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ON CHEEGER CONSTANTS OF HYPERBOLIC SURFACES

Thomas BUDZINSKI* & Nicolas CURIEN† & Bram PETRI‡

Abstract

It is a well-known result due to Bollobas that the maximal Cheeger constant of large d -regular graphs cannot be close to the Cheeger constant of the d -regular tree. We prove analogously that the Cheeger constant of closed hyperbolic surfaces of large genus is bounded from above by $2/\pi \approx 0.63\dots$ which is strictly less than the Cheeger constant of the hyperbolic plane. The proof uses a random construction based on a Poisson–Voronoi tessellation of the surface with a vanishing intensity.

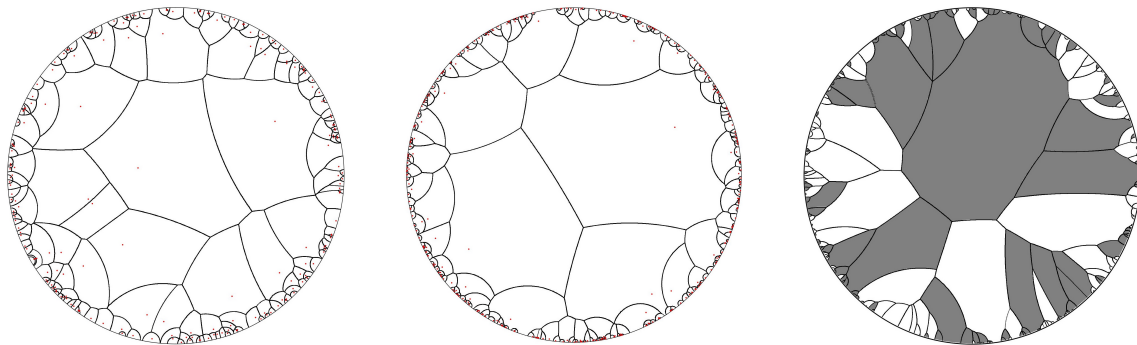


Figure 1 – From left to right: Poisson–Voronoi tessellations of the hyperbolic plane with decreasing intensity. Their limit (on the right) is the *pointless Voronoi tessellation* of the hyperbolic plane whose cells have been colored in black/white uniformly at random. This object has an average "linear" density equal to $2 \times \frac{2}{\pi}$ per unit of area.

1 Introduction

Let \mathcal{S} be a closed hyperbolic surface. If $A \subset \mathcal{S}$ is a subset of \mathcal{S} , we define $h^*(A) = |\partial A|/|A|$ where $|\partial A|$ is the length of its boundary and $|A|$ is its area. If one of these quantities is not well defined, then $h^*(A) = +\infty$ by convention. The *Cheeger constant* of \mathcal{S} is

$$h(\mathcal{S}) = \inf \{h^*(A) \mid A \subset \mathcal{S} \text{ with } |A| \leq |\mathcal{S}|/2\}. \quad (1)$$

Theorem 1. *For $g \geq 2$, let \mathcal{M}_g be the moduli space of all isometry classes of closed hyperbolic surfaces of genus g . Then we have*

$$\limsup_{g \rightarrow \infty} \sup_{\mathcal{S} \in \mathcal{M}_g} h(\mathcal{S}) \leq \frac{2}{\pi}.$$

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We recall that the Cheeger constant of the hyperbolic plane \mathbb{H} is equal to 1, which is asymptotically attained by large disks. As such, our result shows a gap between the maximal Cheeger constant of a large, closed hyperbolic surface and that of its universal cover. The presence of this gap was conjectured by Wright and Lipnowski [27] building upon [7] and inspired by similar results in graph theory. It also follows from our result that Cheeger’s inequality [11] – the original reason for which the Cheeger constant was introduced – cannot be used to show that a closed hyperbolic surface of large genus has an optimal spectral gap. Finally, Theorem 1 implies recent results by Shen–Wu on random Belyĭ surfaces and random covers of the Bolza surface [24, 25].

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2 Context and sketch of proof

Before sketching the proof of Theorem 1, let us place it into context by telling the analogous story in the case of d -regular graphs. We will use similar notation as in the case of hyperbolic surfaces, so that the reader may follow the parallels more easily.

2.1 Warm-up: the graph case

Fix an integer $d \geq 3$ and consider the set $\mathcal{G}_d(n)$ of all connected simple graphs on n vertices having all degree d (we assume that $d \cdot n$ is even so that $\mathcal{G}_d(n)$ is not empty). If $\mathfrak{g}_n \in \mathcal{G}_d(n)$ and A is a subset of the vertices of \mathfrak{g}_n , the isoperimetric constant of A is defined as

$$h^*(A) = \frac{|\partial A|}{|A|},$$

where $|A|$ is the number of vertices of A and $|\partial A|$ is the number of edges having one endpoint in A and the other outside A . The Cheeger constant of \mathfrak{g}_n is then

$$h(\mathfrak{g}_n) := \inf \{h^*(A) \mid A \subset \mathfrak{g}_n \text{ with } |A| \leq n/2\}.$$

It is easy to construct graphs $\mathfrak{g}_n \in \mathcal{G}_d(n)$ with small Cheeger constant, for example if they contain a large piece which looks roughly one-dimensional. On the other hand, we have

$$\mathbf{c}_d := \liminf_{n \rightarrow \infty} \sup_{\mathfrak{g}_n \in \mathcal{G}_d(n)} h(\mathfrak{g}_n) > 0,$$

since there are (families of) graphs, called *expanders*, whose Cheeger constant is uniformly bounded from below. The existence of such graphs has famously been proved by Margulis [19] through an explicit construction and Pinsker [22] using a probabilistic argument. Ramanujan graphs [18, 20] are very good expanders and their existence shows that $\mathbf{c}_d \geq \frac{d}{2} - O(\sqrt{d})$ when $d \rightarrow \infty$, which can also be proved using random graphs [4]. Conversely, an easy argument shows that the Cheeger constant of a large d -regular graph is asymptotically bounded from above by that of the d -regular tree \mathbb{T}_d , which is

equal to $d - 2$, so that in particular $\mathbf{c}_d \leq d - 2$. This bound is not sharp and in fact Bollobas [4], later sharpened by Alon [1], proved that

$$\mathbf{c}_d \leq \frac{1}{2}(d - 2) \quad \text{for all } d \geq 3 \quad (2)$$

(and even $\mathbf{c}_d = \frac{d}{2} - O(\sqrt{d})$ asymptotically as $d \rightarrow \infty$). This gap between the largest Cheeger constant of large d -regular graphs and that of \mathbb{T}_d is the graph analog of our Theorem 1.

