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# On misspecification in cusp-type change-point models

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## Abstract

The problem of parameter estimation by i.i.d. observations of an inhomogeneous Poisson process is considered in situation of misspecification. The model is that of a Poissonian signal observed in presence of a homogeneous Poissonian noise. The intensity function of the process is supposed to have a cusp-type singularity at the change-point (the unknown moment of arrival of the signal), while the supposed (theoretical) and the real (observed) levels of the signal are different. The asymptotic properties of pseudo MLE are described. It is shown that the estimator converges to the value minimizing the Kullback-Leibler divergence, that the normalized error of estimation converges to some limit distribution, and that its polynomial moments also converge.

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*Key words:* misspecification, inhomogeneous Poisson process, parameter estimation, cusp-type change-point.

## 1 Introduction

It is commonplace in statistics that the theoretical models do not coincide with the real models generating the observations. The properties of the estimators constructed on the base of the theoretical models in such situations do not coincide with their real properties. Sometimes this difference between models can be important, and this requires a special study. The study of such situations in statistics was initiated in the work of Huber [11]. There is a large diversity of publications devoted to different statistical models and different types of misspecification. Special attention is paid to the case when the real models are close to the theoretical models. A nice theory of robust estimation was developed in the book of Huber [12]. The large majority of publication is devoted to regular statistical models, i.e.,

to the case when both the theoretical and the real models are sufficiently smooth w.r.t. the unknown parameter. The different cases of misspecification for change-point models (with jump-type changes) were considered as well (see, e.g., the publications [4,5,18,21,23] and the references therein). Note that in the works [4] and [21], the regularities of the theoretical and of the real models are different, e.g., it is supposed that the model is of change-point type, while the real model of observation is regular. It is known that the maximum likelihood estimator (MLE) under misspecification converges to the value which minimizes the Kulback-Leibler divergence between the measure corresponding to the observations and the parametric family of theoretical measures. Usually this value does not coincide with the true value, but in the change-point case, there exists a large class of models admitting consistent estimation even under misspecification (see, e.g., [1,5]).

A recently introduced class of models (*cusp-type change-point* models, or models with *cusp-type singularity*) can be considered as an intermediate between regular (smooth) and change-point (discontinuous) models in the problem of estimation of the moment of arrival of a signal (see [2,3,8]). In the case when the observations are inhomogeneous Poisson processes, the front of the arrival of the signal in such models corresponds to a strongly increasing continuous intensity function with infinite Fisher information. The examples of intensity functions in regular (a), cusp-type change-point (b) and change-point (c) models are given in Fig. 1.

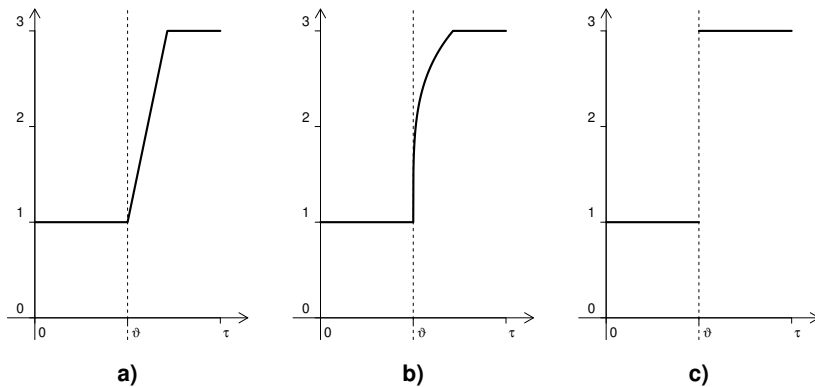


Figure 1: Intensities with three types of fronts of arrival of a signal

The mean-squared errors of the MLE  $\hat{\vartheta}_n$  of the location parameter  $\vartheta$  by  $n$  independent observations of a Poisson process with these three types of fronts are

$$\text{a) } \mathbf{E}_{\vartheta}(\hat{\vartheta}_n - \vartheta)^2 \approx \frac{c}{n}, \quad \text{b) } \mathbf{E}_{\vartheta}(\hat{\vartheta}_n - \vartheta)^2 \approx \frac{c}{n^{\gamma}}, \quad \text{c) } \mathbf{E}_{\vartheta}(\hat{\vartheta}_n - \vartheta)^2 \approx \frac{c}{n^2},$$

where  $1 < \gamma < 2$  and  $c$  are some constants (see, e.g., [16]). That is why the cusp-type change-point models are considered as intermediate between regular and change-point models.

It is reasonable to suppose that the real signals are continuous functions, and that the cusp-type change-point models can provide a better fit of the mathematical model to the real signals. An example of a discontinuous and a close to it cusp-type intensity functions is given in Fig. 2.

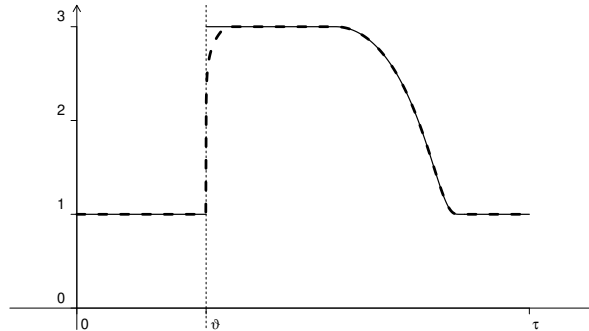


Figure 2: Discontinuous (solid line) and cusp-type (dashed line) intensity functions

Note that the statistical model of i.i.d. observations of a random variable having a cusp-type singularity was first studied in [19]. Afterwards, the parameter estimation problems for models with cusp-type singularities were studied by many authors. For inhomogeneous Poisson processes this was done in [6, 7], for diffusions with small noise in [15], for ergodic diffusions in [9, 10]. Nonparametric estimation of a signal with cusp-type front was considered in [20].

In this work we consider the problem of estimation of the time of arrival of a signal having cusp-type singularity in the situation of misspecification of the level of the signal. An example of intensity functions corresponding to two signals (theoretical and real) with two different levels is given in Fig. 3.

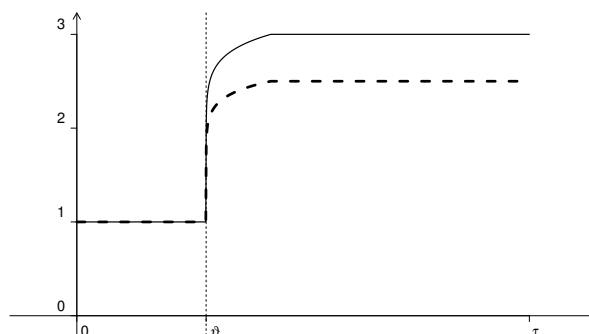


Figure 3: Example of the theoretical (dashed line) and real intensities.

The main result of the paper are the asymptotic properties of the (pseudo) MLE.

## 2 Asymptotic behavior of the pseudo MLE

Suppose that a statistician has  $n$  independent observations  $X^{(n)} = (X_1, \dots, X_n)$ , where  $X_j = (X_j(t), 0 \leq t \leq \tau)$ , for each  $j = 1, \dots, n$ , is an inhomogeneous Poisson process with intensity function  $\lambda_*(\vartheta, \cdot) = (\lambda_*(\vartheta, t), 0 \leq t \leq \tau)$ ,  $\vartheta \in \Theta$ . However, the function  $\lambda_*(\vartheta, \cdot)$  is unknown to the statistician and he uses a different model with an intensity function  $\lambda(\vartheta, \cdot) = (\lambda(\vartheta, t), 0 \leq t \leq \tau)$ ,  $\vartheta \in \Theta$ .

The unknown parameter  $\vartheta$  is the time of arrival of a signal, the latter being observed in presence of a homogeneous Poissonian noise of intensity  $\lambda_0$ . We consider the case when the front of the signal has a cusp-type singularity, i.e., the statistician supposes that the intensity function (called *theoretical*) of the observed Poisson processes is

$$\lambda(\vartheta, t) = S \psi(t - \vartheta) + \lambda_0, \quad t \in [0, \tau], \vartheta \in \Theta, \quad (1)$$

where the front of the signal  $S \psi(t - \vartheta)$  is defined by the function

$$\psi(x) = \left(\frac{x}{\delta}\right)^\kappa \mathbb{1}_{\{0 < x < \delta\}} + \mathbb{1}_{\{x \geq \delta\}}, \quad x \in \mathbb{R}. \quad (2)$$

The parameters  $S > 0$ ,  $\lambda_0 > 0$  and  $\kappa \in (0, 1/2)$  are supposed to be known.

