}(a_1, a_2, r) \cap A \neq \emptyset\} \quad (6)$$

and

$$\text{reach}(A) = \inf \left\{ g_{\|a_2 - a_1\|} \circ \delta_A \left( \frac{a_1 + a_2}{2} \right) : a_1, a_2 \in A \right\}, \quad (7)$$

where  $g_{\|a_2 - a_1\|}$  and  $\mathfrak{b}(a_1, a_2, r)$  are as in (3) and (5) respectively, and  $\delta_A$  is the distance-to-set function defined in Section 2.

See Figure 2 for a geometric interpretation of the function in (3) and its inverse in (4). Evidently from the figure, this function is derived from the height of a spherical cap; it is written about in Attali et al. (2013), Attali and Lieutier (2015), and Divol (2021) in reference to the reach.

*Proof of Theorem 2.* We prove the following two claims separately, which together imply (6).

- (i) If  $r > \text{reach}(A)$ , then there exist  $a_1, a_2 \in A$  such that  $\|a_2 - a_1\| < 2r$  and  $\mathfrak{b}(a_1, a_2, r) \cap A = \emptyset$ .
- (ii) If  $0 < r \leq \text{reach}(A)$ , then for all  $a_1, a_2 \in A$  such that  $\|a_2 - a_1\| < 2r$ , the set  $\mathfrak{b}(a_1, a_2, r) \cap A$  is not empty.

We begin by showing the first claim. Let  $r > \text{reach}(A)$  and let  $\tilde{r} \in (\text{reach}(A), r)$ . By the definition of  $\text{reach}(A)$  (Definition 3),  $\exists p \in A_{\tilde{r}}$  and  $a_1, a_2 \in A$  such that  $\|a_1 - p\| = \|a_2 - p\| = \delta_A(p) \leq \tilde{r}$ . It is easy to check that  $\mathfrak{b}(a_1, a_2, r)$  is contained in the interior of  $B(p, \delta_A(p))$ , which does not intersect  $A$ . Since  $\|a_2 - a_1\| \leq 2\tilde{r} < 2r$  by the triangle inequality, item (i) holds.

Now, to prove the second claim, we recall Theorem 3.8 in Colesanti and Manselli (2010): For  $0 < r \leq \text{reach}(A)$  and any  $a_1, a_2 \in A$  such that  $\|a_2 - a_1\| < 2r$ , it holds that  $\mathfrak{h}(a_1, a_2, r) \cap A$  is connected, where  $\mathfrak{h}(a_1, a_2, r)$  is as in Definition 4.

Let  $0 < r \leq \text{reach}(A)$  and suppose that for some  $a_1, a_2 \in A$ , both  $\|a_2 - a_1\| < 2r$  and  $\mathfrak{b}(a_1, a_2, r) \cap A = \emptyset$  hold. Then

$$\mathfrak{h}(a_1, a_2, r) \cap A = (\mathfrak{h}(a_1, a_2, r) \setminus \mathfrak{b}(a_1, a_2, r)) \cap A$$

is disconnected, since  $a_1$  and  $a_2$  are in two disconnected components of  $\mathfrak{h}(a_1, a_2, r) \setminus \mathfrak{b}(a_1, a_2, r)$ . This contradicts Theorem 3.8 in Colesanti and Manselli (2010). Thus, claim (ii) holds.

Now we show the equality in (7). Since  $g_\alpha(x)$  is monotonically decreasing in  $x$ , it holds for any  $a_1, a_2 \in A$  and  $r > \|a_2 - a_1\|/2$  that,

$$\mathfrak{b}(a_1, a_2, r) \cap A = \emptyset \Leftrightarrow \delta_A \left( \frac{a_1 + a_2}{2} \right) > g_{\|a_2 - a_1\|}^{-1}(r) \Leftrightarrow g_{\|a_2 - a_1\|} \circ \delta_A \left( \frac{a_1 + a_2}{2} \right) < r.$$

Thus, by Claim (i) above, the RHS of (7) is at most  $\text{reach}(A)$ . To finish the proof, we fix  $r \in (-\infty, \text{reach}(A))$ , and aim to show that  $g_{\|a_2 - a_1\|} \circ \delta_A \left( \frac{a_1 + a_2}{2} \right) \geq r$  for all  $a_1, a_2 \in A$ . If  $\|a_2 - a_1\| \geq 2r$ , then the image of  $g_{\|a_2 - a_1\|}$  is contained in  $[r, \infty]$ . If  $\|a_2 - a_1\| < 2r$ , then  $\mathfrak{b}(a_1, a_2, r) \cap A \neq \emptyset$  by Claim (ii), and so  $g_{\|a_2 - a_1\|} \circ \delta_A \left( \frac{a_1 + a_2}{2} \right) \geq r$  as desired.  $\square$

## 4 Methods for point cloud data

In practice, one might be interested in understanding the smoothness of a set  $A \subseteq \mathbb{R}^d$  from a discrete set of points that are known to reside in  $A$ . In this section, we provide methods of bounding the  $r$ -convexity and the reach of  $A$  given such a set of points. We show that as the sampling points becomes dense in  $A$ , these upper bounds converge to their respective quantities. The Hausdorff metric serves as an excellent candidate to quantify the density of points in the set, since it captures the minimal local density of points within the set. Convergence is thus expressed in terms of the Hausdorff distance from the point cloud to the set of interest tending to 0, as has been done in, for example, Cuevas and Fraiman (1997).

### 4.1 A converging upper bound for $r$ -convexity

We have seen in Section 3.1 that in cases where  $\partial A$  is  $(d-1)$ -dimensional and  $C^1$  smooth, the reach is equal to the maximal  $r$  such that the set is  $r$ -convex. More generally, this holds even for some sets that are not  $C^1$  continuous at some “outward pointing” points on their boundaries, such as the filled square in  $\mathbb{R}^2$ . Thus, in many cases, the reach of a set can be obtained through its  $r$ -convexity (see Section 3.1). At the very least, the reach is always bounded above by the maximal  $r$  such that it is  $r$ -convex (see Theorem 1). In this section, we provide a simple and computationally efficient algorithm for bounding  $\text{rconv}(A)$  above for a compact set  $A \subset \mathbb{R}^d$ , based on discrete samples of the space  $\mathbb{R}^d$ . Moreover, if the sampling becomes dense in  $\mathbb{R}^d$ , we show that this method obtains  $\text{rconv}(A)$  exactly.

Here, we introduce the sample points in  $\mathbb{R}^d$ , and analogous operations to those of dilation and erosion for subsets of these points.

**Definition 5.** Let  $\mathcal{P}$  be a point cloud in  $\mathbb{R}^d$ . For a set  $\hat{A} \subseteq \mathcal{P}$ , we make a slight abuse of notation and denote for  $r \in \mathbb{R}$ ,

$$\hat{A}_r := \begin{cases} \{p \in \mathcal{P} : \delta_{\hat{A}}(p) \leq r\}, & \text{for } r \geq 0, \\ \{a \in \hat{A} : \delta_{\mathcal{P} \setminus \hat{A}}(a) > -r\}, & \text{for } r < 0. \end{cases}$$

*Remark 1.* We emphasize that for a point cloud  $\mathcal{P}$ , a subset  $\hat{A} \subseteq \mathcal{P}$ , and a real number  $r \in \mathbb{R}$ , Definition 5 implies that  $\hat{A}_r \subseteq \mathcal{P}$ . Contrast this with the set  $\hat{A} \oplus B(0, |r|)$ , which is not contained in  $\mathcal{P}$  for  $r > 0$ .

