HAL open science

Regularity for rough hypoelliptic equations

Helge Dietert, Jonas Hirsch

▶ To cite this version:

Helge Dietert, Jonas Hirsch. Regularity for rough hypoelliptic equations. 2022. hal-03853087

HAL Id: hal-03853087
https://hal.science/hal-03853087

Preprint submitted on 15 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Regularity for rough hypoelliptic equations

Helge Dietert Jonas Hirsch

November 15, 2022

We present a general approach to obtain a weak Harnack inequality for rough hypoellipitic equations, e.g. kinetic equations. The proof is constructive and does not study the commutator structure but rather compares the rough solution with a smooth problem for which the estimates are assumed.

1. Introduction

1.1. Motivation

One motivation is kinetic theory describing a density $f = f(t, x, v)$ at a time t over the phase space consisting of a spatial position x and a velocity v. For a collisional evolution, like the Boltzmann or Landau equation, the evolution is then given by

$$\partial_t f + v \cdot \nabla_x f = Q(f), \tag{1}$$

where Q is a collision operator. In the most basic form, Q is a diffusion operator in the velocity variable v so that we arrive at

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \nabla_v f. \tag{2}$$

The evolution (2) is not parabolic because there is no diffusion in the spatial position x. The fundamental solution, computed explicitly by Kolmogoroff [18] in 1934, shows that, nevertheless, a solution is smooth in all directions.

In a general setting, Hörmander [15] understood in 1967 this smoothing property. For smooth vector fields $X_0, X_1, \dots, X_m$, he looked at solutions to the equation

$$X_0 u + \sum_{i=1}^{m} (-X_i)^* X_i u = S \tag{3}$$

and called the equation *hypoelliptic* if the smoothness of S implies that u is smooth. He then shows that (3) is hypoelliptic if $X_0, X_1, \dots, X_m$ and their commutators span the full space at every point.

A different development was the question of regularity for elliptic equations with *rough coefficients*. Such a regularity was proved by De Giorgi [9] in 1957 and Nash [23] in 1958, also covering the parabolic case.

The combination of these ideas saw a lot of recent interest [2, 3, 10, 12, 13, 14, 29, 30, 31] as it is a path for regularity results for nonlinear kinetic equations, where the solution satisfies schematically

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (a \nabla_v f) \tag{4}$$

and a depends again on f. On this level, we do not know any regularity on f so that we just assume that a is bounded from below and above, i.e. a is a rough coefficient. If we can still obtain a regularity result, we can use it to bootstrap regularity as explained in [4, 16, 17].

A related direction is the study of sub-Riemannian geometry which asks similar questions without a drift X_0. The lack of the drift seems to simplify several problems and quite general results are available. Along this direction we refer to [7] as a starting point.

1.2. General setting and main results

Our observation is that the smoothing property of an hypoelliptic operator implies in an robust way the key steps of regularity for a rough version: the supremum bound and the weak Harnack inequality.

In this general setting, we study functions $u = u(t,x)$ where $t \in \mathbb{R}$ is a special time-variable and $x \in \mathbb{R}^n$ is a general space. Then suppose smooth vector-fields $\tilde{X}_0, X_1, \dots, X_m$ acting only along the spatial directions x, i.e. $X_i = \sum_{j=1}^n X_i^j(t,x)\partial_{x_j}$ with smooth coefficients $(X_i^j)_{i,j}$. Using the standard $\mathrm{L}^2(\mathbb{R}^n)$ define the adjoints X_i^* of X_i and let

$$X_i^t = -X_i^* \qquad \text{for } i = 1, \dots, m.$$

We consider the smooth operator

$$X_0 - L_0 \quad \text{where} \quad X_0 = \partial_t + \tilde{X}_0 \text{ and } L_0 := \sum_{i=1}^m X_i^t X_i. \tag{5}$$

The natural functional spaces for solutions has already been identified in Hörmander [15], see also [1, 3]. We introduce the space $\mathrm{H}^1_{\mathrm{hyp}}$ defined by the norm

$$\|u\|_{\mathrm{H}^1_{\mathrm{hyp}}} = \|u\|_{\mathrm{L}^2} + \|\vec{X}u\|_{\mathrm{L}^2} \tag{6}$$

where $\vec{X}u = (X_1 u, \dots, X_m u)$ and we denote by $\mathrm{H}^{-1}_{\mathrm{hyp}}$ the dual space of $\mathrm{H}^1_{\mathrm{hyp}}$. Throughout we will consider classical weak solutions u with $u \in \mathrm{H}^1_{\mathrm{hyp}}$ and $X_0 u \in \mathrm{H}^{-1}_{\mathrm{hyp}}$.

For a point $x_0 \in \mathbb{R}^n$, let $B_r(x_0) \subset \mathbb{R}^n$ be the standard open Euclidean ball of radius r. For a parabolic cylinder, we include the drift X_0.

Definition 1 (Parabolic cylinder C^{X_0})**.** *For a point $(t_0, x_0) \in \mathbb{R}^{1+n}$ solve the transport equation*

$$\begin{cases} X_0 \eta = 0 & \text{in } \mathbb{R}^{1+n}, \\ w(t_0, \cdot) = \mathbb{1}_{B_r(x_0)} & \text{on } \{t_0\} \times \mathbb{R}^n. \end{cases} \tag{7}$$

The parabolic cylinder $C^{X_0}_{s,r}(t_0, x_0) \subset \mathbb{R}^{1+n}$ with time size s and space size r is

$$C^{X_0}_{s,r}(t_0, x_0) = \operatorname{supp} \eta \cap (t_0 - s, t_0]. \tag{8}$$

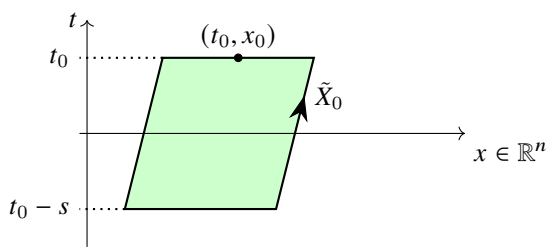

Figure 1: Illustration of Definition 1 of a parabolic cylinder $C^{X_0}_{s,r}(t_0, x_0)$.

Remark 1. If $\tilde{X}_0$ is independent of time, the transport equation is solved by the semigroup $\mathrm{e}^{t\tilde{X}_0}$ and we find

$$C^{X_0}_{s,r}(t_0,x_0) = \{(t,y) \in (t_0-s,t_0] : y \in \mathrm{e}^{(t_0-t)\tilde{X}_0} B_r(x_0)\}.$$

We then capture the hypoelliptic behaviour of $X_0 - L_0$ by supposing estimates gaining local integrability.

Hypothesis 1. *Suppose a parabolic domain $\Omega_t = (t_1,t_2] \times B \subset \mathbb{R}^{1+n}$ for times $-\infty < t_1 < t_2 < \infty$ and a bounded ball $B \subset \mathbb{R}^n$ and suppose an extended domain $\Omega_t^{\mathrm{ext}} = (t_1,t_2] \times B^{\mathrm{ext}}$ for a ball $B \subset B^{\mathrm{ext}}$ (with the possibility $B^{\mathrm{ext}} = \mathbb{R}^n$) and an extended (possible degenerate) elliptic operator L_0^{ext} in divergence form with vanishing lower order terms such that $L_0^{\mathrm{ext}} = L_0$ on Ω_t. Then suppose integrabilities $p_1 > 2$ and $\gamma_0 \le \gamma_1 \le 2$ and a constant $C_0 > 0$ such that for any time $t_{\mathrm{in}} \in (t_1,t_2)$ and functions $G \in \mathrm{L}^{\gamma_0}(\Omega_t^{\mathrm{ext}}), F = (F_1,\dots,F_m) \in \mathrm{L}^{\gamma_0}(\Omega_t^{\mathrm{ext}},\mathbb{R}^m)$ with $\operatorname{supp} G \cup \operatorname{supp} F \subset \Omega_t \cap \{t \ge t_{\mathrm{in}}\}$ there exists a function $w : \Omega_t^{\mathrm{ext}} \cap \{t \ge t_{\mathrm{in}}\} \to \mathbb{R}$ satisfying*

$$\begin{cases} (X_0 - L_0^{\mathrm{ext}})w \ge G + \sum_{i=1}^m X_i^t F^i & \text{in } \Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\}, \\ w \ge 0 & \text{on } \{t_{\mathrm{in}}\} \times B^{\mathrm{ext}} \cup (t_{\mathrm{in}},t_2) \times \partial B^{\mathrm{ext}} \end{cases} \tag{9}$$

and

$$\|w\|_{\mathrm{L}^{p_1}(\Omega_t \cap \{t > t_{\mathrm{in}}\})} \le C_0 \left(\|G\|_{\mathrm{L}^{\gamma_0}(\Omega_t \cap \{t > t_{\mathrm{in}}\})} + \|F\|_{\mathrm{L}^{\gamma_1}(\Omega_t \cap \{t > t_{\mathrm{in}}\})} \right). \tag{10}$$

Remark 2. In the case of kinetic or Kolmogorov equations there exists a fundamental solution of $X_0 - L_0$ over the whole space and we can obtain the sought w and the estimates by the fundamental solution with $L_0^{\mathrm{ext}} = L_0$ and $B^{\mathrm{ext}} = \mathbb{R}^n$.

If we only have local estimates for solutions of $X_0 - L_0$, then it is difficult to construct a solution with boundary condition $w \ge 0$ as it is not clear due to the degeneracy of L_0 what boundary conditions can be imposed. Therefore, we allow a different extension L_0^{ext} which we can take as $L_0^{\mathrm{ext}} = L_0 + \nabla \cdot ((1-\chi)^2 \nabla \cdot)$ with a cutoff χ and the normal gradient ∇ on $\mathbb{R}^n$. This then allows the same local estimates and the imposition of boundary condition $w = 0$ on $\{t_{\mathrm{in}}\} \times B^{\mathrm{ext}} \cup (t_{\mathrm{in}},t_2) \times \partial B^{\mathrm{ext}}$.

In this setting we study the differential operator P with rough coefficients defined by

$$Pu := X_0 u - \sum_{i=1}^m X_i^t A^i(t,x,u,\vec{X}u) - B(t,x,u,\vec{X}u) \tag{11}$$

where

$$A^i(t,x,z,p) = a^{ij}(t,x)p_j + b^i(t,x)z - f^i(t,x), \tag{12}$$

$$B(t,x,z,p) = c^i(t,x)p_i + d(t,x)z - g(t,x). \tag{13}$$

For the diffusion coefficient assume the uniform lower bound λ on the symmetric part

$$\lambda \,\mathrm{id} \le \left(\frac{a^{ij} + a^{ji}}{2} \right)_{ij} \quad \text{in the sense of matrices} \tag{14}$$

and assume that the coefficients are bounded by a function $\Lambda = \Lambda(t,x)$ as

$$|a^{ij}(t,x)| \le \frac{\Lambda(t,x)}{n} \quad \text{for all } i,j = 1,\dots,m. \tag{15}$$

Our first result is a supremum bound for subsolutions.

