



# Quantitative conditions for right-handedness

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# Quantitative conditions for right-handedness

Anna Florio\*, Umberto Hryniewicz†

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## Abstract

We give a numerical condition for right-handedness of a dynamically convex Reeb flow on  $S^3$ . Our condition is stated in terms of an asymptotic ratio between the amount of rotation of the linearised flow and the linking number of trajectories with a periodic orbit that spans a disk-like global surface of section. As an application, we find an explicit constant  $\delta_* < 0.7225$  such that if a Riemannian metric on the 2-sphere is  $\delta$ -pinched with  $\delta > \delta_*$ , then its geodesic flow lifts to a right-handed flow on  $S^3$ . In particular, all finite collections of periodic orbits of such a geodesic flow bind open books whose pages are global surfaces of section.

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# 1 Introduction

Right-handed vector fields, introduced by Ghys in [Ghy09], form a special class of non-singular vector fields on homology 3-spheres. Their flows will also be called right-handed, for simplicity. Roughly speaking, all pairs of trajectories of such a flow link positively. Ghys formalised this definition in terms of positivity of the quadratic linking form, which assigns some kind of linking number to any pair of invariant Borel probability measures; see also [Arn86]. Right-handedness has interesting dynamical implications as the following statement demonstrates.

**Theorem 1.1** (Ghys [Ghy09]). *Every finite collection of periodic orbits of a right-handed flow binds an open book whose pages are global surfaces of section.*

In particular, the above statement imposes strong restrictions on the periodic orbits that can arise in right-handed flows: knots or links that are not fibred can not be realised. For the sake of completeness, and for convenience of the reader, a full proof of Theorem 1.1 is presented. We shall see that both the definition of right-handedness (Definition 2.6) and the proof of Theorem 1.1 (Appendix B) can be given independently of the quadratic linking form.

Right-handedness is difficult to check. It can be checked for certain special flows, like the Hopf flow or other integrable flows. Since in [Ghy09] it is stated that right-handedness is  $C^1$ -open, one can find, in theory, more interesting examples near such special flows, but this abstract reasoning does not give them explicitly. It is precisely this general lack of explicit examples that motivates our work. One exception is provided by the work of Dehornoy in [Deh17]. Sometimes it is more natural to talk about left-handedness, which is equivalent to right-handedness when the ambient orientation is reversed. Dehornoy showed that the geodesic flow on the unit tangent bundle of a hyperbolic  $n$ -conic 2-sphere is left-handed if  $n = 3$ . Then, using a version of Gromov's geodesic rigidity [Gro00], the case of arbitrary negative curvature is reduced to the hyperbolic case.

To state our main application, consider the polynomial  $P(x) = 4x^3 - 2x^2 - 1$ , which has a unique real root  $x_*$ . It satisfies  $0.84 < x_* < 0.85$ . Set  $\delta_* := x_*^2$ .

**Theorem 1.2.** *If  $\delta > \delta_*$ , in particular if  $\delta \geq 0.7225$ , then the geodesic flow of a  $\delta$ -pinched Riemannian metric on  $S^2$  lifts to a right-handed flow on  $S^3$ .*

As mentioned before, the main motivation for such a statement is that it is not a perturbative result, hence it can be used to check right-handedness in an explicit set of flows that are far from integrable.

Other implications of right-handedness besides Theorem 1.1 have been obtained. For instance, Dehornoy and Rechtman showed in [DR20] that if an invariant measure of a right-handed flow can be approximated by long periodic orbits, then their Seifert genera grow proportionally to the square of the period, the proportionality constant being the helicity of the given invariant measure.

More generally, we look for numerical conditions for right-handedness within the class of Reeb flows of dynamically convex contact forms on  $S^3$ . This class was introduced by Hofer, Wysocki and Zehnder in [HWZ98]. They showed that this class is rich enough to provide applications: by [HWZ98, Theorem 3.4] the Hamiltonian flow on a strictly convex energy level in a 4-dimensional symplectic vector space is the Reeb flow of a dynamically convex contact form. Harris and Paternain showed in [HP08] that the geodesic flow of a Finsler metric on the two-sphere with reversibility  $r$  is dynamically convex if the flag curvatures are pinched by more than  $(r/(r+1))^2$ . Theorem 1.2 will be deduced from Theorem 1.13 below, which provides an abstract condition for right-handedness of a dynamically convex Reeb flow on  $S^3$  in terms of a dynamical pinching condition. Theorem 1.13 also implies Theorem 1.14 which gives right-handedness on strictly convex energy levels in terms of a relation between the curvatures and the return times of a disk-like global surface of section.

It was explained to us by Dehornoy in private communication that there are Riemannian metrics on  $S^2$  with strict positive curvature whose geodesic flows do not lift to right-handed flows on  $S^3$ . Consider the infimum  $\delta_0$  of all  $\delta \in (0, 1]$  with the following property: if a Riemannian metric on  $S^2$  has curvatures pinched by at least  $\delta$  then its geodesic flow lifts to a right-handed flow on  $S^3$ . Dehornoy's examples show that  $\delta_0 > 0$ . Together with Theorem 1.2 we get  $0 < \delta_0 \leq \delta_* < 0.7225$ . We are led to ask:

**Question.** What is the value of  $\delta_0$ ?

## 1.1 Global surfaces of section

Let  $X$  be a smooth vector field on a closed and oriented 3-manifold  $M$ . Its flow is denoted by  $\phi^t$ .

**Definition 1.3.** A *global surface of section* (GSS) for  $\phi^t$ , or for  $X$ , is a smooth, embedded and compact surface  $\Sigma \hookrightarrow M$ , such that  $\partial\Sigma$  consists of periodic orbits or is empty,  $X$  is transverse to  $\Sigma \setminus \partial\Sigma$ , and for every  $x \in M$  one finds  $t_- < 0 < t_+$  such that  $\phi^{t_\pm}(x) \in \Sigma$ .

*Remark 1.4.* The orientation of  $M$  and the co-orientation of  $\Sigma \setminus \partial\Sigma$  induced by  $X$  together orient  $\Sigma$ . In this paper we always assume that global surfaces of section are oriented in this way.

*Remark 1.5.* From a dynamical perspective a global surface of section is a valuable tool since one can deduce dynamical properties of the flow from those of the associated *first return map*. To define it we need first to consider the *return time function*  $\tau : \Sigma \setminus \partial\Sigma \rightarrow (0, +\infty)$ ,  $\tau(x) = \min\{t > 0 \mid \phi^t(x) \in \Sigma\}$ . One can use  $\tau$  to define the return map  $\psi : \Sigma \setminus \partial\Sigma \rightarrow \Sigma \setminus \partial\Sigma$  by  $\psi(x) = \phi^{\tau(x)}(x)$ . It follows from Definition 1.3 that  $\psi$  is a smooth diffeomorphism.

Consider the vector bundle  $\xi = TM/\mathbb{R}X \rightarrow M$  and denote by  $\mathbb{P}_+\xi$  the circle bundle  $(\xi \setminus 0)/\mathbb{R}_+ \rightarrow M$ . The linearised flow  $D\phi^t$  induces flows on  $\xi$  and on  $\mathbb{P}_+\xi$ , both denoted by  $D\phi^t$  with no fear of ambiguity. These flows cover  $\phi^t$ . The generating vector field of  $D\phi^t$  on  $\mathbb{P}_+\xi$  is denoted by  $\tilde{X}$ . Both  $\xi$  and  $\mathbb{P}_+\xi$  get oriented as bundles by the flow and the ambient orientation. Now let  $\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow M$  be a periodic orbit of  $\phi^t$ , where  $T > 0$  is the primitive period. The total space  $\mathbb{T}_\gamma$  of the trivial circle bundle  $\gamma(T\cdot)^*\mathbb{P}_+\xi \rightarrow \mathbb{R}/\mathbb{Z}$ , which we see as a submanifold of  $\mathbb{P}_+\xi$ , is a  $D\phi^t$ -invariant torus. The dynamics of  $D\phi^t$  on  $\mathbb{T}_\gamma$  will be referred to as *linearised polar dynamics* along  $\gamma$ . For each boundary orbit  $\gamma$  of a global surface of section  $\Sigma$ , consider

$$\nu^\Sigma\gamma = \{\mathbb{R}_+\nu \mid \nu \in T\Sigma|_\gamma \text{ is outward pointing}\}.$$

It follows that  $\nu^\Sigma\gamma/\mathbb{R}X$  is the graph of a section of  $\gamma(T\cdot)^*\mathbb{P}_+\xi$  and, as such, defines a smooth submanifold of  $\mathbb{T}_\gamma$ .

**Definition 1.6.** The global surface of section  $\Sigma$  is called *strong* if the associated return time function is bounded away from zero and bounded from above. It is called  *$\partial$ -strong* if  $\nu^\Sigma\gamma/\mathbb{R}X$  is a global surface of section for the linearised polar dynamics along every  $\gamma \subset \partial\Sigma$ .

*Remark 1.7.* It is not hard to check that  $\partial$ -strong implies strong.

In [Poi12] Poincaré described annular global surfaces of section for certain regimes of the planar circular restricted three-body problem (PCR3BP). In the same paper one finds his *last geometric theorem*, proved by Birkhoff in [Bir13] and nowadays known as the Poincaré-Birkhoff theorem. Poincaré applied his statement to the return map of the sections he found for the PCR3BP to obtain infinitely many periodic orbits. In [Bir66] Birkhoff explained that positively curved Riemannian geodesic flows on  $S^2$  always have annulus-like global surfaces of section. Birkhoff's result admits a generalisation to Reeb flows in dimension three, see [HSW19]. Birkhoff's section plays an important role in the proof of Theorem 1.2.

## 1.2 Reeb flows

A *contact form*  $\lambda$  on a 3-manifold  $M$  is a 1-form such that  $\lambda \wedge d\lambda$  defines a volume form. We always consider  $M$  equipped with the orientation induced by  $\lambda \wedge d\lambda$ . The associated *Reeb vector field*  $X$  is implicitly determined by the equations

$$i_X d\lambda = 0 \qquad i_X \lambda = 1 \tag{1}$$

and its flow  $\phi^t$  is referred to as the Reeb flow. The plane field  $\xi = \ker \lambda$  is called a *contact structure* and can be seen as a representation of  $TM/\mathbb{R}X$ . It becomes a symplectic vector bundle with  $d\lambda$ . Note that  $\phi^t$  preserves  $\lambda$ , hence also  $\xi$ ,  $d\lambda$  and  $\lambda \wedge d\lambda$ .

A periodic orbit  $\gamma$  of  $\phi^t$  has a *Conley-Zehnder* index relative to a symplectic trivialization of  $(\xi, d\lambda)|_\gamma$ . This index is an integer that can be described in terms of transverse rotation numbers, see Remark 2.1 for details.

**Definition 1.8** (Hofer, Wysocki and Zehnder [HWZ99]). A contact form  $\lambda$  on a closed 3-manifold  $M$  is *dynamically convex* if the first Chern class  $c_1(\xi, d\lambda)$  vanishes on  $\pi_2(M)$ , and every contractible periodic Reeb orbit has Conley-Zehnder index at least equal to three in a symplectic frame that extends to a capping disk.

One of the motivations for the above definition is the following remarkable result.

**Theorem 1.9** (Hofer, Wysocki and Zehnder [HWZ98]). *The Reeb flow of every dynamically convex contact form on  $S^3$  admits a disk-like global surface of section.*

Using the methods from [HWZ98] one can prove a characterisation result for periodic Reeb orbits bounding a disk-like GSS.

**Theorem 1.10** ([Hry14]). *A periodic Reeb orbit of a dynamically convex contact form on  $S^3$  bounds a disk-like global surface of section if, and only if, it is unknotted and has self-linking number  $-1$ .*

*Remark 1.11.* The GSSs obtained from [HWZ98, Hry14] are closures of projections of pseudo-holomorphic planes in symplectisations, and as such are smooth in the interior but only  $C^1$  up to the boundary. In Appendix C it will be proved that they can be  $C^1$ -perturbed to smooth  $\partial$ -strong global surfaces of section.

If  $M = S^3$  then the contact structure  $(\xi, d\lambda)$  is trivial as a symplectic vector bundle. A global symplectic trivialisation  $\sigma$  of  $(\xi, d\lambda)$  induces a trivialisation  $\mathbb{P}_+\xi \simeq S^3 \times \mathbb{R}/2\pi\mathbb{Z}$ . The map obtained by composing this diffeomorphism with the projection onto the second factor will be denoted by

$$\Theta_\sigma : \mathbb{P}_+\xi \rightarrow \mathbb{R}/2\pi\mathbb{Z}. \quad (2)$$

Let the periodic Reeb orbit  $\gamma_0$  bound a  $\partial$ -strong disk-like global surface of section  $\Sigma$ . Stokes theorem implies that  $\Sigma$  orients  $\gamma_0 = \partial\Sigma$  along the flow. For every  $x \in S^3 \setminus \gamma_0$  we denote

$$t_+^\Sigma(x) = \inf\{t \geq 0 \mid \phi^t(x) \in \Sigma\} \quad t_-^\Sigma(x) = \sup\{t \leq 0 \mid \phi^t(x) \in \Sigma\} \quad (3)$$

which are uniformly bounded functions that vanish and are discontinuous on  $\Sigma \setminus \gamma_0$ , and are non-vanishing and smooth on  $S^3 \setminus \Sigma$ . Given  $T > 0$  consider the interval

$$I(T, x; \Sigma) = [t_-^\Sigma(x), T + t_+^\Sigma(\phi^T(x))] \quad (4)$$

and denote by  $k(T, x; \Sigma) \subset S^3 \setminus \gamma_0$  any loop obtained by concatenating to  $\phi^{I(T, x; \Sigma)}(x)$  a path in the interior of  $\Sigma$  from  $\phi^{T+t_+^\Sigma(\phi^T(x))}(x)$  to  $\phi^{t_-^\Sigma(x)}(x)$ . For  $u \in (\xi \setminus 0)/\mathbb{R}_+$  denote by  $t \in \mathbb{R} \mapsto \tilde{\Theta}_\sigma(t, u) \in \mathbb{R}$  a continuous lift of  $t \mapsto \Theta_\sigma(D\phi^t \cdot u)$ . Finally we define

$$\kappa(\gamma_0) = \liminf_{T \rightarrow +\infty} \left( \inf_{x, u} \frac{\tilde{\Theta}_\sigma(T, u) - \tilde{\Theta}_\sigma(0, u)}{\text{link}(k(T, x; \Sigma), \gamma_0)} \right) \quad (5)$$

where the infimum is taken over all pairs  $(x, u)$ , where  $x \in S^3 \setminus \gamma_0$  and  $u \in (\xi_x \setminus 0)/\mathbb{R}_+$ .