However, the *real* intensity function of the observed processes is

$$\lambda_*(\vartheta_0, t) = (S + h) \psi(t - \vartheta_0) + \lambda_0, \quad t \in [0, \tau], \vartheta_0 \in \Theta_0, \quad (3)$$

where  $h$  is the *contamination* of the signal. Here we use different sets  $\Theta_0 = (\alpha, \beta)$  and  $\Theta = (\alpha - \delta, \beta + \delta)$  with  $\Theta_0 \subset \Theta \subset (0, \tau - \delta)$ . The reason to consider in the theoretical model a set  $\Theta$  which is wider than the set  $\Theta_0$  of possible values of the parameter  $\vartheta_0$  will become clear later.

The *pseudo likelihood ratio* (p-LR) used by the statistician (see, e.g., [17]) is

$$L(\vartheta, X^{(n)}) = \exp \left\{ \sum_{j=1}^n \int_{\vartheta}^{\tau} \ln(\lambda(\vartheta, t)) dX_j(t) - n \int_{\vartheta}^{\tau} [\lambda(\vartheta, t) - 1] dt \right\}, \quad \vartheta \in \Theta.$$

We use the word ‘‘pseudo’’ since the intensity of the processes  $X_j$ ,  $j = 1, \dots, n$ , is not  $\lambda(\vartheta_0, \cdot)$ , but  $\lambda_*(\vartheta_0, \cdot)$ .

The *pseudo maximum likelihood estimator* (p-MLE)  $\hat{\vartheta}_n$  is defined by the equation

$$L(\hat{\vartheta}_n, X^{(n)}) = \sup_{\vartheta \in \Theta} L(\vartheta, X^{(n)}).$$

As it is usually the case in misspecified problems, the limit of the p-MLE  $\hat{\vartheta}_n$  will be given by the value  $\hat{\vartheta}$  which minimizes the Kullback-Leibler divergence

$$J_{\text{K-L}}(\vartheta) = \int_{\vartheta \wedge \vartheta_0}^{\tau} \left[ \frac{\lambda(\vartheta, t)}{\lambda_*(\vartheta_0, t)} - 1 - \ln \left( \frac{\lambda(\vartheta, t)}{\lambda_*(\vartheta_0, t)} \right) \right] \lambda_*(\vartheta_0, t) dt, \quad \vartheta \in \Theta.$$

We need to introduce several notations. First, we introduce the random process

$$\hat{Z}(u) = \exp\left(W^H(u) - \frac{u^2}{2}\right), \quad u \in \mathbb{R},$$

where  $H = \kappa + 1/2$ , and  $W^H(\cdot)$  is a two-sided fractional Brownian motion (fBm) with Hurst parameter  $H$ , i.e., a centered Gaussian process with covariance

$$\mathbf{E}\left(W^H(u_1)W^H(u_2)\right) = \frac{1}{2}\left[|u_1|^{2H} + |u_2|^{2H} - |u_1 - u_2|^{2H}\right], \quad u_1, u_2 \in \mathbb{R}.$$

Let us recall here that  $W^H$  admits the representations

$$W^H(u) = \Gamma_\kappa^{-1} \int_{-\infty}^{+\infty} [(v-u)_+^\kappa - v_+^\kappa] dW(v) = \Gamma_\kappa^{-1} \int_{-\infty}^{+\infty} [(u-s)_+^\kappa - (-s)_+^\kappa] d\widetilde{W}(s),$$

where  $W(\cdot)$  and  $\widetilde{W}(\cdot)$  are two-sided Wiener processes (Brownian motions).

Further, we introduce the random variable  $\hat{u}_\kappa$  which is the (almost surely unique) solution of the equation

$$\hat{Z}(\hat{u}_\kappa) = \sup_{u \in \mathbb{R}} \hat{Z}(u).$$

Finally, we introduce the constants

$$\Gamma_\kappa = \int_{\mathbb{R}} [(v-u)_+^\kappa - (v)_+^\kappa]^2 dv \quad \text{and} \quad b = \left( \frac{S \Gamma_\kappa \sqrt{\lambda_*(\vartheta_0, \hat{\vartheta})}}{\lambda_0 \delta^\kappa J''_{\text{K-L}}(\hat{\vartheta})} \right)^{\frac{2}{3-2\kappa}},$$

as well as the set

$$\mathcal{H} = \left\{ h : h > \frac{S}{\ln(1 + \frac{S}{\lambda_0})} - S - \lambda_0 \right\} \quad (4)$$

Note that the explicit expression of  $J''_{\text{K-L}}(\hat{\vartheta})$  is given in Proposition 1 below, where it is equally shown that if the contamination  $h \in \mathcal{H}$ , then the Kullback-Leibler divergence has a unique minimum at some point  $\hat{\vartheta} \in \Theta$ .

Note also that the condition  $h \in \mathcal{H}$  can be rewritten as

$$S + h + \lambda_0 > \frac{(S + \lambda_0) - \lambda_0}{\ln(S + \lambda_0) - \ln(\lambda_0)},$$

and so it coincides with the condition providing the consistent estimation in a similar misspecified problem for the change-point case [16].

Note finally, that the right hand side of the inequality in (4) belongs to the interval  $(-S, 0)$  (this follows immediately from the elementary inequalities  $\ln(x) < x - 1$  and  $\ln(x) > 1 - \frac{1}{x}$  for  $x \neq 1$ ), and so the contamination can be positive or negative. For the case  $\lambda_0 = 1$  (we can always reduce ourselves to this case dividing  $h$  and  $S$  by  $\lambda_0$ ), the region of admissible values of  $h$  (as function of  $S$ ) is represented in Fig. 4.

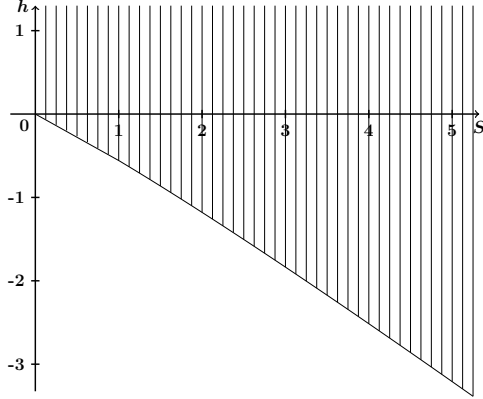


Figure 4: Possible values of the contamination  $h$

Now we can state the main result of the present paper.

**Theorem 1.** *Suppose that we have the model (1)–(3) and  $h \in \mathcal{H} \setminus \{0\}$ . Then the  $p$ -MLE  $\hat{\vartheta}_n$  is “consistent”:  $\hat{\vartheta}_n \xrightarrow{\mathbf{P}} \hat{\vartheta}$ , converges in distribution:*

$$n^{\frac{1}{3-2\kappa}} b^{-1} (\hat{\vartheta}_n - \hat{\vartheta}) \Longrightarrow \hat{u}_\kappa, \quad (5)$$

and we have the convergence of polynomial moments: for any  $p > 0$ , it holds

$$n^{\frac{p}{3-2\kappa}} \mathbf{E} |\hat{\vartheta}_n - \hat{\vartheta}|^p \longrightarrow b^p \mathbf{E} |\hat{u}_\kappa|^p.$$

*Proof.* To prove this theorem we use the approach developed by Ibragimov and Khasminskii in [13], which is based on the weak convergence of the normalized likelihood ratio process to some limit process. The particularity of the misspecified models concerns the study of the corresponding random functions, which are not true likelihood ratios, and hence the direct application of Theorem 1.10.1 of [13] is often impossible. In our case, we follow the modification of this method introduced in [2, 4] (see as well [16]). Let us explain how it works.

We put

$$\varphi_n = bn^{-\frac{1}{3-2\kappa}} \longrightarrow 0$$

and introduce the normalized p-LR

$$Z_n(u) = \frac{L(\hat{\vartheta} + \varphi_n u, X^{(n)})}{L(\hat{\vartheta}, X^{(n)})}, \quad u \in \mathcal{U}_n = \left( \frac{\alpha - \delta - \hat{\vartheta}}{\varphi_n}, \frac{\beta + \delta - \hat{\vartheta}}{\varphi_n} \right).$$

Note that since  $\hat{\vartheta} \in \Theta$ , we have  $\mathcal{U}_n \uparrow \mathbb{R}$ . For any fixed  $u \neq 0$ , we have  $Z_n(u) \longrightarrow \infty$ , that is why we introduce a second normalization: we put

$$\varepsilon_n = \frac{1}{b^2 J''_{\text{K-L}}(\hat{\vartheta})} n^{-\frac{1-2\kappa}{3-2\kappa}} \longrightarrow 0 \quad \text{and} \quad \hat{Z}_n(u) = [Z_n(u)]^{\varepsilon_n}$$

Then we show the following three lemmas (the proofs are in the next section).