#### 4.1.1 Problems with the traditional closing operation on discrete data

In image analysis, if  $\mathcal{P}$  is taken to be the pixels of an image, then the closing of a set  $\hat{A} \subseteq \mathcal{P}$  by a distance  $r > 0$  is usually taken to be  $(\hat{A}_r)_{-r}$ . The closing operation for discrete data, like the closing operation on sets, satisfies  $(\hat{A}_r)_{-r} \supseteq \hat{A}$ . If  $\hat{A}$  is chosen to describe a set  $A \subset \mathbb{R}^2$  such that  $\hat{A} = A \cap \mathcal{P}$ , then one might be interested in understanding the smoothness of  $A$  via  $\hat{A}$ . For  $r > 0$  it is quite clear that  $(\hat{A}_r)_{-r} = \hat{A}$  does not imply  $A_{\bullet r} = A$ , because there may be fine discontinuities of the set  $A$  that are not captured by  $\mathcal{P}$ . One might then ask if the converse holds, *i.e.*, if  $A_{\bullet r} = A$  implies  $(\hat{A}_r)_{-r} = \hat{A}$ . Indeed, it does not. See Figure 3 for a visual example of a set  $A$  satisfying  $A_{\bullet r} = A$  and  $(\hat{A}_r)_{-r} \neq \hat{A}$ .

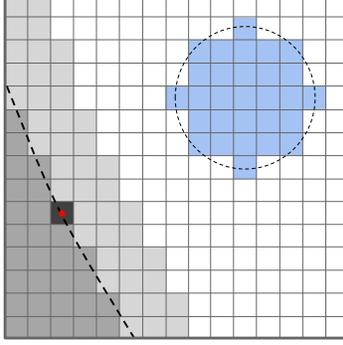


Figure 3: Here is an example where the closing of a discrete set  $\widehat{A}$  by a distance  $r$  (3 pixels) is strictly larger than  $\widehat{A}$ , however the underlying set  $A$  is  $r$ -convex. Medium-grey pixels belong to  $\widehat{A}$ , the dark-grey pixel is the one remaining pixel in  $(\widehat{A}_r)_{-r}$  that is not in  $\widehat{A}$ , and the remaining light-grey pixels are in  $\widehat{A}_r$ .

#### 4.1.2 A corrected closing operation to test for $r$ -convexity

In this section, we introduce a modified closing operation on discrete point clouds that does not suffer from the pitfall discussed in Section 4.1.1.

**Theorem 3.** For  $n \in \mathbb{N}^+$ , let  $\mathcal{P}^{(n)}$  be a point cloud in  $\mathbb{R}^d$ , and suppose that

$$\epsilon_n := \sup \{ \delta_{\mathcal{P}^{(n)}}(q) : q \in \mathbb{R}^d \} \quad (8)$$

is finite. Let  $A$  be a compact subset of  $\mathbb{R}^d$ , and for  $n \in \mathbb{N}^+$ , define  $\widehat{A}^{(n)} := A \cap \mathcal{P}^{(n)}$  and the corresponding approximation of  $\text{rconv}(A)$ ,

$$\widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)}) := \inf \left\{ r > \epsilon_n : (\widehat{A}^{(n)})_{r-\epsilon_n} \not\subseteq \widehat{A}^{(n)} \right\}. \quad (9)$$

It holds that

$$\inf_{n \in \mathbb{N}^+} \widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)}) \geq \text{rconv}(A). \quad (10)$$

Moreover, if  $\epsilon_n \rightarrow 0$  and  $d_H(\widehat{A}^{(n)}, A) \rightarrow 0$  as  $n \rightarrow \infty$ , then

$$\lim_{n \rightarrow \infty} \widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)}) = \text{rconv}(A). \quad (11)$$

*Remark 2.* Note that  $\widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)})$  in (9) can be computed efficiently using entirely available information, since  $\epsilon_n$  in (8) is a feature of the point cloud  $\mathcal{P}^{(n)}$  and not of the unknown set  $A$ . In particular, there is no need to estimate  $d_H(\widehat{A}^{(n)}, A)$  for the construction of  $\widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)})$ . A binary search algorithm, along with the discrete dilation operations in Definition 5, are sufficient for the numerical calculation of  $\widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)})$ . See Figure 4 for an example of a set  $A$  for which  $(\widehat{A}^{(n)})_{r-\epsilon_n} \not\subseteq \widehat{A}^{(n)}$ .

*Remark 3.* The requirement that  $\mathcal{P}^{(n)}$  extends over all of  $\mathbb{R}^d$  allows for a mathematical simplification and is not needed in practice. For applications on data, one only requires that  $d_H(\mathcal{P}^{(n)}, T) = \epsilon_n$  for some compact  $T \subset \mathbb{R}^d$  that contains  $A_{\text{rconv}(A)+2\epsilon_n}$ . In the case where  $A$  is lower dimensional than the embedding space  $\mathbb{R}^d$ , one can add any finite number of points uniformly distributed on  $T$  to  $\mathcal{P}^{(n)}$ , and they will almost surely not belong to  $A$ .

The following result from Rodríguez Casal and Saavedra-Nieves (2016), while interesting on its own, is instrumental in the proof of Theorem 3.

**Proposition 1** (Lemma 8.3 in Rodríguez Casal and Saavedra-Nieves (2016)). Let  $r \in \mathbb{R}^+$  and let  $A \subset \mathbb{R}^d$  be a closed set satisfying  $\text{rconv}(A) < r$ . The set  $A_{\bullet r} \setminus A$  contains an open subset of  $\mathbb{R}^d$ .

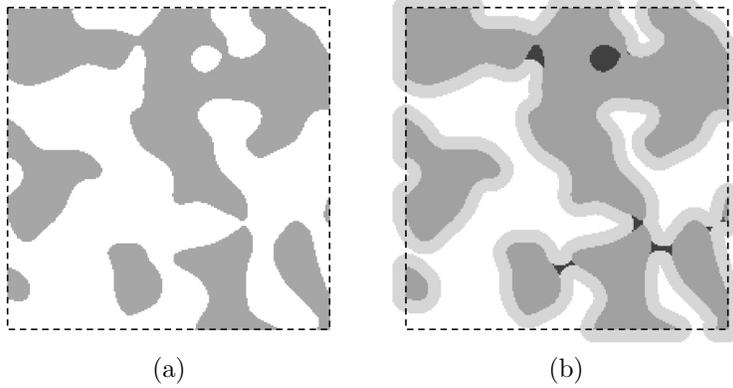


Figure 4: We test the  $r$ -convexity of a set  $A \subset \mathbb{R}^2$  via  $\widehat{A}^{(n)}$  (grey regions in panel (a)) for  $r = 9a_n$ , where  $a_n$  is the lattice spacing of the square lattice  $\mathcal{P}^{(n)}$ . With,  $\epsilon_n = a_n/\sqrt{2}$ , we see that  $(\widehat{A}_{r-\epsilon_n}^{(n)})_{-(r+\epsilon_n)} \setminus \widehat{A}^{(n)}$  (dark grey regions in panel (b)) is not empty, thus,  $\text{rconv}(A) \leq \widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)}) \leq 9a_n$ . The remaining pixels in  $\widehat{A}_{r-\epsilon_n}^{(n)}$  are shaded in light grey in panel (b).