*Remark 1.12.* In Sections A.3 and A.4, we will prove that  $\kappa(\gamma_0)$  does not depend on  $\Sigma$  or on  $\sigma$ ; see Corollary A.10 and Lemma A.11.

Our abstract result reads as follows.

**Theorem 1.13.** *Let  $\lambda$  be a dynamically convex contact form on  $S^3$ , and  $\gamma_0$  be an unknotted periodic Reeb orbit with self-linking number  $-1$ . If  $\kappa(\gamma_0) > 2\pi$  then the Reeb flow of  $\lambda$  is right-handed.*

As mentioned before, a rich source of dynamically convex contact forms are the strictly convex energy levels in  $\mathbb{R}^4$ . Consider  $\mathbb{R}^4$  equipped with coordinates  $(q_1, p_1, q_2, p_2)$  and symplectic form  $\omega_0 = d\lambda_0$ , where  $\lambda_0$  is the 1-form

$$\lambda_0 = \frac{1}{2}(p_1 dq_1 - q_1 dp_1 + p_2 dq_2 - q_2 dp_2).$$

It follows that  $\lambda_0$  is a contact form on hypersurfaces that are star-shaped with respect to the origin.

Let  $C \subset \mathbb{R}^4$  be a smooth convex body with the origin in the interior. Denote by  $\nu_C$  the *gauge function* of  $C$ , which is defined to be the unique 1-homogeneous function  $\nu_C$  satisfying  $\partial C = \nu_C^{-1}(1)$ . Note that  $\nu_C$  is continuous on  $\mathbb{R}^4$  and smooth on  $\mathbb{R}^4 \setminus \{0\}$ . The Hessian of  $\nu_C^2$ , denoted by  $D^2\nu_C^2$ , defines a 0-homogeneous matrix-valued function on  $\mathbb{R}^4 \setminus \{0\}$ . Consider

$$K_{\min}^C = \inf_{z \in \mathbb{R}^4 \setminus \{0\}} \min\{\mu \mid \mu \text{ is an eigenvalue of } D^2\nu_C^2(z)\}. \quad (6)$$

Convexity implies  $K_{\min}^C \geq 0$ . We call  $\partial C$  strictly convex if  $K_{\min}^C > 0$ . The Hamiltonian vector field  $X_H$  of  $H = \nu_C^2$ , defined on  $\mathbb{R}^4 \setminus \{0\}$  by  $i_{X_H}\omega_0 = -dH$ , satisfies  $i_{X_H}\lambda_0 = H$ . Hence  $X_H$  restricts to  $\partial C$  as the Reeb vector field of the contact form induced by  $\lambda_0$ . In [HWZ98] Hofer, Wysocki and Zehnder proved that this contact form is dynamically convex provided  $\partial C$  is strictly convex. In particular, there are  $\partial$ -strong disk-like GSSs for the Hamiltonian dynamics on  $\partial C$  under the assumption of strict convexity.

**Theorem 1.14.** *Assume that  $\partial C$  is strictly convex, and let  $D \subset \partial C$  be a disk-like  $\partial$ -strong global surface of section. Denote by  $\tau_{\min}(D) > 0$  the infimum of the first return time on  $D$ . If the inequality*

$$K_{\min}^C \tau_{\min}(D) > \pi \quad (7)$$

*holds, then the Hamiltonian flow on  $\partial C$  is right handed.*

*Remark 1.15.* If  $C$  is the unit Euclidean ball in  $\mathbb{R}^4$ , then the gauge function is the Euclidean norm,  $K_{\min}^C = 2$ , the Reeb flow on  $\partial C$  is  $\pi$ -periodic and equal to the Hopf flow. In particular,  $\tau_{\min}(D) = \pi$  for any disk-like global surface of section  $D \subset \partial C$ . If  $C'$  is  $C^2$ -close to  $C$  then it can be proved that the Reeb flow on  $\partial C'$  admits a  $\partial$ -strong disk-like global surface of section  $D'$  with return time uniformly close to  $\pi$ . Since the curvatures of  $C'$  are close to those of  $C$  we get

$$K_{\min}^{C'} \tau_{\min}(D') \sim K_{\min}^C \tau_{\min}(D) = 2\pi \quad \Rightarrow \quad K_{\min}^{C'} \tau_{\min}(D') > \pi.$$

By Theorem 1.14 the Reeb flow on  $\partial C'$  is right-handed, as claimed in [Ghy09].

*Remark 1.16.* One gets restrictions of a contact topological nature from the argument given in Appendix B. In the case of Reeb flows, the proof shows that all finite collections of periodic orbits bind open book decompositions that support the contact structure in the sense of Giroux, with pages that are global surfaces of section.

*Organisation of the paper.* In Section 2 we give the definition of right-handedness and prove Theorem 1.13. We also show how Theorem 1.14 follows from Theorem 1.13. Section 3 is devoted to the study of geodesic flows on  $S^2$ : we show how the claimed pinching condition enables us to verify the quantitative condition of Theorem 1.13. The proof uses comparison theorems from Riemannian geometry. In Appendix A, we study the properties of global surfaces of section in generic position and deduce that the quantity  $\kappa(\gamma_0)$  is independent of the choice of the  $\partial$ -strong disk-like global surface of section spanned by  $\gamma_0$ , and of the choice of a global symplectic trivialization. For the sake of completeness, a proof of Theorem 1.1 is given in Appendix B. Finally, in Appendix C, we discuss technical properties of  $\partial$ -strong disk-like global surfaces of section that are used in the proof of Theorem 1.13.

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## 2 Right-handedness from dynamical pinching

### 2.1 Transverse rotation numbers

Let  $\gamma$  be a non-constant periodic orbit of a smooth flow  $\phi^t$  defined on an oriented 3-manifold  $M$ . Denote by  $T > 0$  its primitive period, and think of  $\gamma$  as a map  $\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow M$ . Consider coordinates  $(t, z = x + iy) \in \mathbb{R}/T\mathbb{Z} \times \mathbb{C}$  defined on a small tubular neighbourhood  $N$  of  $\gamma$  such that  $dt \wedge dx \wedge dy > 0$  and  $\phi^t(\gamma(0)) = (t, 0)$ . We shall refer to such coordinates as tubular coordinates around  $\gamma$ . For every  $\theta_0 \in \mathbb{R}$  consider the continuous real valued function  $\theta(t)$  defined by

$$D\phi^t(0, 0) \cdot (0, e^{i\theta_0}) \in \mathbb{R}(1, 0) + \mathbb{R}_+(0, e^{i\theta(t)}) \quad \theta(0) = \theta_0$$



If  $y \in H^1(N \setminus \gamma, \mathbb{R})$  is homologous to  $p \, dt + q \, d\theta$  then we define

$$\rho^y(\gamma) = \frac{T}{2\pi} \left( p + q \lim_{t \rightarrow +\infty} \frac{\theta(t)}{t} \right). \quad (8)$$

This number is called the transverse rotation number of  $\gamma$  with respect to  $y$ . It turns out that  $\rho^y(\gamma)$  does not depend on the choice of tubular coordinates or on the initial condition  $\theta_0$ ; see [Hry20, section 2].

*Remark 2.1.* In the notation above, suppose that  $\phi^t$  is the Reeb flow of a contact form  $\lambda$ , and let  $\sigma$  be a symplectic trivialization of  $(\xi, d\lambda)$  along  $\gamma$ . Denote by  $\gamma_\sigma$  an oriented loop in  $N \setminus \gamma$  obtained by pushing  $\gamma$  in the direction of  $\sigma$ . If  $N$  is small enough then  $d\lambda$  defines an area form on any meridional disk  $D$ . Orient  $D$  by  $d\lambda$ . There is a unique class  $y_\sigma \in H^1(N \setminus \gamma, \mathbb{Z})$  determined by  $\langle y_\sigma, \partial D \rangle = 1$ ,  $\langle y_\sigma, \gamma_\sigma \rangle = 0$ . If no transverse Floquet multiplier of  $\gamma$  is a root of unit of order  $n \geq 1$  then one says that the  $n$ -th iterate  $\gamma^n$  of  $\gamma$  is non-degenerate, and defines

$$\mu_{\text{CZ}}^\sigma(\gamma^n) = 2[n\rho^{y_\sigma}(\gamma)] + 1. \quad (9)$$

Otherwise,  $\mu_{\text{CZ}}^\sigma(\gamma^n)$  is defined to be the lowest possible value of the right-hand side above, obtained from small  $C^2$ -perturbations of  $\lambda$  that keep  $\gamma$  as a periodic Reeb orbit with  $\gamma^n$  non-degenerate. The inequality  $\mu_{\text{CZ}}^\sigma(\gamma) \geq 3$  is equivalent to  $\rho^{y_\sigma}(\gamma) > 1$ .

Suppose further that  $M$  is a homology 3-sphere. Consider any oriented Seifert surface  $S$  spanned by  $\gamma$ . We require that  $\partial S = \gamma$  including orientations, when  $\gamma$  is oriented by the flow. As before, orient the meridional disk  $D \subset N$  by  $d\lambda$ . Let  $S^* \in H^1(M \setminus \gamma, \mathbb{Z})$  denote the class dual to  $S$ . Since  $M$  is a homology 3-sphere, the class  $S^*$  is independent of  $S$ . In fact,  $\langle S^*, \beta \rangle = \text{link}(\gamma, \beta)$  for any oriented loop  $\beta$  in  $M \setminus \gamma$ . After restricting to  $N \setminus \gamma$  we can view it as a class in  $H^1(N \setminus \gamma, \mathbb{Z})$ .

**Definition 2.2.** Under the assumption that  $M$  is a homology 3-sphere, we call  $\rho^y(\gamma)$  the transverse rotation number of  $\gamma$  in a Seifert framing, where  $y$  is the cohomology class dual to some (hence any) oriented Seifert surface  $S$  spanned by  $\gamma$ .

From now on assume that  $M$  is a homology three-sphere and that  $\phi^t$  has no rest points. Let  $\gamma_1, \dots, \gamma_n$  be a collection of periodic orbits oriented along the flow, and consider the oriented link  $L = \gamma_1 \cup \dots \cup \gamma_n$ . Let  $\Sigma_i$  be an oriented Seifert surface satisfying  $\partial \Sigma_i = \gamma_i$ , orientations included. Let  $\Sigma$  be an oriented Seifert surface satisfying  $\partial \Sigma = L$ , orientations included. Consider small tubular neighbourhoods  $N_i$  of  $\gamma_i$  with tubular coordinates as above. Denote by  $\Sigma^* \in H^1(M \setminus L, \mathbb{R})$  and  $\Sigma_i^* \in H^1(M \setminus \gamma_i, \mathbb{R})$  the classes dual to  $\Sigma$  and  $\Sigma_i$  respectively. Each  $\Sigma_i^*$  restricts to a class in  $H^1(N_i \setminus \gamma_i, \mathbb{R})$  still denoted by  $\Sigma_i^*$  with no fear of ambiguity. Similarly  $\Sigma^*$  restricts to a class in  $H^1(N_i \setminus \gamma_i, \mathbb{R})$  for every  $i$ , all of which are denoted by  $\Sigma^*$  with no fear of ambiguity.

**Lemma 2.3.** For every  $i$  we have  $\rho^{\Sigma^*}(\gamma_i) = \rho^{\Sigma_i^*}(\gamma_i) + \frac{1}{2\pi} \sum_{j \neq i} \text{link}(\gamma_i, \gamma_j)$ .