**Lemma 1.** Under the hypotheses of Theorem 1, the finite dimensional distributions of the process  $\hat{Z}_n(\cdot)$  converge to those of the process  $\hat{Z}(\cdot)$ .

**Lemma 2.** Under the hypotheses of Theorem 1, there exist some constants  $c, C > 0$  such that

$$\mathbf{E}\hat{Z}_n^{1/2}(u) \leq C \exp\{-cu^2\} \quad (6)$$

for all  $n \in \mathbb{N}$  and  $u \in \mathbb{U}_n$ .

**Lemma 3.** Under the hypotheses of Theorem 1, there exist some constants  $C > 0$  and  $\gamma > 1$  such that

$$\mathbf{E}[\hat{Z}_n^{1/2}(u_1) - \hat{Z}_n^{1/2}(u_2)]^2 \leq C |u_1 - u_2|^\gamma$$

for all  $n \in \mathbb{N}$  and  $u_1, u_2 \in \mathbb{U}_n$ .

The properties of the normalized p-LR  $\hat{Z}_n(\cdot)$  established in the Lemmas 1-3 allow us to apply a modification of Theorem 1.10.1 in [13] and obtain all the desired properties of the p-MLE. For example, the convergence (5) is proved as follows. Using the change of variable  $\vartheta = \hat{\vartheta} + \varphi_n u$ , we can write

$$\begin{aligned} \mathbf{P}\left(\frac{\hat{\vartheta}_n - \hat{\vartheta}}{\varphi_n} < x\right) &= \mathbf{P}(\hat{\vartheta}_n < \hat{\vartheta} + \varphi_n x) = \mathbf{P}\left(\sup_{\vartheta < \hat{\vartheta} + \varphi_n x} L(\vartheta, X^{(n)}) > \sup_{\vartheta \geq \hat{\vartheta} + \varphi_n x} L(\vartheta, X^{(n)})\right) \\ &= \mathbf{P}\left(\sup_{\vartheta < \hat{\vartheta} + \varphi_n x} \frac{L(\vartheta, X^{(n)})}{L(\hat{\vartheta}, X^{(n)})} > \sup_{\vartheta \geq \hat{\vartheta} + \varphi_n x} \frac{L(\vartheta, X^{(n)})}{L(\hat{\vartheta}, X^{(n)})}\right) \\ &= \mathbf{P}\left(\sup_{u < x} Z_n(u) > \sup_{u \geq x} Z_n(u)\right) = \mathbf{P}\left(\sup_{u < x} \hat{Z}_n(u) > \sup_{u \geq x} \hat{Z}_n(u)\right) \\ &\longrightarrow \mathbf{P}\left(\sup_{u < x} \hat{Z}(u) > \sup_{u \geq x} \hat{Z}(u)\right) = \mathbf{P}(\hat{u}_\kappa < x), \end{aligned}$$

as soon as

$$\mathbf{P}\left(\sup_{u < x} \hat{Z}(u) = \sup_{u \geq x} \hat{Z}(u)\right) = 0,$$

that is, as soon as  $\mathbf{P}(\hat{u}_\kappa = x) = 0$ . □

### 3 Proof of the lemmas

We start with the following proposition which gives some properties of the Kullback-Leibler divergence  $J_{\text{K-L}}(\cdot)$  in our case.

**Proposition 1.** Suppose that  $h \in \mathcal{H}$ . Then the function  $J_{\text{K-L}}(\cdot)$  has the following properties.



i) The function  $J_{\text{K-L}}(\cdot)$  is continuously differentiable, and

$$J'_{\text{K-L}}(\vartheta) = \begin{cases} \lambda_0 \ln\left(1 + \frac{S}{\lambda_0}\right) - S, & \text{if } \vartheta \leq \vartheta_0 - \delta, \\ \lambda_0 \ln\left(1 + \frac{S}{\lambda_0} \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^\kappa\right) - S \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^\kappa + I_1(\vartheta), & \text{if } \vartheta \in [\vartheta_0 - \delta, \vartheta_0], \\ \lambda_+ \ln\left(\frac{S + \lambda_0}{S \left(1 - \frac{\vartheta_0 - \vartheta}{\delta}\right)^\kappa + \lambda_0}\right) + S \left(1 - \frac{\vartheta_0 - \vartheta}{\delta}\right)^\kappa - S + I_2(\vartheta), & \text{if } \vartheta \in [\vartheta_0, \vartheta_0 + \delta], \\ \lambda_+ \ln\left(1 + \frac{S}{\lambda_0}\right) - S, & \text{if } \vartheta \geq \vartheta_0 + \delta, \end{cases}$$

where  $\lambda_+ = S + h + \lambda_0$  and

$$I_1(\vartheta) = \int_{\frac{\vartheta_0 - \vartheta}{\delta}}^1 S \kappa x^{\kappa-1} \frac{(S + h) \left(x - \frac{\vartheta_0 - \vartheta}{\delta}\right)^\kappa - S x^\kappa}{S x^\kappa + \lambda_0} dx,$$

$$I_2(\vartheta) = \int_0^{1 - \frac{\vartheta - \vartheta_0}{\delta}} S \kappa x^{\kappa-1} \frac{(S + h) \left(x + \frac{\vartheta - \vartheta_0}{\delta}\right)^\kappa - S x^\kappa}{S x^\kappa + \lambda_0} dx.$$

In particular,

- $J'_{\text{K-L}}(\vartheta) < 0$  for  $\vartheta \leq \vartheta_0 - \delta$ ,
- $J'_{\text{K-L}}(\vartheta) > 0$  for  $\vartheta \geq \vartheta_0 + \delta$ ,
- $J'_{\text{K-L}}(\vartheta_0) = Ah$  with  $A = 1 - \frac{\lambda_0}{S} \ln\left(1 + \frac{S}{\lambda_0}\right) > 0$ .

ii) The function  $J_{\text{K-L}}(\cdot)$  is twice continuously differentiable everywhere except at the point  $\vartheta_0$  (in which  $J''_{\text{K-L}}(\vartheta_0) = +\infty$ ), and

$$J''_{\text{K-L}}(\vartheta) = \begin{cases} \frac{S(S + h)\kappa^2}{\delta} \int_{\frac{\vartheta_0 - \vartheta}{\delta}}^1 \frac{x^{\kappa-1} \left(x - \frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa-1}}{S x^\kappa + \lambda_0} dx, & \text{if } \vartheta \in [\vartheta_0 - \delta, \vartheta_0), \\ \frac{S(S + h)\kappa^2}{\delta} \int_0^{1 - \frac{\vartheta - \vartheta_0}{\delta}} \frac{x^{\kappa-1} \left(x + \frac{\vartheta - \vartheta_0}{\delta}\right)^{\kappa-1}}{S x^\kappa + \lambda_0} dx, & \text{if } \vartheta \in (\vartheta_0, \vartheta_0 + \delta], \\ 0, & \text{if } \vartheta \notin (\vartheta_0 - \delta, \vartheta_0 + \delta). \end{cases}$$

In particular,  $J''_{\text{K-L}}(\cdot) > 0$  on  $(\vartheta_0 - \delta, \vartheta_0) \cup (\vartheta_0, \vartheta_0 + \delta)$ , and hence the function  $J'_{\text{K-L}}(\cdot)$  is strictly increasing on  $(\vartheta_0 - \delta, \vartheta_0 + \delta)$ .

iii) The function  $J_{\text{K-L}}(\cdot)$  attains its unique minimum at the point  $\hat{\vartheta}$  which is the (unique) solution of the equation  $J'_{\text{K-L}}(\hat{\vartheta}) = 0$ . Moreover, if  $h > 0$ , we have  $\hat{\vartheta} \in (\vartheta_0 - \delta, \vartheta_0)$ , and if  $h < 0$ , we have  $\hat{\vartheta} \in (\vartheta_0, \vartheta_0 + \delta)$  (of course  $\hat{\vartheta} = \vartheta_0$  for  $h = 0$ ).

iv) If  $h \neq 0$ , there exist some constants  $m, M > 0$  such that we have (for all  $\vartheta \in \Theta$ ) the estimates

$$m|\vartheta - \hat{\vartheta}| \leq |J'_{\text{K-L}}(\vartheta)| \leq M|\vartheta - \hat{\vartheta}|, \quad (7)$$

and consequently

$$\frac{m}{2} (\vartheta - \hat{\vartheta})^2 \leq J_{\text{K-L}}(\vartheta) - J_{\text{K-L}}(\hat{\vartheta}) \leq \frac{M}{2} (\vartheta - \hat{\vartheta})^2. \quad (8)$$

*Proof.* Throughout the proof,  $C$  denotes a “generic” quantity not depending on  $\vartheta$ , which can vary from formula to formula (and even in the same formula). Note also that since we have supposed  $\vartheta_0 \in \Theta_0 = (\alpha, \beta)$ , we have  $\vartheta_0 - \delta, \vartheta_0 + \delta \in \Theta = (\alpha - \delta, \beta + \delta)$ .