Theorem 2 (Supremum bound)**.** *Assume a parabolic cylinder $C^{X_0}_{S,R}(t_0,x_0)$ around a point $(t_0,x_0) \in \mathbb{R}^{1+n}$ with $0 < S$ and $0 < R$ containing the cylinder $C^{X_0}_{s,r}(t_0,x_0)$ with $0 < s < S$ and $0 < r < R$ and assume **(H1)** is satisfied for the smooth problem on a domain Ω_t containing the closure of $C^{X_0}_{S,R}(t_0,x_0)$. Take $2 < p_0 < p_1$ and integrabilities q_Λ, q_b, q_c, q_d satisfying*

$$\frac{1}{q_\Lambda} \le \min\left\{\frac{1}{2} - \frac{1}{p_0}, \frac{1}{\gamma_1} - \frac{1}{2}\right\}, \qquad \frac{1}{q_b} \le \min\left\{\frac{1}{2}\left(\frac{1}{\gamma_0} - \frac{1}{p_0}\right), \frac{1}{2} - \frac{1}{p_0}\right\},$$
$$\frac{1}{q_c} \le \min\left\{\frac{1}{\gamma_0} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_0}\right\}, \qquad \frac{1}{q_d} \le \min\left\{\frac{1}{\gamma_0} - \frac{1}{p_0}, 1 - \frac{2}{p_0}\right\}.$$

Then there exist constants $C_S, \beta > 0$ such that a function u satisfying $Pu \le 0$ on $C^{X_0}_{S,R}(t_0,x_0)$ for a differential operator P of (11) is bounded as

$$\sup_{C^{X_0}_{s,r}(t_0,x_0)} u \le C_S\,(1+\delta_S)^\beta \left(\|u\|_{L^1(C^{X_0}_{S,R}(t_0,x_0))} + \|f\|_{L^{q_b}(C^{X_0}_{S,R}(t_0,x_0))} + \|g\|_{L^{q_d}(C^{X_0}_{S,R}(t_0,x_0))}\right)$$

where

$$\delta_S = \|\Lambda\|_{L^{q_\Lambda}(C^{X_0}_{S,R}(t_0,x_0))} + \|b\|_{L^{q_b}(C^{X_0}_{S,R}(t_0,x_0))} + \|c\|_{L^{q_c}(C^{X_0}_{S,R}(t_0,x_0))} + \|d\|_{L^{q_d}(C^{X_0}_{S,R}(t_0,x_0))}.$$

The next step for the regularity of solutions to the rough operator P is a weak Harnack inequality. For a nonnegative solution u and a cylinder $C^{X_0}_{1,1}(0,x_0)$ we want to conclude that u is strictly positive in $C^{X_0}_{1,1}(0,x_0) \cap \{-1/3 \le t \le 0\}$ if $\{u \ge 1\}$ is a set of positive measure in $C^{X_0}_{1,1}(0,x_0) \cap \{-1/3 \le t \le 0\}$ beforehand. The idea is to again use a similar property for the smooth dual problem. For the conclusion, we need a larger domain with an arbitrary smooth cutoff. This is captured in the following hypothesis, cf. Fig. 2.

Hypothesis 2. *From the point $(0,x_0) \in \mathbb{R}^{1+n}$, there exists for every $R > 1$ bounded domains $C^{X_0}_{1,2}(0,x_0) \subset \tilde{\Sigma}_R \subset \Sigma_R \subset [-1,0] \times \mathbb{R}^n$ and smooth cutoffs $\tilde{\eta}_R, \eta_r : [-1,0] \times \mathbb{R}^n \to [0,1]$ with $\operatorname{supp} \tilde{\eta}_R \subset \tilde{\Sigma}_R$ and $\operatorname{supp} \eta_R \subset \Sigma_R$ and $\tilde{\eta}_R \equiv 1$ on $C^{X_0}_{1,1}(0,x_0)$ and $\eta_R \equiv 1$ on $\tilde{\Sigma}_r$ such that*

$$X_0\tilde{\eta}_R = X_0\eta_R = 0$$

and

$$\|\vec{X}\tilde{\eta}_R\|_{L^\infty} \le 1, \quad \|\vec{X}\vec{X}\tilde{\eta}_R\|_{L^\infty} \le \frac{1}{R}, \quad \|\vec{X}\eta_R\|_{L^\infty} \le 1.$$

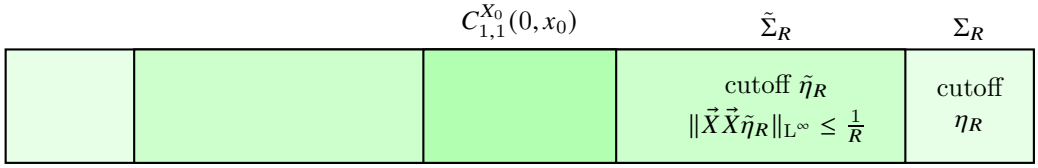

Figure 2: Illustration of the enlarged domain with the control of cutoff in **(H2)**.

Remark 3. In simple hypoelliptic cases like in kinetic theory, the sets $\tilde{\Sigma}_R$ and Σ_R can be taken as parabolic cylinders $C^{X_0}_{\bar{R},1}(0,x_0)$ for large enough $\bar{R}$ and $\tilde{\eta}, \eta$ can be taken as solutions of $X_0\tilde{\eta} = X_0\eta = 0$ with a prescribed standard cutoff at $\{t = 0\}$. For general operators, it can be assumed locally by using the underlying scaling of the vector fields $X_0, X_1, \ldots, X_m$.

We now state the assumption of the smooth dual problem assuming **(H2)**, cf. Fig. 3.

Hypothesis 3. *For the point $x_0 \in \mathbb{R}^n$, assume constants $\eta, \mu_0 > 0$ such that the problem*

$$\begin{cases} (X_0^t - L_0)w = \mathbb{1}_E & \text{in } (-1,0] \times \mathbb{R}^n, \\ w(-1,\cdot) = 0 & \text{on } \{t = -1\} \times \mathbb{R}^n \end{cases} \tag{16}$$

for a set $E \subset C_{1,1}^{X_0}(0,x_0) \cap \{t \le -2/3\}$ with $|E| \ge \eta |C_{1,1}^{X_0}(0,x_0) \cap \{t \le -2/3\}|$ has a solution $w \ge 0$ satisfying

$$w(t,x) \ge \mu_0 \quad \text{for} \quad (t,x) \in C_{1/2,2}^{X_0}(0,x_0) \tag{17}$$

and

$$\|w\|_{\mathrm{L}^1((-1,0)\times\mathbb{R}^n)} \lesssim 1.$$

Assume further an integrability $p_2 \ge 2$. Then for any $R > 0$, there exists a constant $C_d(R)$ such that

$$\|w\|_{\mathrm{L}^{p_2}(\Sigma_R)} + \|\vec{X}w\|_{\mathrm{L}^{p_2}(\Sigma_R)} \le C_d(R).$$

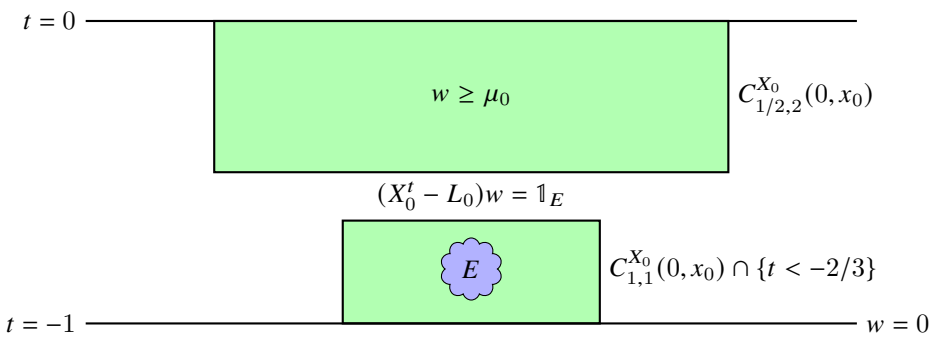

Figure 3: Illustration of **(H3)**.

We then satisfy a weak Harnack inequality for the rough problem, cf. Fig. 4.

Theorem 3 (Weak Harnack inequality)**.** *Assume that Theorem 2 applies to $C_{1/2,2}^{X_0}(0,x_0)$ and additionally **(H2)** and **(H3)**. Then there exists C_R such that for $\delta_S, \Delta > 0$ there exists $\mu, \epsilon > 0$ such that with $R = C_R(1+\delta_S)^\beta$ a nonnegative supersolution $u \ge 0$ of*

$$Pu \ge 0 \text{ over } \Sigma_R$$

with

$$|E| \ge \eta |C_{1,1}^{X_0}(0,x_0) \cap \{t \le -2/3\}| \text{ for } E = \{u \ge 1\} \cap C_{1,1}^{X_0}(0,x_0) \cap \{t \le -2/3\} \tag{18}$$

satisfies

$$u(t,x) \ge \mu \text{ in } C_{1/3,1}^{X_0}(0,x_0)$$

if P is an operator of the form (11) with

$$\|\Lambda\|_{\mathrm{L}^{q_\Lambda}(C_{\frac{1}{2},2}^{X_0}(t_0,x_0))} + \|c\|_{\mathrm{L}^{q_c}(C_{\frac{1}{2},2}^{X_0}(t_0,x_0))} \le \delta_S$$

and

$$\|\Lambda\|_{\mathrm{L}^{\bar{q}_2}(\Sigma_R)} + \|\tilde{c}\|_{\mathrm{L}^{\bar{q}_2}(\Sigma_R)} \le \Delta \qquad \text{with } \frac{1}{\bar{q}_2} + \frac{1}{p_2} = \frac{1}{2}$$

and

$$\|f - ub\|_{\mathrm{L}^{q_b}(\Sigma_R)} + \|g - ud\|_{\mathrm{L}^{q_d}(\Sigma_R)} \le \epsilon.$$

After some preliminary remarks in Section 2, we will prove Theorem 2 in Section 3 and Theorem 3 in Section 4. Both proofs are quantitative. The proofs are presented as a priori estimates and we briefly discuss the required function space in Appendix A.