The idea of the proof of (2) is surprisingly simple: fix a large graph $\mathfrak{g}_n \in \mathcal{G}_d(n)$ and color *uniformly at random* half of its vertices black and the other half white. Consider then the subset A of size $n/2$ consisting of the black vertices. Since an edge is counted in $|\partial A|$ if and only its endpoints have different colors, the probability that a given edge contributes to $|\partial A|$ is approximately $\frac{1}{2}$. We thus have

$$\mathbb{E}[|\partial A|] \approx \frac{1}{2} |\text{Edges}(\mathfrak{g}_n)| = \frac{1}{2} \cdot \frac{dn}{2}, \quad (3)$$

and so $h(\mathfrak{g}_n) \leq \mathbb{E}[h^*(A)] \leq \frac{d}{2}$ asymptotically when n is large. To get the better bound $h(\mathfrak{g}_n) \leq \frac{d-2}{2}$, the idea is to first regroup the vertices of \mathfrak{g}_n into connected “regions” R_1, \dots, R_k of vertices which are all large (i.e. $1 \ll |R_i|$), but not too large (i.e. $|R_i| \ll n$). In particular k must be large. In the graph case, one can perform such a splitting in various deterministic ways, e.g. using a spanning tree of \mathfrak{g}_n as in [1, Section 3]. Since those regions are large and connected we have the crude bound valid on d -regular trees

$$h^*(R_i) \leq d - 2 + o(1). \quad (4)$$

Hence, the number of edges whose endpoints lie in two different regions $R_i \neq R_j$ is roughly $\frac{n(d-2)}{2}$. We can then proceed as above and color each region uniformly at random in black or white. Since the regions are not too large, a standard concentration argument shows that the black vertices form a subset \tilde{A} of approximately $n/2$ vertices. Again, a given edge is counted in $|\partial \tilde{A}|$ if its endpoints lie in two different regions with different colors. By the same computation as in (3) for $\mathbb{E}[|\partial \tilde{A}|]$, we deduce

$$h^*(\tilde{A}) \leq \frac{1}{2} \cdot \frac{n(d-2)}{2} \cdot \frac{2}{n} = \frac{d-2}{2}.$$

2.2 Hyperbolic surfaces

Let us now draw the parallel with the case of hyperbolic surfaces. A first difficulty is that the *a priori* crude bound (4) does not hold in the continuous setting: there is no upper bound on the Cheeger constant of a connected set since there are such sets with a nasty fractal boundary having a large one-dimensional measure. However, the global inequality

$$h(\mathcal{S}_g) \leq h(\mathbb{H}) + o(1) = 1 + o(1) \quad \text{as } g \rightarrow \infty$$

is still true for any hyperbolic surface \mathcal{S}_g of genus $g \geq 2$. This can e.g. be derived by spectral considerations, combining works of Cheeger [11] and Cheng [12]. Here also, the existence of “expander

surfaces” of large genus with Cheeger constant bounded away from 0 is known. The known constructions essentially fall into three (overlapping) categories. First of all, there are multiple ways to compare the Cheeger constant of a hyperbolic surface to that of a graph [8, 5]. Secondly, just like in the case of graphs, it is known, due to Buser [9], that spectral expansion implies isoperimetric expansion. As such, the many examples of surfaces with a spectral gap – the first examples are due to Selberg [23] and recently near optimal spectral expanders were found by Hide and Magee [15] – also give rise to surfaces with large Cheeger constants. Finally, multiple random constructions [21, 6] are known to provide examples.

Let us now try to mimic the proof of the graph case to prove a strict inequality in the last display. Obviously, taking half of the points of \mathcal{S}_g does not make sense, and one wants to first split our deterministic surface \mathcal{S}_g into regions R_1, \dots, R_k satisfying $1 \ll |R_i| \ll g$ before coloring them in black and white with equal probability. In [7] Brooks and Zuk roughly speaking postulated the existence of such a decomposition based on balls where the isoperimetric constants $h^*(R_i)$ of the regions in question are close to 1. Given those hypothetical decompositions, the coloring argument would yield

$$h(\mathcal{S}_g) \leq \frac{1}{2}$$

when $g \rightarrow \infty$. We were not able to prove that such decompositions actually exist on every hyperbolic surface and rather use a *random splitting of \mathcal{S}_g based on a Voronoi decomposition*. More precisely, we consider a Poisson point process with points X_1, \dots, X_N of intensity given by $\lambda \cdot \mu_{\mathcal{S}_g}$, where $\mu_{\mathcal{S}_g}$ is the hyperbolic area measure on \mathcal{S}_g with total mass $4\pi(g-1)$, and $\lambda > 0$ is a small constant. Those points decompose \mathcal{S}_g into the Voronoi regions $\text{Vor}_\lambda(\mathcal{S}_g) := \{C_1, \dots, C_N\}$ where

$$C_i = \left\{ x \in \mathcal{S}_g : d_{\mathcal{S}_g}(x, X_i) = \min_{1 \leq j \leq N} d_{\mathcal{S}_g}(x, X_j) \right\}.$$

When the intensity parameter λ is small, the typical area of a region is of order $1/\lambda$ and so one can expect (see Lemma 1 below) that indeed we have $1 \ll |C_i| \ll |\mathcal{S}_g|$ at least for most regions. Now recall that we have no *a priori* upper bound for the Cheeger constant of these regions (contrary to the graph case). The crux of the argument boils down to showing that, at least on average, we have $h^*(C_i) = \frac{4}{\pi}$. More precisely, we will show (see Proposition 2) that the expected length of the union $\partial \text{Vor}_\lambda(\mathcal{S}_g)$ of the boundaries of the Voronoi cells C_1, \dots, C_N satisfies

$$\limsup_{\lambda \rightarrow 0} \sup_{g \geq 2} \sup_{\mathcal{S}_g \in \mathcal{M}_g} \frac{1}{|\mathcal{S}_g|} \mathbb{E} [|\partial \text{Vor}_\lambda(\mathcal{S}_g)|] \leq \frac{2}{\pi}. \quad (5)$$

Given the above display, one can then run the same proof as in the graph case: we color uniformly the Voronoi cells in black and white and consider the resulting black component. Its volume is concentrated around $\frac{1}{2}|\mathcal{S}_g|$ and its boundary size is less than $|\mathcal{S}_g| \cdot \frac{1}{2} \cdot (\frac{2}{\pi} + \varepsilon)$ for g large when λ is small, so that $h(\mathcal{S}_g) \leq (\frac{2}{\pi} + \varepsilon)$ asymptotically as claimed in Theorem 1.

2.3 The pointless Voronoi tessellation of \mathbb{H}

Let us gain some intuition on the proof of (5) done in Proposition 2. Suppose first that the systole of \mathcal{S}_g tends to ∞ as $g \rightarrow \infty$. In that situation, the neighborhood of each point in \mathcal{S}_g looks like a piece

of the hyperbolic plane and the Voronoi tessellation $\text{Vor}_\lambda(\mathcal{S}_g)$ converges in distribution (in the local Hausdorff sense) towards the Voronoi tessellation $\text{Vor}_\lambda(\mathbb{H})$ of the hyperbolic plane with intensity λ , see Figure 1.