We provide an alternative proof of Proposition 1 in Section 5.

*Proof of Theorem 3.* We begin by showing (10). Let  $n \in \mathbb{N}^+$  and fix  $p \in \mathcal{P}^{(n)} \setminus \widehat{A}^{(n)}$ . If  $\epsilon_n \geq \text{rconv}(A)$ , then (10) holds trivially. Now, let  $r \in \mathbb{R}^+$  be such that  $\epsilon_n < r < \text{rconv}(A)$  so that  $A_{\bullet, r} = A$ . We aim to show that  $p \notin (\widehat{A}_{r-\epsilon_n}^{(n)})_{-(r+\epsilon_n)}$ . Indeed, by the  $r$ -convexity of  $A$ , there exists  $x \in (A_r)^c$  such that  $\|x - p\| < r$ . In addition,  $B(x, \epsilon_n) \cap A_{r-\epsilon_n} = \emptyset$ , so there exists  $q \in \mathcal{P}^{(n)} \setminus \widehat{A}_{r-\epsilon_n}^{(n)}$  such that  $\|q - x\| \leq \epsilon_n$ . Therefore, by the triangle inequality,  $\|q - p\| \leq r + \epsilon_n$ , which implies  $\delta_{\mathcal{P}^{(n)} \setminus \widehat{A}_{r-\epsilon_n}^{(n)}}(p) \leq r + \epsilon_n$ . Thus,  $p \notin (\widehat{A}_{r-\epsilon_n}^{(n)})_{-(r+\epsilon_n)}$  as desired.

Now, to prove (11), first fix  $r > \text{rconv}(A)$ . To simplify notation, let  $\delta_n := d_H(\widehat{A}^{(n)}, A)$ . By Proposition 1, there exists an open subset  $O \subseteq A_{\bullet, r} \setminus A$  that contains a closed ball of radius  $\delta_{n_0} + 3\epsilon_{n_0}$  for sufficiently large  $n_0 \in \mathbb{N}^+$ . Let  $n \geq n_0$ . The point cloud  $\mathcal{P}^{(n)}$  is sufficiently dense such that there exists  $q \in O_{-(\delta_n + 2\epsilon_n)} \cap \mathcal{P}^{(n)}$ . Importantly, this implies that  $q \in (A_r)_{-(r+\delta_n+2\epsilon_n)} \cap \mathcal{P}^{(n)}$ . By Lemma 1, for all  $s > r$ , we have  $(A_r)_{-(r+\delta_n+2\epsilon_n)} \subseteq ((A_r)_{\bullet, (s-r)})_{-(r+\delta_n+2\epsilon_n)} = (A_s)_{-(s+\delta_n+2\epsilon_n)}$ , and therefore  $q \in (A_s)_{-(s+\delta_n+2\epsilon_n)} \cap \mathcal{P}^{(n)}$ . Notice that  $A_s \cap \mathcal{P}^{(n)} \subseteq \widehat{A}_{\delta_n+s}^{(n)}$ , which implies  $(A_s)^c \supseteq \mathcal{P}^{(n)} \setminus \widehat{A}_{\delta_n+s}^{(n)}$  and thus  $(A_s)_{-(s+\delta_n+2\epsilon_n)} \subseteq (\widehat{A}_{\delta_n+s}^{(n)})_{-(s+\delta_n+2\epsilon_n)}$ . Summarizing, we have shown that there is a point  $q$  in  $(\widehat{A}_{\delta_n+s}^{(n)})_{-(s+\delta_n+2\epsilon_n)}$  that is not in  $\widehat{A}^{(n)}$ . By the change of variables  $\tilde{s} := s + \delta_n + \epsilon_n$ , it follows that  $(\widehat{A}_{\tilde{s}-\epsilon_n}^{(n)})_{-(\tilde{s}+\epsilon_n)} \setminus \widehat{A}^{(n)} \neq \emptyset$  for all  $\tilde{s} > r + \delta_n + \epsilon_n$ , which implies  $\widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)}) \leq r + \delta_n + \epsilon_n$ . Sending  $n \rightarrow \infty$  yields  $\lim_{n \rightarrow \infty} \widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)}) \leq r$ , and since  $r \in (\text{rconv}(A), \infty)$  was chosen freely,  $\lim_{n \rightarrow \infty} \widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)}) \leq \text{rconv}(A)$ . This result, along with (10), gives the convergence in (11).  $\square$

*Remark 4.* Equation (11) is provided without a rate of convergence. This is due to the fact that Proposition 1 provides no guarantees on the size of the open subset that can be found in  $A_{\bullet, r} \setminus A$  for  $r > \text{rconv}(A)$ . To provide a rate of convergence, a deeper analysis is needed to understand the rate at which the size of the largest open ball in  $A_{\bullet, r} \setminus A$  decreases as  $r \rightarrow \text{rconv}(A)$ . By contrast, the bound that we construct for the reach in Section 4.2 converges to the reach at a known rate (see Theorem 4 below).

## 4.2 A converging upper bound for the reach

In the general case where the boundary of a compact set  $A \subset \mathbb{R}^d$ , is not  $(d-1)$ -dimensional, or perhaps not  $C^1$  smooth, one does not have any guarantees that  $\widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)})$  (the upper bound introduced in Section 4.1.2) converges to the reach of  $A$  as  $\mathcal{P}^{(n)}$  becomes dense in  $\mathbb{R}^d$ . Nonetheless, one can leverage the fact that  $\widehat{\text{r}}^{(\epsilon_n)}(\widehat{A}^{(n)})$  is an upper-bound for  $\text{reach}(A)$  by Theorems 1 and 3. In this section we introduce a method of approximating  $\text{reach}(A)$ , based on Theorem 2. To guarantee a rate of convergence, we impose assumptions on a quantity which we define here as the  $\beta$ -reach of  $A$ .

**Definition 6.** For a closed set  $A \subseteq \mathbb{R}^d$  and  $\beta \in \mathbb{R}^+ \cup \{0\}$ , let the  $\beta$ -reach of  $A$  be defined as

$$\text{reach}_\beta(A) := \inf \left\{ g_{\|a_2 - a_1\|}(x) : a_1, a_2 \in A, x = \delta_A \left( \frac{a_1 + a_2}{2} \right) \geq \beta \right\},$$

where  $g_{\|a_2 - a_1\|}(x)$  is defined in (3).

Importantly, the  $\beta$ -reach of a set for  $\beta = 0$  is identified with the reach, *i.e.*,  $\text{reach}_0(A) = \text{reach}(A)$  for closed  $A \subseteq \mathbb{R}^d$  (see Theorem 2, Equation (7)). Another easily verifiable property of the  $\beta$ -reach is that for any closed set  $A$ , the  $\beta$ -reach satisfies  $\text{reach}_\beta(A) \geq \beta$ .

*Remark 5.* The  $\beta$ -reach of a set is closely related to the *spherical distortion radius* (with parameter  $\delta$ ) introduced in Aamari et al. (2022), or the  $\mu$ -reach (with parameter  $\mu$ ) introduced in Chazal et al. (2009). Indeed, all three notions give a measure of reach that excludes ‘‘local’’ effects, where locality is measured by the corresponding parameter  $\beta$ ,  $\delta$ , or  $\mu$ .