*Proof.* Let  $(t, z = |z|e^{i\theta}) \in \mathbb{R}/T_i\mathbb{Z} \times \mathbb{C}$  be tubular coordinates on a small tubular neighbourhood  $N_i$  of  $\gamma_i$ . With  $\epsilon > 0$  small we consider the loops  $e_i(t) = (t, \epsilon)$ ,  $f_i(\theta) = (0, \epsilon e^{i\theta})$ . Then  $\{e_i, f_i\}$  is a basis for  $H_1(N_i \setminus \gamma_i, \mathbb{Z})$  whose dual basis is  $\{dt/T_i, d\theta/2\pi\}$ . Then on  $N_i \setminus \gamma_i$  one has

$$\begin{aligned}\Sigma_i^* &\equiv \langle \Sigma_i^*, e_i \rangle \frac{dt}{T_i} + \langle \Sigma_i^*, f_i \rangle \frac{d\theta}{2\pi} = \langle \Sigma_i^*, e_i \rangle \frac{dt}{T_i} + \frac{d\theta}{2\pi} \\ \Sigma^* &\equiv \langle \Sigma^*, e_i \rangle \frac{dt}{T_i} + \langle \Sigma^*, f_i \rangle \frac{d\theta}{2\pi} = \langle \Sigma^*, e_i \rangle \frac{dt}{T_i} + \frac{d\theta}{2\pi}\end{aligned}$$

Denote by  $\nu_i$  a section of the normal bundle of  $\Sigma_i$  along  $\gamma_i$ . Let  $\gamma'_i$  be the loop obtained by pushing  $\gamma_i$  in the direction of  $\nu_i$ . Note that  $\gamma'_i$  gets an orientation from  $\gamma_i$ . Then

$$\gamma'_i \equiv \left\langle \frac{dt}{T_i}, \gamma'_i \right\rangle e_i + \left\langle \frac{d\theta}{2\pi}, \gamma'_i \right\rangle f_i = e_i + \left\langle \frac{d\theta}{2\pi}, \gamma'_i \right\rangle f_i$$

From these formulas it follows that

$$\text{int}(\gamma'_i, \Sigma) - \text{int}(\gamma'_i, \Sigma_i) = \langle \Sigma^*, \gamma'_i \rangle - \langle \Sigma_i^*, \gamma'_i \rangle = \langle \Sigma^*, e_i \rangle - \langle \Sigma_i^*, e_i \rangle$$

Consider the 2-cycle  $S = \Sigma - \Sigma_1 - \dots - \Sigma_n$ . Since the ambient space is a homology sphere (over  $\mathbb{Z}$ ),  $S$  is a boundary and we get

$$\begin{aligned}0 &= \text{int}(\gamma'_i, S) = \text{int}(\gamma'_i, \Sigma) - \text{int}(\gamma'_i, \Sigma_i) - \sum_{j \neq i} \text{link}(\gamma'_i, \gamma_j) \\ &= \langle \Sigma^*, e_i \rangle - \langle \Sigma_i^*, e_i \rangle - \sum_{j \neq i} \text{link}(\gamma_i, \gamma_j)\end{aligned}\tag{10}$$

Hence, if  $\theta(t)$  denotes a lift of the polar angle of the linearised flow along  $\gamma_i$  in the given tubular coordinates we get

$$\begin{aligned}\rho^{\Sigma^*}(\gamma_i) &= \frac{T_i}{2\pi} \left( \frac{\langle \Sigma^*, e_i \rangle}{T_i} + \frac{1}{2\pi} \lim_{t \rightarrow +\infty} \frac{\theta(t)}{t} \right) \\ &= \frac{T_i}{2\pi} \left( \frac{\langle \Sigma_i^*, e_i \rangle}{T_i} + \frac{1}{T_i} \sum_{j \neq i} \text{link}(\gamma_i, \gamma_j) + \frac{1}{2\pi} \lim_{t \rightarrow +\infty} \frac{\theta(t)}{t} \right) \\ &= \rho^{\Sigma_i^*}(\gamma_i) + \frac{1}{2\pi} \sum_{j \neq i} \text{link}(\gamma_i, \gamma_j)\end{aligned}\tag{11}$$

as desired.  $\square$

**Corollary 2.4.** *If  $n \geq 2$  and  $\text{link}(\gamma_i, \gamma_j) \geq 1$  for all  $i \neq j$  then  $\rho^{\Sigma^*}(\gamma_i) > \rho^{\Sigma_i^*}(\gamma_i)$  for every  $i$ .*

*Remark 2.5.* Let the periodic orbit  $\gamma$  be oriented by the flow, and let  $\Sigma, \hat{\Sigma}$  be oriented Seifert surfaces such that  $\partial\Sigma = \gamma$ ,  $\partial\hat{\Sigma} = \gamma$  (including orientations). Lemma 2.3 proves the previously claimed fact that  $\rho^\Sigma(\gamma) = \rho^{\hat{\Sigma}}(\gamma)$ .

## 2.2 Right-handedness

Fix a smooth non-vanishing vector field  $X$  on an oriented homology 3-sphere, and denote its flow by  $\phi^t$ . Let  $\mathcal{P}$  be the set of  $\phi^t$ -invariant Borel probability measures. Denote by  $\mathcal{R}$  the set of recurrent points, and consider the following measurable set:

$$R = \{(x, y) \in \mathcal{R} \times \mathcal{R} \mid \phi^{\mathbb{R}}(x) \cap \phi^{\mathbb{R}}(y) = \emptyset\}. \quad (12)$$

Let  $\mu_1, \mu_2 \in \mathcal{P}$  be ergodic, and denote by  $\mu_1 \times \mu_2$  the product measure. There are two cases:

(A)  $\mu_1 \times \mu_2(R) = 1$ .

(B)  $\mu_1 \times \mu_2(R) = 0$  and  $\text{supp}(\mu_1) \cup \text{supp}(\mu_2) \subset \gamma$  for some periodic orbit  $\gamma$ .

Each case needs to be treated separately. Fix an auxiliary Riemannian metric  $g$ .

**Case A.** Consider  $(p, q) \in R$  and let  $\mathcal{S}(p, q)$  denote the set of ordered pairs of sequences  $(\{T_n\}, \{S_n\})$  satisfying  $T_n, S_n \rightarrow +\infty$ ,  $\phi^{T_n}(p) \rightarrow p$  and  $\phi^{S_n}(q) \rightarrow q$ . For  $n$  large enough let  $\alpha_n$  and  $\beta_n$  be the (unique) shortest geodesic arcs from  $\phi^{T_n}(p)$  to  $p$  and from  $\phi^{S_n}(q)$  to  $q$ , respectively. Consider  $C^1$ -small<sup>1</sup> perturbations  $\hat{\alpha}, \hat{\beta}$  of  $\alpha_n, \beta_n$ , keeping end points fixed, such that the closed loops  $k(T_n, p)$  and  $k(S_n, q)$  obtained by concatenating  $\hat{\alpha}$  to  $\phi^{[0, T_n]}(p)$  and  $\hat{\beta}$  to  $\phi^{[0, S_n]}(q)$ , respectively, do not intersect each other. Define

$$\text{link}_-(\phi^{[0, T_n]}(p), \phi^{[0, S_n]}(q)) = \liminf_{\hat{\alpha} \xrightarrow{C^1} \alpha_n, \hat{\beta} \xrightarrow{C^1} \beta_n} \text{link}(k(T_n, p), k(S_n, q)) \quad (13)$$

and

$$\ell(p, q) = \inf_{(\{T_n\}, \{S_n\}) \in \mathcal{S}(p, q)} \liminf_{n \rightarrow \infty} \frac{1}{T_n S_n} \text{link}_-(\phi^{[0, T_n]}(p), \phi^{[0, S_n]}(q)). \quad (14)$$

Note that (13) and (14) belong to  $[-\infty, +\infty]$ . One says that  $\mu_1, \mu_2$  are positively linked if for  $\mu_1 \times \mu_2$ -almost all points  $(p, q)$  in  $R$  the inequality  $\ell(p, q) > 0$  holds.

**Case B.** One says that  $\mu_1, \mu_2$  are positively linked if the transverse rotation number of the periodic orbit  $\gamma$  containing the supports of  $\mu_1, \mu_2$  computed in a Seifert framing is strictly positive.

**Definition 2.6.** The flow  $\phi^t$  is said to be right-handed if all pairs of ergodic measures in  $\mathcal{P}$  are positively linked.

*Remark 2.7.* The above definition is equivalent to the one explained in [Ghy09]. Of course, Ghys defines right-handedness in a much more elegant way by explaining that in case A the ergodicity assumption can be used to prove that for  $\mu_1 \times \mu_2$ -almost all  $(p, q) \in R$ , all possible sequences

$$\frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n}$$

---

<sup>1</sup>The assumption that  $\hat{\alpha}, \hat{\beta}$  are  $C^1$ -close to  $\alpha_n, \beta_n$  is important. Being  $C^0$ -close is not enough.

as above will converge to a common limit. This limit is defined to be the value of the quadratic linking form evaluated at the pair  $(\mu_1, \mu_2)$ . There is also a way of assigning a number in case **B**. The advantage of the definition explained here is that it avoids dealing with details on the existence of the quadratic linking form.

## 2.3 Proof of Theorem 1.13

We fix a dynamically convex contact form  $\lambda$  on  $S^3$ , denote by  $X$  the associated Reeb vector field and by  $\phi^t$  the Reeb flow. The following technical statement, which is based on the main result from [HWZ98], is proved in appendix C.

**Proposition 2.8.** *Let  $\lambda$  be a dynamically convex contact form on  $S^3$ , and denote by  $X$  the Reeb vector field of  $\lambda$ . Let  $\gamma_0$  be any unknotted periodic Reeb orbit with self-linking number  $-1$ . Denote the primitive period by  $T_0 > 0$ . There exists a map of class  $C^\infty$*

$$\Psi : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \rightarrow S^3 \quad (15)$$

with the following properties:

- (a)  $\Psi(0, e^{is}) = \gamma_0(T_0 s / 2\pi)$  for all  $s \in \mathbb{R}/2\pi\mathbb{Z}$ , and  $\Psi(0, \cdot) : \mathbb{D} \rightarrow S^3$  is an embedding that defines a  $\partial$ -strong global surface of section for the flow of  $X$ .
- (b)  $\Psi$  defines an orientation preserving diffeomorphism  $\mathbb{R}/\mathbb{Z} \times \mathring{\mathbb{D}} \rightarrow S^3 \setminus \gamma_0$ .
- (c) There exists a smooth vector field  $W$  on  $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$  that coincides with the pull-back of  $X$  by  $\Psi|_{\mathbb{R}/\mathbb{Z} \times \mathring{\mathbb{D}}}$  on  $\mathbb{R}/\mathbb{Z} \times \mathring{\mathbb{D}}$ , and is tangent to  $\mathbb{R}/\mathbb{Z} \times \partial\mathbb{D}$ .
- (d) For every  $t \in \mathbb{R}/\mathbb{Z}$  the disk  $\{t\} \times \mathbb{D}$  is transverse to  $W$  up to the boundary, and defines a global section for the flow of  $W$ .
- (e) There is non-vanishing vector field  $Z$  on  $S^3$  satisfying  $i_Z \lambda = 0$  with the following property. Let  $Z_0$  be the unique smooth vector field on  $\mathring{\mathbb{D}}$  defined by

$$Z(\Psi(0, z)) - d\Psi(0, z) \cdot Z_0(z) \in \mathbb{R}X(\Psi(0, z)).$$

If  $\phi : \mathring{\mathbb{D}} \rightarrow \mathbb{R}$  is continuous and satisfies  $Z_0 = |Z_0|e^{i\phi}$ , then  $\phi \in L^\infty(\mathring{\mathbb{D}})$ .

Denote by  $\varphi^t$  the flow of  $W$ , so that on  $\mathbb{R}/\mathbb{Z} \times \mathring{\mathbb{D}}$  we have

$$\varphi^t = \Psi^{-1} \circ \phi^t \circ \Psi.$$

The first return time back to  $\{0\} \times \mathbb{D}$  is

$$\tau : \mathbb{D} \rightarrow (0, +\infty) \quad \tau(z) = \inf\{t > 0 \mid \varphi^t(0, z) \in \{0\} \times \mathbb{D}\}. \quad (16)$$

It follows from (c) and (d) that  $\tau$  is bounded away from zero and bounded from above. Denote

$$0 < \tau_{\min} = \min \tau \quad \tau_{\max} = \max \tau < +\infty$$

The return map is denoted by

$$h : \mathbb{D} \rightarrow \mathbb{D} \quad (0, h(z)) = \varphi^{\tau(z)}(0, z) \quad (17)$$

**Lemma 2.9.** *For every  $c \in (0, 1)$  there exists  $t : [0, 1] \times \mathbb{D} \rightarrow [0, +\infty)$  smooth such that:*

- (i)  $t(0, z) = 0$  and  $t(1, z) = \tau(z)$  for all  $z \in \mathbb{D}$ .
- (ii)  $D_1 t(s, z) \geq c\tau_{\min}$  for all  $(s, z) \in [0, 1] \times \mathbb{D}$ .
- (iii)  $\exists 0 < \epsilon \ll 1$  such that  $D_1 t(s, z) = \tau_{\max}$  if  $s \in [0, \epsilon) \cup (1 - \epsilon, 1]$ , for all  $z \in \mathbb{D}$ .