This classical object has been studied in stochastic geometry [16, 10] and in particular in relation to its percolation properties [3, 13, 14]. In particular, Isokawa [16] computed the mean characteristics of a typical¹ cell C_λ in $\text{Vor}_\lambda(\mathbb{H})$: this cell is an almost surely finite convex hyperbolic polygon satisfying

$$\mathbb{E}[C_\lambda] = \frac{1}{\lambda} \quad \text{and} \quad \mathbb{E}[|\partial C_\lambda|] = \frac{8}{\sqrt{\pi\lambda}} \int_0^\infty e^{-u} \sqrt{u + \frac{u^2}{4\pi\lambda}} \, du.$$

In particular, the ‘‘average Cheeger constant’’ of cells of $\text{Vor}_\lambda(\mathbb{H})$ satisfies in the small intensity limit

$$\frac{\mathbb{E}[|\partial C_\lambda|]}{\mathbb{E}[C_\lambda]} \xrightarrow{\lambda \rightarrow 0} \frac{4}{\pi},$$

which explains (5) in the case when $\text{Systole}(\mathcal{S}_g) \rightarrow \infty$. Note that recently percolation on hyperbolic Poisson–Voronoi tessellation with small intensity has been studied in [13]. Underneath the convergence of the above display as the intensity tends to 0 lies the fact that $\text{Vor}_\lambda(\mathbb{H})$ converges (in distribution for the Hausdorff topology on compact sets of \mathbb{H}) towards a limiting object that we name the *pointless Poisson–Voronoi tessellation* of the hyperbolic disk (see Figure 1) and whose construction and properties will be studied in a forthcoming work.

Finally, in order to deal with the case in which the systole is not large, we will need to study $\partial\text{Vor}_\lambda(\mathcal{S}_g)$ in the neighbourhood of a point x of \mathcal{S}_g . To do so, we will replace the hyperbolic plane by the *Dirichlet domain* – a specific fundamental domain – of \mathcal{S}_g around x . We will prove that in order to prove (5), it will be enough to understand $\partial\text{Vor}_\lambda(\mathcal{S})$ inside the Dirichlet domain, thus also reducing the general case to computations in the hyperbolic plane.

3 Proof

Let us recall some basic notions and fix notation before starting the proof.

3.1 Preliminaries

The hyperbolic plane. We recall that the *hyperbolic plane* \mathbb{H} is the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, equipped with the Riemannian metric $\frac{4|dz|^2}{(1-|z|^2)^2}$. This metric has constant curvature -1 and is invariant under Möbius transformations. We will denote by $d_{\mathbb{H}}(x, y)$ the hyperbolic distance in \mathbb{H} . For several computations in the paper, it will be useful to use polar coordinates on \mathbb{H} . More precisely, for $r > 0$ and $\theta \in [0, 2\pi)$, we denote by $[r; \theta]$ the point of \mathbb{H} of the form $z = \rho(r)e^{i\theta}$, where $\rho(r) > 0$ is such that $d_{\mathbb{H}}(0, z) = r$. We note that the area measure on \mathbb{H} can be written as $\sinh(r) \, dr \, d\theta$.

¹By Palm calculus, such a cell can be obtained by adding the point 0 to the Poisson process and considering the associated region.

Hyperbolic surfaces and Dirichlet domains. We recall that a closed hyperbolic surface \mathcal{S} of genus $g \geq 2$ is a Riemannian surface which is locally isometric to \mathbb{H} , or equivalently which is the quotient of \mathbb{H} by a discrete, torsion-free, cocompact group G of isometries. We denote by $d_{\mathcal{S}}$ and $\mu_{\mathcal{S}}$ (or just d and μ if no confusion is possible) the hyperbolic metric and area measure on \mathcal{S} . If $A \subset \mathcal{S}$ is a Borel subset we write $|A|$ for $\mu(A)$ and $|\partial A|$ for the length of its boundary. If x is a point of a hyperbolic surface and $r \geq 0$, we denote by $B_r(x)$ the closed ball of radius r around x .

We will make important use of the *Dirichlet domain*, which is a particular choice of a fundamental domain for the action of G on \mathbb{H} , see [2, Section 9.4]. More precisely, let x be a point of a hyperbolic surface $\mathcal{S} = \mathbb{H}/G$, and let $p : \mathbb{H} \rightarrow \mathcal{S}$ be its universal cover, so that $p(0) = x$. We denote by $D(\mathcal{S}, x)$ the set of those points $y' \in \mathbb{H}$ such that $d_{\mathbb{H}}(0, y') = d_{\mathcal{S}}(x, p(y'))$. In other words, this means that the image under p of the geodesic from 0 to y' is still a shortest path in \mathcal{S} . Dirichlet domains are convex polygons of \mathbb{H} [2, Theorem 9.4.2]. For (almost) every point $y \in \mathcal{S}$, we will denote by $p^{-1}(y)$ the unique point of $D(\mathcal{S}, x)$ which is sent to y . Note that the maps $p : D(\mathcal{S}, x) \rightarrow \mathcal{S}$ and $p^{-1} : \mathcal{S} \rightarrow D(\mathcal{S}, x)$ are measure-preserving.

Poisson point processes. The surface \mathcal{S} or the hyperbolic plane \mathbb{H} both carry a Borel measure $\mu_{\mathcal{S}}$ which is of finite mass $4\pi(g-1)$ in the case of \mathcal{S} and $\mu_{\mathbb{H}}$ which is σ -finite in the case of \mathbb{H} . They have no atoms, so one can define a Poisson point process on those spaces with intensity $\lambda \cdot \mu$. This is a cloud of distinct random points that is characterized by the fact that for any disjoint measurable subsets A_1, \dots, A_k , the number of points falling inside A_i are independent Poisson random variables of mean $|A_i|$, see [17] for details.

Poisson–Voronoi tessellation. Let \mathcal{S} be a hyperbolic surface of (large) genus g . As announced above, to bound its Cheeger constant from above, we shall build a random subset A_{λ} of \mathcal{S} in the following way. We first throw a Poisson point process $\Pi = \{X_1, \dots, X_N\}$ on \mathcal{S} with intensity $\lambda \cdot \mu_{\mathcal{S}}$, where $\lambda > 0$ is a small constant. In particular the random variable N follows a Poisson law with mean $\lambda \cdot 4\pi(g-1)$. We then consider the closed Voronoi cells $\text{Vor}_{\lambda}(\mathcal{S}) = \{C_1, \dots, C_N\}$ it defines and denote by $C(x)$ the Voronoi cell containing the point $x \in \mathcal{S}$ (with ties broken arbitrarily). Conditionally on Π , each cell is colored black or white independently with probability $\frac{1}{2}$ and we let $A_{\lambda} \subset \mathcal{S}$ be (the closure of) the union of the black cells. On the one hand, we will prove (Lemma 1) that unless $h(\mathcal{S})$ is very small (in which case our main result is trivial), the area $|A_{\lambda}|$ is close to $\frac{|\mathcal{S}|}{2}$ if the surface is large enough. On the other hand, we will show (Proposition 2) that $\mathbb{E}[|\partial A_{\lambda}|]$ is close to $\frac{1}{\pi}|\mathcal{S}|$ if the intensity λ is chosen small enough.