For  $\vartheta \leq \vartheta_0 - \delta$ , we can write

$$\begin{aligned} J_{\text{K-L}}(\vartheta) &= \int_{\vartheta}^{\vartheta_0} \left[ \frac{S \left( \frac{t-\vartheta}{\delta} \right)^\kappa + \lambda_0}{\lambda_0} - 1 - \ln \left( \frac{S \left( \frac{t-\vartheta}{\delta} \right)^\kappa + \lambda_0}{\lambda_0} \right) \right] \lambda_0 dt \\ &\quad + \int_{\vartheta_0}^{\vartheta_0} \left[ \frac{S + \lambda_0}{\lambda_0} - 1 - \ln \left( \frac{S + \lambda_0}{\lambda_0} \right) \right] \lambda_0 dt + C \\ &= \delta \int_0^1 \left[ S x^\kappa - \lambda_0 \ln \left( 1 + \frac{S}{\lambda_0} x^\kappa \right) \right] dx + (\vartheta_0 - \vartheta - \delta) \left[ S - \lambda_0 \ln \left( 1 + \frac{S}{\lambda_0} \right) \right] + C \\ &= \vartheta \left[ \lambda_0 \ln \left( 1 + \frac{S}{\lambda_0} \right) - S \right] + C. \end{aligned}$$

Hence, in this case,  $J'_{\text{K-L}}(\vartheta) = \lambda_0 \ln \left( 1 + \frac{S}{\lambda_0} \right) - S$  and  $J''_{\text{K-L}}(\vartheta) = 0$ . The fact that  $J'_{\text{K-L}}(\vartheta) < 0$  follows immediately from the elementary inequality  $\ln(x) < x - 1$  for  $x \neq 0$ .

Similarly, for  $\vartheta \geq \vartheta_0 + \delta$ , we have

$$\begin{aligned} J_{\text{K-L}}(\vartheta) &= \int_{\vartheta_0+\delta}^{\vartheta} \left[ \frac{\lambda_0}{\lambda_+} - 1 - \ln \left( \frac{\lambda_0}{\lambda_+} \right) \right] \lambda_+ dt \\ &\quad + \int_{\vartheta}^{\vartheta_0+\delta} \left[ \frac{S \left( \frac{t-\vartheta}{\delta} \right)^\kappa + \lambda_0}{\lambda_+} - 1 - \ln \left( \frac{S \left( \frac{t-\vartheta}{\delta} \right)^\kappa + \lambda_0}{\lambda_+} \right) \right] \lambda_+ dt \\ &\quad + \int_{\vartheta_0+\delta}^{\tau} \left[ \frac{S + \lambda_0}{\lambda_+} - 1 - \ln \left( \frac{S + \lambda_0}{\lambda_+} \right) \right] \lambda_+ dt + C \\ &= (\vartheta - \vartheta_0 - \delta) \left[ -S - h - \lambda_+ \ln \left( \frac{\lambda_0}{\lambda_+} \right) \right] \\ &\quad + \delta \int_0^1 \left[ S x^\kappa - S - h - \lambda_+ \ln \left( \frac{S x^\kappa + \lambda_0}{\lambda_+} \right) \right] dx \\ &\quad + (\tau - \vartheta - \delta) \left[ -h - \lambda_+ \ln \left( \frac{S + \lambda_0}{\lambda_+} \right) \right] + C \\ &= \vartheta \left[ \lambda_+ \ln \left( 1 + \frac{S}{\lambda_0} \right) - S \right] + C. \end{aligned}$$

Hence, in this case,  $J'_{\text{K-L}}(\vartheta) = \lambda_+ \ln \left( 1 + \frac{S}{\lambda_0} \right) - S$  and  $J''_{\text{K-L}}(\vartheta) = 0$ . The fact that  $J'_{\text{K-L}}(\vartheta) > 0$  follows immediately from the condition  $h \in \mathcal{H}$ .

Now, in the case  $\vartheta \in [\vartheta_0 - \delta, \vartheta_0]$ , denoting for shortness  $\phi(y) = (S + h)y^\kappa + \lambda_0$ , it comes

$$\begin{aligned}
J_{\text{K-L}}(\vartheta) &= \int_{\vartheta}^{\vartheta_0} \left[ \frac{S\left(\frac{t-\vartheta}{\delta}\right)^\kappa + \lambda_0}{\lambda_0} - 1 - \ln\left(\frac{S\left(\frac{t-\vartheta}{\delta}\right)^\kappa + \lambda_0}{\lambda_0}\right) \right] \lambda_0 dt \\
&+ \int_{\vartheta_0}^{\vartheta_0+\delta} \left[ \frac{S\left(\frac{t-\vartheta}{\delta}\right)^\kappa + \lambda_0}{\phi\left(\frac{t-\vartheta_0}{\delta}\right)} - 1 - \ln\left(\frac{S\left(\frac{t-\vartheta}{\delta}\right)^\kappa + \lambda_0}{\phi\left(\frac{t-\vartheta_0}{\delta}\right)}\right) \right] \phi\left(\frac{t-\vartheta_0}{\delta}\right) dt \\
&+ \int_{\vartheta_0+\delta}^{\vartheta_0+\delta} \left[ \frac{S + \lambda_0}{\phi\left(\frac{t-\vartheta_0}{\delta}\right)} - 1 - \ln\left(\frac{S + \lambda_0}{\phi\left(\frac{t-\vartheta_0}{\delta}\right)}\right) \right] \phi\left(\frac{t-\vartheta_0}{\delta}\right) dt + C \\
&= \delta \int_0^{\frac{\vartheta_0-\vartheta}{\delta}} \left[ Sx^\kappa - \lambda_0 \ln\left(1 + \frac{S}{\lambda_0} x^\kappa\right) \right] dx \\
&+ \delta \int_0^{1-\frac{\vartheta_0-\vartheta}{\delta}} \left[ S\left(y + \frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa - (S+h)y^\kappa - \phi(y) \ln\left(\frac{S\left(y + \frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa + \lambda_0}{\phi(y)}\right) \right] dy \\
&+ \delta \int_{1-\frac{\vartheta_0-\vartheta}{\delta}}^1 \left[ S - (S+h)y^\kappa - \phi(y) \ln\left(\frac{S + \lambda_0}{\phi(y)}\right) \right] dy + C.
\end{aligned}$$

Therefore, differentiating with respect to  $\vartheta$ , we get in this case

$$\begin{aligned}
J'_{\text{K-L}}(\vartheta) &= - \left[ S\left(\frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa - \lambda_0 \ln\left(1 + \frac{S}{\lambda_0} \left(\frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa\right) \right] \\
&+ \left[ S - (S+h) \left(1 - \frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa - \phi\left(1 - \frac{\vartheta_0-\vartheta}{\delta}\right) \ln\left(\frac{S + \lambda_0}{\phi\left(1 - \frac{\vartheta_0-\vartheta}{\delta}\right)}\right) \right] \\
&+ \delta \int_0^{1-\frac{\vartheta_0-\vartheta}{\delta}} \left[ -\frac{S\kappa}{\delta} \left(y + \frac{\vartheta_0-\vartheta}{\delta}\right)^{\kappa-1} + \phi(y) \frac{\frac{S\kappa}{\delta} \left(y + \frac{\vartheta_0-\vartheta}{\delta}\right)^{\kappa-1}}{S\left(y + \frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa + \lambda_0} \right] dy \\
&- \left[ S - (S+h) \left(1 - \frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa - \phi\left(1 - \frac{\vartheta_0-\vartheta}{\delta}\right) \ln\left(\frac{S + \lambda_0}{\phi\left(1 - \frac{\vartheta_0-\vartheta}{\delta}\right)}\right) \right] \\
&= \lambda_0 \ln\left(1 + \frac{S}{\lambda_0} \left(\frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa\right) - S\left(\frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa \\
&+ \int_{\frac{\vartheta_0-\vartheta}{\delta}}^1 \left[ -S\kappa x^{\kappa-1} + \phi\left(x - \frac{\vartheta_0-\vartheta}{\delta}\right) \frac{S\kappa x^{\kappa-1}}{Sx^\kappa + \lambda_0} \right] dx \\
&= \lambda_0 \ln\left(1 + \frac{S}{\lambda_0} \left(\frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa\right) - S\left(\frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa \\
&+ \int_{\frac{\vartheta_0-\vartheta}{\delta}}^1 S\kappa x^{\kappa-1} \frac{(S+h)\left(x - \frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa - Sx^\kappa}{Sx^\kappa + \lambda_0} dx \\
&= \lambda_0 \ln\left(1 + \frac{S}{\lambda_0} \left(\frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa\right) - S\left(\frac{\vartheta_0-\vartheta}{\delta}\right)^\kappa + I_1(\vartheta).
\end{aligned}$$