**Lemma 2.** For  $A \subset \mathbb{R}^d$  closed, the function  $\varphi : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$  that maps  $\beta \mapsto \text{reach}_A(\beta)$  is non-decreasing, and is continuous at  $\beta = 0$ , so that  $\lim_{\beta \rightarrow 0} \text{reach}_\beta(A) = \text{reach}(A)$ .

*Proof of Lemma 2.* The non-decreasing property is seen via the inclusion

$$\left\{ (a_1, a_2) \in A^2 : \delta_A \left( \frac{a_1 + a_2}{2} \right) \geq \beta_2 \right\} \subseteq \left\{ (a_1, a_2) \in A^2 : \delta_A \left( \frac{a_1 + a_2}{2} \right) \geq \beta_1 \right\},$$

for all  $\beta_1, \beta_2 \in \mathbb{R}$  satisfying  $\beta_2 > \beta_1$ . Moreover, this implies  $\lim_{\beta \searrow 0} \text{reach}_\beta(A) \geq \text{reach}_0(A)$ . Now, it suffices to show the reverse inequality. Let  $\epsilon > 0$ . By Theorem 2, Equation (7), there exists  $a_1, a_2 \in A$  satisfying

$$g_{\|a_2 - a_1\|} \circ \delta_A \left( \frac{a_1 + a_2}{2} \right) < \text{reach}(A) + \epsilon = \text{reach}_0(A) + \epsilon.$$

Any such pair  $(a_1, a_2)$  must satisfy  $\delta_A \left( \frac{a_1 + a_2}{2} \right) > 0$ , and so for  $\beta \in (0, \delta_A \left( \frac{a_1 + a_2}{2} \right))$ , it holds that  $\text{reach}_\beta(A) < \text{reach}_0(A) + \epsilon$ . Thus,  $\lim_{\beta \searrow 0} \text{reach}_\beta(A) \leq \text{reach}_0(A)$ .  $\square$

The following assumption, imposed on the  $\beta$ -reach of a set, serves as a regularity condition that is used to prove a convergence rate for the approximation of the reach in Theorem 4.

**Assumption 1.** Suppose that the set  $A \subset \mathbb{R}^d$  is compact, and that there exists  $\delta > 0$  such that  $\varphi : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$  defined by  $\beta \mapsto \text{reach}_A(\beta)$  is either constant or strictly increasing on  $[0, \delta]$ . In addition, suppose that

$$K_A := \lim_{\beta \searrow 0} \frac{\text{reach}_\beta(A) - \text{reach}(A)}{\beta} \tag{12}$$

exists and is finite.

Assumption 1, crucial for the proof of Theorem 4 below, is satisfied by a large class of sets. Consider the following examples.

*Example 1.* For two line segments with ends joined by an angle  $\theta \in (0, \pi)$ , the  $\beta$ -reach of their union is  $\frac{\beta}{2} (1 + \sec^2(\frac{\theta}{2}))$ , for sufficiently small  $\beta > 0$ .

*Example 2.* The  $\beta$ -reach of a circular arc is equal to the radius of the arc, for sufficiently small  $\beta > 0$ .

*Example 3.* Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^2$  smooth function with  $h'(0) > 0$ . Suppose that the graph of the function  $f : [-1, 1] \rightarrow \mathbb{R}$  defined by  $f(x) := h(x^2)$  obtains its maximal curvature at  $x = 0$ . Then, the set  $G := \{(x, f(x)) : -1 \leq x \leq 1\} \subset \mathbb{R}^2$  satisfies

$$\text{reach}(G) = \frac{1}{2h'(0)} \tag{13}$$

and

$$\text{reach}_\beta(G) = \frac{1}{2h'(0)} + \left( \frac{1}{2} - \frac{h''(0)}{4h'(0)^3} \right) \beta + o(\beta), \tag{14}$$

for sufficiently small  $\beta > 0$ . Justifications for Equation (13) and (14) are provided at the end of Section 5.

Assumption 1 is imposed on the  $\beta$ -reach of  $A$  alternatively to the regularity of  $\partial A$  since the regularity of  $\beta$ -reach lends very naturally to the proof of Theorem 4. Moreover, Assumption 1 allows for most compact sets in  $\mathbb{R}^d$ , excluding some pathological counterexamples.

**Theorem 4.** *Let  $A \subset \mathbb{R}^d$  satisfy Assumption 1. For  $n \in \mathbb{N}^+$ , let  $\mathcal{P}^{(n)}$  be a point cloud in  $\mathbb{R}^d$ , and denote  $\widehat{A}^{(n)} := A \cap \mathcal{P}^{(n)}$ . Let  $(\epsilon_n)_{n \geq 1}$  be a sequence in  $\mathbb{R}^+$  tending to 0 as  $n \rightarrow \infty$ , and suppose that  $\epsilon_n \geq d_H(\widehat{A}^{(n)}, A)$  for all but finitely many  $n$ . For each  $n \in \mathbb{N}^+$ , define the corresponding approximation of  $\text{reach}(A)$  by*

$$\widehat{\text{rch}}^{(\epsilon_n)}(\widehat{A}^{(n)}) := \sup \left\{ r \in \mathbb{R}^+ : \forall a_1, a_2 \in \widehat{A}^{(n)}, \|a_2 - a_1\| < 2r \Rightarrow \mathfrak{b}(a_1, a_2, r)_{\epsilon_n} \cap \widehat{A}^{(n)} \neq \emptyset \right\}, \quad (15)$$

where  $\mathfrak{b}(a_1, a_2, r)_{\epsilon_n}$  is the ball in (5) dilated by an amount  $\epsilon_n$ . Then there exists  $n_0 \in \mathbb{N}^+$  such that

$$0 \leq \widehat{\text{rch}}^{(\epsilon_n)}(\widehat{A}^{(n)}) - \text{reach}(A) \leq \widetilde{K} \sqrt{\epsilon_n}, \quad (16)$$

for all  $n \geq n_0$ , where  $\widetilde{K} := (2 \text{reach}(A) + \frac{K_A}{8} + \frac{1}{2})$ , with  $K_A$  defined in (12).

*Proof of Theorem 4.* In the following, we assume that  $A$  is not convex; for if it is convex, then  $\widehat{\text{rch}}^{(\epsilon_n)}(\widehat{A}^{(n)}) = \sup \mathbb{R}^+ = \infty$  for all  $n \in \mathbb{N}^+$ . Fix  $n_0 \in \mathbb{N}^+$  be such that  $\epsilon_n \geq d_H(\widehat{A}^{(n)}, A)$  for all  $n \geq n_0$ . We begin by showing first inequality in (16). If  $\text{reach}(A) = 0$ , then  $0 \leq \widehat{\text{rch}}^{(\epsilon_n)}(\widehat{A}^{(n)})$  holds trivially. Now, suppose that  $\text{reach}(A) > 0$  and let  $r \in (0, \text{reach}(A))$ . By Equation (6) in Theorem 2, there exists an  $s \in [r, \text{reach}(A)]$  such that for all  $a_1, a_2 \in \widehat{A}^{(n)}$ , the implication  $\|a_2 - a_1\| < 2s \Rightarrow \mathfrak{b}(a_1, a_2, s) \cap A \neq \emptyset$  holds. Moreover, for  $n \geq n_0$ , the implication  $\mathfrak{b}(a_1, a_2, s) \cap A \neq \emptyset \Rightarrow \mathfrak{b}(a_1, a_2, s)_{\epsilon_n} \cap \widehat{A}^{(n)} \neq \emptyset$  holds. Therefore, for such  $n$ , we have  $\widehat{\text{rch}}^{(\epsilon_n)}(\widehat{A}^{(n)}) \geq s \geq r$ , and since  $r \in (0, \text{reach}(A))$  was chosen freely, the first inequality in (16) holds.