3.2 Area estimate

We start with the area estimate. We believe that Lemma 1 below should be true even without the Cheeger constant assumption, but we could not find a short argument to prove the general statement.

Lemma 1 (Area estimate). *For any $\lambda > 0$ and $\delta > 0$, there is $g_0 \geq 2$ with the following property. For every hyperbolic surface \mathcal{S} of genus $g \geq g_0$ such that $h(\mathcal{S}) \geq \delta$, if the random subset A_{λ} of \mathcal{S} is built*

as described above, we have

$$\mathbb{P} \left(\left| \frac{|A_\lambda|}{|\mathcal{S}|} - \frac{1}{2} \right| > \delta \right) < \delta.$$

Proof. Let \mathcal{S} be a hyperbolic surface with Cheeger constant at least δ . It follows immediately from our setup that $\mathbb{E}[|A_\lambda|] = \frac{|\mathcal{S}|}{2}$, so we only need to bound the variance of $|A_\lambda|$. By conditioning on $\text{Vor}_\lambda(\mathcal{S})$, we have

$$\text{Var}(|A_\lambda|) = \mathbb{E} \left[\frac{1}{4} \sum_{i=1}^N |C_i|^2 \right] = \frac{1}{4} \int_{\mathcal{S}^2} \mathbb{P}(C(x) = C(y)) \mu(dx) \mu(dy).$$

To bound $\mathbb{P}(C(x) = C(y))$ from above, we will first argue that the assumption $h(\mathcal{S}) \geq \delta$ implies a lower bound on the volume of balls around x and y for "most" points $x, y \in \mathcal{S}$.

More precisely, for all $x \in \mathcal{S}$ and $r > 0$, by the Cheeger constant assumption we have

$$\frac{d}{dr} |B_r(x)| = |\partial B_r(x)| \geq \delta |B_r(x)|.$$

Therefore, let $r_1 > r_0 > 0$ (the values of r_0 and r_1 , depending only on δ and λ , will be specified later), and assume that g is large enough to have $2\pi(\cosh(r_1) - 1) < \frac{|\mathcal{S}|}{2}$. We have

$$|B_{r_1}(x)| \geq e^{\delta(r_1 - r_0)} |B_{r_0}(x)|.$$

On the other hand, by the collar lemma, we know that there is an absolute constant $C > 0$ such that, if r_0 is small enough, we have

$$|\{x \in \mathcal{S} \mid \text{InjRad}(x) \leq r_0\}| \leq Cr_0 \cdot |\mathcal{S}|.$$

But if the injectivity radius around x is larger than r_0 , then $|B_{r_0}(x)| = 2\pi(\cosh(r_0) - 1)$ and we get a lower bound on $|B_{r_1}(x)|$. In particular, by taking r_0 small enough, we find

$$\left| \left\{ x \in \mathcal{S} \mid |B_{r_1}(x)| < e^{\delta(r_1 - r_0)} 2\pi(\cosh(r_0) - 1) \right\} \right| \leq \delta^3 |\mathcal{S}|. \quad (6)$$

We denote by $K_{r_1} \subset \mathcal{S}^2$ the set of pairs (x, y) such that $d(x, y) > 2r_1$ and neither x nor y satisfies the event in the left-hand side of (6). Let $(x, y) \in K_{r_1}$. If x and y belong to the same Voronoi cell, since $B_{r_1}(x)$ and $B_{r_1}(y)$ are disjoint, at least one of these two balls contains none of the points X_i . It follows that

$$\begin{aligned} \mathbb{P}(C(x) = C(y)) &\leq \exp(-\lambda |B_{r_1}(x)|) + \exp(-\lambda |B_{r_1}(y)|) \\ &\leq 2 \exp\left(-2\pi\lambda e^{\delta(r_1 - r_0)} (\cosh(r_0) - 1)\right). \end{aligned}$$

In particular, if we have chosen a large enough value for r_1 , this probability is smaller than δ^3 . Therefore, we get

$$\text{Var}(|A_\lambda|) \leq \frac{\delta^3 |\mathcal{S}|^2}{4} + \frac{|\mathcal{S}^2 \setminus K_{r_1}|}{4} \leq \frac{\delta^3 |\mathcal{S}|^2}{4} + 2 \frac{\delta^3 |\mathcal{S}|^2}{4} + \frac{2\pi}{4} (\cosh(2r_1) - 1) |\mathcal{S}|,$$

where the second term comes from (6), and the third term counts the pairs (x, y) with $d(x, y) < 2r_1$. In particular, for $|\mathcal{S}| = 4\pi(g - 1)$ large enough, the variance of $|A_\lambda|$ is smaller than $\delta^3 |\mathcal{S}|^2$ and the conclusion follows by the Chebychev inequality. \square

3.3 Perimeter estimate

Let us now pass to the perimeter estimate which is the most technical part of the proof.

Proposition 2 (Perimeter estimate). *Recall that $\partial\text{Vor}_\lambda(\mathcal{S})$ is the union of the sides of the Poisson-Voronoi cells C_1, \dots, C_N with intensity $\lambda \cdot \mu_{\mathcal{S}}$. Then we have*

$$\limsup_{\lambda \rightarrow 0} \sup_{g \rightarrow \infty} \sup_{\mathcal{S} \in \mathcal{M}_g} \frac{1}{|\mathcal{S}|} \mathbb{E} [|\partial\text{Vor}_\lambda(\mathcal{S})|] \leq \frac{2}{\pi}.$$

An optimization lemma. Before proving this estimate, we state two intermediate results that will be useful for us. The first is a nice optimization lemma, for which a very short and direct proof is provided in the last page of [26].

Lemma 3. *Let ν be a probability measure on $[0, 2\pi]$. Then we have*

$$\int_0^{2\pi} \int_0^{2\pi} \left| \sin \frac{\theta_1 - \theta_2}{2} \right| \nu(d\theta_1) \nu(d\theta_2) \leq \frac{2}{\pi},$$

with equality if ν is the uniform measure.

The proof of Lemma 3 consists of assuming by density that ν has a smooth density f with respect to Lebesgue, and expressing the integral in terms of the Fourier coefficients of f . We will use this lemma to handle the fact that a Dirichlet domain $D(\mathcal{S}, x)$ is not as isotropic as \mathbb{H} .