*Proof.* Fix  $c \in (0, 1)$  and let  $\delta := c\tau_{\min}$ . Consider the following set

$$\Omega := \{(t, z) \mid z \in \mathbb{D}, 0 \leq t \leq \tau(z)\} \subset \mathbb{R} \times \mathbb{D},$$

and define the piecewise smooth function  $s : \Omega \rightarrow [0, 1]$  by

$$s(t, z) := \begin{cases} \frac{t}{\tau_{\max}} & 0 \leq t \leq \tau(z) - \delta \\ \frac{\tau(z) - \delta}{\tau_{\max}} + \frac{\tau_{\max} - \tau(z) + \delta}{\delta\tau_{\max}} (t - \tau(z) + \delta) & \text{for } \tau(z) - \delta \leq t \leq \tau(z). \end{cases}$$

Let us now define  $\hat{t} : [0, 1] \times \mathbb{D} \rightarrow [0, +\infty)$  by the identity

$$\hat{t}(s(t, z), z) = t.$$

Observe that  $\hat{t}(0, z) = 0$  and  $\hat{t}(1, z) = \tau(z)$ . For every  $(s, z) \in [0, 1] \times \mathring{\mathbb{D}}$  it holds

$$\min \{D_1^+ \hat{t}(s, z), D_1^- \hat{t}(s, z)\} \geq \min \left\{ \tau_{\max}, \frac{\delta\tau_{\max}}{\tau_{\max} - \tau(z) + \delta} \right\} = \frac{\delta\tau_{\max}}{\tau_{\max} - \tau(z) + \delta}$$

where  $D_1^\pm$  denote one-sided derivatives in the first variable. In particular

$$\frac{\delta\tau_{\max}}{\tau_{\max} - \tau(z) + \delta} > \delta = c\tau_{\min}.$$

Using a parametrised splining method we get the a smooth  $t : [0, 1] \times \mathbb{D} \rightarrow [0, +\infty)$  satisfying  $D_1 t(s, z) \in [D_1^+ \hat{t}(s, z), D_1^- \hat{t}(s, z)]$ ,  $t(0, z) = 0$  and  $t(1, z) = \tau(z)$ . In particular  $D_1 t(s, z) \geq c\tau_{\min}$  for all  $(s, z) \in [0, 1] \times \mathbb{D}$ . Extend  $t$  to  $[0, +\infty) \times \mathbb{D}$  by

$$t(s, z) = \sum_{j=0}^{\lfloor s \rfloor - 1} \tau \circ h^j(z) + t(s - \lfloor s \rfloor, h^{\lfloor s \rfloor}(z)) \quad (s > 1) \quad (18)$$

It follows that  $t$  is continuous on  $[0, +\infty) \times \mathbb{D}$  and smooth on  $[n, n+1] \times \mathbb{D}$  for each  $n \geq 0$ , that  $D_1 t$  has only discontinuities possibly at  $\mathbb{N} \times \mathbb{D}$ , and that  $D_1 t \geq c\tau_{\min}$  on  $[n, n+1] \times \mathring{\mathbb{D}}$  for every  $n \geq 0$ . Again by a parametrized splining method, we can ask also that the extended function  $t$  is smooth at  $\mathbb{N} \times \mathbb{D}$  and that there exists  $0 < \epsilon \ll 1$  such that  $D_1 t(s, z) = \tau_{\max}$  if  $s \in [0, \epsilon) \cup (1 - \epsilon, 1]$ .  $\square$

Consider disks  $D_s$  defined by

$$D_s = \{\varphi^{t(s,z)}(0, z) \mid z \in \mathbb{D}\} \quad (19)$$

where  $s$  varies on  $[0, 1]$ . Note that  $D_1 = D_0 = \{0\} \times \mathbb{D}$  in view of (17). Hence we get a continuous  $\mathbb{R}/\mathbb{Z}$ -family  $\{D_s\}_{s \in \mathbb{R}/\mathbb{Z}}$  of smooth disks.

**Lemma 2.10.** *The family  $\{D_s\}_{s \in \mathbb{R}/\mathbb{Z}}$  defines a smooth foliation of  $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$ .*

*Proof.* It suffices to study smoothness of the foliation near the disk  $D_0$ . We claim that there exists  $\epsilon > 0$  small enough such that  $D_{1+u} = \varphi^{\tau_{\max} u}(D_0)$  for all  $u \in (-\epsilon, 0]$ , and  $D_u = \varphi^{\tau_{\max} u}(D_0)$  for all  $u \in [0, \epsilon)$ . In fact, let  $\epsilon > 0$  be small enough so that (iii) in Lemma 2.9 holds. It follows from (i) and (iii) in Lemma 2.9 that  $t(s, z) = \tau_{\max} s$  for all  $s \in [0, \epsilon)$  and that  $t(s, z) = \tau(z) + \tau_{\max}(s - 1)$  for all  $s \in (1 - \epsilon, 1]$ , for all  $z \in \mathbb{D}$ . In particular, the  $D_u = \varphi^{\tau_{\max} u}(D_0)$  holds for all  $u \in [0, \epsilon)$ . With  $u \in (-\epsilon, 0]$  we have

$$\begin{aligned} p \in D_{1+u} &\Leftrightarrow p = \varphi^{t(1+u,z)}(0, z) \quad (\text{for some } z \in \mathbb{D}) \\ &= \varphi^{\tau(z) + \tau_{\max}(1+u-1)}(0, z) \\ &= \varphi^{\tau_{\max} u}(\varphi^{\tau(z)}(0, z)) \\ &= \varphi^{\tau_{\max} u}(0, h(z)) \in \varphi^{\tau_{\max} u}(D_0) \end{aligned} \quad (20)$$

It follows that the disks fit around  $D_0$  as the smooth foliation  $\{\varphi^{\tau_{\max} u}(D_0)\}_{|u| < \epsilon}$ .  $\square$

**Lemma 2.11.** *In addition to properties (a)-(e) from Proposition 2.8 one can assume that the map  $\Psi$  in (15) satisfies in addition the following property.*

(f) *If  $\varphi^t$  denotes the flow of  $W$  then for every  $c \in (0, 1)$  there exists a smooth function  $t : [0, 1] \times \mathbb{D} \rightarrow [0, +\infty)$  satisfying properties (i)-(iii) in Lemma 2.9, and a smooth isotopy  $\{h_s : \mathbb{D} \rightarrow \mathbb{D}\}_{s \in [0, 1]}$  such that*

$$\varphi^{t(s,z)}(0, z) = (s, h_s(z)) \quad (21)$$

*for all  $(s, z) \in [0, 1] \times \mathbb{D}$ .*

*Proof.* As a consequence of lemmas 2.9 and 2.10 one finds an orientation preserving diffeomorphism  $\Phi : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{D}$  satisfying  $\Phi(\{s\} \times \mathbb{D}) = D_s$  for all  $s \in \mathbb{R}/\mathbb{Z}$ . If  $\widetilde{W} = \Phi^* W$  and  $\widetilde{\varphi}^t$  denotes the flow of  $\widetilde{W}$  then we find a smooth isotopy  $\{h_s\}_{s \in [0, 1]}$  of self-diffeomorphisms of  $\mathbb{D}$  satisfying  $h_0 = id$ ,  $h_1 = h$ , uniquely determined by  $\widetilde{\varphi}^{t(s,z)}(0, z) = (s, h_s(z))$ . Here  $h$  is the return map (17). The proof is concluded if we revert the notation from  $\widetilde{\varphi}^t$ ,  $\widetilde{W}$  and  $\Phi \circ \Psi$  back to  $\varphi^t$ ,  $W$  and  $\Psi$ .  $\square$

**Notation 2.12.** If  $z : [a, b] \rightarrow \mathbb{C} \setminus \{0\}$  is continuous then we denote

$$\text{wind}_{[a,b]}(z) = \frac{\theta(b) - \theta(a)}{2\pi} \quad (22)$$

where  $\theta : [a, b] \rightarrow \mathbb{R}$  is continuous and satisfies  $z(s) = |z(s)|e^{i\theta(s)}$ .

Up to rotating the coordinate system on  $S^3 \setminus \gamma_0$  obtained via the map  $\Psi$ , i.e. up to changing  $\Psi(t, z)$  by  $\Psi(t, e^{2\pi kt} z)$  for some  $k \in \mathbb{Z}$ , we may assume that the following property holds: for any pair of loops  $\alpha, \beta : \mathbb{R}/\mathbb{Z} \rightarrow \mathring{\mathbb{D}}$  satisfying  $\alpha(t) \neq \beta(t) \forall t$ , the linking number in  $S^3$  between the loops  $t \mapsto \Psi(t, \alpha(t))$  and  $t \mapsto \Psi(t, \beta(t))$  is equal to  $\text{wind}_{[0,1]}(\beta(t) - \alpha(t))$ . From now on we assume that  $\Psi$  satisfies this property.

**Lemma 2.13.** *If  $\kappa(\gamma_0) > 2\pi$  then every periodic orbit has strictly positive transverse rotation in a Seifert framing.*

*Proof.* Let  $\gamma \subset S^3 \setminus \gamma_0$  be a periodic orbit with primitive period  $T > 0$ . Fix  $p \in \gamma$  and  $u$  a non-zero vector of the contact plane at  $p$ . Since any periodic orbit that links once with  $\gamma_0$  has self-linking number  $-1$ , see [Hry14, section 4], it follows that if  $\eta$  is a section of the contact structure along  $\gamma$  that has winding number zero in a Seifert framing  $\tau$  for  $\gamma$ , then  $\eta$  makes  $\text{link}(\gamma, \gamma_0)$  positive turns in a global symplectic frame  $\sigma$  of the contact structure. Hence for any  $k \in \mathbb{N}$

$$\Delta \tilde{\Theta}_\tau(kT, u) = \Delta \tilde{\Theta}_\sigma(kT, u) - 2\pi k \text{link}(\gamma, \gamma_0)$$

where, for  $* = \tau, \sigma$ ,  $\Delta \tilde{\Theta}_*(t, u)$  denotes the angular variations of  $s \in [0, t] \mapsto D\phi^s(p)u$  in the frame  $*$ . Let  $\epsilon > 0$  satisfy  $\kappa(\gamma_0) - \epsilon > 2\pi$ . In view of (5), we find  $k_*$  such that

$$k \geq k_* \quad \Rightarrow \quad \frac{\Delta \tilde{\Theta}_\sigma(kT, u)}{k \text{link}(\gamma, \gamma_0)} \geq \kappa(\gamma_0) - \epsilon.$$

Hence the transverse rotation number  $\rho(\gamma)$  in a Seifert framing is

$$\begin{aligned} \frac{T}{2\pi} \lim_{k \rightarrow \infty} \frac{\Delta \tilde{\Theta}_\tau(kT, u)}{kT} &= \lim_{k \rightarrow \infty} \frac{\Delta \tilde{\Theta}_\tau(kT, u)}{2\pi k} \\ &= \lim_{k \rightarrow \infty} \frac{\Delta \tilde{\Theta}_\sigma(kT, u) - 2\pi k \text{link}(\gamma, \gamma_0)}{2\pi k} \\ &\geq \lim_{k \rightarrow \infty} \frac{(\kappa(\gamma_0) - \epsilon)k \text{link}(\gamma, \gamma_0) - 2\pi k \text{link}(\gamma, \gamma_0)}{2\pi k} \\ &= \frac{1}{2\pi} (\kappa(\gamma_0) - 2\pi - \epsilon) \text{link}(\gamma, \gamma_0) > 0 \end{aligned} \tag{23}$$

as desired. The only periodic orbit left to be analysed is  $\gamma_0$ , but the positivity of its transverse rotation number (in a Seifert framing) follows from dynamical convexity and the fact that  $\gamma_0$  has self-linking number  $-1$ .  $\square$

Let  $I = \{f_t\}_{t \in [0,1]} \subset \text{Diff}^1(\mathring{\mathbb{D}})$  be a  $C^1$ -isotopy joining  $f_0 = \text{id}_{\mathring{\mathbb{D}}}$  to  $f_1 = f$ . Extend the isotopy on  $\mathbb{R}_+ = [0, +\infty)$  by asking that  $f_{1+t} = f_t \circ f$ .

**Theorem 2.14** ([Flo19b]). *Let  $x, y \in \mathring{\mathbb{D}}$ ,  $x \neq y$ , and  $T \geq 0$  be fixed arbitrarily. Then*

$$\text{wind}_{[0,T]}(f_t(y) - f_t(x)) = \text{wind}_{[0,T]}(Df_t(z)(y - x))$$

*holds for some  $z$  in the segment  $[x, y]$  joining  $x$  to  $y$ .*

From the standing assumption that  $\kappa(\gamma_0) > 2\pi$  we can choose  $0 < \epsilon \ll 1$  such that

$$\kappa(\gamma_0) - \epsilon > 2\pi. \quad (24)$$

Our main technical statement reads as follows.

**Proposition 2.15.** *Let  $p, q \in \Psi(\{0\} \times \mathring{\mathbb{D}})$  be recurrent points for  $\phi^t$  in distinct trajectories. Then*

$$\ell(p, q) \geq \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi\tau_{\max}^2} > 0 \quad (25)$$

where  $\ell(p, q)$  is the number (14).

From now on we are concerned with the proof of Proposition 2.15. With  $n$  large enough consider  $k(T_n, p), k(S_n, q)$  loops as constructed in case **A** described in subsection 2.2. Their construction is as follows. Fix any auxiliary Riemannian metric on  $S^3$ , for instance, we can use a metric which is equal to  $\Psi_*(ds^2 + |dz|^2)$  on a neighbourhood of the set  $\{p, q\}$  fixed *a priori*. Denote the coordinates on the domain  $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$  of the map  $\Psi$  by  $(s, z)$ . The sequences  $T_n, S_n$  satisfy  $T_n, S_n \rightarrow +\infty$ ,  $\phi^{T_n}(p) \rightarrow p$  and  $\phi^{S_n}(q) \rightarrow q$ . Hence we find real numbers  $s_n, \hat{s}_n \rightarrow 0$  and  $z_n, \hat{z}_n \in \mathring{\mathbb{D}}$  such that  $\phi^{T_n}(p) = \Psi(s_n, z_n)$ ,  $\phi^{S_n}(q) = \Psi(\hat{s}_n, \hat{z}_n)$ . When  $n$  is large enough the short closing paths  $\alpha_n$  and  $\beta_n$  from  $\phi^{T_n}(p)$  to  $p$  and from  $\phi^{S_n}(q)$  to  $q$ , respectively, are just straight line segments in  $(s, z)$  coordinates, in view of the form of the metric. There exist  $m_n(p), m_n(q) \geq 1$  and

$$t_0^p = 0 < t_1^p < \dots < t_{m_n(p)}^p \leq T_n \quad t_0^q = 0 < t_1^q < \dots < t_{m_n(q)}^q \leq S_n$$

characterized by

$$\begin{aligned} \{t_0^p, \dots, t_{m_n(p)}^p\} &= \{t \in [0, T_n] \mid \phi^t(p) \in \Psi(\{0\} \times \mathring{\mathbb{D}})\} \\ \{t_0^q, \dots, t_{m_n(q)}^q\} &= \{t \in [0, S_n] \mid \phi^t(q) \in \Psi(\{0\} \times \mathring{\mathbb{D}})\} \end{aligned}$$