Now, we show the second inequality in (16). Let  $n \geq n_0$ , and suppose that there exist  $a_1, a_2 \in \widehat{A}^{(n)}$  such that  $\delta_A(\frac{a_1+a_2}{2}) > \epsilon_n$ . Then  $\mathfrak{b}(a_1, a_2, s)_{\epsilon_n} \cap A = \emptyset$  for  $s \geq g_{\|a_2-a_1\|}(\delta_A(\frac{a_1+a_2}{2}) - \epsilon_n)$ , and so  $\widehat{\text{rch}}^{(\epsilon_n)}(\widehat{A}^{(n)}) \leq s$  for all such  $s$ . Indeed,

$$\widehat{\text{rch}}^{(\epsilon_n)}(\widehat{A}^{(n)}) \leq \inf \left\{ g_{\|a_2-a_1\|} \left( \delta_A \left( \frac{a_1+a_2}{2} \right) - \epsilon_n \right) : a_1, a_2 \in \widehat{A}^{(n)}, \delta_A \left( \frac{a_1+a_2}{2} \right) > \epsilon_n \right\}.$$

Now, we recall Equation (7) in Theorem 2, and split the analysis into two cases.

**Case 1:** *There exists  $a_1, a_2 \in A$  such that  $a_1 \neq a_2$  and  $\text{reach}(A) = g_{\|a_2-a_1\|} \circ \delta_A(\frac{a_1+a_2}{2})$ .*

Let  $n \geq n_0$ , then for this pair  $a_1, a_2 \in A$  there exists  $p_1, p_2 \in \widehat{A}^{(n)}$  satisfying  $\|p_2 - a_2\|, \|p_1 - a_1\| \leq \epsilon_n$ . This, in turn, implies

$$\left\| \frac{p_1 + p_2}{2} - \frac{a_1 + a_2}{2} \right\| \leq \epsilon_n.$$

Let  $n$  be sufficiently large such that  $2\epsilon_n < x := \delta_A(\frac{a_1+a_2}{2})$  so that  $\epsilon_n < x - \epsilon_n \leq \delta_A(\frac{p_1+p_2}{2}) \leq x + \epsilon_n$ . Then,

$$\begin{aligned} \widehat{\text{rch}}^{(\epsilon_n)}(\widehat{A}^{(n)}) - \text{reach}(A) &\leq g_{\|p_2-p_1\|} \left( \delta_A \left( \frac{p_1+p_2}{2} \right) - \epsilon_n \right) - g_{\|a_2-a_1\|}(x) \\ &\leq \left( \frac{(\|a_2 - a_1\| + 2\epsilon_n)^2}{8(x - 2\epsilon_n)} + \frac{x}{2} \right) - \left( \frac{\|a_2 - a_1\|^2}{8x} + \frac{x}{2} \right) = \mathcal{O}(\epsilon_n) \end{aligned}$$

which is stronger than required.

**Case 2:**  *$\text{reach}(A) < g_{\|a_2-a_1\|} \circ \delta_A(\frac{a_1+a_2}{2})$  for all distinct  $a_1, a_2 \in A$ .*

By Assumption 1, for all sufficiently large  $n$ , the function  $\text{reach}_\beta(A)$  is strictly increasing for  $\beta$  in a neighbourhood of  $\beta_n := \sqrt{\epsilon_n} + 2\epsilon_n$ . For each of these values of  $n$ , there exists a sequence  $\{(\psi_{1k}, \psi_{2k})\}_{k \geq 1}$  in  $\Psi_n := \{(a_1, a_2) \in A^2 : \delta_A(\frac{a_1+a_2}{2}) \geq \beta_n\}$  such that

$$g_{\|\psi_{2k}-\psi_{1k}\|} \circ \delta_A \left( \frac{\psi_{1k} + \psi_{2k}}{2} \right) \rightarrow \text{reach}_{\beta_n}(A), \quad (17)$$

as  $k \rightarrow \infty$ . Since  $\Psi_n$  is compact,  $\{(\psi_{1k}, \psi_{2k})\}_{k \geq 1}$  has a convergent subsequence in  $\Psi_n$ , and by the strict monotonicity of  $\text{reach}_\beta(A)$  around  $\beta_n$ , the subsequence converges to an element  $(a_{1n}, a_{2n}) \in A^2$  satisfying  $\delta_A\left(\frac{a_{1n}+a_{2n}}{2}\right) = \beta_n$ . Thus, for all  $n$  sufficiently large, we have shown the existence of an element  $(a_{1n}, a_{2n}) \in A^2$  that satisfies  $\delta_A\left(\frac{a_{1n}+a_{2n}}{2}\right) = \beta_n$  and  $g_{\|a_{2n}-a_{1n}\|} \circ \delta_A\left(\frac{a_{1n}+a_{2n}}{2}\right) = \text{reach}_{\beta_n}(A)$ .

By Assumption 1,

$$\text{reach}_{\beta_n}(A) = \frac{\|a_{2n} - a_{1n}\|^2}{8\beta_n} + \frac{\beta_n}{2} = \text{reach}(A) + K_A\beta_n + o(\beta_n).$$

Therefore,

$$\|a_{2n} - a_{1n}\|^2 = 8\beta_n \text{reach}(A) + K_A\beta_n^2 + o(\beta_n^2) = 8\sqrt{\epsilon_n} \text{reach}(A) + (16 \text{reach}(A) + K_A)\epsilon_n + o(\epsilon_n).$$

For sufficiently large  $n$ , the points in  $\mathcal{P}^{(n)}$  are sufficiently dense so that there exists  $p_1, p_2 \in \widehat{A}^{(n)}$  satisfying  $\|p_2 - a_{2n}\|, \|p_1 - a_{1n}\| \leq \epsilon_n$ . As argued before in Case 1,

$$\widehat{\text{rch}}^{(\epsilon_n)}(\widehat{A}^{(n)}) \leq \frac{(\|a_{2n} - a_{1n}\| + 2\epsilon_n)^2}{8(\beta_n - 2\epsilon_n)} + \frac{\beta_n}{2} = \text{reach}(A) + \left(2 \text{reach}(A) + \frac{K_A}{8} + \frac{1}{2}\right) \sqrt{\epsilon_n} + o(\sqrt{\epsilon_n}).$$

□

*Remark 6.* The split of the proof of Theorem 4 into two cases is natural in the context of reach estimation. Recall Theorem 3.4 in Aamari et al. (2019): For a compact set  $A$  with  $0 < \text{reach}(A) < \infty$ , there is either a bottleneck structure (two distinct points  $a_1, a_2 \in A$  such that  $\|a_2 - a_1\| = 2\delta_A\left(\frac{a_1+a_2}{2}\right) = 2 \text{reach}(A)$ ) or an arc-length parametrized geodesic of  $A$  with curvature  $1/\text{reach}(A)$ . The convergence rate developed in Case 1 applies to sets whose reach is decided by a bottleneck structure, and that of Case 2 applies to sets whose reach is decided by a region of high curvature.