Intersection of thin rings. The second lemma will help ruling out some pathological behaviours of Dirichlet domains. For $0 < a < b$ and $x \in \mathbb{H}$, we denote by $R_a^b(x)$ the set of points $z \in \mathbb{H}$ such that

$$a \leq d(x, z) \leq b.$$

Lemma 4. *Let $K > 0$ and let x, y be two distinct points of \mathbb{H} . Then for all $a > 0$, we have*

$$|R_a^{a+\varepsilon}(x) \cap R_a^{a+\varepsilon}(y)| = o(\varepsilon)$$

as $\varepsilon \rightarrow 0$. Moreover, the $o(\varepsilon)$ is uniform in (x, y, a) provided $a \leq K$ and $d(x, y) \geq K^{-1}$.

Proof. We first note that if $d(x, y) > 2K + 2$, then the intersection $R_a^{a+\varepsilon}(x) \cap R_a^{a+\varepsilon}(y)$ is empty as soon as $\varepsilon < 1$, so we may assume $d(x, y) \leq 2K + 2$. For the same reason, since $d(x, y) \geq K^{-1}$, we may assume $a \geq \frac{1}{2K}$. Moreover, by invariance under Möbius transformations, we may assume that $x = 0$ and that $y = [d; 0]$ in polar coordinates, with $K^{-1} \leq d \leq 2K + 2$.

Let us express $R_a^{a+\varepsilon}(0) \cap R_a^{a+\varepsilon}(y)$ in polar coordinates. First, if $z = [r; \theta] \in R_a^{a+\varepsilon}(0) \cap R_a^{a+\varepsilon}(y)$, then we must have $r \in [a, a + \varepsilon]$. Second, by the hyperbolic law of cosines, we have

$$\cosh(d(y, z)) = \cosh(d) \cosh(r) - \cos(\theta) \sinh(d) \sinh(r).$$

Since $a \leq d(y, z) \leq a + \varepsilon$, we deduce

$$\frac{\cosh(d) \cosh(r) - \cosh(a + \varepsilon)}{\sinh(d) \sinh(r)} \leq \cos(\theta) \leq \frac{\cosh(d) \cosh(r) - \cosh(a)}{\sinh(d) \sinh(r)}.$$

That is, $\cos(\theta)$ lies in an interval $I(a, d, r, \varepsilon)$ of length $\frac{\cosh(a+\varepsilon) - \cosh(a)}{\sinh(d)\sinh(r)} \leq K \frac{\varepsilon \sinh(a+\varepsilon)}{\sinh(r)} \leq C(K)\varepsilon$ by the assumption $r \geq \frac{1}{2K}$. This implies that θ must lie in a set $S(a, d, r, \varepsilon) \subset [0, 2\pi]$ of measure at most $C(K)\sqrt{\varepsilon}$. Therefore, we have

$$\begin{aligned} |R_a^{a+\varepsilon}(x) \cap R_a^{a+\varepsilon}(y)| &= \int_a^{a+\varepsilon} |S(a, d, r, \varepsilon)| \sinh(r) \, dr \\ &\leq \sinh(a + \varepsilon) C(K) \varepsilon^{3/2} \leq \sinh(K + 1) C(K) \varepsilon^{3/2}, \end{aligned}$$

which proves the lemma. \square

Proof of Proposition 2. In the rest of the proof we write $\partial C \equiv \partial \text{Vor}_\lambda(\mathcal{S})$ to lighten notation. For all $\varepsilon > 0$, we denote by $\partial^\varepsilon C$ the set of points of \mathcal{S} lying at hyperbolic distance at most ε from ∂C . Since ∂C is a.s. the union of finitely many geodesic segments, we have

$$|\partial C| = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} |\partial^\varepsilon C|.$$

It follows from Fatou's lemma that

$$\mathbb{E}[|\partial C|] \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \mathbb{E}[|\partial^\varepsilon C|] = \liminf_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{\mathcal{S}} \mathbb{P}(x \in \partial^\varepsilon C) \mu_{\mathcal{S}}(dx). \quad (7)$$

Hence, let $0 < \varepsilon < 1$ and $x \in \mathcal{S}$, and let i_1 be such that $C(x) = C_{i_1}$. We first note that if $x \in \partial^\varepsilon C$, then there are at least two Voronoi cells which intersect the ball $B_\varepsilon(x)$, and one of these cells is C_{i_1} . Hence, there is an $i_2 \neq i_1$ such that the bisector between X_{i_1} and X_{i_2} intersects $B_\varepsilon(x)$. Therefore we introduce, for any point y , the set $A_{\mathcal{S}}^\varepsilon(x, y) \subset \mathcal{S}$ consisting of those points z such that $d(x, z) \geq d(x, y)$ and such that the bisector between y and z intersects the closed ball $B_\varepsilon(x)$. We first note that if $z \in A_{\mathcal{S}}^\varepsilon(x, y)$, then by the triangle inequality, we have²

$$d(x, y) \leq d(x, z) \leq d(x, y) + 2\varepsilon. \quad (8)$$

The event $x \in \partial^\varepsilon C$ is equivalent to saying that at least one other point of the Poisson process lands in the region $A_{\mathcal{S}}^\varepsilon(x, X_{i_1})$. Therefore, by conditioning on the point X_{i_1} closest to x , we have

$$\mathbb{P}(x \in \partial^\varepsilon C) = \lambda \int_{\mathcal{S}} \mu_{\mathcal{S}}(dy) \exp\left(-\lambda |B_{d(x, y)}(x)|\right) \times (1 - \exp(-\lambda |A_{\mathcal{S}}^\varepsilon(x, y)|)) \quad (9)$$

$$\leq \lambda^2 \int_{\mathcal{S}} \mu_{\mathcal{S}}(dy) \exp\left(-\lambda |B_{d(x, y)}(x)|\right) |A_{\mathcal{S}}^\varepsilon(x, y)|. \quad (10)$$

If the injectivity radius at $x \in \mathcal{S}$ is much larger than $d(x, y)$, then $|A_{\mathcal{S}}^\varepsilon(x, y)|$ can be computed as in the hyperbolic plane (and one would recover the estimates of Isokawa [16]). In the general case, we will work with the Dirichlet domain $D(\mathcal{S}, x) \subset \mathbb{H}$ defined above. We recall that $D(\mathcal{S}, x)$ is a fundamental domain for the projection $p: \mathbb{H} \rightarrow \mathcal{S}$ with $p(0) = x$. We claim that for $y \in \mathcal{S}$, the subset $A_{\mathcal{S}}^\varepsilon(x, y)$ of \mathcal{S} is very close to the subset $A_{D(\mathcal{S}, x)}^\varepsilon(0, p^{-1}(y))$ of \mathbb{H} .

²Using this inequality to crudely bound $|A_{\mathcal{S}}^\varepsilon(x, y)|$, we would obtain Proposition 2 (and therefore Theorem 1) with a constant 1 instead of $\frac{2}{\pi}$. This is why we will need the more accurate description given by Lemma 6.