For  $\vartheta \in [\vartheta_0 - \delta, \vartheta_0)$ , differentiating once more with respect to  $\vartheta$ , we obtain

$$\begin{aligned}
J''_{\text{K-L}}(\vartheta) &= \frac{-\frac{S\kappa}{\delta} \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa-1}}{1 + \frac{S}{\lambda_0} \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa}} + \frac{S\kappa}{\delta} \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa-1} + I'_1(\vartheta) \\
&= \frac{S\kappa}{\delta} \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa-1} \left[1 - \frac{1}{1 + \frac{S}{\lambda_0} \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa}}\right] + \frac{S\kappa}{\delta} \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa-1} \frac{-S \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa}}{S \left(\frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa} + \lambda_0} \\
&\quad + \int_{\frac{\vartheta_0 - \vartheta}{\delta}}^1 S\kappa x^{\kappa-1} \frac{\frac{(S+h)\kappa}{\delta} \left(x - \frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa-1}}{Sx^{\kappa} + \lambda_0} dx \\
&= \frac{S(S+h)\kappa^2}{\delta} \int_{\frac{\vartheta_0 - \vartheta}{\delta}}^1 \frac{x^{\kappa-1} \left(x - \frac{\vartheta_0 - \vartheta}{\delta}\right)^{\kappa-1}}{Sx^{\kappa} + \lambda_0} dx.
\end{aligned}$$

Note that for  $\vartheta \in (\vartheta_0 - \delta, \vartheta_0)$  the last integral is strictly positive, and that for  $\vartheta = \vartheta_0$  it would become  $\int_0^1 \frac{x^{2\kappa-2}}{Sx^{\kappa} + \lambda_0} dx$  and diverge to  $+\infty$  (since  $2\kappa - 2 < -1$ ).

The calculation of  $J'_{\text{K-L}}(\vartheta)$  and  $J''_{\text{K-L}}(\vartheta)$  in the remaining case  $\vartheta \in [\vartheta_0, \vartheta_0 + \delta]$  can be carried out in a similar way.

So, to conclude the proof of the parts *i*) and *ii*) of the lemma, it remains to show that  $J'_{\text{K-L}}(\vartheta_0) = Ah$ . Indeed,

$$\begin{aligned}
J'_{\text{K-L}}(\vartheta_0) &= I_1(\vartheta_0) = I_2(\vartheta_0) = \int_0^1 S\kappa x^{\kappa-1} \frac{hx^{\kappa}}{Sx^{\kappa} + \lambda_0} dx = \int_0^1 \frac{Shy}{Sy + \lambda_0} dy \\
&= h \int_0^1 \left[1 - \frac{\lambda_0}{Sy + \lambda_0}\right] dy = h \left[ y - \frac{\lambda_0}{S} \ln\left(1 + \frac{S}{\lambda_0} y\right) \right]_0^1 = h \left[1 - \frac{\lambda_0}{S} \ln\left(1 + \frac{S}{\lambda_0}\right)\right].
\end{aligned}$$

The parts *iii*) of the lemma follows directly from the parts *i*) and *ii*). So, it remains to prove the part *iv*).

As  $h \neq 0$ , we have  $\hat{\vartheta} \neq \vartheta_0$ , and hence  $J''_{\text{K-L}}(\hat{\vartheta}) > 0$ . So, there exist some vicinity of  $\hat{\vartheta}$  and some constants  $m, M > 0$ , such that we have  $m < J''_{\text{K-L}}(\vartheta) < M$  for  $\vartheta$  belonging to this vicinity. Hence, as

$$|J'_{\text{K-L}}(\vartheta)| = |J'_{\text{K-L}}(\vartheta) - J'_{\text{K-L}}(\hat{\vartheta})| = J''_{\text{K-L}}(\tilde{\vartheta})|\vartheta - \hat{\vartheta}|,$$

where  $\tilde{\vartheta}$  is some intermediate value between  $\vartheta$  and  $\hat{\vartheta}$ , the estimates (7) are valid for  $\vartheta$  belonging to this vicinity. Noting that the function  $J'_{\text{K-L}}(\cdot)$  is non-decreasing and bounded, this inequalities can be clearly extended to the whole  $\Theta$  by adjusting the constants  $m$  and  $M$ .

The estimates (8) follow easily from the estimates (7). For example, the upper estimate in the case  $\vartheta < \hat{\vartheta}$  can be obtained as follows

$$J_{\text{K-L}}(\vartheta) - J_{\text{K-L}}(\hat{\vartheta}) = - \int_{\vartheta}^{\hat{\vartheta}} J'_{\text{K-L}}(t) dt = \int_{\vartheta}^{\hat{\vartheta}} |J'_{\text{K-L}}(t)| dt \leq M \int_{\vartheta}^{\hat{\vartheta}} (\hat{\vartheta} - t) dt = \frac{M}{2} (\hat{\vartheta} - \vartheta)^2.$$

The proposition is proved.  $\square$

Now we turn to the proof of the lemmas.

*Proof of Lemma 1.* Let us note that the theoretical intensity function can be rewritten as

$$\lambda(\vartheta, t) = S \left( \frac{t - \vartheta}{\delta} \right)^\kappa \mathbb{1}_{\{0 < t - \vartheta < \delta\}} + S \mathbb{1}_{\{t - \vartheta \geq \delta\}} + \lambda_0 = S \left( \frac{t - \vartheta}{\delta} \right)_+^\kappa + \tilde{\psi}(t - \vartheta),$$

where the function

$$\tilde{\psi}(x) = S \left[ 1 - \left( \frac{x}{\delta} \right)^\kappa \right] \mathbb{1}_{\{x \geq \delta\}} + \lambda_0, \quad x \in \mathbb{R},$$

is Lipschitz continuous:

$$|\tilde{\psi}(x) - \tilde{\psi}(y)| \leq C|x - y|, \quad x, y \in \mathbb{R},$$

with  $C = |\tilde{\psi}'(\delta)| = \frac{S\kappa}{\delta}$ .

Denoting  $\vartheta_u = \hat{\vartheta} + \varphi_n u$  and

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left[ X_j(t) - \int_0^t \lambda_*(\vartheta_0, s) ds \right],$$

we can write

$$\begin{aligned} \ln \hat{Z}_n(u) &= \varepsilon_n \sum_{j=1}^n \int_0^\tau \ln \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right) dX_j(t) - n\varepsilon_n \int_0^\tau [\lambda(\vartheta_u, t) - \lambda(\hat{\vartheta}, t)] dt \\ &= \varepsilon_n \sum_{j=1}^n \int_0^\tau \ln \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right) [dX_j(t) - \lambda_*(\vartheta_0, t) dt] \\ &\quad - n\varepsilon_n \int_0^\tau \left[ \lambda(\vartheta_u, t) - \lambda(\hat{\vartheta}, t) - \lambda_*(\vartheta_0, t) \ln \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right) \right] dt \\ &= \varepsilon_n \sqrt{n} \int_0^\tau \ln \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right) dW_n(t) - n\varepsilon_n [J_{\text{K-L}}(\vartheta_u) - J_{\text{K-L}}(\hat{\vartheta})] \\ &= A_n(u) - B_n(u) \end{aligned}$$

with evident notations.

For  $B_n(u)$ , we have

$$B_n(u) = n\varepsilon_n [J_{\text{K-L}}(\hat{\vartheta} + u\varphi_n) - J_{\text{K-L}}(\hat{\vartheta})] = n\varepsilon_n \frac{J''_{\text{K-L}}(\hat{\vartheta})}{2} (u\varphi_n)^2 + o(n\varepsilon_n \varphi_n^2) = \frac{u^2}{2} + o(1).$$

Here we used the Taylor expansion of the function  $J_{\text{K-L}}$  in the vicinity of the point  $\hat{\vartheta}$ , the fact that  $J'_{\text{K-L}}(\hat{\vartheta}) = 0$ , and the equality  $n\varepsilon_n \varphi_n^2 J''_{\text{K-L}}(\hat{\vartheta}) = 1$ .