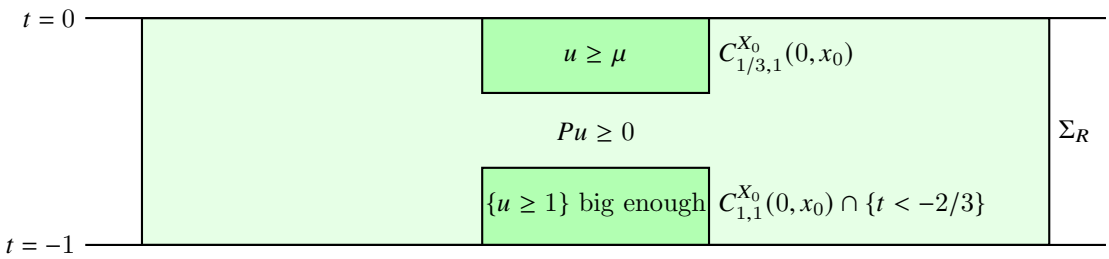

Figure 4: Illustration of weak Harnack inequality (Theorem 3).

1.3. Application to hypoelliptic operator

In the first work of hypoellipticity by Hörmander [15], the key estimate is that a solution u to $(X_0 - L_0)u = G + X_i^t F_i$ satisfies $\|u\|_{\mathrm{H}^s} \lesssim \|u\|_2 + \|F\|_2 + \|G\|_2$ for some $s > 0$ under the commutator condition. This then shows the estimate **(H1)** by Sobolev embedding with $\gamma_0 = \gamma_1 = 2$, see Appendix B.

In this general setting, Bony [5] proved a strong maximum principle which yields the claimed spreading of positivity in **(H3)** by a compactness argument.

In the kinetic or general Kolmogorov setting, there is an explicit fundamental solution from which all estimates on the smooth problem can be easily verified. Using the best possible integrabilities, the assumed integrabilities on the lower order terms are as expected from the classical parabolic case arbitrary close to the integrabilities expected from scaling. For the upper bound Λ on the diffusion coefficients a^{ij}, our result matches Trudinger [28] in the classical case.

The kinetic or Kolmogorov equation have an underlying scaling and group structure (corresponding to Galilean transformation in the kinetic theory) which allows to conclude from the weak Harnack result (Theorem 3) a Hölder regularity by a standard argument, see e.g. [13, Appendix B].

In the general setting, Rothschild and Stein [26] show that every hypoelliptic operator can be approximated locally by an operator with a suitable scaling and group structure, see also [6, 27] for use of this idea in order to obtain estimates on the smooth problem. The application to this general setting will be explained in a forthcoming paper.

1.4. Comparison with literature

As far as we are aware, there are no results in this general setting for rough coefficients. Even in the more studied special case of kinetic (or Kolmogorov) equations our proofs appear to be new and for the supremum bound (Theorem 2) it appears that we require less integrability on the coefficients as e.g. in [3] (other works for the supremum bound are [2, 8, 12, 14, 24, 25, 29, 30]).

For the proof of the weak Harnack inequality we use a log transform as it already appears in the early work by Nash [23] on rough coefficients. This has been used heavily for the study of equations with rough coefficients [19, 20, 21, 22] and has also been used in the kinetic and Kolmogorv setting [3, 13]. Here we differ by using the dual problem to conclude the result (instead of a Poincaré inequality inspired by the framework of [1, 15]).

2. Preliminaries

For the rough operator P, we define the principal part of P as

$$L_p = X_i^t a^{ij} X_j.$$

When deriving estimates, note that X_i^t satisfies the chain rule

$$X_i^t(\alpha\,\beta) = X_i(\alpha)\,\beta + \alpha\, X_i^t(\beta).$$

As a first step, we note how a subsolution behaves under a composition.

Lemma 4. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\Phi' \geq 0$ and $\Phi'' \geq 0$. Suppose the operator P of the form* (11) *and a subsolution $Pu \leq 0$. Then $v = \Phi \circ u$ satisfies*

$$\tilde{P}v := X_0 v - X_i^t \tilde{A}^i(t, x, v + h, \vec{X}v) - \tilde{B}(t, x, v + h, \vec{X}v) \leq -\Phi''(u)\,\frac{\lambda}{2}|\vec{X}u|^2 \leq 0$$

where $\tilde{A}$ and $\tilde{B}$ are of the same form (12) *and* (13)*, respectively, with new coefficients*

$$\begin{aligned}
\tilde{a}^{ij} &= a^{ij}, & \tilde{b}^i &= 0,\\
\tilde{c}^i &= c^i, & \tilde{d} &= 0,\\
\tilde{f}^i &= \Phi'(u)\left(f^i - b^i u\right), & \tilde{g} &= \Phi'(u)\,(g - du) - \frac{\Phi''(u)}{2\lambda}|f - bu|^2.
\end{aligned}$$

The same result holds if $Pu \geq 0$ and $\Phi' \leq 0$ and $\Phi'' \geq 0$.

Proof. Using that $Pu \leq 0$ and $\Phi' \geq 0$, we find

$$\begin{aligned}
&(\partial_t + X_0 - L_p)(v)\\
&= \Phi'(u)\,(X_0 - L_p)(u) - \Phi''(u)\,a^{ij}X_i u X_j u\\
&\leq \Phi'(u)\left[X_i^t(b^i u - f^i) + (c^i X_i u + du - g)\right] - \Phi''(u)\,a^{ij}X_i u X_j u\\
&= -X_i^t(\tilde{f}) - (b^i u - f^i)\Phi''(u)X_i u + c^i X_i v + \Phi'(u)\,(du - g) - \Phi''(u)a^{ij}X_i u X_j u.
\end{aligned}$$

Using the square control $\Phi''(u)a^{ij}X_i u X_j u$, we estimate

$$-\Phi''(u)\left[\left(b^i u - f^i\right)X_i u - a^{ij}X_i u X_j u\right] \leq \Phi''(u)\left[\frac{|f - bu|^2}{2\lambda} - \frac{\lambda}{2}|\vec{X}u|^2\right],$$

which then yields the result. The case $Pu \geq 0$ and $\Phi' \leq 0$ and $\Phi'' \geq 0$ follows in the same way. □

In the proof of the supremum bound (Theorem 2), we need several spatial cutoffs η and temporal cutoffs τ within the overall set $C_{S,R}^{X_0}(t_0, x_0)$.

For the temporal cutoff τ between times $s_1 < s_2$, i.e. $\tau : \mathbb{R} \to [0, 1]$ with $\tau(t) = 0$ for $t \leq s_1$ and $\tau(t) = 1$ for $t \geq s_2$, we can rescale a standard cutoff and therefore have uniformly

$$\|\partial_t \tau\|_\infty \lesssim \frac{1}{s_2 - s_1}.$$

For the spatial cutoff $\eta : \mathbb{R}^{1+n} \to [0, 1]$ between radii $r_1 < r_2$ around (t, x) over a time length S, we impose that $X_0\eta = 0$ and $\eta = 0$ outside $C_{S,r_2}^{X_0}(t, x)$ and $\eta = 1$ inside $C_{S,r_1}^{X_0}(t, x)$. These can be constructed by taking a cutoff $\bar{\eta}$ between the balls $B_{r_1}(x)$ and $B_{r_2}(x)$ and taking η as solution to

$$\begin{cases} X_0\eta = 0,\\ \eta(t, \cdot) = \bar{\eta}. \end{cases}$$

By the definition of the parabolic cylinder, this yields a required cutoff. Moreover, as it is always constructed within a fixed bounded set, the smoothness of the vector fields implies

$$\|\vec{X}\eta\|_\infty \lesssim \frac{1}{r_2 - r_1}.$$

3. Local supremum bound

In this section we prove Theorem 2 by the de Giorgi method using the bound **(H1)** on the smooth problem. In the special setting of kinetic or Kolmogorov equations, the knowledge of the fundamental solution for the smooth problem has been used in [3, 14, 25]. In this setting the main difference is that these works use a Moser iteration and do not obtain the integrability assumptions on the coefficients.

The classical idea is to consider $(u-h_k)_+$ for a sequence of cutoffs $(h_k)_k$ on nested cylinders $C_1 \supset C_2 \supset \dots$ and deduce that $\|(u-h_k)_+\| \to 0$ for a suitable norm while $h_k \uparrow D < \infty$. In the non-degenerate setting, this convergence is obtained by a direct energy estimate which yields by Sobolev embedding a gain of integrability.

In our setting, we not only perform a direct energy estimate but also compare the subsolution of the rough problem to a solution of the smooth problem. Hence a simple truncation is not sufficient and we need a smoothed cutoff.

Let $\rho \in \mathscr{C}^\infty(\mathbb{R})$ be a non-negative mollification kernel with $\operatorname{supp}\rho \subset [-1,1]$ and set for $\epsilon > 0$

$$\rho_\epsilon(z) = \frac{1}{\epsilon}\rho\left(\frac{z}{\epsilon}\right).$$

As replacement for the truncation, we then define for $h \in \mathbb{R}$ and $\epsilon > 0$ the function $K_{\epsilon,h} \in \mathscr{C}^\infty(\mathbb{R})$ by

$$K_{\epsilon,h}(z) = \rho_\epsilon * (z-h)_+.$$

By considering $K_{\epsilon,h}(u)$ instead of the truncation $(u-h)_+$, we find the gain of integrability in the following lemma.