Since these are the returning times we have estimates

$$m_n(p)\tau_{\min} \leq t_{m_n(p)}^p \leq T_n < t_{m_n(p)+1}^p \leq t_{m_n(p)}^p + \tau_{\max} \leq (m_n(p) + 1)\tau_{\max}$$

where  $t_{m_n(p)+1}^p$  denotes the hitting time following  $t_{m_n(p)}^p$ . It follows that

$$\frac{m_n(p)}{T_n} \leq \frac{1}{\tau_{\min}}, \quad \frac{T_n}{\tau_{\max}} - 1 \leq m_n(p) \rightarrow +\infty. \quad (26)$$

Analogously

$$\frac{m_n(q)}{S_n} \leq \frac{1}{\tau_{\min}}, \quad \frac{S_n}{\tau_{\max}} - 1 \leq m_n(q) \rightarrow +\infty. \quad (27)$$

The loop  $k(T_n, p) = \phi^{[0, T_n]}(p) + \hat{\alpha}_n$  is obtained by concatenating to  $\phi^{[0, T_n]}(p)$  a  $C^1$ -small perturbation  $\hat{\alpha}_n$  of  $\alpha_n$  (with end points fixed), and  $k(S_n, q) = \phi^{[0, S_n]}(q) + \hat{\beta}_n$



is obtained by concatenating to  $\phi^{[0,S_n]}(q)$  a  $C^1$ -small perturbation  $\hat{\beta}_n$  of  $\beta_n$  (with end points fixed). But the choices of  $\hat{\alpha}_n, \hat{\beta}_n$  need to be made so that  $k(T_n, p)$  and  $k(S_n, q)$  do not intersect. Observe that different choices of  $\hat{\alpha}_n, \hat{\beta}_n$  determine an ambiguity by an additive integer in  $[-m_n(p) - m_n(q), m_n(p) + m_n(q)]$  of the value of  $\text{link}(k(T_n, p), k(S_n, q))$ . This is so because  $\hat{\alpha}_n, \hat{\beta}_n$  are  $C^1$ -close enough to  $\alpha_n, \beta_n$ . In view of (26)-(27) we see that the ambiguity of the value of

$$\frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n}$$

is at most

$$\frac{m_n(p) + m_n(q)}{T_n S_n} \leq \frac{1}{\tau_{\min}} \left( \frac{1}{S_n} + \frac{1}{T_n} \right) \rightarrow 0$$

provided  $\hat{\alpha}_n, \hat{\beta}_n$  are  $C^1$ -close enough to  $\alpha_n, \beta_n$ . In particular

$$\begin{aligned} & \left| \frac{\text{link}_-(\phi^{[0,T_n]}(p), \phi^{[0,S_n]}(q))}{T_n S_n} - \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} \right| \\ & \leq \frac{1}{\tau_{\min}} \left( \frac{1}{S_n} + \frac{1}{T_n} \right) \end{aligned} \quad (28)$$

provided  $\hat{\alpha}_n, \hat{\beta}_n$  are  $C^1$ -close enough to  $\alpha_n, \beta_n$ .

Similarly, consider another sequence  $T'_n \rightarrow +\infty$  satisfying  $|T_n - T'_n| \rightarrow 0$ . Let  $\alpha'_n, \alpha_n$  be shortest geodesic arcs from  $\phi^{T'_n}(p), \phi^{T_n}(p)$  to  $p$ , respectively, and let  $\hat{\alpha}'_n, \hat{\alpha}_n$  be small  $C^1$ -perturbations of  $\alpha'_n, \alpha_n$  with end points fixed so that the corresponding loops  $k(T_n, p), k(T'_n, p)$  as described before do not intersect  $k(S_n, q)$ . Construct the loop  $c = (-\hat{\alpha}'_n) + \phi^{T'_n \rightarrow T_n}(p) + \hat{\alpha}_n$ . For  $n$  large enough we can estimate

$$|\text{link}(c, k(S_n, q))| \leq 2m_n(q) \quad (29)$$

provided  $\hat{\alpha}'_n, \hat{\alpha}_n$  are  $C^1$ -close enough to  $\alpha'_n, \alpha_n$ . The identity  $k(T'_n, p) + c = k(T_n, p)$  implies that

$$e_n = \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} - \frac{\text{link}(k(T'_n, p), k(S_n, q))}{T'_n S_n}$$

can be estimated by

$$\begin{aligned} |e_n| &= \left| \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} - \frac{\text{link}(k(T'_n, p), k(S_n, q))}{T'_n S_n} \right| \\ &= \left| \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} - \frac{\text{link}(k(T_n, p), k(S_n, q)) - \text{link}(c, k(S_n, q))}{T_n S_n} \frac{T_n}{T'_n} \right| \\ &\leq \left| 1 - \frac{T_n}{T'_n} \right| \left| \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} \right| + \left| \frac{\text{link}(c, k(S_n, q))}{S_n T'_n} \right| \\ &\leq \left| 1 - \frac{T_n}{T'_n} \right| \left| \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} \right| + \frac{2m_n(q)}{S_n T'_n} \end{aligned}$$

provided  $\hat{\alpha}'_n, \hat{\alpha}_n, \hat{\beta}_n$  are  $C^1$ -close enough to  $\alpha'_n, \alpha_n, \beta_n$ . Together with (26) and (28) this implies

$$\begin{aligned}
& \frac{\text{link}_-(\phi^{[0, T'_n]}(p), \phi^{[0, S_n]}(q))}{T'_n S_n} \\
& \geq \frac{\text{link}(k(T'_n, p), k(S_n, q))}{T'_n S_n} - \frac{1}{\tau_{\min}} \left( \frac{1}{S_n} + \frac{1}{T'_n} \right) \\
& \geq \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} - |e_n| - \frac{1}{\tau_{\min}} \left( \frac{1}{S_n} + \frac{1}{T'_n} \right) \\
& \geq \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} - \left| 1 - \frac{T_n}{T'_n} \right| \left| \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} \right| \\
& \quad - \frac{2}{\tau_{\min}} \frac{1}{T'_n} - \frac{1}{\tau_{\min}} \left( \frac{1}{S_n} + \frac{1}{T'_n} \right) \\
& \geq \frac{\text{link}_-(\phi^{[0, T_n]}(p), \phi^{[0, S_n]}(q))}{T_n S_n} - \left| 1 - \frac{T_n}{T'_n} \right| \left| \frac{\text{link}_-(\phi^{[0, T_n]}(p), \phi^{[0, S_n]}(q))}{T_n S_n} \right| \\
& \quad - \frac{2}{\tau_{\min}} \frac{1}{T'_n} - \frac{1}{\tau_{\min}} \left( \frac{1}{S_n} + \frac{1}{T'_n} \right) - \frac{1}{\tau_{\min}} \left( \frac{1}{S_n} + \frac{1}{T_n} \right) \left| 1 - \frac{T_n}{T'_n} \right| \\
& \quad - \frac{1}{\tau_{\min}} \left( \frac{1}{S_n} + \frac{1}{T_n} \right)
\end{aligned} \tag{30}$$

provided  $\hat{\alpha}'_n, \hat{\alpha}_n, \hat{\beta}_n$  are  $C^1$ -close enough to  $\alpha'_n, \alpha_n, \beta_n$ . Now note that if two sequences  $A_n, A'_n \in \mathbb{R}$  satisfy  $A'_n \geq A_n - \delta_n |A_n| - \tilde{\delta}_n \forall n$  where  $\delta_n, \tilde{\delta}_n \rightarrow 0$  then the inequality  $\liminf_n A_n > 0$  implies that  $\liminf_n A'_n \geq \liminf_n A_n > 0$ . This fact combined with (30) and with an analogous argument interchanging the roles of  $p$  and  $q$ , shows the following: if there exists  $a > 0$  such that for every  $\{T_n\}, \{S_n\}$  as above one finds sequences  $\{T'_n\}, \{S'_n\}$  satisfying

$$|T_n - T'_n| \rightarrow 0 \quad |S_n - S'_n| \rightarrow 0 \quad \liminf_{n \rightarrow \infty} \frac{\text{link}_-(\phi^{[0, T'_n]}(p), \phi^{[0, S'_n]}(q))}{T'_n S'_n} \geq a$$

then

$$\ell(p, q) \geq a.$$

Thus, we can assume that  $\phi^{T_n}(p), \phi^{S_n}(q)$  belong to  $\Psi(\{0\} \times \mathring{\mathbb{D}})$ . We proceed under this assumption.

The form of the metric near  $p, q$  implies that  $\alpha_n, \beta_n$  are images under  $\Psi$  of straight line segments in  $\{0\} \times \mathring{\mathbb{D}}$ . Denote by  $z_0^p, \dots, z_{m_n(p)}^p$  and by  $z_0^q, \dots, z_{m_n(q)}^q \in \mathring{\mathbb{D}}$  the points in  $\mathring{\mathbb{D}}$  uniquely determined by

$$\Psi(0, z_i^p) = \phi^{t_i^p}(p), \quad \Psi(0, z_j^q) = \phi^{t_j^q}(q).$$

It follows that

$$\alpha_n = \Psi(\{0\} \times [z_{m_n(p)}^p, z_0^p]) \quad \beta_n = \Psi(\{0\} \times [z_{m_n(q)}^q, z_0^q])$$

where  $[z, w]$  denotes the path  $u \in [0, 1] \mapsto (1 - u)z + uw$ . By transversality of the flow to  $\Psi(\{0\} \times \mathring{\mathbb{D}})$  we can choose  $\hat{\alpha}_n, \hat{\beta}_n$  to be contained in  $\Psi(\{0\} \times \mathring{\mathbb{D}})$ , i.e. there are  $C^1$ -small perturbations  $c_n^p, c_n^q : [0, 1] \rightarrow \mathring{\mathbb{D}}$  of  $[z_{m_n(p)}^p, z_0^p], [z_{m_n(q)}^q, z_0^q]$  respectively, such that  $c_n^p$  misses the points  $z_0^q, \dots, z_{m_n(q)}^q$  and  $c_n^q$  misses the points  $z_0^p, \dots, z_{m_n(p)}^p$ , and such that we can choose

$$\hat{\alpha}_n = \Psi(0 \times c_n^p), \quad \hat{\beta}_n = \Psi(0 \times c_n^q).$$

Define paths

$$k_0^p, \dots, k_{m_n(p)-1}^p, k_0^q, \dots, k_{m_n(q)-1}^q : [0, 1] \rightarrow \mathring{\mathbb{D}}$$

by

$$\Psi(s, k_i^p(s)) = \phi^{t(s, z_i^p)}(\Psi(0, z_i^p)) \quad \Psi(s, k_j^q(s)) = \phi^{t(s, z_j^q)}(\Psi(0, z_j^q))$$

where  $t(s, z)$  is the function given by Lemma 2.9. It follows that

$$\begin{aligned} i \in \{0, \dots, m_n(p) - 1\} &\Rightarrow k_i^p(0) = z_i^p, & k_i^p(1) = z_{i+1}^p \\ j \in \{0, \dots, m_n(q) - 1\} &\Rightarrow k_j^q(0) = z_j^q, & k_j^q(1) = z_{j+1}^q \end{aligned}$$

Fix  $0 < \delta < 1$ . Consider paths

$$\hat{k}_i^p, \hat{k}_j^q : [0, 1] \rightarrow \mathring{\mathbb{D}} \quad (i, j) \in \{0, \dots, m_n(p) - 1\} \times \{0, \dots, m_n(q) - 1\}$$

defined by

$$\begin{aligned} \text{If } 0 \leq i \leq m_n(p) - 2 : \hat{k}_i^p(s) &= \begin{cases} k_i^p(\frac{s}{1-\delta}) & \text{if } s \in [0, 1 - \delta] \\ z_{i+1}^p & \text{if } s \in [1 - \delta, 1] \end{cases} \\ \text{If } 0 \leq j \leq m_n(q) - 2 : \hat{k}_j^q(s) &= \begin{cases} k_j^q(\frac{s}{1-\delta}) & \text{if } s \in [0, 1 - \delta] \\ z_{j+1}^q & \text{if } s \in [1 - \delta, 1] \end{cases} \end{aligned} \tag{31}$$

and

$$\begin{aligned} \hat{k}_{m_n(p)-1}^p(s) &= \begin{cases} k_{m_n(p)-1}^p(\frac{s}{1-\delta}) & \text{if } s \in [0, 1 - \delta] \\ c_n^p(\frac{s-1+\delta}{\delta}) & \text{if } s \in [1 - \delta, 1] \end{cases} \\ \hat{k}_{m_n(q)-1}^q(s) &= \begin{cases} k_{m_n(q)-1}^q(\frac{s}{1-\delta}) & \text{if } s \in [0, 1 - \delta] \\ c_n^q(\frac{s-1+\delta}{\delta}) & \text{if } s \in [1 - \delta, 1] \end{cases} \end{aligned}$$

In the following we extend  $k_i^p, \hat{k}_i^p$  to all  $i \in \mathbb{Z}$  in a  $m_n(p)$ -periodically way. Similarly, we extend  $k_j^q, \hat{k}_j^q$  to all  $j \in \mathbb{Z}$  in a  $m_n(q)$ -periodically way. With this convention we have

$$\hat{k}_i^p(1) = \hat{k}_{i+1}^p(0) \quad \forall i \quad \hat{k}_j^q(1) = \hat{k}_{j+1}^q(0) \quad \forall j$$

Let  $L = \text{lcm}(m_n(p), m_n(q))$  and  $r_n(p), r_n(q) \in \mathbb{N}$  satisfy

$$L = r_n(p)m_n(p) = r_n(q)m_n(q)$$

Let us denote by  $k(T_n, p)^{r_n(p)}, k(S_n, q)^{r_n(q)}$  the  $r_n(p)$ -fold iteration of  $k(T_n, p)$  and the  $r_n(q)$ -fold iteration of  $k(S_n, q)$ , respectively. Then, by the transversality of  $\phi^t$  to each  $\Psi(\{s\} \times \mathring{\mathbb{D}})$ , it holds

$$\text{link}(k(T_n, p)^{r_n(p)}, k(S_n, q)^{r_n(q)}) = \sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(\hat{k}_i^p(s) - \hat{k}_j^q(s)). \quad (32)$$

With  $i, j \in \{0, \dots, L-1\}$  consider

$$e_{ij} = \text{wind}_{s \in [1-\delta, 1]}(\hat{k}_i^p(s) - \hat{k}_j^q(s)). \quad (33)$$

For each pair  $(i, j)$  there are two cases to be studied separately:

- (i)  $m_n(p)$  does not divide  $i+1$  and  $m_n(q)$  does not divide  $j+1$ .
- (ii)  $m_n(p)$  divides  $i+1$  or  $m_n(q)$  divides  $j+1$ .