Lemma 5. For $x, y \in \mathcal{S}$, we have

$$|A_{\mathcal{S}}^{\varepsilon}(x, y)| \leq \left| A_{D(\mathcal{S}, x)}^{\varepsilon}(0, p^{-1}(y)) \right| + o(\varepsilon)$$

as $\varepsilon \rightarrow 0$, uniformly in $x, y \in \mathcal{S}$.

Proof. Let us fix x and y , and write $r = d_{\mathcal{S}}(x, y)$. Since $p^{-1} : \mathcal{S} \rightarrow D(\mathcal{S}, x)$ is measure-preserving, it is sufficient to prove

$$\left| p^{-1}(A_{\mathcal{S}}^{\varepsilon}(x, y)) \setminus A_{D(\mathcal{S}, x)}^{\varepsilon}(0, p^{-1}(y)) \right| = o(\varepsilon), \quad (11)$$

uniformly in $(x, y) \in \mathcal{S}^2$. We fix a Fuchsian group G such that $\mathcal{S} = G \backslash \mathbb{H}$. Moreover, to avoid heavy notation involving the projection map, we will use $w' \in D(\mathcal{S}, x)$ to denote the unique (up to a set of measure 0) pre-image of $w \in \mathcal{S}$ under p . In particular, $x' = 0$.

Now assume that $\varepsilon < \frac{1}{2} \text{Systole}(\mathcal{S})$ so that $B_{\varepsilon}(0) \subset D(\mathcal{S}, x)$, and let z' be in the difference of sets of (11). This implies that there is a point $a \in B_{\varepsilon}(x)$ such that

$$d_{\mathcal{S}}(a, z) < d_{\mathcal{S}}(a, y) \quad \text{but} \quad d_{\mathbb{H}}(a', z') > d_{\mathbb{H}}(a', y').$$

Now let $g \cdot a'$ (for some $g \in G$) be the translate of a' such that $d_{\mathcal{S}}(a, z) = d_{\mathbb{H}}(g \cdot a', z')$. The last display means that $d_{\mathbb{H}}(g \cdot a', z') < d_{\mathbb{H}}(a', y')$. Observe that $g \cdot a' \in B_{\varepsilon}(g \cdot x')$. We get

$$d_{\mathbb{H}}(g \cdot x', z') \leq d_{\mathbb{H}}(g \cdot x', g \cdot a') + d_{\mathbb{H}}(g \cdot a', z') \leq \varepsilon + d_{\mathcal{S}}(a, z) \leq 2\varepsilon + d_{\mathcal{S}}(x, z) \leq r + 4\varepsilon,$$

where we have used (8) for the last inequality. On the other hand, by the definition of the Dirichlet domain, we have

$$d_{\mathbb{H}}(g \cdot x', z') \geq d_{\mathbb{H}}(x', z') = d_{\mathcal{S}}(x, z) \geq d_{\mathcal{S}}(x, y) = r.$$

Finally, we have

$$d_{\mathbb{H}}(x', g \cdot x') \leq d_{\mathbb{H}}(x', z') + d_{\mathbb{H}}(z', g \cdot x') = d_{\mathcal{S}}(x, z) + d_{\mathbb{H}}(z', g \cdot x') \leq 2r + 6\varepsilon,$$

where we have used (8) again. Putting the last three inequalities together, we find that the set described in the left-hand side of (11) is contained in

$$\bigcup_{\substack{w \in G \cdot 0 \setminus \{0\} \\ d_{\mathbb{H}}(0, w) \leq 2r+1}} \left\{ z' \in \mathbb{H} \mid \begin{array}{l} r \leq d(w, z') \leq r + 4\varepsilon \text{ and} \\ r \leq d(0, z') \leq r + 2\varepsilon \end{array} \right\}.$$

The sets in this union are intersections of two annuli of width 4ε , centered around 0 and a translate w of 0. Such an intersection has area $o(\varepsilon)$, by Lemma 4. Note that in order to apply this lemma, we use the fact that $r \leq \text{Diameter}(\mathcal{S})$ and that a translate $w = g \cdot 0$ satisfies $d(w, 0) \geq \text{Systole}(\mathcal{S})$.

Finally, the number of sets in the union can be bounded in terms of r , and hence in terms of $\text{Diameter}(\mathcal{S})$. Indeed, two points in $G \cdot 0$ are least $\text{Systole}(\mathcal{S})$ apart and as such only so many of them fit in a disk of radius $2r + 1$. This concludes the proof of Lemma 5. \square

The next step is to give a precise description of the set $A_{D(\mathcal{S},x)}^\varepsilon(0,y)$. It will be particularly natural to express this description in polar coordinates. For all $r > 0$, we denote by $I_r(x)$ the set of angles $\theta \in [0, 2\pi)$ such that the point $[r; \theta]$ belongs to $D(\mathcal{S}, x)$. We note that $I_{r_2}(x) \subset I_{r_1}(x)$ when $r_1 \leq r_2$ by convexity of $D(\mathcal{S}, x)$. Moreover, the area measure on $D(\mathcal{S}, x)$ can be written as $\sinh(r) \mathbb{1}_{\theta \in I_r(x)} dr d\theta$. Then we have the following good approximation of $A_{D(\mathcal{S},x)}^\varepsilon(0,y)$ (see also Figure 2).

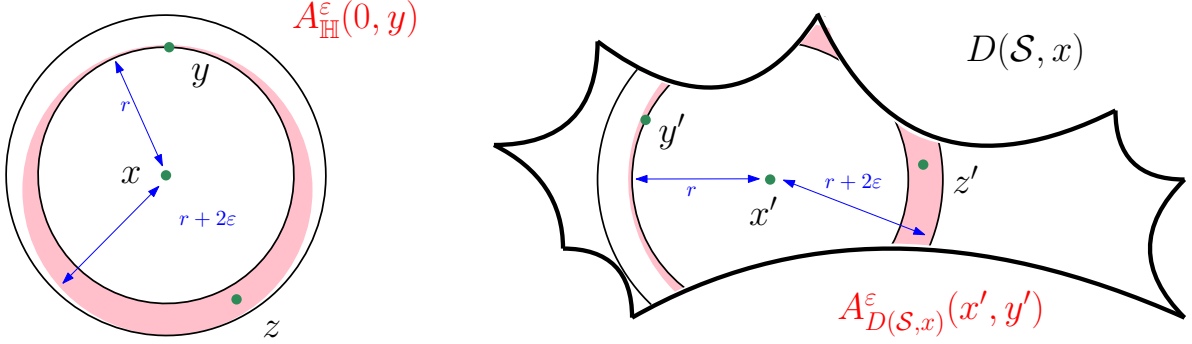


Figure 2 – On the left, in pink the set $A_{\mathbb{H}}^\varepsilon(x,y)$, see Lemma 6. On the right the set $A_{D(\mathcal{S},x)}^\varepsilon(x',y')$ which by Lemma 5 is a very good approximation of $A_{\mathcal{S}}^\varepsilon(x,y)$.