Lemma 5. *Assume **(H1)** on Ω_t and let q_0, q_1 be the integrabilities given by*

$$\frac{1}{2} + \frac{1}{q_0} = \frac{1}{\gamma_0} \quad \text{and} \quad \frac{1}{2} + \frac{1}{q_1} = \frac{1}{\gamma_1}.$$

Then there exists a constant C_2 with the following gain of integrability: for nested parabolic cylinder $C^{X_0}_{s,r}(t_0,x_0) \subset C^{X_0}_{S,R}(t_0,x_0) \subset\subset \Omega_t$ and u with $Pu \le 0$ over $C^{X_0}_{S,R}(t_0,x_0)$, the composition

$$v = K_{\epsilon,h}(u)$$

satisfies for any $\epsilon > 0$ and $h \in \mathbb{R}$ that

$$\begin{aligned}\|v\|_{\mathrm{L}^{p_1}(C^{X_0}_{s,r})} \le{}& C_2\left[1 + \frac{1}{S-s} + \frac{1}{R-r}\right]^2 \left[1 + \|\Lambda\|_{\mathrm{L}^{q_1}(M)} + \|c\|_{\mathrm{L}^{q_0}(M)}\right] \\ &\left\{\|(1+\Lambda)v\|_{\mathrm{L}^2(M)} + \|\bar f\|_{\mathrm{L}^2(M)} + \|cv\|_{\mathrm{L}^2(M)} + \sqrt{\|\bar g v\|_{\mathrm{L}^1(M)}}\right\} \\ &+ C_2\left[1 + \frac{1}{S-s} + \frac{1}{R-r}\right]\left(\|\bar g\|_{\mathrm{L}^{\gamma_0}(M)} + \frac{1}{\epsilon}\|\bar f \mathbb{1}_{v\le 2\epsilon}\|^2_{\mathrm{L}^{2\gamma_0}(M)}\right)\end{aligned}$$

where $\bar f = f - bu$ and $\bar g = g - du$ and

$$M = C^{X_0}_{S,R}(t_0,x_0) \cap \{v > 0\}.$$

Proof. First note that

$$K'_{\epsilon,h}(z) = \rho_\epsilon * \mathbb{1}_{[h,\infty)} \ge 0, \qquad \text{and} \qquad K''_{\epsilon,h}(z) = \rho_\epsilon * \delta_h = \frac{1}{\epsilon}\rho\left(\frac{z-h}{\epsilon}\right) \ge 0.$$

Hence we can apply Lemma 4 to find that v satisfies with new coefficients $\tilde a, \tilde b, \tilde c, \tilde d, \tilde f, \tilde g$

$$\tilde P v \le 0. \tag{19}$$

The results is then obtained in two steps as illustrated in Fig. 5 with intermediate scale

$$s < s_1 < S \qquad \text{and} \qquad r < r_1 < R$$

and corresponding cylinders $C^{X_0}_{s,r} \subset C^{X_0}_{s_1,r_1} \subset C^{X_0}_{S,R}$ (always with the base point (t_0, x_0) which we therefore omit within this proof). Performing a L^2 energy estimate with a cutoff from $C^{X_0}_{s_1,r_1}$ to $C^{X_0}_{S,R}$ we obtain the control

$$\|\vec{X}v\|^2_{\mathrm{L}^2(C^{X_0}_{s_1,r_1})} \lesssim \left(1 + \frac{1}{R-r_1} + \frac{1}{S-s_1}\right)^2 \|(1+\Lambda)v\|^2_{\mathrm{L}^2(C^{X_0}_{s_1,r_1})} + \|\tilde{f}\|^2_{\mathrm{L}^2(C^{X_0}_{s_1,r_1})} + \|\tilde{c}v\|^2_{\mathrm{L}^2(C^{X_0}_{s_1,r_1})} + \|\tilde{g}v\|_{\mathrm{L}^1(C^{X_0}_{s_1,r_1})}. \tag{20}$$

By this gained control, we can compare with the solution of the smooth problem and gain the claimed control in $C^{X_0}_{s,r}$.

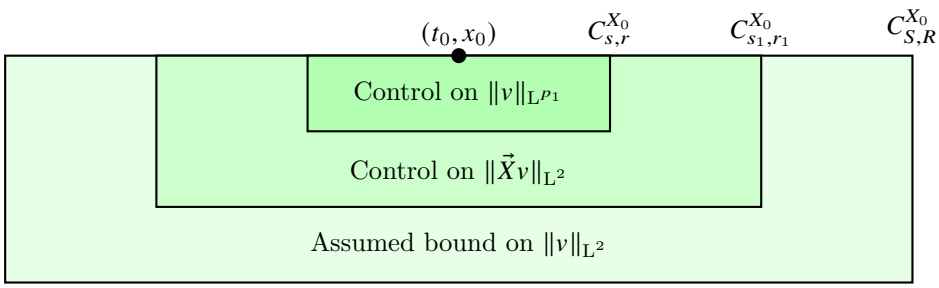

Figure 5: Illustration of the strategy of proof for Lemma 5.

Step 1: L^2 estimate

As discussed in Section 2 take a spatial cutoff η_1 between r_1 and R and a temporal cutoff τ_1 between $t_0 - s_1$ and $t_0 - S$.

We now test (19) against $\tau_1\eta_1^2 v$. For the drift note that (using $X_0\eta_1 = 0$)

$$\int_{C^{X_0}_{S,R}} X_0(v)\,\tau_1\eta_1^2 v = \int_{\{t_0\}\times B_R(x_0)} \eta_1^2 \frac{v^2}{2} - \int_{C^{X_0}_{S,R}} \tau_1'\eta_1^2 \frac{v^2}{2} - \int_{C^{X_0}_{S,R}} \tau_1\eta_1^2 \frac{v^2}{2}\,\operatorname{div} X_0.$$

For the operator $\tilde{A}$ note that (recalling $\tilde{b} \equiv 0$)

$$p_i\tilde{A}^i(t,x,z,p) \geq \lambda|p|^2 - |p|\,|\tilde{f}| \geq \frac{3\lambda}{4}|p|^2 - \lambda^{-1}|\tilde{f}|^2$$

and

$$|q_i\tilde{A}^i(t,x,z,p)| \leq |q|\left(\Lambda(t,x)\,|p| + |\tilde{f}|\right).$$

Hence

$$\begin{aligned}
&\int_{C^{X_0}_{S,R}} -X_i^t\tilde{A}^i(t,x,v+h,\vec{X}v)\,\tau_1\eta_1^2 v \\
&= \int_{C^{X_0}_{S,R}} X_i(v)\tilde{A}^i(t,x,v+h,\vec{X}v)\,\tau_1\eta_1^2 + 2\int_{C^{X_0}_{S,R}} X_i(\eta_1)\tilde{A}^i(t,x,v+h,\vec{X}v)\,\tau_1\eta_1 v \\
&\geq \frac{\lambda}{2}\int_{C^{X_0}_{S,R}} |\vec{X}v|^2\tau_1\eta_1^2 - \int_{C^{X_0}_{S,R}} |\tilde{f}|^2\tau_1\eta_1^2 - \int_{C^{X_0}_{S,R}} \left(1 + 4\frac{\Lambda(t,x)^2}{\lambda}\right)|\vec{X}\eta|^2\tau_1 v^2.
\end{aligned}$$

Finally for $\tilde{B}$, we find that (recalling $\tilde{d} \equiv 0$)

$$\int_{C^{X_0}_{S,R}} -B(t,x,v+h,\vec{X}v)\,\tau_1\eta_1^2 v \geq -\int_{C^{X_0}_{S,R}} \left[\frac{\lambda}{4}\delta|\vec{X}v|^2 + |\tilde{c}v|^2 + |\tilde{g}v|\right]\tau_1\eta_1^2.$$

Hence combining the different parts yields the claimed control (20) on $C^{X_0}_{s_1,r_1}$.

Step 2: comparison with smooth problem

For the next step, take a spatial cutoff η_1 between r and r_1 and a temporal cutoff τ_1 between $t_0 - s$ and $t_0 - s_1$.

The idea is to rewrite (19) for v as

$$(X_0 - L_p)(\tau_2\eta_2 v) \leq \tilde{G} + X_i^t \tilde{F}^i. \tag{21}$$

By **(H1)**, we then find a function w solving

$$\begin{cases} (X_0 - L_0^{\text{ext}})(w) \geq \tilde{G} + X_i^t\tilde{F}^i + (L_p - L_0^{\text{ext}})(\tau_2\eta_2 v) & \text{in } \Omega_t^{\text{ext}} \cap \{t > t_0 - s_1\}, \\ w \geq 0 & \text{on } \{t_0 - s_1\} \times B^{\text{ext}} \cup (t_0 - s_1, t_2) \times \partial B^{\text{ext}} \end{cases} \tag{22}$$

By the weak maximum principle for $X_0 - L_0^{\text{ext}}$, we find

$$\tau_2\eta_2 v \leq w \qquad \text{in } \Omega_t^{\text{ext}} \cap \{t > t_0 - s_1\}$$

so that $\|v\|_{\mathrm{L}^{p_1}(C^{X_0}_{s,r})} \leq \|w\|_{\mathrm{L}^{p_1}(C^{X_0}_{s,r})}$. Then the result follows by the bound (10) in **(H1)**.

Hence we first compute (recalling $\tilde{b} \equiv 0$ and $\tilde{d} \equiv 0$)

$$\begin{aligned}(X_0 - L_p)(\tau_2\eta_2 v) &= \tau_2\eta_2(X_0 - L_p)(v) + v\,X_0(\tau_2\eta_2) - \tau_2 a^{ij}X_i\eta_2 X_j v - X_i^t(a^{ij}vX_j\eta_2)\tau_2 \\ &\leq \tau_2\eta_2\left[-X_i^t\tilde{f}^i + \tilde{c}^i X_i v - \tilde{g}\right] + v\eta_2\tau_2' - \tau_2 a^{ij}X_i\eta_2 X_j v - X_i^t(a^{ij}vX_j\eta_2)\tau_2\end{aligned}$$

so that we verify (21) with

$$\tilde{G} = \tau_2 X_i\eta_2\tilde{f}^i + \tau_2\eta_2(\tilde{c}^i X_i v - \tilde{g}) + v\eta_2\tau_2' - \tau_2 a^{ij}X_i\eta_2 X_j v \qquad \text{and} \qquad \tilde{F}^i = -\tau_2\eta_2\tilde{f}^i - \tau_2 a^{ij}vX_j\eta_2.$$

For the additional term in (22) note that $L_0^{\text{ext}} = L_0$ in Ω_t and $\operatorname{supp}\tau_2\eta_2 v \subset \Omega_t$ to get

$$(L_p - L_0^{\text{ext}})(\tau_2\eta_2 v) = X_i^t\left((a^{ij} - \delta^{ij})X_j(\tau_2\eta_2 v)\right)$$

so that we find

$$(X_0 - L_0^{\text{ext}})(w) \geq G + X_i^t F^i$$

where

$$G = \tilde{G} \text{ and } F^i = \tilde{F}^i + (a^{ij} - \delta^{ij})X_j(\tau_2\eta_2 v) = -\tau_2\eta_2\tilde{f}^i + \tau_2\eta_2(a^{ij} - \delta^{ij})X_j v - \tau_2 v\delta^{ij}X_j\eta_2.$$