For  $A_n(u)$ , using the Taylor expansion of the function  $x \mapsto \ln(1+x)$ , we get

$$\begin{aligned}
A_n(u) &= \varepsilon_n \sqrt{n} \int_0^\tau \ln\left(\frac{\lambda(\hat{\vartheta} + u\varphi_n, t)}{\lambda(\hat{\vartheta}, t)}\right) dW_n(t) \\
&= \varepsilon_n \sqrt{n} \int_0^\tau \frac{\lambda(\hat{\vartheta} + u\varphi_n, t) - \lambda(\hat{\vartheta}, t)}{\lambda(\hat{\vartheta}, t)} dW_n(t) (1 + o_{\mathbf{P}}(1)) \\
&= \varepsilon_n \sqrt{n} \int_0^\tau \frac{S\left(\frac{t-\hat{\vartheta}-u\varphi_n}{\delta}\right)_+^\kappa - S\left(\frac{t-\hat{\vartheta}}{\delta}\right)_+^\kappa}{\lambda(\hat{\vartheta}, t)} dW_n(t) (1 + o_{\mathbf{P}}(1)) \\
&\quad + \varepsilon_n \sqrt{n} \int_0^\tau \frac{\tilde{\psi}(t - \hat{\vartheta} - u\varphi_n) - \tilde{\psi}(t - \hat{\vartheta})}{\lambda(\hat{\vartheta}, t)} dW_n(t) (1 + o_{\mathbf{P}}(1)).
\end{aligned}$$

Taking into account the Lipschitz continuity of  $\tilde{\psi}$ , the inequality  $\lambda(\hat{\vartheta}, t) \geq \lambda_0$  and the fact that  $\varepsilon_n \sqrt{n} \varphi_n \rightarrow 0$ , the last term clearly converges to zero in probability.

So,  $A_n(u)$  has the same limit as

$$\begin{aligned}
\tilde{A}_n(u) &= \varepsilon_n \sqrt{n} \int_0^\tau \frac{S\left(\frac{t-\hat{\vartheta}-u\varphi_n}{\delta}\right)_+^\kappa - S\left(\frac{t-\hat{\vartheta}}{\delta}\right)_+^\kappa}{\lambda(\hat{\vartheta}, t)} dW_n(t) \\
&= \varepsilon_n \sqrt{n} \varphi_n^{\kappa+1/2} \frac{S\sqrt{\lambda_*(\vartheta_0, \hat{\vartheta})}}{\delta^\kappa} \int_{-\hat{\vartheta}/\varphi_n}^{(\tau-\hat{\vartheta})/\varphi_n} \frac{(v-u)_+^\kappa - v_+^\kappa}{\lambda(\hat{\vartheta}, \hat{\vartheta} + v\varphi_n)} dw_n(v) \\
&= \varepsilon_n \sqrt{n} \varphi_n^{\kappa+1/2} \frac{S\sqrt{\lambda_*(\vartheta_0, \hat{\vartheta})}}{\lambda_0 \delta^\kappa} \int_{-\hat{\vartheta}/\varphi_n}^{(\tau-\hat{\vartheta})/\varphi_n} [(v-u)_+^\kappa - v_+^\kappa] dw_n(v) (1 + o_{\mathbf{P}}(1)) \\
&= \Gamma_\kappa^{-1} \int_{-\hat{\vartheta}/\varphi_n}^{(\tau-\hat{\vartheta})/\varphi_n} [(v-u)_+^\kappa - v_+^\kappa] dw_n(v) (1 + o_{\mathbf{P}}(1)),
\end{aligned}$$

where we used the change of variable  $t = \hat{\vartheta} + v\varphi_n$  and denoted

$$w_n(v) = \frac{W_n(\hat{\vartheta} + v\varphi_n) - W_n(\hat{\vartheta})}{\sqrt{\lambda_*(\vartheta_0, \hat{\vartheta})\varphi_n}}.$$

Therefore, noting that  $w_n(\cdot) \Rightarrow W(\cdot)$ , where  $W$  is a two-sided Wiener process, we obtain

$$A_n(u) \Rightarrow \Gamma_\kappa^{-1} \int_{-\infty}^{+\infty} [(v-u)_+^\kappa - v_+^\kappa] dW(v) = W^H(u),$$

and hence

$$\ln \hat{Z}_n(u) \Rightarrow W^H(u) - \frac{u^2}{2} = \ln \hat{Z}(u),$$

which yields the convergence of one-dimensional distributions of  $\hat{Z}_n(\cdot)$  to those of  $\hat{Z}(\cdot)$ . Clearly, the convergence of multi-dimensional distributions equally holds.  $\square$

*Proof of Lemma 2.* Throughout the proof,  $c$  and  $C$  denote “generic” strictly positive constants, which can vary from formula to formula (and even in the same formula).

Using the Taylor-Lagrange formula

$$y^\varepsilon = 1 + \varepsilon \ln y + \frac{\varepsilon^2}{2} (\ln y)^2 y^{\gamma\varepsilon},$$

where  $y > 0$  and  $0 < \gamma < 1$ , and denoting for shortness

$$D = \frac{\varepsilon_n^2}{8} \left[ \ln \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right) \right]^2 \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right)^{\gamma\varepsilon_n/2},$$

we can write

$$\begin{aligned} \mathbf{E} \hat{Z}_n^{1/2}(u) &= \mathbf{E} \exp \left\{ \frac{\varepsilon_n}{2} \sum_{j=1}^n \int_0^\tau \ln \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right) dX_j(t) - n \frac{\varepsilon_n}{2} \int_0^\tau [\lambda(\vartheta_u, t) - \lambda(\hat{\vartheta}, t)] dt \right\} \\ &= \exp \left\{ n \int_0^\tau \left[ \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right)^{\varepsilon_n/2} - 1 \right] \lambda_*(\vartheta_0, t) dt - n \frac{\varepsilon_n}{2} \int_0^\tau [\lambda(\vartheta_u, t) - \lambda(\hat{\vartheta}, t)] dt \right\} \\ &= \exp \left\{ n \int_0^\tau \left[ \frac{\varepsilon_n}{2} \ln \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right) + D \right] \lambda_*(\vartheta_0, t) dt - n \frac{\varepsilon_n}{2} \int_0^\tau [\lambda(\vartheta_u, t) - \lambda(\hat{\vartheta}, t)] dt \right\} \\ &= \exp \left\{ -n \frac{\varepsilon_n}{2} [J_{\text{K-L}}(\vartheta_u) - J_{\text{K-L}}(\hat{\vartheta})] + n \int_0^\tau D \lambda_*(\vartheta_0, t) dt \right\} \\ &= \exp \{-F_n(u) + G_n(u)\} \end{aligned}$$

with evident notations.

For the first term, using (8), we obtain

$$-F_n(u) \leq -n \frac{\varepsilon_n}{2} \frac{m}{2} (\vartheta_u - \hat{\vartheta})^2 = -c u^2 n \varepsilon_n \varphi_n^2 = -c u^2.$$

For the second term, we have

$$\begin{aligned} G_n(u) &= n \frac{\varepsilon_n^2}{8} \int_0^\tau \left[ \ln \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right) \right]^2 \left( \frac{\lambda(\vartheta_u, t)}{\lambda(\hat{\vartheta}, t)} \right)^{\gamma\varepsilon_n/2} \lambda_*(\vartheta_0, t) dt \\ &\leq C n \varepsilon_n^2 \int_0^\tau [\ln(\lambda(\vartheta_u, t)) - \ln(\lambda(\hat{\vartheta}, t))]^2 dt \\ &= C n \varepsilon_n^2 \int_0^\tau \left[ \frac{\lambda(\vartheta_u, t) - \lambda(\hat{\vartheta}, t)}{\tilde{\lambda}} \right]^2 dt \\ &\leq C n \varepsilon_n^2 \int_0^\tau [\lambda(\vartheta_u, t) - \lambda(\hat{\vartheta}, t)]^2 dt \\ &\leq C n \varepsilon_n^2 |\vartheta_u - \hat{\vartheta}|^{2\kappa+1} \\ &= C |u|^{2\kappa+1} n \varepsilon_n^2 \varphi_n^{2\kappa+1} = C |u|^{2\kappa+1}. \end{aligned} \tag{9}$$

Here  $\tilde{\lambda}$  is some intermediate value between  $\lambda(\vartheta_u, t)$  and  $\lambda(\hat{\vartheta}, t)$ , we use the fact that the intensities  $\lambda$  and  $\lambda_*$  are bounded and separated from zero, and the inequality (9) is a particular case of the following more general inequality (which will be also needed in the proof of the next lemma): for any  $p \geq 1$ , there exist a constant  $C$  such that

$$\int_0^\tau |\lambda(\vartheta_1, t) - \lambda(\vartheta_2, t)|^{2p} dt \leq C |\vartheta_1 - \vartheta_2|^{2p\kappa+1} \quad (10)$$

for all  $\vartheta_1, \vartheta_2 \in \Theta$ .