In case (i) both paths  $\hat{k}_i^p|_{[1-\delta, 1]}$  and  $\hat{k}_j^q|_{[1-\delta, 1]}$  are constant paths, hence  $e_{ij} = 0$ . In case (ii) there are two subcases: either one of the paths is a  $C^1$ -perturbation of a straight line segment in  $\mathring{\mathbb{D}}$  and the other is a point in the complement, or both are contained in disjoint open balls (assuming  $n$  is large enough). In both subcases we get  $|e_{ij}| \leq 1$ . The number of multiples of  $m_n(p)$  in  $[0, L-1]$  is  $r_n(p)$ , and the number of multiples of  $m_n(q)$  in  $[0, L-1]$  is  $r_n(q)$ . Hence there at most  $r_n(p) + r_n(q)$  pairs  $(i, j)$  falling in case (ii), and this yields the estimate

$$\sum_{i,j=0}^{L-1} |e_{ij}| \leq r_n(p) + r_n(q). \quad (34)$$

Now note that

$$\begin{aligned} \sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(\hat{k}_i^p(s) - \hat{k}_j^q(s)) &= \sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1-\delta]}(\hat{k}_i^p(s) - \hat{k}_j^q(s)) + e_{ij} \\ &= \sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_j^q(s)) + e_{ij} \end{aligned} \quad (35)$$

Combining (32), (34) and (35)

$$\begin{aligned} &\left| \text{link}(k(T_n, p)^{r_n(p)}, k(S_n, q)^{r_n(q)}) - \sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_j^q(s)) \right| \\ &\leq r_n(p) + r_n(q). \end{aligned} \quad (36)$$

Now define maps

$$\begin{aligned} \Gamma^p : \mathbb{R}/L\mathbb{Z} &\rightarrow \mathring{\mathbb{D}} & \Gamma^p(s) &= k_{\lfloor s \rfloor}^p(s - \lfloor s \rfloor) \\ \Gamma^q : \mathbb{R}/L\mathbb{Z} &\rightarrow \mathring{\mathbb{D}} & \Gamma^q(s) &= k_{\lfloor s \rfloor}^q(s - \lfloor s \rfloor) \end{aligned} \quad (37)$$

Note that  $\Gamma^p$  is discontinuous, unless  $p$  lies on a periodic orbit. Analogously,  $\Gamma^q$  is discontinuous, unless  $q$  lies on a periodic orbit. For each  $j \in \mathbb{Z}/L\mathbb{Z}$  define:

$$j * \Gamma^q(s) = \Gamma^q(s + j) \quad (38)$$

Define

$$\begin{aligned} E_p &= \{0, m_n(p), \dots, (r_n(p) - 1)m_n(p)\} \\ E_q &= \{0, m_n(q), \dots, (r_n(q) - 1)m_n(q)\} \end{aligned}$$

We see  $E_p, E_q$  as subsets of  $\mathbb{Z}/L\mathbb{Z}$ . The set of discontinuity points of  $\Gamma^p$  is contained in  $E_p$  whose cardinality is equal to  $r_n(p)$ . The set of discontinuity points of  $j * \Gamma^q$  is contained in  $E_q - j$  whose cardinality is  $r_n(q)$ . We then write

$$\begin{aligned} \sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_j^q(s)) &= \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_{i+j}^q(s)) \\ &= \sum_{j=0}^{L-1} \text{wind}_{s \in [0,L]}(\Gamma^p(s) - j * \Gamma^q(s)) \end{aligned} \quad (39)$$

Note that the set  $E_p \cup (E_q - j)$  divides the circle  $\mathbb{R}/L\mathbb{Z}$  into  $N_j$  closed intervals  $I_1^j, \dots, I_{N_j}^j$ . Note that  $N_j \leq r_n(p) + r_n(q)$ . We denote the end points of these intervals by  $I_\lambda^j = [a_\lambda^j, b_\lambda^j]$  and their lengths by  $|I_\lambda^j| = b_\lambda^j - a_\lambda^j \in \mathbb{N}$ . The end points of these intervals belong to  $\mathbb{Z}/L\mathbb{Z}$ . For each  $I_\lambda^j = [a_\lambda^j, b_\lambda^j]$  we find points  $\zeta_\lambda, \zeta'_\lambda \in \mathring{\mathbb{D}}$  such that

$$h_s(\zeta_\lambda) = \Gamma^p(s + a_\lambda^j) \quad h_s(\zeta'_\lambda) = j * \Gamma^q(s + a_\lambda^j) \quad \forall s \in [0, b_\lambda^j - a_\lambda^j] \quad (40)$$

where  $h_s$  is the unique extension of the isotopy (21) to  $s \in [0, +\infty)$  determined by the identity

$$h_{s+1} = h_s \circ h \quad s \in [0, +\infty). \quad (41)$$

We know from Lemma 2.10 that the disks  $\{D_s\}_{s \in \mathbb{R}/\mathbb{Z}}$  form a smooth foliation of  $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$ . It follows that (41) is a smooth isotopy.

In particular, observe that, for every  $j = 0, \dots, L-1$ , it holds

$$\text{wind}_{s \in [0,L]}(\Gamma^p(s) - j * \Gamma^q(s)) = \sum_{\lambda=1}^{N_j} \text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)). \quad (42)$$

Recall that, from (24),  $\kappa(\gamma_0) - \epsilon > 2\pi$ . Note that, for  $x \in S^3 \setminus \gamma_0$ ,

$$\text{link}(k(T, x; D), \gamma_0) = \#\{t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \mathring{\mathbb{D}})\} + \ell,$$

for some  $\ell \in \{-1, 0, 1\}$ , where  $\#\{t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \mathring{\mathbb{D}})\}$  is the number of times that  $\phi^{[0,T]}(x)$  meets  $\Psi(\{0\} \times \mathring{\mathbb{D}})$ . Observe that

$$\kappa(\gamma_0) = \liminf_{T \rightarrow +\infty} \inf_{\substack{x \in S^3 \setminus \gamma_0 \\ u \in \xi_x \setminus 0}} \frac{\Delta \tilde{\Theta}_\sigma(T; x, u)}{\#\{t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \mathring{\mathbb{D}})\}}.$$

Here we used that for every  $x \in S^3 \setminus \gamma_0$  we can estimate

$$\frac{T}{\tau_{\max}} \leq \#\{t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \mathring{\mathbb{D}})\} \leq \frac{T}{\tau_{\min}},$$

and, as a consequence,  $\#\{t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \mathring{\mathbb{D}})\}$  grows linearly to  $+\infty$  as  $T \rightarrow +\infty$ . Thus, there exists  $M = M(\varepsilon) > 0$  such that for every  $x \in S^3 \setminus \gamma_0$  and every  $u \in \xi_x \setminus 0$ ,

$$T \geq M \quad \Rightarrow \quad \frac{\Delta \tilde{\Theta}_\sigma(T; x, u)}{\#\{t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \mathring{\mathbb{D}})\}} > \kappa(\gamma_0) - \varepsilon. \quad (43)$$

Observe that, by the extension of the function  $t$  in (18) and by (ii) of Lemma 2.9, for every  $z \in \mathbb{D}$ , if  $s \geq \frac{M}{c\tau_{\min}}$ , then  $t(s, z) \geq M$ . Let us define

$$\mathcal{M} := \sup_{\substack{z \in \mathbb{D} \\ u \in T_z \mathbb{D} \setminus 0}} \max_{1 \leq i \leq \lfloor \frac{M}{c\tau_{\min}} \rfloor + 1} |\text{wind}_{s \in [0, i]}(Dh_s(z)u)| < +\infty. \quad (44)$$

*Remark 2.16.* We deduce that for every  $\lambda$  such that  $|I_\lambda^j| < \frac{M}{c\tau_{\min}}$

$$\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) \geq -\mathcal{M}.$$

Indeed, by Theorem 2.14 there exists  $z_\lambda \in [\zeta'_\lambda, \zeta_\lambda]$  such that

$$\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) = \text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(Dh_s(z_\lambda)(\zeta_\lambda - \zeta'_\lambda))$$

Thus, by (44), it holds  $\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(Dh_s(z_\lambda)(\zeta_\lambda - \zeta'_\lambda)) \geq -\mathcal{M}$ .

**Lemma 2.17.** *There exists a constant  $C > 0$  independent of  $p, q, \{S_n\}, \{T_n\}$  such that for each  $\lambda \in \{1, \dots, N_j\}$  so that  $|I_\lambda^j| = b_\lambda^j - a_\lambda^j \geq \frac{M}{c\tau_{\min}}$ , the estimate*

$$\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) \geq (\kappa(\gamma_0) - \varepsilon - 2\pi) \frac{|I_\lambda^j|}{2\pi} - C$$

*holds.*

*Proof.* We start by introducing some notation. Let  $\tilde{X}_1, \tilde{X}_2$  be the nowhere vanishing vector fields given by  $\sigma \circ \tilde{X}_j \equiv e_j$  where  $e_1 = (1, 0)$ ,  $e_2 = (0, 1)$ . We choose  $\tilde{X}_1$  to be equal to the vector field  $Z$  given by (e) in Proposition 2.8. Observe that  $i_{\tilde{X}_j} \lambda \equiv 0$ . Denote by  $\hat{X}_1, \hat{X}_2$  the pull-back of  $\tilde{X}_1|_{S^3 \setminus \gamma_0}, \tilde{X}_2|_{S^3 \setminus \gamma_0}$  respectively under the diffeomorphism  $\Psi|_{\mathbb{R}/\mathbb{Z} \times \mathring{\mathbb{D}}}$ . For every  $s \in \mathbb{R}/\mathbb{Z}$  let  $\Pi^s : T(\mathbb{R}/\mathbb{Z} \times \mathring{\mathbb{D}})|_{\{s\} \times \mathring{\mathbb{D}}} \rightarrow T\mathring{\mathbb{D}}$  be the vector bundle map determined by projecting along the direction of  $W$ , where  $\{s\} \times \mathring{\mathbb{D}}$  is identified with  $\mathring{\mathbb{D}}$ . Then  $X_1^s, X_2^s$  denote the images of  $\hat{X}_1|_{\{s\} \times \mathring{\mathbb{D}}}, \hat{X}_2|_{\{s\} \times \mathring{\mathbb{D}}}$  under  $\Pi^s$ , respectively. We get two smooth families  $X_1^s, X_2^s$  of smooth vector fields on  $\mathring{\mathbb{D}}$  parametrized by  $s \in \mathbb{R}/\mathbb{Z}$ . At times it might be convenient to think of  $s$  as

variable in  $\mathbb{R}$  and the families  $X_1^s, X_2^s$  as 1-periodic in  $s$ . By Theorem 2.14, there exists  $z_\lambda \in [\zeta'_\lambda, \zeta_\lambda]$  such that

$$\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) = \text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(Dh_s(z_\lambda)(\zeta_\lambda - \zeta'_\lambda)). \quad (45)$$

By definition we have

$$\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(Dh_s(z_\lambda)(\zeta_\lambda - \zeta'_\lambda)) = \frac{\theta(b_\lambda^j - a_\lambda^j) - \theta(0)}{2\pi}$$

where  $\theta : [0, b_\lambda^j - a_\lambda^j] \rightarrow \mathbb{R}$  is a continuous argument of  $Dh_s(z_\lambda)(\zeta_\lambda - \zeta'_\lambda)$ . Consider now the smooth path  $c : [0, b_\lambda^j - a_\lambda^j] \rightarrow \mathbb{C} \setminus \{0\}$

$$s \mapsto c(s) := X_1^s(s, h_s(z_\lambda))$$

and let  $\Theta : [0, b_\lambda^j - a_\lambda^j] \rightarrow \mathbb{R}$  be a continuous choice of argument of  $c(s)$ . Thus we can write  $\theta(s) = \vartheta(s) + \Theta(s)$ , where  $\vartheta : [0, b_\lambda^j - a_\lambda^j] \rightarrow \mathbb{R}$  is a unique choice of continuous angular coordinate of the vector  $Dh_s(z_\lambda)(\zeta_\lambda - \zeta'_\lambda)$  in the frame  $\{c(s), ic(s)\}$  determined by  $\theta$  and  $\Theta$ . Hence

$$\begin{aligned} \frac{\theta(b_\lambda^j - a_\lambda^j) - \theta(0)}{2\pi} &= \frac{\Theta(b_\lambda^j - a_\lambda^j) - \Theta(0)}{2\pi} + \frac{\vartheta(b_\lambda^j - a_\lambda^j) - \vartheta(0)}{2\pi} \\ &= \text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(c(s)) + \frac{\vartheta(b_\lambda^j - a_\lambda^j) - \vartheta(0)}{2\pi} \end{aligned} \quad (46)$$