We now estimate

$$\begin{aligned}\|G\|_{\mathrm{L}^{\gamma_0}(\Omega_t^{\text{ext}})} \lesssim\ & \frac{1}{r_1 - r}\left(\|\tilde{f}\|_{\mathrm{L}^{\gamma_0}(C^{X_0}_{s_1,r_1})} + \|\Lambda\vec{X}v\|_{\mathrm{L}^{\gamma_0}(C^{X_0}_{s_1,r_1})}\right) \\ & + \frac{1}{s_1 - s}\|v\|_{\mathrm{L}^{\gamma_0}(C^{X_0}_{s_1,r_1})} + \|\tilde{c}\vec{X}v\|_{\mathrm{L}^{\gamma_0}(C^{X_0}_{s_1,r_1})} + \|\tilde{g}\|_{\mathrm{L}^{\gamma_0}(C^{X_0}_{s_1,r_1})}\end{aligned}$$

and

$$\|F\|_{\mathrm{L}^{\gamma_1}(\Omega_t^{\text{ext}})} \lesssim \|\tilde{f}\|_{\mathrm{L}^{\gamma_1}(C^{X_0}_{s_1,r_1})} + \|(1+\Lambda)\vec{X}v\|_{\mathrm{L}^{\gamma_1}(C^{X_0}_{s_1,r_1})} + \frac{1}{r_1 - r}\|v\|_{\mathrm{L}^{\gamma_1}(C^{X_0}_{s_1,r_1})}.$$

Recalling $\gamma_0 \le \gamma_1 \le 2$ we therefore find

$$\|w\|_{\mathrm{L}^{p_1}(C^{X_0}_{s_1,r_1})} \lesssim \left(1 + \frac{1}{r_1 - r} + \frac{1}{s_1 - s}\right)$$
$$\left(\|\tilde{f}\|_{\mathrm{L}^2(C^{X_0}_{s_1,r_1})} + \|\tilde{g}\|_{\mathrm{L}^{\gamma_0}(C^{X_0}_{s_1,r_1})} + \|v\|_{\mathrm{L}^2(C^{X_0}_{s_1,r_1})} + \left(1 + \|\Lambda\|_{\mathrm{L}^{q_1}(M)} + \|\tilde{c}\|_{\mathrm{L}^{q_0}(M)}\right)\|\vec{X}v\|_{\mathrm{L}^2(C^{X_0}_{s_1,r_1})}\right).$$

This shows the claimed result (setting $s_1 = (s+S)/2$ and $r_1 = (r+R)/2$) by using the expressions for $\tilde{f}$ and $\tilde{g}$ and noting that if $K''_{\epsilon,h}(z)$ is zero unless $|z-h| \le \epsilon$ so that

$$\|K''_{\epsilon,h}(f-bu)^2 v\|_{\mathrm{L}^1(M)} \lesssim \|\bar{f}\mathbb{1}_{v\le 2\epsilon}\|^2_{\mathrm{L}^2(M)} \text{ and } \|K''_{\epsilon,h}(f-bu)^2\|_{\mathrm{L}^{\gamma_0}(M)} \lesssim \frac{1}{\epsilon}\|\bar{f}\mathbb{1}_{v\le 2\epsilon}\|^2_{\mathrm{L}^{2\gamma_0}(M)}.$$

The restriction of $\bar{f}$ and $\bar{g}$ to $M = \{v > 0\}$ follows from the fact that the factor $K'_{\epsilon,h}(u)$ and $K''_{\epsilon,h}(u)$ in $\tilde{f}$ and $\tilde{g}$ vanish otherwise. □

By interpolation we can start from the L^1 norm.

Lemma 6. *Assume the setup of Lemma 5 with $h > 0$ and take as in the statement of Theorem 2 exponents p_0 and q_d and let $\bar{q}_0$ be the exponent given by*

$$\frac{1}{\bar{q}_0} + \frac{1}{p_0} = \frac{1}{2}.$$

Then there exists a constant C_3 and exponent α such that for $0 < s < S$ and $0 < r < R$ it holds that

$$\|v\|_{\mathrm{L}^{p_1}(C^{X_0}_{s,r})} \le C_3 P^{\alpha}\|v\|_{\mathrm{L}^1(C^{X_0}_{S,R})} + C_3 Q$$

where (with p_0^* as the dual exponent of p_0)

$$P = \left(1 + \frac{1}{S-s} + \frac{1}{R-r}\right)^2 \left[1 + \|\Lambda\|_{\mathrm{L}^{q_1}(M)} + \|c\|_{\mathrm{L}^{q_0}(M)}\right]$$
$$\left[1 + \|\Lambda\|_{\mathrm{L}^{\bar{q}_0}(M)} + \|c\|_{\mathrm{L}^{\bar{q}_0}(M)} + \|b\|_{\mathrm{L}^{\bar{q}_0}(M)} + \|d\|_{\mathrm{L}^{q_d}(M)} + \|g\|_{\mathrm{L}^{p_0^*}(M)}\right]$$

and

$$Q = \left(1 + \frac{1}{S-s} + \frac{1}{R-r}\right)^2 \left[1 + \|\Lambda\|_{\mathrm{L}^{q_1}(M)} + \|c\|_{\mathrm{L}^{q_0}(M)}\right]$$
$$\left[\|f\|_{\mathrm{L}^2(M)} + \frac{1}{\epsilon}\|f\|^2_{\mathrm{L}^{2\gamma_0}(M)} + (\epsilon + h)\left(\|b\|_{\mathrm{L}^2(M)} + \frac{\epsilon + h}{\epsilon}\|b\|^2_{\mathrm{L}^{2\gamma_0}(M)} + \|d\|_{\mathrm{L}^{\gamma_0}(M)}\right) + \|g\|_{\mathrm{L}^{\gamma_0}(M)}\right].$$

Proof. For $\sigma \in [0,1]$ define the cylinder $C^{X_0}_{\sigma} = C^{X_0}_{s_\sigma, r_\sigma}$ with

$$s_\sigma = s + \sigma(S-s) \quad \text{and} \quad r_\sigma = r + \sigma(R-r)$$

and let

$$Z(\sigma) = \|v\|_{\mathrm{L}^{p_1}(C^{X_0}_{\sigma})}.$$

For $0 \le \sigma_1 < \sigma_2 \le 1$ apply Lemma 5 to find with a constant $\tilde{C}_2$

$$Z(\sigma_1) \le \tilde{C}_2 \left[1 + \frac{1}{\sigma_2 - \sigma_1}\right]^2 \left[P\|v\|_{\mathrm{L}^{p_0}(C^{X_0}_{\sigma_2})} + Q\right].$$

As $1 < p_0 < p_1$ there exist interpolation parameter $\theta \in (0,1)$ such that for all $\delta > 0$

$$\|v\|_{\mathrm{L}^{p_0}} \le \|v\|^{1-\theta}_{\mathrm{L}^1}\|v\|^{\theta}_{\mathrm{L}^{p_1}} \le \theta\delta\|v\|_{\mathrm{L}^{p_1}} + (1-\theta)\delta^{\frac{-\theta}{1-\theta}}\|v\|_{\mathrm{L}^1}.$$

By the interpolation we find with a constant $\tilde{C}_3$

$$Z(\sigma_1) \leq \theta Z(\sigma_2) + \tilde{C}_3 \left[1 + \frac{1}{\sigma_2 - \sigma_1}\right]^{\frac{2}{1-\theta}} \left[P^{\frac{1}{1-\theta}} \|v\|_{\mathrm{L}^1(M)} + Q\right].$$

The result now follows by a standard argument for geometric series, see e.g. [11, Lemma 6.1]. Consider for some $\beta < 1$ the sequence

$$\sigma_i = 1 - \beta^i.$$

Then the previous argument shows

$$Z(\sigma_{i-1}) \leq \theta Z(\sigma_i) + \tilde{C}_3 \left[P^{\frac{1}{1-\theta}} \|v\|_{\mathrm{L}^1(M)} + Q\right] \left(1 + \frac{1}{(1-\beta)\beta^{i-1}}\right)^{\frac{2}{1-\theta}}$$

and iterating the argument shows that for any $k \in \mathbb{N}$

$$Z(0) = Z(\sigma_0) \leq \theta^k Z(\sigma_k) + \tilde{C}_3 \left[P^{\frac{1}{1-\theta}} \|v\|_{\mathrm{L}^1(M)} + Q\right] \sum_{i=1}^{k} \theta^{i-1} \left(1 + \frac{1}{(1-\beta)\beta^{i-1}}\right)^{\frac{2}{1-\theta}}.$$

For β sufficiently close to 1, the series converges and the result follows. □

We can now collect the different parts.

Proof of Theorem 2. By considering u/N instead of u where $N = \|u\|_{\mathrm{L}^1(C^{X_0}_{S,R}(t_0,x_0))} + \|f\|_{\mathrm{L}^{q_b}(C^{X_0}_{S,R}(t_0,x_0))} + \|g\|_{\mathrm{L}^{q_d}(C^{X_0}_{S,R}(t_0,x_0))}$ it suffices to prove

$$\sup_{C^{X_0}_{s,r}(t_0,x_0)} u \leq C_S\,(1+\delta_S)^{\beta}$$

under the assumption that

$$\|u\|_{\mathrm{L}^1(C^{X_0}_{S,R}(t_0,x_0))} + \|f\|_{\mathrm{L}^{q_b}(C^{X_0}_{S,R}(t_0,x_0))} + \|g\|_{\mathrm{L}^{q_d}(C^{X_0}_{S,R}(t_0,x_0))} \leq 1.$$

For the proof, consider a sequence of cylinders $C^{X_0}_k = C^{X_0}_{s_k,r_k}(t_0,x_0)$ for $k \in \mathbb{N}$ where

$$s_k = s + 2^{-k}(S-s) \text{ and } r_k = r + 2^{-k}(R-r).$$

On the cylinders consider the regularised cutoffs $v_k = K_{\epsilon_k,h_k}(u)$ where

$$h_k = D(1-2^{-k}) \text{ and } \epsilon_k = \frac{D}{4}2^{-k}$$

for a parameter $D \geq 1$. We then study

$$Z_k = \|v_k\|_{\mathrm{L}^{p_0}(C^{X_0}_{k+1})} \text{ and } M_k = C_k \cap \{v_k > 0\}.$$

As a first step we will then show for an exponent $\alpha_1 > 0$ the initial bound

$$Z_1 \lesssim (1+\delta_S)^{\alpha_1} D^{1-\frac{p_1}{p_0}}. \tag{23}$$

The second step is to show for a constant W and exponents $\alpha_2, \delta > 0$ that

$$Z_k \lesssim W^k (1+\delta_S)^2 Z_{k-1}^{1+\delta}. \tag{24}$$

Hence for $D \gtrsim (1+\delta_S)^{\beta}$ for some exponent $\beta > 0$ we have that $Z_k \to 0$ as $k \to \infty$ which implies the result.