Before continuing the proof of the lemma, let us prove the inequality (10). Without loss of generality we can suppose that  $\vartheta_1 > \vartheta_2$ . Using the elementary inequality

$$|a + b|^q \leq 2^{q-1} [|a|^q + |b|^q] \quad (11)$$

(valid for all  $a, b \in \mathbb{R}$  and  $q \geq 1$ ), the Lipschitz continuity of  $\tilde{\psi}$  and the change of variable  $t = \vartheta_2 + v(\vartheta_1 - \vartheta_2)$ , we get

$$\begin{aligned} \int_0^\tau |\lambda(\vartheta_1, t) - \lambda(\vartheta_2, t)|^{2p} dt &\leq C \int_0^\tau \left| S\left(\frac{t - \vartheta_1}{\delta}\right)_+^\kappa - S\left(\frac{t - \vartheta_2}{\delta}\right)_+^\kappa \right|^{2p} dt \\ &\quad + C \int_0^\tau |\tilde{\psi}(t - \vartheta_1) - \tilde{\psi}(t - \vartheta_2)|^{2p} dt \\ &\leq C \int_0^\tau |(t - \vartheta_1)_+^\kappa - (t - \vartheta_2)_+^\kappa|^{2p} dt + C (\vartheta_1 - \vartheta_2)^{2p} \\ &= C (\vartheta_1 - \vartheta_2)^{2p\kappa+1} \int_{-\frac{\vartheta_2}{\vartheta_1 - \vartheta_2}}^{\frac{\tau - \vartheta_2}{\vartheta_1 - \vartheta_2}} |(v - 1)_+^\kappa - v_+^\kappa|^{2p} dv + C (\vartheta_1 - \vartheta_2)^{2p}. \end{aligned}$$

As  $\kappa < 1/2$  and  $p \geq 1$ , we have

$$\int_{-\frac{\vartheta_2}{\vartheta_1 - \vartheta_2}}^{\frac{\tau - \vartheta_2}{\vartheta_1 - \vartheta_2}} |(v - 1)_+^\kappa - v_+^\kappa|^{2p} dv \leq \int_{-\infty}^{+\infty} |(v - 1)_+^\kappa - v_+^\kappa|^{2p} dv < +\infty$$

and (noting that  $\vartheta_1 - \vartheta_2 \leq \tau - \delta$  and  $2p - 2p\kappa - 1 > 0$ )

$$C (\vartheta_1 - \vartheta_2)^{2p} = C (\vartheta_1 - \vartheta_2)^{2p\kappa+1} (\vartheta_1 - \vartheta_2)^{2p-2p\kappa-1} \leq C (\vartheta_1 - \vartheta_2)^{2p\kappa+1},$$

which yields the inequality (10).

Now, combining the bounds obtained for  $-F_n(u)$  and  $G_n(u)$ , we have

$$\mathbf{E}\hat{Z}_n^{1/2}(u) \leq \exp\{-cu^2 + C|u|^{2\kappa+1}\}. \quad (12)$$

This concludes the proof of the lemma, since taking  $c' = c/2$  and noting that the function  $-\frac{c}{2}u^2 + C|u|^{2\kappa+1}$  is bounded, we obtain (6).

Note also that the moments  $\mathbf{E}\hat{Z}_n^q(u)$ ,  $u \in \mathbb{U}_n$ , of an arbitrary order  $q > 0$  can be bounded by the same inequalities (6) and (12) (with constants depending on  $q$ ). Indeed,



it is clear from the proof above that only the order of the rate at which  $\varepsilon_n \rightarrow 0$  is important, and so it is sufficient to apply the lemma to  $\tilde{Z}_n(u) = [Z_n(u)]^{\varepsilon'_n}$  with  $\varepsilon'_n = 2q\varepsilon_n$ , and note that  $\tilde{Z}_n^{1/2}(u) = [Z_n(u)]^{q\varepsilon_n} = \hat{Z}_n^q(u)$ . In particular, for any  $q > 0$ , there exist some constants  $c' = c'(q) > 0$  and  $C' = C'(q) > 0$  such that

$$\mathbf{E}\hat{Z}_n^q(u) \leq C' \exp\{-c'u^2\} \quad (13)$$

for all  $n \in \mathbb{N}$  and  $u \in \mathbb{U}_n$ .  $\square$

*Proof of Lemma 3.* Throughout the proof, once more  $c$  and  $C$  denote “generic” strictly positive constants, which can vary from formula to formula (and even in the same formula).

First of all, let us note that in the case  $|u_1 - u_2| \geq 1$ , using the inequality (13), we get

$$\mathbf{E}[\hat{Z}_n^{1/2}(u_1) - \hat{Z}_n^{1/2}(u_2)]^2 \leq 2\mathbf{E}\hat{Z}_n(u_1) + 2\mathbf{E}\hat{Z}_n(u_2) \leq C \leq C|u_1 - u_2|^\gamma.$$

Hence, we can suppose from now on that  $|u_1 - u_2| \leq 1$  and, without loss of generality, that  $|u_1| \geq |u_2|$  (and, henceforth,  $|u_1| \leq |u_2| + 1$ ).

Using the elementary inequality

$$|e^x - e^y| \leq |x - y| \max\{e^x, e^y\}$$

(valid for all  $x, y \in \mathbb{R}$ ), we obtain

$$\mathbf{E}[\hat{Z}_n^{1/2}(u_1) - \hat{Z}_n^{1/2}(u_2)]^2 \leq \mathbf{E}\left(|\ln \hat{Z}_n^{1/2}(u_1) - \ln \hat{Z}_n^{1/2}(u_2)|^2 \max\{\hat{Z}_n(u_1), \hat{Z}_n(u_2)\}\right).$$

Now, let us fix some  $p > 1$  (the choice of  $p$  will be precised later) and put  $q = \frac{p}{p-1} > 1$  (so that  $\frac{1}{p} + \frac{1}{q} = 1$ ). Using the Hölder inequality, we can write

$$\begin{aligned} \mathbf{E}[\hat{Z}_n^{1/2}(u_1) - \hat{Z}_n^{1/2}(u_2)]^2 &\leq \left[\mathbf{E}|\ln \hat{Z}_n^{1/2}(u_1) - \ln \hat{Z}_n^{1/2}(u_2)|^{2p}\right]^{\frac{1}{p}} \left[\mathbf{E}\max\{\hat{Z}_n^q(u_1), \hat{Z}_n^q(u_2)\}\right]^{\frac{1}{q}} \\ &\leq \left[\left(\frac{\varepsilon_n}{2}\right)^{2p} \mathbf{E}|\ln Z_n(u_1) - \ln Z_n(u_2)|^{2p}\right]^{\frac{1}{p}} \left[\mathbf{E}(\hat{Z}_n^q(u_1) + \hat{Z}_n^q(u_2))\right]^{\frac{1}{q}} \\ &\leq C \left[\varepsilon_n^{2p} \mathbf{E}|\ln Z_n(u_1) - \ln Z_n(u_2)|^{2p}\right]^{\frac{1}{p}} \left[e^{-cu_1^2} + e^{-cu_2^2}\right]^{\frac{1}{q}} \\ &\leq C e^{-cu_2^2} \left[\varepsilon_n^{2p} \mathbf{E}|\ln Z_n(u_1) - \ln Z_n(u_2)|^{2p}\right]^{\frac{1}{p}}, \end{aligned}$$

where we used again the inequality (13).

Introducing a centered Poisson process of intensity function  $n\lambda_*(\vartheta_0, t)$ ,  $t \in [0, \tau]$ , by

$$\pi_n(t) = \sum_{j=1}^n X_j(t) - n \int_0^t \lambda_*(\vartheta_0, s) ds,$$

we can write

$$\begin{aligned}
\ln Z_n(u_1) - \ln Z_n(u_2) &= \sum_{j=1}^n \int_0^\tau \ln \left( \frac{\lambda(\vartheta_{u_1}, t)}{\lambda(\vartheta_{u_2}, t)} \right) dX_j(t) - n \int_0^\tau [\lambda(\vartheta_{u_1}, t) - \lambda(\vartheta_{u_2}, t)] dt \\
&= \int_0^\tau \ln \left( \frac{\lambda(\vartheta_{u_1}, t)}{\lambda(\vartheta_{u_2}, t)} \right) d\pi_n(t) \\
&\quad - n \int_0^\tau \left[ \lambda(\vartheta_{u_1}, t) - \lambda(\vartheta_{u_2}, t) - \lambda_*(\vartheta_0, t) \ln \left( \frac{\lambda(\vartheta_{u_1}, t)}{\lambda(\vartheta_{u_2}, t)} \right) \right] dt \\
&= \int_0^\tau \ln \left( \frac{\lambda(\vartheta_{u_1}, t)}{\lambda(\vartheta_{u_2}, t)} \right) d\pi_n(t) - n [J_{\text{K-L}}(\vartheta_{u_1}) - J_{\text{K-L}}(\vartheta_{u_2})] \\
&= A_n(u_1, u_2) - B_n(u_1, u_2)
\end{aligned}$$

with evident notations. Therefore, using the inequality (11), it comes

$$\begin{aligned}
\mathbf{E}[\hat{Z}_n^{1/2}(u_1) - \hat{Z}_n^{1/2}(u_2)]^2 &\leq C e^{-cu_2^2} \left[ \varepsilon_n^{2p} \mathbf{E} |A_n(u_1, u_2) - B_n(u_1, u_2)|^{2p} \right]^{\frac{1}{p}} \\
&\leq C e^{-cu_2^2} \left[ \varepsilon_n^{2p} \mathbf{E} |A_n(u_1, u_2)|^{2p} + \varepsilon_n^{2p} |B_n(u_1, u_2)|^{2p} \right]^{\frac{1}{p}}.
\end{aligned}$$