Lemma 6. *Let $\delta > 0$. Then there are $r_0(\delta) > 1$ and $\varepsilon_0(\delta) > 0$ with the following property. For all $0 < \varepsilon < \varepsilon_0(\delta)$ and any point $y = [r; \theta] \in \mathbb{H}$ such that $d(0,y) = r > r_0(\delta)$, we have the inclusion*

$$A_{\mathbb{H}}^\varepsilon(0,y) \subset \left\{ [r'; \theta'] \in \mathbb{H} \mid r \leq r' \leq r + 2(1 + \delta) \left| \sin \frac{\theta' - \theta}{2} \right| \varepsilon \right\}. \quad (12)$$

In particular, under those assumptions, if $x \in \mathcal{S}$ and $y \in D(\mathcal{S}, x)$, we have

$$\left| A_{D(\mathcal{S},x)}^\varepsilon(0,y) \right| \leq 2(1 + \delta) \varepsilon \sinh(r + 3\varepsilon) \int_{I_r(x)} \left| \sin \frac{\theta' - \theta}{2} \right| d\theta'.$$

Proof. This is just a calculation using hyperbolic trigonometry. We write $z = [r'; \theta'] \in A_{\mathbb{H}}^\varepsilon(0,y)$. We want to prove that $r' \leq r + \left(2(1 + \delta) \left| \sin \frac{\theta' - \theta}{2} \right| \right) \varepsilon$. Because $z \in A_{\mathbb{H}}^\varepsilon(0,y)$, there is a point a of the form $[\varepsilon; \varphi]$ with $\varphi \in [0, 2\pi)$ which is closer to z than to y . Using the hyperbolic cosine law, we can write down the distances $d(a,y)$ and $d(a,z)$ in terms of $r, r', \varepsilon, \theta, \theta'$ and φ . We find

$$\cosh(r') \cosh(\varepsilon) - \cos(\varphi - \theta') \sinh(r') \sinh(\varepsilon) \leq \cosh(r) \cosh(\varepsilon) - \cos(\varphi - \theta) \sinh(r) \sinh(\varepsilon),$$

or equivalently

$$\cos(\varphi - \theta') \sinh(r') - \cos(\varphi - \theta) \sinh(r) \geq \coth(\varepsilon) (\cosh(r') - \cosh(r)) \geq \frac{\cosh(r') - \cosh(r)}{\varepsilon}. \quad (13)$$

Moreover, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
& \cos(\varphi - \theta') \sinh(r') - \cos(\varphi - \theta) \sinh(r) \\
&= (\cos(\theta') \sinh(r') - \cos(\theta) \sinh(r)) \cos(\varphi) + (\sin(\theta') \sinh(r') - \sin(\theta) \sinh(r)) \sin(\varphi) \\
&\leq \sqrt{(\cos(\theta') \sinh(r') - \cos(\theta) \sinh(r))^2 + (\sin(\theta') \sinh(r') - \sin(\theta) \sinh(r))^2} \\
&= \sqrt{\sinh^2(r') + \sinh^2(r) - 2 \cos(\theta' - \theta) \sinh(r) \sinh(r')} \\
&= \sqrt{(\sinh(r') - \sinh(r))^2 + 2(1 - \cos(\theta' - \theta)) \sinh(r) \sinh(r')}.
\end{aligned}$$

Hence (13) squared becomes

$$\varepsilon^2 (\sinh(r') - \sinh(r))^2 + 4\varepsilon^2 \sin^2 \frac{\theta' - \theta}{2} \sinh(r) \sinh(r') \geq (\cosh(r') - \cosh(r))^2.$$

If ε is small enough and r large enough (depending only on δ), the first term in the left-hand side is small compared to the right-hand side, so we have

$$4\varepsilon^2 \sin^2 \frac{\theta' - \theta}{2} \sinh(r) \sinh(r') \geq (1 + \delta/2)^{-2} (\cosh(r') - \cosh(r))^2,$$

which becomes

$$2\varepsilon \left| \sin \frac{\theta' - \theta}{2} \right| \geq \left(1 + \frac{\delta}{2}\right)^{-1} \frac{\cosh(r') - \cosh(r)}{\sqrt{\sinh(r) \sinh(r')}} \geq \left(1 + \frac{\delta}{2}\right)^{-1} \frac{(r' - r) \sinh(r)}{\sqrt{\sinh(r) \sinh(r')}},$$

which implies

$$r' - r \leq \left(1 + \frac{\delta}{2}\right) 2\varepsilon \left| \sin \frac{\theta' - \theta}{2} \right| \sqrt{\frac{\sinh(r')}{\sinh(r)}}.$$

Finally, recalling $r' \leq r + 2\varepsilon$, if r is large enough (depending only on δ), this last expression is smaller than $(1 + \delta)2\varepsilon \left| \sin \frac{\theta' - \theta}{2} \right|$, which concludes the proof of (12).

Let us move on to the second point. We know that $A_{D(\mathcal{S}, x)}^\varepsilon(0, y) \subset D(\mathcal{S}, x) \cap A_{\mathbb{H}}^\varepsilon(0, y)$. Using the expression of the area measure in polar coordinates, Equation (12) translates into

$$\begin{aligned}
\left| A_{D(\mathcal{S}, x)}^\varepsilon(0, y) \right| &\leq \int_0^{2\pi} \int_r^{r+2(1+\delta)|\sin \frac{\theta' - \theta}{2}| \varepsilon} \mathbb{1}_{\theta' \in I_{r'}(x)} \sinh(r') \, dr' \, d\theta' \\
&\leq \int_{I_r(x)} \int_r^{r+2(1+\delta)|\sin \frac{\theta' - \theta}{2}| \varepsilon} \sinh(r') \, dr' \, d\theta' \\
&\leq 2(1 + \delta)\varepsilon \sinh(r + 3\varepsilon) \int_{I_r(x)} \left| \sin \frac{\theta' - \theta}{2} \right| \, d\theta',
\end{aligned}$$

where the second inequality uses the inclusion $I_{r'}(x) \subset I_r(x)$. This concludes the proof of Lemma 6. \square