(I) *Lower bound for  $\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(c(s))$ .* Consider a path  $\rho : [0, 1] \rightarrow \mathring{\mathbb{D}}$  of the form  $\rho(0) = h_{b_\lambda^j - a_\lambda^j}(z_\lambda)$  and  $\rho(1) = z_\lambda$ . Build then the loop  $\Gamma : [0, b_\lambda^j - a_\lambda^j + 1] \rightarrow \mathbb{R}/\mathbb{Z} \times \mathring{\mathbb{D}}$

$$\Gamma(s) := \begin{cases} (s, h_s(z_\lambda)) & s \in [0, b_\lambda^j - a_\lambda^j] \\ (0, \rho(s - b_\lambda^j + a_\lambda^j)) & s \in [b_\lambda^j - a_\lambda^j, b_\lambda^j - a_\lambda^j + 1] \end{cases}$$

Observe that  $\text{link}(\Psi \circ \Gamma, \gamma_0) = |I_\lambda^j| = b_\lambda^j - a_\lambda^j$ . We associate to  $\Gamma$  the path

$$D\Gamma : [0, b_\lambda^j - a_\lambda^j + 1] \rightarrow \mathbb{C} \setminus \{0\}$$

defined by

$$D\Gamma(s) := \begin{cases} X_1^s(\Gamma(s)) & s \in [0, b_\lambda^j - a_\lambda^j] \\ X_1^0(\Gamma(s)) & s \in [b_\lambda^j - a_\lambda^j, b_\lambda^j - a_\lambda^j + 1]. \end{cases}$$

Then it holds that

$$\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j + 1]}(D\Gamma(s)) = \text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(c(s)) + \text{wind}_{s \in [0, 1]}(X_1^0(\rho(s))).$$

Since  $\tilde{X}_1$  satisfies (e) of Proposition 2.8 we deduce that there exists a constant  $C_1 > 0$ , independent from  $p, q, \{T_n\}, \{S_n\}$ , such that

$$|\text{wind}_{s \in [0, 1]}(X_1^0(\rho(s)))| \leq C_1. \quad (47)$$

Brouwer's translation theorem gives a fixed point  $\Psi(0, \bar{z})$  of the first return map to  $\Psi(\{0\} \times \mathring{\mathbb{D}})$  corresponding to a periodic Reeb orbit  $\bar{\gamma} \subset S^3 \setminus \gamma_0$ . We denote by  $\bar{\gamma}^{|I_\lambda^j|}$  the  $|I_\lambda^j|$ -fold iteration of  $\bar{\gamma}$ . The loop  $\beta := \Psi^{-1}(\bar{\gamma}^{|I_\lambda^j|})$  is homotopic to  $\Gamma$  in  $\mathbb{R}/\mathbb{Z} \times \mathring{\mathbb{D}}$ . The loop  $\beta$  can be parametrised by  $s \in [0, b_\lambda^j - a_\lambda^j]$  so that  $\beta(s) = (s, \hat{\beta}(s - \lfloor s \rfloor))$  where  $\hat{\beta} : [0, 1] \rightarrow \mathring{\mathbb{D}}$  satisfies  $\hat{\beta}(1) = \hat{\beta}(0)$ . Then

$$\begin{aligned} \text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j + 1]}(D\Gamma(s)) &= \text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(X_1^s(\hat{\beta}(s - \lfloor s \rfloor))) \\ &= (b_\lambda^j - a_\lambda^j) \text{wind}_{s \in [0, 1]}(X_1^s(\hat{\beta}(s))). \end{aligned} \quad (48)$$

The quantity  $\text{wind}_{s \in [0, 1]}(X_1^s(\hat{\beta}(s)))$  is equal to the self-linking number of  $\bar{\gamma}$ , see Definition 1.5 in [Hry14]. By [Hry14, Theorem 1.8], the self-linking number of  $\bar{\gamma}$  is  $-1$ . Thus, from (48) we get

$$\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j + 1]}(D\Gamma(s)) = -(b_\lambda^j - a_\lambda^j). \quad (49)$$

From (47) and (49) we deduce that

$$\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(c(s)) \geq -(b_\lambda^j - a_\lambda^j) - C_1 = -|I_\lambda^j| - C_1 \quad (50)$$

(II) *Lower bound for  $\frac{\vartheta(b_\lambda^j - a_\lambda^j) - \vartheta(0)}{2\pi}$ .* With respect to the frame  $\{c(s), ic(s)\}$  the angle coordinate of the vector  $Dh_s(z_\lambda)(\zeta_\lambda - \zeta'_\lambda)$  is  $\vartheta(s)$ . We are interested in the variation of the angle coordinate with respect to the frame

$$\{c(s) = X_1^s(s, h_s(z_\lambda)), X_2^s(s, h_s(z_\lambda))\} \quad (51)$$

obtained from  $\sigma$ . If we denote by  $\tilde{\vartheta}(s)$  the angle coordinate of  $Dh_s(z_\lambda)(\zeta_\lambda - \zeta'_\lambda)$  with respect to the frame (51), we can invoke [Flo19a, Claim 1.1.1] to conclude that<sup>2</sup>

$$\frac{\vartheta(b_\lambda^j - a_\lambda^j) - \vartheta(0)}{2\pi} > \frac{\tilde{\vartheta}(b_\lambda^j - a_\lambda^j) - \tilde{\vartheta}(0)}{2\pi} - 1. \quad (52)$$

Recall the generating vector field  $\tilde{X}$  of the flow on  $(\xi \setminus 0)/\mathbb{R}_+$  induced by the linearised flow of  $\phi^t$  on  $\xi$ , and the  $\mathbb{R}/2\pi\mathbb{Z}$  coordinate in  $(\xi \setminus 0)/\mathbb{R}_+$  induced by the frame  $\sigma$ . Let  $v \in \xi|_{\Psi(0, z_\lambda)}$  be uniquely characterised by

$$D\Psi(0, z_\lambda)(0, \zeta_\lambda - \zeta'_\lambda) \in v + \mathbb{R}X.$$

From Lemma 2.9, the strictly increasing function  $t(\cdot, z_\lambda) : [0, b_\lambda^j - a_\lambda^j] \rightarrow [0, +\infty)$  satisfies

$$t(0, z_\lambda) = 0 \quad \Psi(s, h_s(z_\lambda)) = \phi^{t(s, z_\lambda)}(\Psi(0, z_\lambda)),$$

---

<sup>2</sup>Comparing  $\vartheta$  and  $\tilde{\vartheta}$  corresponds to compare oriented angles with respect to different Riemannian metrics.



with

$$t(s, z_\lambda) = \sum_{k=0}^{\lfloor s \rfloor - 1} t(1, h^k(z_\lambda)) + t(s - \lfloor s \rfloor, h^{\lfloor s \rfloor}(z_\lambda)). \quad (53)$$

See also (18). The coordinates of the vector  $Dh_s(z_\lambda)(\zeta_\lambda - \zeta'_\lambda)$  in the basis (51) are the same of

$$D\phi^{t(s, z_\lambda)}(\Psi(0, z_\lambda)) \cdot v$$

in the basis  $\{\tilde{X}_1, \tilde{X}_2\}$ . Thus

$$\begin{aligned} & \tilde{\vartheta}(b_\lambda^j - a_\lambda^j) - \tilde{\vartheta}(0) \\ &= \tilde{\Theta}_\sigma(t(b_\lambda^j - a_\lambda^j, z_\lambda), \Psi(0, z_\lambda), v) - \tilde{\Theta}_\sigma(0, \Psi(0, z_\lambda), v) \\ &=: \Delta\Theta_\sigma(t(b_\lambda^j - a_\lambda^j, z_\lambda), \Psi(0, z_\lambda), v). \end{aligned} \quad (54)$$

Since, by assumption,  $b_\lambda^j - a_\lambda^j \geq \frac{M}{c\tau_{\min}}$ , then  $t(b_\lambda^j - a_\lambda^j, z_\lambda) \geq M$  and, from (43), it holds that

$$\begin{aligned} & \Delta\Theta_\sigma(t(b_\lambda^j - a_\lambda^j, z_\lambda), \Psi(0, z_\lambda), v) \\ & > (\kappa(\gamma_0) - \epsilon) \# \{t \in [0, t(b_\lambda^j - a_\lambda^j, z_\lambda)] \mid \phi^t(\Psi(0, z_\lambda)) \in \Psi(\{0\} \times \mathring{\mathbb{D}})\} \\ & = (\kappa(\gamma_0) - \epsilon) (|I_\lambda^j| + 1). \end{aligned} \quad (55)$$

Consequently

$$\frac{\vartheta(b_\lambda^j - a_\lambda^j) - \vartheta(0)}{2\pi} > \frac{(\kappa(\gamma_0) - \epsilon) (|I_\lambda^j| + 1)}{2\pi} - 1 \quad (56)$$

With the help of (I) and (II) we can conclude the proof, since from (45), (46), (50), (56) we have

$$\text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) > \left( \kappa(\gamma_0) - \epsilon - 2\pi \right) \frac{|I_\lambda^j|}{2\pi} - C,$$

where  $C := C_1 + 1 - \frac{(\kappa(\gamma_0) - \epsilon)}{2\pi}$ . □

We now give the final estimate to complete the proof of Proposition 2.15. For every  $j = 0, \dots, L-1$  denote

$$\mathcal{G}_j = \left\{ \lambda \in \{1, \dots, N_j\} \mid |I_\lambda^j| \geq \frac{M}{c\tau_{\min}} \right\},$$

$$\mathcal{B}_j = \left\{ \lambda \in \{1, \dots, N_j\} \mid |I_\lambda^j| < \frac{M}{c\tau_{\min}} \right\}.$$

With Lemma 2.17 and Remark 2.16 we can estimate (39) as (see (42))

$$\sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_j^q(s)) =$$

$$\begin{aligned}
&= \sum_{j=0}^{L-1} \left( \sum_{\lambda \in \mathcal{G}_j} \text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) + \sum_{\lambda \in \mathcal{B}_j} \text{wind}_{s \in [0, b_\lambda^j - a_\lambda^j]}(h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) \right) \\
&> \sum_{j=0}^{L-1} \left( \sum_{\lambda \in \mathcal{G}_j} \left[ (\kappa(\gamma_0) - \epsilon - 2\pi) \frac{|I_\lambda^j|}{2\pi} - C \right] - \sum_{\lambda \in \mathcal{B}_j} \mathcal{M} \right)
\end{aligned}$$

and, denoting  $\tilde{C} := \max(C, \mathcal{M})$ , we have

$$\begin{aligned}
\sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_j^q(s)) &\geq \sum_{j=0}^{L-1} \left( \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi} \sum_{\lambda \in \mathcal{G}_j} |I_\lambda^j| - \sum_{\lambda=1}^{N_j} \tilde{C} \right) \\
&\geq \sum_{j=0}^{L-1} \left( \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi} \sum_{\lambda \in \mathcal{G}_j} |I_\lambda^j| \right) - \tilde{C} L(r_n(p) + r_n(q)).
\end{aligned}$$

For every  $\lambda \in \mathcal{B}_j$  recall that  $|I_\lambda^j| < \frac{M}{c\tau_{\min}}$ . Observe now that

$$0 \leq \sum_{\lambda \in \mathcal{B}_j} |I_\lambda^j| < \frac{M}{c\tau_{\min}} N^j \leq \frac{M}{c\tau_{\min}} (r_n(p) + r_n(q)) = L \frac{M}{c\tau_{\min}} \frac{m_n(p) + m_n(q)}{m_n(q)m_n(p)}.$$

Moreover, it holds that  $\sum_{\lambda \in \mathcal{G}_j} |I_\lambda^j| + \sum_{\lambda \in \mathcal{B}_j} |I_\lambda^j| = L$ . Consequently

$$\sum_{\lambda \in \mathcal{G}_j} |I_\lambda^j| > \left( 1 - \frac{M}{c\tau_{\min}} \frac{m_n(p) + m_n(q)}{m_n(q)m_n(p)} \right) L. \tag{57}$$

Therefore, we have

$$\begin{aligned}
&\sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_j^q(s)) \\
&> \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi} \left( 1 - \frac{M}{c\tau_{\min}} \frac{m_n(p) + m_n(q)}{m_n(q)m_n(p)} \right) L^2 - \tilde{C} L(r_n(p) + r_n(q)),
\end{aligned}$$

which together with (36) yields

$$\begin{aligned}
&\text{link}(k(T_n, p)^{r_n(p)}, k(S_n, q)^{r_n(q)}) \\
&> \frac{L^2}{2\pi} (\kappa(\gamma_0) - \epsilon - 2\pi) \left( 1 - \frac{M}{c\tau_{\min}} \frac{m_n(p) + m_n(q)}{m_n(q)m_n(p)} \right) - (1 + \tilde{C}L)(r_n(p) + r_n(q)).
\end{aligned} \tag{58}$$

Combining with (26)-(27) we get

$$\begin{aligned}
& \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} \\
&= \frac{\text{link}(k(T_n, p)^{r_n(p)}, k(S_n, q)^{r_n(q)})}{r_n(p) T_n r_n(q) S_n} \\
&> \frac{(\kappa(\gamma_0) - \epsilon - 2\pi)}{2\pi} \frac{m_n(p) m_n(q)}{T_n S_n} - \frac{M}{c\tau_{\min}} \frac{(\kappa(\gamma_0) - \epsilon - 2\pi)}{2\pi} \frac{m_n(p) + m_n(q)}{T_n S_n} + \\
&\quad - (1 + \tilde{C}L) \frac{r_n(p) + r_n(q)}{L^2} \frac{m_n(q) m_n(p)}{T_n S_n} \\
&\geq \frac{(\kappa(\gamma_0) - \epsilon - 2\pi)}{2\pi\tau_{\max}^2} \frac{m_n(p) m_n(q)}{(m_n(p) + 1)(m_n(q) + 1)} \\
&\quad - \frac{M}{c\tau_{\min}} \frac{(\kappa(\gamma_0) - \epsilon - 2\pi)}{2\pi\tau_{\min}^2} \left( \frac{1}{m_n(q)} + \frac{1}{m_n(p)} \right) + \\
&\quad - \frac{(1 + \tilde{C}L)}{\tau_{\min}^2} \frac{1}{m_n(p) m_n(q)} \left( \frac{1}{r_n(q)} + \frac{1}{r_n(p)} \right),
\end{aligned}$$

from where it follows that

$$\liminf_{n \rightarrow \infty} \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} \geq \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi\tau_{\max}^2} > 0 \quad (59)$$

Here we made use of (24). The proof of Proposition 2.15 is complete.