Initial bound (23)
This follows from applying Lemma 6 twice. As a first step apply it between the cylinders $C_1^{X_0}$ and $C_0^{X_0}$ with $K_{0,1}(u)$ to find

$$\|K_{0,1}(u)\|_{\mathrm{L}^{p_1}(C_1^{X_0})} \lesssim (1+\delta_S)^{2\alpha}.$$

By Hölder this implies for D large enough that

$$|M_1| \lesssim D^{-\frac{1}{p_1}}(1+\delta_S)^{2\alpha}.$$

Using again Hölder, this shows by the choice of the integrabilities that for D large enough

$$(\epsilon_1+h_1)\left(\|b\|_{\mathrm{L}^2(M_1)} + \frac{\epsilon+h}{\epsilon}\|b\|^2_{\mathrm{L}^{2\gamma_0}(M_1)} + \|d\|_{\mathrm{L}^{\gamma_0}(M_1)}\right) \lesssim (1+\delta_S)^2.$$

Hence we can apply Lemma 6 again to find with some exponent α_1

$$\|v_1\|_{\mathrm{L}^{p_1}(C_1^{X_0})} \lesssim (1+\delta_S)^{\alpha_1}.$$

This yields (23) by another application of Hölder.

Iteration step (24)
Note that the regularisations ϵ_k are chosen such that $\epsilon_k+\epsilon_{k+1} \le (h_{k+1}-h_k)/2$ so that

$$\{v_{k+1}>0\} \subset \{v_k > (h_{k+1}-h_k)/2\} \text{ and } v_{k+1} \ge v_k. \tag{25}$$

Hence we find by Hölder that

$$|M_k|^{1/p_0} \le \frac{2}{h_k-h_{k-1}} Z_{k-1}. \tag{26}$$

and

$$Z_k \le |M_k|^{\frac{1}{p_0}-\frac{1}{p_1}} \|v_k\|_{\mathrm{L}^{p_1}(C_{k+1}^{X_0})}. \tag{27}$$

By Lemma 5 we estimate $\|v_k\|_{\mathrm{L}^{p_1}(C_{k+1}^{X_0})}$ by going to $C_k^{X_0}$ as

$$\|v_k\|_{\mathrm{L}^{p_1}(C_{k+1}^{X_0})} \lesssim 8^k(1+\delta_S)^2\left[(1+D)|M_k|^{\frac{1}{p_0}} + \|v_k\|_{\mathrm{L}^{p_0}(C_k^{X_0})}\right] \tag{28}$$

by noting the bounds

$$\begin{aligned}
\|(1+\Lambda)v_k\|_{\mathrm{L}^2(M_k)} &\le (1+\|\Lambda\|_{\mathrm{L}^{\bar q_0}})\|v_k\|_{\mathrm{L}^{p_0}(C_k^{X_0})}\\
\|\bar f_k\|_{\mathrm{L}^2(M_k)} &\le \|f\|_{\mathrm{L}^{\bar q_0}}|M_k|^{\frac12-\frac{1}{\bar q_0}} + \|b\|_{\mathrm{L}^{\bar q_0}}\left((1+D)|M_k|^{\frac12-\frac{1}{\bar q_0}} + \|v_k\|_{\mathrm{L}^{p_0}(M_k)}\right)\\
\|cv_k\|_{\mathrm{L}^2(M_k)} &\le \|c\|_{\mathrm{L}^{\bar q_0}}|M_k|^{\frac12-\frac{1}{\bar q_0}}\\
\|\bar g_k v_k\|^{1/2}_{\mathrm{L}^1(M_k)} &\lesssim \|\bar g_k\|_{\mathrm{L}^{p_0^*}(M_k)} + \|v_k\|_{\mathrm{L}^{p_0}(M_k)}\\
&\lesssim (\|g\|_{\mathrm{L}^{q_d}} + (1+D)\|d\|_{\mathrm{L}^{q_d}})\,|M_k|^{\frac{1}{p_0^*}-\frac{1}{q_d}} + (1+\|d\|_{\mathrm{L}^{q_d}})\|v_k\|_{\mathrm{L}^{p_0}(M_k)}\\
\|\bar g_k\|_{\mathrm{L}^{\gamma_0}(M_k)} &\le \|g\|_{\mathrm{L}^{q_d}}|M_k|^{\frac{1}{\gamma_0}-\frac{1}{q_d}} + \|d\|_{\mathrm{L}^{q_d}}\left((1+D)|M_k|^{\frac{1}{\gamma_0}-\frac{1}{q_d}} + \|v_k\|_{\mathrm{L}^{p_0}(M_k)}\right)\\
\frac{1}{\epsilon_k}\|\bar f_k \mathbb{1}_{v_k\le 2\epsilon_k}\|^2_{\mathrm{L}^{2\gamma_0}(M_k)} &\lesssim 2^k(1+D)\left(\|f\|^2_{\mathrm{L}^{q_b}} + \|b\|^2_{\mathrm{L}^{q_b}}\right)|M_k|^{2\left(\frac{1}{2\gamma_0}-\frac{1}{q_b}\right)}.
\end{aligned}$$

Chaining (26), (27) and (28) then yields the required bound (24). □

4. Weak Harnack inequality

In this section we prove Theorem 3. We first introduce a regularised version of $(-\log(z))_+$ as $G \in \mathscr{C}^2((0,\infty))$ by

$$G(z) = (-\log z + z - 1)\mathbb{1}_{z\le 1}$$

so that

$$G'(z) = \left(-\frac{1}{z}+1\right)\mathbb{1}_{z\le 1} \le 0 \text{ and } G''(z) = \frac{1}{z^2}\mathbb{1}_{z\le 1} \ge 0.$$

This implies

$$G''(z) \ge [G'(z)]^2 . \tag{29}$$

As already used in Nash [23], then consider

$$v = G_\delta(u) \quad \text{where} \quad G_\delta(z) = G\left(\frac{z+\delta}{1+\delta}\right)$$

for a small enough $\delta > 0$. As $u \ge 0$ we have the trivial bound

$$|v| \le G_\delta(0). \tag{30}$$

The strategy is to use (29) and (30) in order to gain a control of the rough form $\|\vec{X}v\|^2_{\mathrm{L}^2} \lesssim G_\delta(0)$. In the parabolic case, we can reinterpret the argument by Nash [23] as using the information that $u \ge 1$ in E to conclude by a variation of Poincaré and the supremum bound that $\|v\|_{\mathrm{L}^\infty} \lesssim \|\vec{X}v\|_{\mathrm{L}^2} \lesssim \sqrt{G_\delta(0)}$. As we have gained a square root, we can then make δ sufficiently small to conclude a non-trivial bound on $\|v\|_{\mathrm{L}^\infty}$ which yields the statement. In the classical non-degenerate setting, this ideas has been used in [19, 20, 21, 22] and in the kinetic and Kolmogorv setting [3, 13, 29, 30]. Here we differ by using the dual problem to conclude the result (instead of a Poincaré inequality inspired by the framework of [1, 15]).

As u is a supersolution, we apply Lemma 4 to conclude together with (29) that

$$\tilde{P}v + \frac{\lambda}{2}|\vec{X}v|^2 \le 0 \tag{31}$$

where $\tilde{P}$ is the operator in (11) with the new coefficients $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{f}, \tilde{g}$.

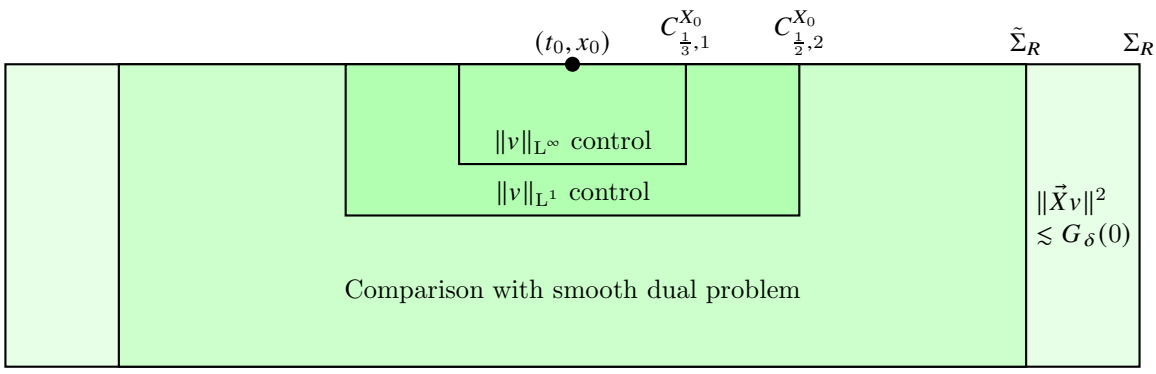

Figure 6: Illustration of the overview of proof for Theorem 3.

For a parameter R which is chosen large enough later, we take by **(H2)** the sets $\tilde{\Sigma}_R$ and Σ_R. Then we obtain the result in the following steps, cf. Fig. 6. By integrating the trivial L^1 bound we find that

$$\|\vec{X}v\|^2_{\mathrm{L}^2(\tilde{\Sigma}_R)} \le \tilde{C}_1 \left[C(R)G_\delta(0) + \|\Lambda\|^2_{\mathrm{L}^2(\Sigma_R)} + \|\tilde{f}\|_{\mathrm{L}^1(\Sigma_R)} + \|\tilde{c}\|^2_{\mathrm{L}^2(\Sigma_R)} + \|\tilde{g}\|_{\mathrm{L}^1(\Sigma_R)}\right] \tag{32}$$

for a constant $\tilde{C}_1$ and a constant $C(R)$ depending on R. Then integrating v with the solution of the dual problem w solving (16) from **(H1)**, we find for $\bar{q}_2$ from the statement of Theorem 3 and a constant $\tilde{C}_2$ and a new constant $C(R)$ depending on R that

$$\|v\|^2_{\mathrm{L}^1(C^{X_0}_{1/2,2})} \leq \tilde{C}_2 \left(\frac{G_\delta(0)}{R}\right)^2 + C(R)\left[(1+\|\Lambda\|_{\mathrm{L}^{\bar{q}_2}(\tilde{\Sigma}_R)} + \|\tilde{c}\|_{\mathrm{L}^{\bar{q}_2}(\tilde{\Sigma}_R)})^2 \|\vec{X}v\|^2_{\mathrm{L}^2(\tilde{\Sigma}_R)} + \|\tilde{f}\|^2_{\mathrm{L}^2(\tilde{\Sigma}_R)} + \|\tilde{g}\|^2_{\mathrm{L}^2(\tilde{\Sigma}_R)}\right] \tag{33}$$

and by the supremum bound we finally conclude that by choosing $R = C_R(1+\delta_S)^\beta$ for C_R sufficiently large that

$$\begin{aligned}&\|v\|^2_{\mathrm{L}^\infty(C^{X_0}_{1/3,1})} - \left(\frac{G_\delta(0)}{2}\right)^2 \\ &\lesssim (1+\delta_S)^{2\beta}(1+\Delta)^2\left[G_\delta(0) + 1 + \Delta^2 + \frac{\|f-ub\|^2_{\mathrm{L}^{q_b}(\Sigma_R)}}{\delta^2} + \frac{\|g-ud\|^2_{\mathrm{L}^{q_d}(\Sigma_R)}}{\delta^2} + \frac{\|f-ub\|^4_{\mathrm{L}^{2q_d}(\Sigma_R)}}{\delta^4}\right].\end{aligned} \tag{34}$$

By taking δ small enough, we will then conclude the result.