We can now finish the proof of Proposition 2. We re-express (9) as an integral over $D(\mathcal{S}, x)$ (since the projection p preserves the measure), and write it down in polar coordinates:

$$\begin{aligned}
\mathbb{P}(x \in \partial^\varepsilon C) &\leq \lambda^2 \int_0^{+\infty} \exp(-\lambda |B_r(x)|) \int_{I_r(x)} |A_{\mathcal{S}}^\varepsilon(x, p([r; \theta]))| \sinh(r) \, d\theta \, dr \\
&= o(\varepsilon) + \lambda^2 \int_0^{+\infty} \exp(-\lambda |B_r(x)|) \int_{I_r(x)} \left| A_{D(\mathcal{S}, x)}^\varepsilon(0, [r; \theta]) \right| \sinh(r) \, d\theta \, dr,
\end{aligned}$$

where the $o(\varepsilon)$ is uniform in x , and the last part comes from Lemma 5 and the fact that $D(\mathcal{S}, x)$ is bounded. We now assume that ε is smaller than the $\varepsilon_0(\delta)$ of Lemma 6. For r larger than the $r_0(\delta)$ of Lemma 6, we bound $|A_{D(\mathcal{S}, x)}^\varepsilon(0, [r; \theta])|$ using Lemma 6. For $r \leq r_0(\delta)$, we use the crude bound $|A_{D(\mathcal{S}, x)}^\varepsilon(0, [r; \theta])| \leq 4\pi\varepsilon \sinh(r + 2\varepsilon)$ coming from (8). We obtain

$$\begin{aligned} \mathbb{P}(x \in \partial^\varepsilon \mathcal{C}) &\leq o(\varepsilon) + \lambda^2 \int_0^{r_0(\delta)} \int_0^{2\pi} 4\pi\varepsilon \sinh(r + 2\varepsilon) \sinh(r) \, d\theta \, dr \\ &\quad + 2\lambda^2(1 + \delta)\varepsilon \int_{r_0}^{+\infty} \exp(-\lambda|B_r(x)|) \int_{I_r(x)^2} \left| \sin \frac{\theta' - \theta}{2} \right| \sinh(r) \sinh(r + 3\varepsilon) \, d\theta \, d\theta' \, dr. \end{aligned}$$

The first integral is bounded by $C(\delta)\lambda^2\varepsilon$, so if λ is chosen smaller than some $\lambda_0(\delta)$ it is smaller than $\delta\varepsilon$. Moreover, up to increasing the value $r_0(\delta)$, we may assume $\sinh(r + 3\varepsilon) \leq (1 + \delta)\sinh(r)$. By these remarks and Lemma 3 to handle the integral over $I_r(x)^2$, we find

$$\mathbb{P}(x \in \partial^\varepsilon \mathcal{C}) \leq o(\varepsilon) + \delta\varepsilon + \frac{4}{\pi}\lambda^2(1 + \delta)^2\varepsilon \int_{r_0}^{+\infty} \exp(-\lambda|B_r(x)|) |I_r(x)|^2 \sinh^2(r) \, dr. \quad (14)$$

Our goal is now to write this in a form which can be directly integrated. For this, we notice that

$$|I_r(x)| \sinh(r) = |\partial B_r(x)| = \frac{d}{dr} |B_r(x)|.$$

Therefore, for all $r \geq r_0(\delta)$, we have

$$|B_r(x)| = \int_0^r |I_s(x)| \sinh(s) \, ds \geq |I_r(x)|(\cosh(r) - 1).$$

Up to increasing the value $r_0(\delta)$, we may assume $\cosh(r) - 1 \geq (1 + \delta)^{-1} \sinh(r)$, so that $|I_r(x)| \sinh(r) \leq (1 + \delta)|B_r(x)|$. Therefore, replacing one of the two factors $|I_r(x)| \sinh(r)$ in (14), we obtain

$$\begin{aligned} \mathbb{P}(x \in \partial^\varepsilon \mathcal{C}) &\leq o(\varepsilon) + \delta\varepsilon + \frac{4}{\pi}\lambda^2(1 + \delta)^3\varepsilon \int_0^{+\infty} \left(\frac{d}{dr} |B_r(x)| \right) |B_r(x)| \exp(-\lambda|B_r(x)|) \, dr \\ &= o(\varepsilon) + \delta\varepsilon + \frac{4}{\pi}\lambda^2(1 + \delta)^3\varepsilon \left[- \left(\frac{1}{\lambda^2} + \frac{|B_r(x)|}{\lambda} \right) \exp(-\lambda|B_r(x)|) \right]_0^{+\infty} \\ &= o(\varepsilon) + \delta\varepsilon + \frac{4}{\pi}(1 + \delta)^3\varepsilon, \end{aligned}$$

where the $o(\varepsilon)$ is still uniform in x . Plugging this into (7), we obtain

$$\mathbb{E}[|\partial \mathcal{C}|] \leq \frac{2}{\pi}(1 + \delta)^4 |\mathcal{S}|$$

for $\lambda < \lambda_0(\delta)$, which finally proves Proposition 2. \square

Remark. Let us briefly compare our approach for Proposition 2 to the computations of Isokawa [16] in the hyperbolic plane. The main difference is that Isokawa takes a point of the Poisson process as the center of polar coordinates, whereas we center them around a typical point of \mathcal{S} . In particular, Isokawa does not need to consider an ε -thickening of the boundary. However, if we tried to adapt the arguments of [16] to our Dirichlet domains, we would need to carefully study the interplay between the sets $I_r(x)$ and $I_s(x)$ for some $r \neq s$, which seems complex. On the other hand, our argument only uses the interplay between $I_r(x)$ and itself (this is the role of Lemma 3).

3.4 Proof of Theorem 1

Let $0 < \delta < \frac{1}{2} < \frac{2}{\pi}$ and let \mathcal{S} be a surface with genus g and Cheeger constant larger than δ (otherwise our result is trivial). Let $\lambda > 0$ be small enough so that we have by Proposition 2

$$\mathbb{E} [|\partial \text{Vor}_\lambda(\mathcal{S})|] \leq \frac{2}{\pi}(1 + \delta)|\mathcal{S}|.$$

Recall that A_λ is obtained by coloring each cell of $\text{Vor}_\lambda(\mathcal{S})$ with probability $1/2$ independently. In particular, each side of $\partial \text{Vor}_\lambda(\mathcal{S})$ is in ∂A_λ with probability $1/2$ (conditionally on $\text{Vor}_\lambda(\mathcal{S})$), so

$$\mathbb{E} [|\partial A_\lambda|] \leq \frac{1}{\pi}(1 + \delta)|\mathcal{S}|.$$

Therefore, by the Markov inequality

$$\mathbb{P} \left(|\partial A_\lambda| \leq \frac{1}{\pi}(1 + \delta)^2|\mathcal{S}| \right) \geq 1 - \frac{(1 + \delta)|\mathcal{S}|/\pi}{(1 + \delta)^2|\mathcal{S}|/\pi} = \frac{\delta}{1 + \delta}.$$

Now suppose that g is large enough so that Lemma 1 holds with δ replaced by δ^2 . In this case, since $\frac{\delta}{1 + \delta} + (1 - \delta^2) > 1$ there is a positive probability that both $(1 - \delta)\frac{|\mathcal{S}|}{2} \leq |A_\lambda| \leq (1 + \delta)\frac{|\mathcal{S}|}{2}$ and $|\partial A_\lambda| \leq \frac{1}{\pi}(1 + \delta)^2|\mathcal{S}|$. This implies

$$h(\mathcal{S}) \leq \frac{(1 + \delta)^2}{1 - \delta} \frac{2}{\pi},$$

which proves Theorem 1. Et voilà.

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