### 4.3 Simulations

The compact subsets of  $\mathbb{R}^2$  that we study empirically are constructed such that their reach is unity and that they are not  $r$ -convex for  $r > 1$ . This choice does not limit the generality of these simulation studies, since we sample the sets on randomly oriented square lattices with varying lattice spacing. Indeed, to consider the lattice of sampling points at various scales and orientations is equivalent to considering a fixed lattice and various scales and orientations of the underlying set.

As discussed in Remark 6, there are two fundamental phenomena that limit the reach of set: maximal curvature and bottleneck structures. In this section, to each of these two cases, we dedicate its own simulation study to assess the rate of convergence of the approximation methods developed in Sections 4.1 and 4.2.

#### 4.3.1 When the reach is determined by maximal curvature

Here, we study the compact set  $U := \{(x, y) \in \mathbb{R}^2 : y \leq x^2/2\} \cap B(0, 10)$ . It is easy to check that  $\text{reach}(U) = \text{rconv}(U) = 1$ . Moreover, the reach is determined by a point of maximal curvature at  $\mathbf{0} \in U$ . For this simulation study, we suppose that  $U$  is sampled on a sequence of point clouds  $(\mathcal{P}^{(n)})_{n \geq 1}$ , where  $\mathcal{P}^{(n)}$  is a square lattice with a lattice spacing  $a_n = 0.7/n$ , and uniformly random position and orientation. For  $n \in \mathbb{N}^+$ , let  $\widehat{U}^{(n)} := U \cap \mathcal{P}^{(n)}$ . See Figure 5 for a depiction of particular realizations of  $\widehat{U}^{(2)}$  and  $\widehat{U}^{(10)}$ .

For each  $n \in \{2, \dots, 12\}$ , we compute 50 independent realizations of  $\widehat{U}^{(n)}$ , from which we obtain 50 independent realizations of  $\widehat{\text{rch}}^{(a_n/\sqrt{2})}(\widehat{U}^{(n)})$  in (9). Here,  $\epsilon_n = a_n/\sqrt{2}$  was chosen since  $d_H(\mathcal{P}^{(n)}, \mathbb{R}^2) = a_n/\sqrt{2}$ . From the same samples of  $\widehat{U}^{(n)}$ , we obtain 50 independent realizations  $\widehat{\text{rch}}^{(\sqrt{1.25}a_n)}(\widehat{U}^{(n)})$ , where  $\epsilon_n = \sqrt{1.25}a_n$  was chosen since  $U \cap B(0, 9) \subset \widehat{U}^{(n)} \oplus B(0, \sqrt{1.25}a_n)$  almost surely. The mean estimates, along with the empirical 95% confidence intervals, are plotted in Figure 7 (a). A linear regression on the log-log plot of the sample means shows that

$$\mathbb{E}[\widehat{\text{rch}}^{(a_n/\sqrt{2})}(\widehat{U}^{(n)})] \approx 1 + 1.71n^{-0.59} \quad \text{and} \quad \mathbb{E}[\widehat{\text{rch}}^{(\sqrt{1.25}a_n)}(\widehat{U}^{(n)})] \approx 1 + 1.54n^{-0.62}.$$

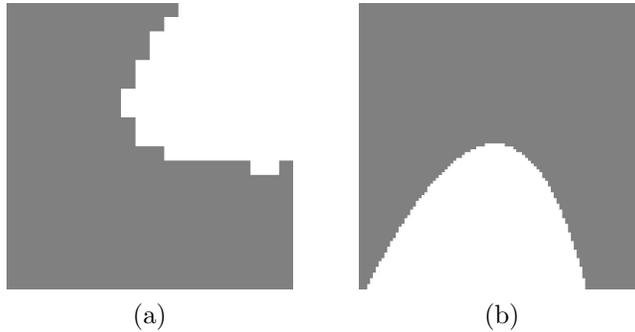


Figure 5: The set  $\{(x, y) \in \mathbb{R}^2 : y \leq x^2/2\}$  sampled on a randomly oriented square lattice with lattice spacing (a)  $a_n = 0.35$  and (b)  $a_n = 0.07$ . The orientation of the lattice is aligned with the orientation of each image.

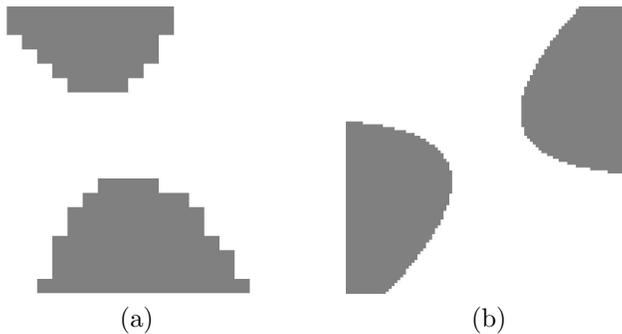


Figure 6: The set  $\{(x, y) \in \mathbb{R}^2 : |y| \geq x^2/2 + 1\}$  sampled on a randomly oriented square lattice with lattice spacing (a)  $a_n = 0.35$  and (b)  $a_n = 0.07$ . The orientation of the lattice is aligned with the orientation of each image.

These empirical convergence rates —  $\mathcal{O}(\epsilon_n^{0.59})$  for the bound on  $\text{rconv}(U)$  and  $\mathcal{O}(\epsilon_n^{0.62})$  for the bound on  $\text{reach}(U)$  — are both slightly faster than the predicted rate of  $\mathcal{O}(\sqrt{\epsilon_n})$  established in (16) for the bound on the reach. Since the reach of  $U$  is determined by a region of maximal curvature, the convergence rate is governed by the analysis in Case 2 in the proof of Theorem 4.

### 4.3.2 When the reach is determined by a bottleneck structure

Now, we study the compact set  $W := \{(x, y) \in \mathbb{R}^2 : |y| \geq x^2/2 + 1\} \cap B(0, 10)$ . Like the set  $U$  in Section 4.3.1, the set  $W$  satisfies  $\text{reach}(W) = \text{rconv}(W) = 1$ . This time, the reach is determined by the minimal distance between disconnected components of  $W$ . For this simulation study, we suppose that  $W$  is sampled on the same sequence of point clouds  $(\mathcal{P}^{(n)})_{n \geq 1}$  described in Section 4.3.1. For  $n \in \mathbb{N}^+$ , let  $\widehat{W}^{(n)} := W \cap \mathcal{P}^{(n)}$ . See Figure 6 for a depiction of particular realizations of  $\widehat{W}^{(2)}$  and  $\widehat{W}^{(10)}$ .

The experiment outlined in Section 4.3.1 is repeated for the samples  $\widehat{W}^{(n)}$ , and the results are plotted in Figure 7 (b). A linear regression on the data shows that

$$\mathbb{E}[\widehat{\text{r}}^{(a_n/\sqrt{2})}(\widehat{W}^{(n)})] \approx 1 + 2.81n^{-1.20} \quad \text{and} \quad \mathbb{E}[\widehat{\text{rch}}^{(\sqrt{1.25}a_n)}(\widehat{W}^{(n)})] \approx 1 + 0.65n^{-1.50}.$$

Again, these empirical convergence rates are faster than the anticipated rate of  $\mathcal{O}(\epsilon_n)$  established in Case 1 in the proof of Theorem 4 for the bound on the reach, *i.e.*, in the case where the reach is determined by a bottleneck structure.