Integrating L^1 bound

Take the cutoff η_R from **(H2)** and consider the localised L^1 norm

$$E(t) = \int_{\Sigma_t \cap (\{t\}\times\mathbb{R}^n)} v(t,x)\,\eta_1^2(t,x)\,\mathrm{d}x.$$

By (30) we find as Σ_R is bounded the trivial bound

$$E(t) \lesssim C(R)G_\delta(0).$$

By (31) we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}E \leq -\frac{\lambda}{4}\int |\vec{X}v|^2\eta_R^2 + \|\operatorname{div} X_0\|_\infty E + \frac{2}{\lambda}\int \Lambda^2|\vec{X}\eta_R|^2 + \int |\tilde{f}|\,|\vec{X}\eta_R^2| + \frac{2}{\lambda}\int |\tilde{c}|^2\,\eta_R^2 + \int |\tilde{g}|.$$

Integrating over time hence yields the claimed control (32).

Comparison with dual problem

Let w be the solution of the smooth dual problem (16) where E is given in (18).

Then consider with the cutoff $\tilde{\eta}_R$ from **(H2)**

$$K(t) = \int_{\Sigma_t \cap (\{t\}\times\mathbb{R}^n)} v\,w\,\tilde{\eta}_R\,\mathrm{d}x.$$

We then find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}K \leq &-\int X_j v a^{ij} X_i(w\tilde{\eta}_R) - \int X_j v \delta^{ij}\,[\tilde{\eta}_R X_i w - w X_i \tilde{\eta} R] + \int v w \delta^{ij} X_i X_j \tilde{\eta}_R \\ &+ \int \tilde{f}^i X_i(w\tilde{\eta}_R) + \int \tilde{c}^i X_i v\, w\tilde{\eta}_R + \int \tilde{g} w \tilde{\eta}_R + \int v \mathbb{1}_E.\end{aligned}$$

By construction $v(x) = 0$ if $x \in E$ so that the last term vanishes and $w \equiv 0$ for $t = -1$. Hence integrating $t \in [0,1]$ yields (using **(H2)** for bounding $\delta^{ij}X_iX_j\tilde{\eta}_R$)

$$\begin{aligned}K(t) \lesssim &\|\vec{X}v\|_{\mathrm{L}^2(\tilde{\Sigma}_R)}\left(\|(1+\Lambda)\vec{X}w\|_{\mathrm{L}^2(\tilde{\Sigma}_R)} + \|(1+\Lambda)w\|_{\mathrm{L}^2(\tilde{\Sigma}_R)} + \|\tilde{c}w\|_{\mathrm{L}^2(\tilde{\Sigma}_R)}\right) + G_\delta(0)\,\|w\|_{\mathrm{L}^1([-1,0]\times\mathbb{R}^n)}R^{-1} \\ &+ \|\tilde{f}\|_{\mathrm{L}^2(\Sigma_R)}\left(\|w\|_{\mathrm{L}^{p_2}(\Sigma_R)} + \|Xw\|_{\mathrm{L}^{p_2}(\Sigma_R)}\right) + \|\tilde{g}\|_{\mathrm{L}^2(\Sigma_R)}\|w\|_{\mathrm{L}^{p_2}(\Sigma_R)}\end{aligned}$$

By **(H3)** we find that $\|v\|_{\mathrm{L}^1(C^{X_0}_{1/2,2})} \leq \mu_0^{-1}\sup_{t\in[-1/2,0]} K(t)$ so that the claimed estimate (33) follows by the bounds of **(H3)**.

Using the supremum bound and conclusion

By using the supremum bound (Theorem 2) between $C^{X_0}_{\frac{1}{3},1}$ and $C^{X_0}_{\frac{1}{2},2}$, we find with a constant $\tilde{C}_3$ and a new constant $C(R)$ depending on R that

$$\|v\|^2_{\mathrm{L}^\infty(C^{X_0}_{\frac{1}{3},1})} - \tilde{C}_3(1+\delta_S)^{2\beta}\left(\frac{G_\delta(0)}{R}\right)^2 \leq C(R)(1+\delta_S)^{2\beta}\left[(1+\Delta)^2\|\vec{X}v\|_2^2 + \|\tilde{f}\|^2_{\mathrm{L}^{q_b}(\Sigma_R)} + \|\tilde{g}\|^2_{\mathrm{L}^{q_d}(\Sigma_R)}\right].$$

By choosing C_R sufficiently large and setting $R = C_R(1+\delta_S)^\beta$, we find

$$\|v\|^2_{\mathrm{L}^\infty(C^{X_0}_{\frac{1}{3},1})} - \left(\frac{G_\delta(0)}{2}\right)^2 \lesssim (1+\delta_S)^{2\beta}\left[(1+\Delta)^2\|\vec{X}v\|_2^2 + \|\tilde{f}\|^2_{\mathrm{L}^{q_b}(\Sigma_R)} + \|\tilde{g}\|^2_{\mathrm{L}^{q_d}(\Sigma_R)}\right].$$

Plugging in (32) then gives the claimed estimate (34).

As $G_\delta(0) \to \infty$ as $\delta \to 0$, we can then find a sufficiently small δ such that (34) becomes

$$\|v\|_{\mathrm{L}^\infty(C^{X_0}_{\frac{1}{3},1})} - \frac{3}{4}G_\delta(0) \leq \epsilon.$$

By letting ϵ small enough this shows that $v \leq (5/4)G_\delta(0)$ in $C^{X_0}_{\frac{1}{3},1}$. As the relation $G_\delta(u) = v \leq (5/4)G_\delta(0)$ implies $u \geq \mu$ for a constant μ, this shows the result.

Acknowledgements

The first author would like to thank the mathematical department of the University of Leipzig for the possibility of a long visit during which this work has started and the continued hospitality. He would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Frontiers in Kinetic Theory". This was supported by EPSRC grant no EP/R014604/1 and a grant from the Simons Foundations. He also acknowledges the grant ANR-18-CE40-0027 of the French National Research Agency (ANR). The second author was partially supported by the German Science Foundation DFG in context of the Priority Program SPP 2026 "Geometry at Infinity".

A. Notes on function spaces

We first note that $u \in \mathrm{H}^1_{\mathrm{hyp}}$ and $X_0 u \in \mathrm{H}^{-1}_{\mathrm{hyp}}$ implies with **(H1)** more regularity.

Lemma 7. *For a domain Ω_t suppose **(H1)** and suppose $u \in \mathrm{H}^1_{\mathrm{hyp}}(\Omega_t)$ with $X_0 u \in \mathrm{H}^{-1}_{\mathrm{hyp}}(\Omega_t)$. Then $u \in L^{p_1}_{loc}(\Omega_t)$.*

Proof. Take a compactly supported subset Ω'_t of Ω_t and let φ be a smooth cutoff. Then $\varphi u \in \mathrm{H}^1_{\mathrm{hyp}}$ and $X_0(\varphi u) \in \mathrm{H}^{-1}_{\mathrm{hyp}}$ as we can note

$$\|X_0(\varphi u)\|_{\mathrm{H}^{-1}_{\mathrm{hyp}}} \leq (\|\varphi\|_\infty + \|X_0\varphi\|_\infty + \|\vec{X}\varphi\|_\infty)(\|u\|_{\mathrm{L}^2} + \|X_0 u\|_{\mathrm{H}^{-1}_{\mathrm{hyp}}}).$$

Due to embedding $u \in \mathrm{H}^1_{\mathrm{hyp}} \mapsto (u, \vec{X}u) \in (\mathrm{L}^2)^{m+1}$ every element $\lambda \in \mathrm{H}^{-1}_{\mathrm{hyp}}$ can be represented as $\lambda = -X_i^t f^i + f^0$. Hence **(H1)** yields the result. □

After the above described control of $u \in \mathrm{L}^{p_1}_{loc}$ all the a priori estimates can be defined by standard methods.

Furthermore we shortly want to recall a simple argument for a weak maximum principle in the setting of hypoelliptic operators:

Lemma 8. *Let $w \in \mathrm{H}^1_{\mathrm{hyp}}(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})$ be a weak subsolution of*

$$\begin{cases} (X_0 - L_0^{\mathrm{ext}})w \le 0 & \text{in } \Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\}, \\ w = 0 & \text{on } \{t_{\mathrm{in}}\} \times B^{\mathrm{ext}} \cup (t_{\mathrm{in}}, t_2) \times \partial B^{\mathrm{ext}} \end{cases} \tag{35}$$

then $w \le 0$ a.e. in $\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})$.

Proof. Let $\epsilon > 0$ and consider the non-decreasing convex function $K(z) = K_{\epsilon,2\epsilon}(z) = \rho_\epsilon * (z - 2\epsilon)_+$. Since by assumptions $X_i w \in \mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})$ for $i = 1, \dots, m$, it is not hard to check that $w_\epsilon = K(w)$ is still a weak subsolution of (35) with $X_i w_\epsilon \in \mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})$ for $i = 1, \dots, m$.