For the term containing  $B_n(u_1, u_2)$ , using the mean value theorem and the upper bound of (7), we get

$$\begin{aligned}
\varepsilon_n^{2p} |B_n(u_1, u_2)|^{2p} &= \left| n\varepsilon_n [J_{\text{K-L}}(\vartheta_{u_1}) - J_{\text{K-L}}(\vartheta_{u_2})] \right|^{2p} = \left| n\varepsilon_n (\vartheta_{u_1} - \vartheta_{u_2}) J'_{\text{K-L}}(\vartheta_{\tilde{u}}) \right|^{2p} \\
&\leq C \left| n\varepsilon_n \varphi_n(u_1 - u_2) (\vartheta_{\tilde{u}} - \hat{\vartheta}) \right|^{2p} = C \left| n\varepsilon_n \varphi_n^2(u_1 - u_2) \tilde{u} \right|^{2p} \\
&\leq C |u_1 - u_2|^{2p} (\max\{|u_1|, |u_2|\})^{2p} \leq C |u_1 - u_2|^{2p} (1 + |u_2|)^{2p}.
\end{aligned}$$

Here  $\tilde{u}$  is some intermediate value between  $u_1$  and  $u_2$ .

For the term containing  $A_n(u_1, u_2)$ , using Rosenthal's inequality (see, for example, [22]) and proceeding similarly as while bounding  $G_n(u)$  in the proof of the previous lemma, we have

$$\begin{aligned}
\varepsilon_n^{2p} \mathbf{E} |A_n(u_1, u_2)|^{2p} &\leq C \varepsilon_n^{2p} \left( n \int_0^\tau \left[ \ln \left( \frac{\lambda(\vartheta_{u_1}, t)}{\lambda(\vartheta_{u_2}, t)} \right) \right]^2 \lambda_*(\vartheta_0, t) dt \right)^p \\
&\quad + C n \varepsilon_n^{2p} \int_0^\tau \left| \ln \left( \frac{\lambda(\vartheta_{u_1}, t)}{\lambda(\vartheta_{u_2}, t)} \right) \right|^{2p} \lambda_*(\vartheta_0, t) dt \\
&\leq C \left( n \varepsilon_n^2 \int_0^\tau [\lambda(\vartheta_{u_1}, t) - \lambda(\vartheta_{u_2}, t)]^2 dt \right)^p \\
&\quad + C n \varepsilon_n^{2p} \int_0^\tau |\lambda(\vartheta_{u_1}, t) - \lambda(\vartheta_{u_2}, t)|^{2p} dt \\
&\leq C (n \varepsilon_n^2 |\vartheta_{u_1} - \vartheta_{u_2}|^{2\kappa+1})^p + C n \varepsilon_n^{2p} |\vartheta_{u_1} - \vartheta_{u_2}|^{2p\kappa+1} \\
&\leq C |u_1 - u_2|^{(2\kappa+1)p} + C |u_1 - u_2|^{2p\kappa+1}.
\end{aligned}$$

Here we equally used the inequality (10), the fact that  $n\varepsilon_n^2\varphi_n^{2\kappa+1} = C$  and the boundedness of  $n\varepsilon_n^{2p}\varphi_n^{2p\kappa+1} = o(n\varepsilon_n^2\varphi_n^{2\kappa+1}) = o(1)$ .

So, finally, we obtain

$$\begin{aligned} \mathbf{E}[\hat{Z}_n^{1/2}(u_1) - \hat{Z}_n^{1/2}(u_2)]^2 &\leq C e^{-cu_2^2} \left[ |u_1 - u_2|^{(2\kappa+1)p} + |u_1 - u_2|^{2p\kappa+1} \right. \\ &\quad \left. + |u_1 - u_2|^{2p} (1 + |u_2|)^{2p} \right]^{\frac{1}{p}} \\ &\leq C e^{-cu_2^2} |u_1 - u_2|^{2\kappa + \frac{1}{p}} (1 + |u_2|)^2 \\ &\leq C |u_1 - u_2|^{2\kappa + \frac{1}{p}}, \end{aligned}$$

since the function  $u \mapsto e^{-cu^2} (1 + |u|)^2$  is bounded.

To conclude the proof of the lemma, it remains to notice that choosing  $p > 1$  sufficiently close to 1, we can make  $\gamma = 2\kappa + \frac{1}{p} < 2\kappa + 1$  arbitrary close to  $2\kappa + 1$  and, in particular, strictly greater than 1.  $\square$

## 4 Discussion

Recall that if we have a cusp-type singularity of order  $\kappa \in (0, 1/2)$  and there is no misspecification, the mean square error of the MLE has the following asymptotics (see [6]):

$$\mathbf{E}(\hat{\vartheta}_n - \vartheta)^2 = c n^{-\frac{2}{2\kappa+1}} (1 + o(1)).$$

Therefore, the smaller is the value of  $\kappa$ , the better is the rate of convergence. It is interesting to compare this rate with the rate of convergence for the model with misspecification. According to Theorem 1, the corresponding mean square error is

$$\mathbf{E}(\hat{\vartheta}_n - \vartheta)^2 = c n^{-\frac{2}{3-\kappa}} (1 + o(1)),$$

and so we have an opposite situation: the smaller is the value of  $\kappa$ , the worse is the rate of convergence.

The plots of the rate exponents  $\gamma = \frac{2}{2\kappa+1}$  and  $\gamma = \frac{2}{3-\kappa}$  with and without misspecification are given in Fig. 5. Note that for  $\kappa > 1/2$  (regular case), the plotted value is  $\gamma = 1$ , since in this case the mean square error goes to zero at rate  $1/n$  both with and without misspecification (see [16]).

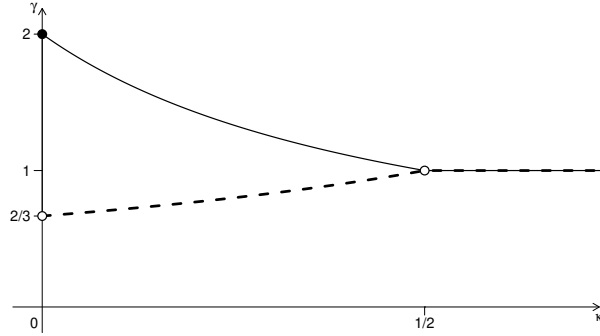


Figure 5: Rate exponents  $\gamma$  with (dashed line) and without (solid line) misspecification

The limit at  $\kappa = 0$  of the solid line corresponds well to the rate exponent  $\gamma = 2$  of the change-point problem with discontinuous intensity function (see [14]). In the case of misspecification the situation is essentially different. If the intensity function is discontinuous, then the p-MLE converges to the true value (is consistent) and  $\gamma = 2$  (see [16]), while the limit at  $\kappa = 0$  of the dashed curve is only  $2/3$ .

Note that the case  $\kappa = 1/2$  was not included in this study. If there is no misspecification and  $\kappa = 1/2$ , we are in the *almost smooth* case, and the error is

$$\mathbf{E}(\hat{\vartheta}_n - \vartheta)^2 = \frac{c}{n \ln n} (1 + o(1))$$

(see [13,16]). The properties of the p-MLE for the model with misspecification and  $\kappa = 1/2$  were not yet studied. Of course, this can be done with the help of the developed in this work approach.

Note also that it is possible to generalize the presented in this work results to the case of non constant signals, i.e., when the theoretical and real intensity functions of the observed inhomogeneous Poisson processes are given by

$$\begin{aligned} \lambda(\vartheta, t) &= S(t) \psi(t - \vartheta) + \lambda_0, \\ \lambda_*(\vartheta, t) &= S_*(t) \psi(t - \vartheta) + \lambda_0, \end{aligned}$$

with functions  $S(\cdot)$  and  $S_*(\cdot)$  satisfying the condition

$$\inf_{t \in \Theta} |S(t) - S_*(t)| > 0.$$

The main difference will be in the proof of Proposition 1.

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