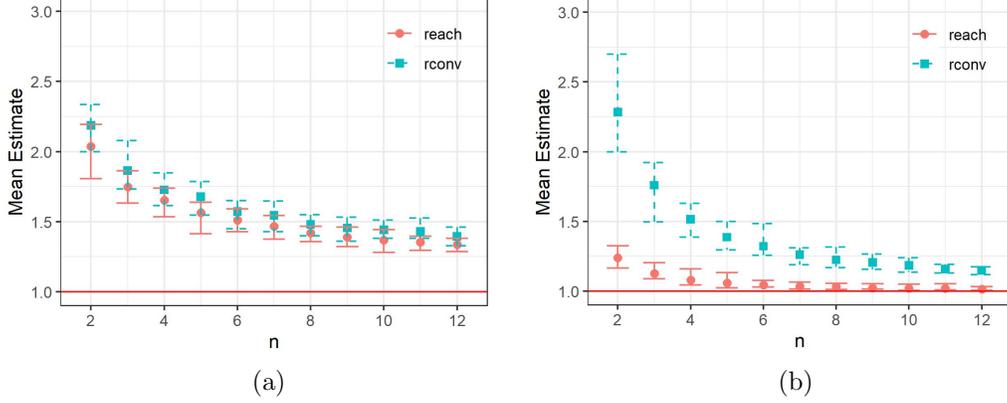


Figure 7: For each  $n \in \{2, \dots, 12\}$ , we compute 50 independent realizations of  $\widehat{U}^{(n)}$  (see Section 4.3.1), and plot the mean and 95% confidence interval for the 50 corresponding values of  $\widehat{r}^{(a_n/\sqrt{2})}(\widehat{U}^{(n)})$  (squares) and  $\widehat{rch}^{(\sqrt{1.25}a_n)}(\widehat{U}^{(n)})$  (circles) in panel (a). In panel (b), we show the values of  $\widehat{r}^{(a_n/\sqrt{2})}(\widehat{W}^{(n)})$  (squares) and  $\widehat{rch}^{(\sqrt{1.25}a_n)}(\widehat{W}^{(n)})$  for 50 independent realizations of  $\widehat{W}^{(n)}$  (see Section 4.3.2). In both panels,  $a_n = 0.7/n$ .

## 5 Proofs and technical results

Here, we provide the proofs and auxiliary lemmas that support the claims in Theorem 1, and Corollaries 1 and 2 in Section 3.1.

*Proof of Theorem 1.* We begin by showing (1). Suppose that  $r \in (0, \text{reach}(A))$ . By Lemma 1,  $A \subseteq A_{\bullet r}$ , so it suffices to show that  $A^c \subseteq (A_{\bullet r})^c$ . Let  $x \in A^c$ . If  $x \in (A_r)^c$ , then clearly,  $x \in (A_r)^c \oplus B(0, r) = (A_{\bullet r})^c$ . If  $x \in A_r \setminus A$ , then by Federer (1959, Corollary 4.9),

$$\delta_{(A_r)^c}(x) = r - \delta_A(x) < r,$$

and so  $x \in (A_r)^c \oplus B(0, r) = (A_{\bullet r})^c$ , which proves (1). What remains to be shown is that if  $\partial A$  is  $C^1$  smooth and  $(d-1)$ -dimensional, then  $\text{rconv}(A) \leq \text{reach}(A)$ . This inequality is shown via proof by contradiction. Suppose that

- (i)  $r > \text{reach}(A)$  and
- (ii)  $A_{\bullet r} = A$ .

By (i), there exists  $p \in A_{\frac{r+\text{reach}(A)}{2}}$  with no unique point in  $A$  closest to  $p$ . In particular, since  $A$  is closed, there exists two non-identical points  $a_1, a_2 \in \partial A \subseteq A$  such that  $\|p - a_1\| = \|p - a_2\| = \delta_A(p) < r$ . Let  $n_1$  be the unit normal vector to  $\partial A$  at  $a_1$ , pointing towards  $p$ . Since  $a_1$  is a limit point of  $A$ ,  $A^c \cap B(a_1, \epsilon) \neq \emptyset$  for all  $\epsilon > 0$ . By (ii),  $A^c = \bigcup_{x \in (A_r)^c} B(x, r)$ , and so for all  $\epsilon > 0$ , there exists  $x \in (A_r)^c$  such that

- (iii)  $B(x, r) \cap B(a_1, \epsilon) \neq \emptyset$  and
- (iv)  $B(x, r) \cap A = \emptyset$ .

The boundary  $\partial A$  is  $C^1$  smooth at  $a_1$ , so it is easily checked that for  $\epsilon$  close to 0, the locations  $x$  that satisfy both (iii) and (iv) are necessarily contained in a small neighbourhood around  $a_1 + rn_1$ . That is, there exists a mapping  $\theta : \epsilon \mapsto \theta(\epsilon) \in \mathbb{R}^+$  that tends to 0 as  $\epsilon \rightarrow 0$ , such that if  $x \in (A_r)^c$  and  $\epsilon \in \mathbb{R}^+$  satisfy (iii) and (iv), then  $x \in B(a_1 + rn_1, \theta(\epsilon))$ . Note that  $\|(a_1 + rn_1) - a_2\| < r$  since  $a_2 \in B(p, \delta_A(p)) \subset B(a_1 + rn_1, r)$ . Choose  $\epsilon$  such that  $\theta(\epsilon) < r - \|(a_1 + rn_1) - a_2\|$ . Then for any  $x \in (A_r)^c \cap B(a_1 + rn_1, \theta(\epsilon))$ ,

$$\begin{aligned} \|x - a_2\| &\leq \|x - (a_1 + rn_1)\| + \|(a_1 + rn_1) - a_2\| \\ &\leq \theta(\epsilon) + \|(a_1 + rn_1) - a_2\| < r. \end{aligned}$$

This contradicts (iv). □

**Lemma 3.** *Let  $U$  be an open set in  $\mathbb{R}^d$ , and let  $r \geq 0$ . Then,*

$$\text{cl}(U)_{\bullet r} = \text{cl}(U) \quad \implies \quad U_{\bullet r} = U.$$

*Proof of Lemma 3.* By Lemma 1, item (b),  $U \subseteq U_{\bullet r} \subseteq \text{cl}(U)_{\bullet r}$ . Note also that  $U_{\bullet r}$  is open. Suppose that  $\text{cl}(U)_{\bullet r} = \text{cl}(U)$ . Then  $U_{\bullet r}$  is an open subset of  $\text{cl}(U)$  that contains all of the interior points of  $\text{cl}(U)$ . Therefore,  $U_{\bullet r} = U$ .  $\square$