Testing the equation with w_ϵ and using a classical Gronwall argument one obtains

$$\sup_{t > t_{\mathrm{in}}} \|w_\epsilon\|_{\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t = t_{\mathrm{in}}\})} \le 0 .$$

But this clearly implies $w_\epsilon = 0$ a.e. Since $\epsilon > 0$ was chosen arbitrary the conclusion follows. □

B. Construction of comparision function

In this section, we will discuss how the Hörmander estimates can be used to verify **(H1)**. Here we take bounded balls B and B^{ext} and the corresponding parabolic domains Ω_t and Ω_t^{ext}. Between the balls B and B^{ext}, find a smooth cutoff χ and consider

$$L_0^{\mathrm{ext}} = L_0 + \nabla \cdot ((1 - \chi)^2 \nabla \cdot).$$

For any $\varepsilon > 0$, we can then find by standard parabolic theory or the method of continuity a solution w^ε of

$$\begin{cases} (X_0 - L_0^{\mathrm{ext}} - \epsilon\Delta)w^\varepsilon = G + \sum_{i=1}^m X_i^t F^i & \text{in } \Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\}, \\ w^\varepsilon = 0 & \text{on } \{t_{\mathrm{in}}\} \times B^{\mathrm{ext}} \cup (t_{\mathrm{in}}, t_2) \times \partial B^{\mathrm{ext}} \end{cases}$$

with the uniform L^2 estimate

$$\begin{aligned} &\|w^\varepsilon\|_{\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})} + \|\vec{X} w^\varepsilon\|_{\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})} + \|(1 - \chi)\nabla w^\varepsilon\|_{\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})} \\ &\quad \lesssim \|F\|_{\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})} + \|G\|_{\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})}. \end{aligned}$$

By compactness, we can therefore find a weak limit $w \in \mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})$ with the same bound solving

$$\begin{cases} (X_0 - L_0^{\mathrm{ext}})w = G + \sum_{i=1}^m X_i^t F^i & \text{in } \Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\}, \\ w = 0 & \text{on } \{t_{\mathrm{in}}\} \times B^{\mathrm{ext}} \cup (t_{\mathrm{in}}, t_2) \times \partial B^{\mathrm{ext}} \end{cases}$$

Here the bound on $\|(1 - \chi)\nabla w^\varepsilon\|_{\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})}$ imply by the trace theorem that $w = 0$ on $(t_{\mathrm{in}}, t_2) \times \partial B^{\mathrm{ext}}$ has a well-defined meaning and still holds for the limit w^ϵ.

Going back to the equation, this shows on Ω_t that

$$\|w\|_{\mathrm{H}^1_{\mathrm{hyp}}(\Omega_t \cap \{t > t_{\mathrm{in}}\}} + \|X_0 w\|_{\mathrm{H}^{-1}_{\mathrm{hyp}}(\Omega_t \cap \{t > t_{\mathrm{in}}\}} \lesssim \|F\|_{\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})} + \|G\|_{\mathrm{L}^2(\Omega_t^{\mathrm{ext}} \cap \{t > t_{\mathrm{in}}\})}.$$

Under the commutator condition, Hörmander [15] shows for some $s > 0$ that $\|w\|_{\mathrm{H}^s(\Omega_t \cap \{t > t_{\mathrm{in}}\}} \lesssim \|w\|_{\mathrm{H}^1_{\mathrm{hyp}}(\Omega_t \cap \{t > t_{\mathrm{in}}\}} + \|X_0 w\|_{\mathrm{H}^{-1}_{\mathrm{hyp}}(\Omega_t \cap \{t > t_{\mathrm{in}}\}}$, which implies the thought bound (10) by Sobolev embedding for some $p_1 > 2$ and $\gamma_0 = \gamma_1 = 2$.

Remark 4. The discussion of local hypoelliptic operator to the whole space with uniform bounds is discussed in Bramanti, Brandolini, Lanconelli, and Uguzzoni [6, Part 1].

References

[1] D. Albritton, S. Armstrong, J. C. Mourrat, and M. Novack. *Variational methods for the kinetic Fokker-Planck equation*. 2019. arXiv: 1902.04037v2 [math.AP].

[2] Francesca Anceschi, Sergio Polidoro, and Maria Alessandra Ragusa. "Moser's estimates for degenerate Kolmogorov equations with non-negative divergence lower order coefficients". English. In: *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods* 189 (2019). Id/No 111568, p. 19. DOI: 10.1016/j.na.2019.07.001.

[3] Francesca Anceschi and Annalaura Rebucci. *A note on the weak regularity theory for degenerate Kolmogorov equations*. 2021. arXiv: 2107.04441v2 [math.AP].

[4] Francesca Anceschi and Yuzhe Zhu. *On a spatially inhomogeneous nonlinear Fokker-Planck equation: Cauchy problem and diffusion asymptotics*. 2021. arXiv: 2102.12795v2 [math.AP].

[5] Jean-Michel Bony. "Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérées". French. In: *Ann. Inst. Fourier* 19.1 (1969), pp. 277–304. DOI: 10.5802/aif.319.

[6] Marco Bramanti, Luca Brandolini, Ermanno Lanconelli, and Francesco Uguzzoni. *Non-divergence equations structured on Hörmander vector fields: heat kernels and Harnack inequalities*. English. Vol. 961. Mem. Am. Math. Soc. Providence, RI: American Mathematical Society (AMS), 2010. DOI: 10.1090/S0065-9266-09-00605-

[7] Luca Capogna, Giovanna Citti, and Garrett Rea. "A subelliptic analogue of Aronson–Serrin's Harnack inequality". In: *Mathematische Annalen* 357.3 (2013), 1175–1198. DOI: 10.1007/s00208-013-0937-y. URL: http://dx.doi.org/10.1007/s00208-013-0937-y.

[8] Chiara Cinti and Sergio Polidoro. "Pointwise local estimates and Gaussian upper bounds for a class of uniformly subelliptic ultraparabolic operators". In: *Journal of Mathematical Analysis and Applications* 338.2 (2008), 946–969. DOI: 10.1016/j.jmaa.2007.05.059. URL: http://dx.doi.org/10.1016/j.jmaa.2007.05.059.

[9] Ennio De Giorgi. *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*. Italian. Mem. Accad. Sci. Torino, P. I., III. Ser. 3, 25-43 (1957). 1957.

[10] Prashanta Garain and Kaj Nyström. *On regularity and existence of weak solutions to nonlinear Kolmogorov-Fokker-Planck type equations with rough coefficients*. 2022. arXiv: 2204.12277v1 [math.AP].

[11] Enrico Giusti. *Direct methods in the calculus of variations*. English. Singapore: World Scientific, 2003.

[12] François Golse, Cyril Imbert, and Alexis Vasseur. "Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation". English. In: *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* 19.1 (2019), pp. 253–295. DOI: 10.2422/2036-2145.201702_001.

[13] Jessica Guerand and Cyril Imbert. *Log-transform and the weak Harnack inequality for kinetic Fokker-Planck equations*. 2021. arXiv: 2102.04105v1 [math.AP].

[14] Jessica Guerand and Clément Mouhot. *Quantitative de Giorgi Methods in Kinetic Theory*. 2021. arXiv: 2103.09646v1 [math.AP].

[15] Lars Hörmander. "Hypoelliptic second order differential equations". English. In: *Acta Math.* 119 (1967), pp. 147–171. DOI: 10.1007/BF02392081.

[16] Cyril Imbert and Clément Mouhot. "The Schauder estimate in kinetic theory with application to a toy nonlinear model". In: *Annales Henri Lebesgue* 4 (2021), 369–405. DOI: 10.5802/ahl.75. URL: http://dx.doi.org/10.5802/ahl.75.

[17] Cyril Imbert and Luis Silvestre. "The weak Harnack inequality for the Boltzmann equation without cut-off". English. In: *J. Eur. Math. Soc. (JEMS)* 22.2 (2020), pp. 507–592. DOI: 10.4171/JEMS/928.

[18] A. Kolmogoroff. "Zufällige Bewegungen. (Zur Theorie der Brownschen Bewegung.)" German. In: *Ann. Math. (2)* 35 (1934), pp. 116–117. DOI: 10.2307/1968123.

[19] S. N. Kruzhkov. "A priori estimates for generalized solutions of second-order elliptic and parabolic equations". English. In: *Sov. Math., Dokl.* 4 (1963), pp. 757–761.

[20] S. N. Kruzhkov. "Apriori estimates and certain properties of the solutions of elliptic and parabolic equations". English. In: *Transl., Ser. 2, Am. Math. Soc.* 68 (1968), pp. 169–220. DOI: 10.1090/trans2/068/08.

[21] Jürgen Moser. "A Harnack inequality for parabolic differential equations". English. In: *Commun. Pure Appl. Math.* 17 (1964), pp. 101–134. DOI: 10.1002/cpa.3160170106.

[22] Jürgen Moser. "On Harnack's theorem for elliptic differential equations". English. In: *Commun. Pure Appl. Math.* 14 (1961), pp. 577–591. DOI: 10.1002/cpa.3160140329.

[23] John F. Nash. "Continuity of solutions of parabolic and elliptic equations". English. In: *Am. J. Math.* 80 (1958), pp. 931–954. DOI: 10.2307/2372841.

[24] Andrea Pascucci and Sergio Polidoro. "A Gaussian upper bound for the fundamental solutions of a class of ultraparabolic equations". In: *Journal of Mathematical Analysis and Applications* 282.1 (2003), 396–409. DOI: 10.1016/s0022-247x(03)00159-8. URL: http://dx.doi.org/10.1016/s0022-247x(03)00159-8.

[25] Andrea Pascucci and Sergio Polidoro. "The Moser's iterative method for a class of ultraparabolic equations." English. In: *Commun. Contemp. Math.* 6.3 (2004), pp. 395–417. DOI: 10.1142/S0219199704001355.

[26] Linda Preiss Rothschild and Elias M. Stein. "Hypoelliptic differential operators and nilpotent groups". English. In: *Acta Math.* 137 (1977), pp. 247–320. DOI: 10.1007/BF02392419.

[27] Antonio Sánchez-Calle. "Fundamental solutions and geometry of the sum of squares of vector fields". English. In: *Invent. Math.* 78 (1984), pp. 143–160. DOI: 10.1007/BF01388721.

[28] N. S. Trudinger. "On the regularity of generalized solutions of linear, non-uniformly elliptic equations". English. In: *Arch. Ration. Mech. Anal.* 42 (1971), pp. 50–62. DOI: 10.1007/BF00282317.

[29] WenDong Wang and LiQun Zhang. "The C^α regularity of a class of non-homogeneous ultraparabolic equations". English. In: *Sci. China, Ser. A* 52.8 (2009), pp. 1589–1606. DOI: 10.1007/s11425-009-0158-8.

[30] Wendong Wang and Liqun Zhang. "The C^α regularity of weak solutions of ultraparabolic equations". English. In: *Discrete Contin. Dyn. Syst.* 29.3 (2011), pp. 1261–1275. DOI: 10.3934/dcds.2011.29.1261.

[31] Yuzhe Zhu. *Velocity averaging and Hölder regularity for kinetic Fokker-Planck equations with general transport operators and rough coefficients.* 2020. arXiv: 2010.03867v2 [math.AP].