**Lemma 4.** *Let  $A \subseteq \mathbb{R}^d$  be closed. It holds that*

$$\text{rconv}(\text{cl}(A^c)) = \sup\{r \in \mathbb{R} : A_{\bullet -r} = A\}$$

*Proof of Lemma 4.* By Lemma 3 and item (a) in Lemma 1,  $\{r \in \mathbb{R} : \text{cl}(A^c)_{\bullet r} = \text{cl}(A^c)\} \subseteq \{r \in \mathbb{R} : (A^c)_{\bullet r} = A^c\} = \{r \in \mathbb{R} : A_{\bullet -r} = A\}$ . Therefore,  $\text{rconv}(\text{cl}(A^c)) \leq \sup\{r \in \mathbb{R} : A_{\bullet -r} = A\}$ . Now we show the reverse inequality. Let  $r, \tilde{r} \in \mathbb{R}^+$  satisfy  $r > \tilde{r} > \text{rconv}(\text{cl}(A^c))$ . By Proposition 1,  $\text{cl}(A^c)_{\bullet \tilde{r}} \setminus \text{cl}(A^c)$  contains a ball of radius  $\epsilon$  for some  $\epsilon \in (0, r)$ . Now, let  $\delta = \min(\epsilon, 2(r - \tilde{r}))$ . Note that  $(A^c)_{\delta/2} \supseteq \text{cl}(A^c)$ , and so by Lemma 1,  $((A^c)_{\delta/2})_{\bullet r - \delta/2} = ((A^c)_r)_{\bullet -r + \delta/2} \supseteq \text{cl}(A^c)_{\bullet r - \delta/2} \supseteq \text{cl}(A^c)_{\bullet \tilde{r}}$ , which contains the ball of radius  $\epsilon$ . Now, eroding by  $\delta/2$  preserves a ball of radius  $\epsilon - \delta/2 \geq \epsilon/2$ . That is,  $((A^c)_{\delta/2})_{\bullet r - \delta/2} \supseteq (A^c)_{\bullet r}$  contains a ball of radius  $\epsilon/2$  that does not intersect  $\text{cl}(A^c)$ , and so  $(A^c)_{\bullet r} \neq A^c$  which implies the desired  $A_{\bullet -r} \neq A$  by Lemma 1, item (a).  $\square$

*Proof of Corollary 1.* If  $\text{rconv}(A) = 0$ , then by Theorem 1,  $\text{reach}(A) = 0$  and we are done. Now assume  $\text{rconv}(A) > 0$  and  $\text{rconv}(\text{cl}(A^c)) > 0$ , then by Lemma 4, there exists  $\delta > 0$  such that  $A_{\bullet r} = A$  for all  $r \in (-\delta, \delta)$ . Theorem 1 in Walther (1999) states that  $\partial A$  is  $(d-1)$ -dimensional and  $C^1$  smooth, which implies Equation 2 in our Theorem 1.  $\square$

*Proof of Corollary 2.* By Theorem 1, the statement holds if  $\partial(A_\epsilon)$  is  $C^1$  smooth and  $(d-1)$ -dimensional. Otherwise, if there is indeed a point  $c$  on the boundary of  $A_\epsilon = \bigcup_{a \in A} B(a, \epsilon)$  that does not have a continuous derivative, then  $c$  is an intersection point of two distinct  $d$ -spheres centered in  $A$  of radius  $\epsilon$ . In other words,  $c$  is at cusp that points inwards towards the interior of  $A_\epsilon$ , making  $\text{reach}(A_\epsilon) = \text{rconv}(A_\epsilon) = 0$ .  $\square$

*Remark 7.* Remark that  $(A_\epsilon)_{\bullet -\epsilon} = A_\epsilon$ . Therefore, if the hypotheses of Corollary 2 are strengthened to those of Corollary 1, then the proof of Corollary 2 holds by applying Theorem 1 in Walther (1999) followed by our Theorem 1.

Here, we present an auxiliary lemma, and use it in our proof of Proposition 1 in Section 4.1.

**Lemma 5.** *For closed  $A \subseteq \mathbb{R}^d$  and  $r \in \mathbb{R}^+$ , it holds that*

$$A_{\bullet r} = \{p \in \mathbb{R}^d : \forall x \in B(0, r), B(p+x, r) \cap A \neq \emptyset\}. \quad (18)$$

*Proof of Lemma 5.* By manipulating the expressions in Definition 1, one obtains

$$A_{\bullet r} = \left( \bigcup_{y \in (A \oplus B(0, r))^c} B(y, r) \right)^c, \quad (19)$$

which reads: *the elements of  $A_{\bullet r}$  are the  $p \in \mathbb{R}^d$  such that  $p$  is not contained in any closed ball of radius  $r$  that does not intersect  $A$ . This statement is equivalent to:  $p \notin A_{\bullet r}$  if and only if there exists a closed ball of radius  $r$  that contains  $p$  but does not intersect  $A$ . Thus, we have shown that  $(A_{\bullet r})^c$  is equal to the compliment of the RHS of (18), which proves the desired result.  $\square$*

*Proof of Proposition 1.* Let  $\tilde{r} := (r + \text{rconv}(A))/2$  and fix  $p \in A_{\bullet \tilde{r}} \setminus A$ . Since  $A$  is closed, there is an open neighbourhood containing  $p$  that does not intersect  $A$ . Choose  $\epsilon \in \mathbb{R}^+$  such that  $B(p, \epsilon) \cap A = \emptyset$ . There exists a sufficiently small  $\delta \in (0, \epsilon)$  such that for every  $y \in \mathbb{R}^d$  satisfying  $B(y, r) \cap B(p, \delta) \neq \emptyset$ , there exists an  $x \in B(0, \tilde{r})$  that satisfies  $B(p+x, \tilde{r}) \setminus B(p, \epsilon) \subset B(y, r)$ . By Lemma 5, each  $B(p+x, \tilde{r}) \setminus B(p, \epsilon)$  contains an element of  $A$ , and therefore so does each  $B(y, r)$ . With a final application of Lemma 5, we have shown that  $B(p, \delta) \subseteq A_{\bullet r} \setminus A$ .  $\square$

**Justification for Equations (13) and (14).** Recall (Aamari et al., 2019, Theorem 3.4). Equation (13) holds since the curvature of  $f$  at  $x = 0$  is  $f''(0) = 2h'(0)$ . Without loss of generality, suppose  $h(0) = f(0) = 0$ . Consider the points  $a_1 = (x, f(x))$  and  $a_2 = (-x, f(x))$  in  $G$  for  $x$  close to 0, and remark that  $g_{\|a_2 - a_1\|} \circ \delta_G \left( \frac{a_1 + a_2}{2} \right) = \frac{x^2}{2f(x)} + \frac{f(x)}{2}$ . For sufficiently small  $x$ , we claim without proof that the choice of  $a_1$  and  $a_2$  is minimal in the sense that for all  $\tilde{a}_1, \tilde{a}_2 \in G$  satisfying  $\delta_G \left( \frac{\tilde{a}_1 + \tilde{a}_2}{2} \right) \geq f(x)$ , it holds that  $g_{\|\tilde{a}_2 - \tilde{a}_1\|} \circ \delta_G \left( \frac{\tilde{a}_1 + \tilde{a}_2}{2} \right) \geq g_{\|a_2 - a_1\|} \circ \delta_G \left( \frac{a_1 + a_2}{2} \right)$ . There is a  $\delta > 0$  such that  $h$  has an inverse on  $[0, \delta]$ , and so for  $\beta \in [0, \delta]$ , choose  $x$  such that  $f(x) = \beta$ . Remark that

$$x^2 = h^{-1}(\beta) = \frac{\beta}{h'(0)} - \frac{\beta^2 h''(0)}{2h'(0)^3} + o(\beta^2)$$

by Taylor's theorem. Thus, we have found the minimal pair  $a_1, a_2 \in G$  satisfying  $\delta_G \left( \frac{a_1 + a_2}{2} \right) = \beta$  and with  $g_{\|a_2 - a_1\|} \circ \delta_G \left( \frac{a_1 + a_2}{2} \right)$  corresponding to the RHS of (14).

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