By Proposition 2.15 and Lemma 2.13 the assumption  $\kappa(\gamma_0) > 2\pi$  implies right-handedness of the Reeb flow. The proof of Theorem 1.13 is complete.

## 2.4 Right-handedness on strictly convex energy levels

We start by studying criteria to estimate the invariant  $\kappa$  in (5). Let the contact form  $\lambda$  on  $S^3$  be dynamically convex. The contact structure and the Reeb flow are  $\xi$  and  $\phi^t$ , respectively. Consider

$$K_\sigma := \inf_{\mathbb{P}_+\xi} i_{\tilde{X}} d\Theta_\sigma \quad (60)$$

where  $\sigma$  is the global frame,  $\Theta_\sigma : \mathbb{P}_+\xi \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  is the global angle coordinate induced by  $\sigma$  as in (2), and  $\tilde{X}$  is the vector field generating the linearised Reeb flow on  $\mathbb{P}_+\xi$ . Let  $\gamma_0$  be an unknotted periodic orbit with self-linking number  $-1$ , and let  $D \subset S^3$  be the disk-like global surface of section given by Theorem 1.7 in [Hry14]. Denote  $0 < \tau_{\min}(D) \leq \tau_{\max}(D) < +\infty$  the infimum and the supremum of the return time function on  $D$ .

**Theorem 2.18.** *If  $K_\sigma \tau_{\min}(D) > 2\pi$  then the Reeb flow of  $\lambda$  is right-handed.*

*Proof.* We show that the hypothesis implies  $\kappa(\gamma_0) > 2\pi$  and then apply Theorem 1.13. By the fundamental theorem of calculus, for every  $x \in S^3 \setminus \gamma_0$  and  $u \in \xi|_x$ ,  $u \neq 0$ , we have for  $T > 0$

$$\begin{aligned} \tilde{\Theta}_\sigma(T, u) - \tilde{\Theta}_\sigma(0, u) &= \int_0^T \frac{d}{dt} \Theta_\sigma(\mathbb{R}_+ D \phi^t(x) u) dt \\ &\geq T \inf_{\substack{x \in S^3 \setminus \gamma_0 \\ u \in \xi_x \setminus 0}} i_{\tilde{X}} d\Theta_\sigma(x, u) = TK_\sigma. \end{aligned} \quad (61)$$

Moreover

$$\text{link}(k(T, x; D), \gamma_0) = \#\{t \in [t_-^D(x), T + t_+^D(\phi^T(x))] \mid \phi^t(x) \in D\} - 1. \quad (62)$$

Observe that

$$\#\{t \in [t_-^D(x), T + t_+^D(\phi^T(x))] \mid \phi^t(x) \in D\} - 1 \leq \frac{T + t_+^D(\phi^T(x)) - t_-^D(x)}{\tau_{\min}(D)}. \quad (63)$$

Hence

$$\frac{\tilde{\Theta}_\sigma(T, u) - \tilde{\Theta}_\sigma(0, u)}{\text{link}(k(T, x; D), \gamma_0)} \geq \frac{TK_\sigma}{\frac{T + t_+^D(\phi^T(x)) - t_-^D(x)}{\tau_{\min}(D)}} = K_\sigma \tau_{\min}(D) \frac{T}{T + t_+^D(\phi^T(x)) - t_-^D(x)}.$$

Since  $0 < t_+^D(\phi^T(x)) - t_-^D(x) \leq 2\tau_{\max}(D)$  for every  $x \in S^3 \setminus \gamma_0$ , we finally get

$$\frac{\tilde{\Theta}_\sigma(T, u) - \tilde{\Theta}_\sigma(0, u)}{\text{link}(k(T, x; D), \gamma_0)} \geq K_\sigma \tau_{\min}(D) \frac{T}{T + 2\tau_{\max}(D)}.$$

It follows immediately by first taking the infimum on  $x, u$  and then taking the limit as  $T \rightarrow +\infty$  that  $\kappa(\gamma_0) \geq K_\sigma \tau_{\min}(D) > 2\pi$ .  $\square$

Our final task in this section is to prove Theorem 1.14. Let  $J_0 : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  be the complex structure defined by the matrix

$$J_0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (64)$$

with respect to coordinates  $(q_1, q_2, p_1, p_2)$  of  $\mathbb{R}^4$ . Then  $\omega_0(u, v) = \langle u, J_0 v \rangle$  holds for every  $u, v \in \mathbb{R}^4$ , where  $\langle \cdot, \cdot \rangle$  is the standard Euclidean inner product. We will denote as  $\|\cdot\|$  the inherited norm. As explained in the introduction, the Hamiltonian vector field of  $H = \nu_C^2$ , defined as  $i_{X_H} \omega_0 = -dH$ , is the Reeb vector field of the contact form on  $\partial C$  induced by  $\lambda_0$ . The associated contact structure is denoted by  $\xi \subset T\partial C$ .

Denote as  $\varphi_H^t$  the Hamiltonian flow on  $\partial C$ , and  $\tilde{X}$  the vector field on  $\mathbb{P}_+\xi$  generating the linearised flow. Together with the matrix  $J_0$ , consider the following

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then  $J_i^2 = -I$  and  $J_i^T = -J_i$  hold for all  $i = 0, 1, 2$  where  $I$  denotes the identity matrix. Moreover,  $J_0 J_1 = J_2$ ,  $J_1 J_2 = J_0$  and  $J_2 J_0 = J_1$ . On  $\partial C$  consider the following vector fields

$$X_0 = \frac{\nabla H}{\|\nabla H\|}, \quad X_1 = J_2 X_0, \quad X_2 = J_1 X_0, \quad X_3 = -J_0 X_0. \quad (65)$$

The frame  $\{X_1, X_2, X_3\}$  is an orthonormal global frame of  $T\partial C$ . Observe that the vector field  $X_3$  is positively colinear with  $X_H$  and so it is tangent to the trajectories of the Reeb flow on  $\partial C$ . Let  $\sigma$  the global symplectic trivialisation of  $(\xi, d\lambda_0)$  determined by projecting  $X_1, X_2$  onto  $\xi$  along the Reeb direction  $\mathbb{R}X_3$ . As explained in the introduction, it induces a trivialisation  $\mathbb{P}_+\xi \simeq \partial C \times \mathbb{R}/2\pi\mathbb{Z}$  of circle bundles. Denote by  $\Theta_\sigma$  the  $\mathbb{R}/2\pi\mathbb{Z}$ -component of this map. A crucial ingredient is the following statement.

**Lemma 2.19** ([GRS06]). *The inequality  $i_{\tilde{X}} d\Theta_\sigma \geq 2K_{\min}^C$  holds everywhere on  $\mathbb{P}_+\xi$ .*

*Proof.* Let  $z_0 \in \partial C$  and  $z(t) = \varphi_H^t(z_0)$ . For any  $v_0 \in T_{z_0}\partial C$ , the trajectory of the linearised flow is  $v(t) = D\varphi_H^t(z_0)v_0$  and satisfies

$$\dot{v}(t) = DX_H(z(t))v(t) = -J_0 D^2 H(z(t))v(t) \quad v(0) = v_0. \quad (66)$$

Write  $v(t) = \sum_{i=1}^3 \alpha_i(t) X_i(z(t))$ , where  $\alpha_i(t) \in \mathbb{R}$ . Using (66) we obtain

$$\langle \dot{v}, X_1 \rangle = - \sum_{i=1}^3 \alpha_i \langle D^2 H X_i, X_2 \rangle \quad \text{and} \quad \langle \dot{v}, X_2 \rangle = \sum_{i=1}^3 \alpha_i \langle D^2 H X_i, X_1 \rangle. \quad (67)$$

Differentiating  $v = \sum_{i=1}^3 \alpha_i X_i$  we get

$$\dot{v} = \sum_{i=1}^3 \dot{\alpha}_i X_i + \alpha_i DX_i X_H. \quad (68)$$

In order to understand this linear flow we need to analyse terms  $\langle DX_i X_H, X_j \rangle$ . If  $i = j$  then the corresponding term vanishes since  $\langle DX_i X_H, X_i \rangle = \frac{d}{dt} \|X_i\|^2$  and all  $X_i$  have unit norm. When  $i \neq j$  the corresponding term is of the form  $\langle A DX_0 X_H, B X_0 \rangle$  with  $A, B$  distinct matrices in  $\{-J_0, J_1, J_2\}$ . We get

$$\langle DX_i X_H, X_j \rangle = -\langle DX_0 X_H, AB X_0 \rangle = -\langle D^2 H X_3, AB X_0 \rangle. \quad (69)$$

Here it was used that  $DX_0(z)$  acts as  $\frac{D^2H(z)}{\|\nabla H(z)\|}$  on  $T_z\partial C$  whenever  $z \in \partial C$ , and that  $\frac{X_H}{\|\nabla H\|} = X_3$ . Using (68) and (69) we get

$$\begin{aligned}\langle \dot{v}, X_1 \rangle &= \dot{\alpha}_1 + \alpha_2 \langle D^2H X_3, X_3 \rangle - \alpha_3 \langle D^2H X_3, X_2 \rangle \\ \langle \dot{v}, X_2 \rangle &= \dot{\alpha}_2 - \alpha_1 \langle D^2H X_3, X_3 \rangle + \alpha_3 \langle D^2H X_3, X_1 \rangle.\end{aligned}\tag{70}$$

Combining (70) and (67) we obtain

$$\begin{aligned}\dot{\alpha}_1 &= -\alpha_1 \langle D^2H X_1, X_2 \rangle - \alpha_2 \langle D^2H X_2, X_2 \rangle - \alpha_3 \langle D^2H X_3, X_3 \rangle, \\ \dot{\alpha}_2 &= \alpha_1 \langle D^2H X_1, X_1 \rangle + \alpha_2 \langle D^2H X_2, X_1 \rangle + \alpha_3 \langle D^2H X_3, X_3 \rangle.\end{aligned}$$

Consider the symmetric  $2 \times 2$  matrix defined by

$$M(t) = \begin{pmatrix} \langle D^2H X_1, X_1 \rangle & \langle D^2H X_2, X_1 \rangle \\ \langle D^2H X_1, X_2 \rangle & \langle D^2H X_2, X_2 \rangle \end{pmatrix} + \langle D^2H X_3, X_3 \rangle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With an abuse of notation and via the standard identification  $\mathbb{C} \simeq \mathbb{R}^2$ ,  $M(t)$  can be seen as an  $\mathbb{R}$ -linear map on  $\mathbb{C}$ . The crucial observation is that for every  $u$

$$\langle M(t)u, u \rangle \geq 2K_{\min}^C \|u\|^2.$$

By the previous calculations we have

$$\begin{pmatrix} \dot{\alpha}_2(t) \\ -\dot{\alpha}_1(t) \end{pmatrix} = M(t) \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}.$$

Consequently, for every  $z_0 \in \partial C$ ,  $v_0 \in \xi|_{z_0}$  non-zero, and  $t \in \mathbb{R}$  we have

$$\begin{aligned}i_{\tilde{X}}d\Theta_\sigma(z(t), v(t)) &= \frac{\alpha_1(t)\dot{\alpha}_2(t) - \alpha_2(t)\dot{\alpha}_1(t)}{\left\| \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} \right\|^2} = \\ \frac{\left\langle \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}, \begin{pmatrix} \dot{\alpha}_2(t) \\ -\dot{\alpha}_1(t) \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} \right\|^2} &= \frac{\left\langle \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}, M(t) \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} \right\|^2} \geq 2K_{\min}^C.\end{aligned}$$

□

*Proof of Theorem 1.14.* Combine Lemma 2.19 with Theorem 2.18. □

### 3 Pinched two-spheres

The aim of this section is to prove Theorem 1.2. The strategy is, first, to lift the geodesic flow on the unit tangent bundle of a Riemannian two-sphere  $(S^2, g)$  to the Reeb flow of a contact form  $f_g\lambda_0$  on  $S^3$  for some smooth  $f_g : S^3 \rightarrow (0, +\infty)$ , where  $\lambda_0$  is the restriction to  $S^3 \subset \mathbb{C}^2$  of the standard Liouville form  $\frac{1}{4i}(\bar{z}dz - z d\bar{z} + \bar{w}dw - w d\bar{w})$ . Here  $(z, w)$  denote the complex coordinates on  $\mathbb{C}^2$ . The second step is to use the pinching condition on the curvatures to show that the assumptions of Theorem 1.13 are fulfilled.

### 3.1 Lifting geodesic flows on $S^2$ and Birkhoff annuli

Here we establish notation, and recall well-known facts about geodesic flows. Let  $(S^2, g)$  be an oriented Riemannian two-sphere. The foot-point projection of its tangent bundle is denoted by  $\pi : TS^2 \rightarrow S^2$ . Consider the unit tangent bundle  $T_g^1 S^2$  defined as the set of  $(x, v) \in TS^2$  such that  $g(x)(v, v) = 1$ .

There are two important vector bundle maps covering  $\pi$ , namely, the differential of the projection  $d\pi : TTS^2 \rightarrow TS^2$  and the connection map  $K : TTS^2 \rightarrow TS^2$ . The connection map  $K$  can be written in local coordinates  $x = (x^1, x^2)$  as

$$K(x, v, \delta x, \delta v) = (x, \delta v + \Gamma(x)(v, \delta x))$$

where  $\Gamma(x)(u, w) = \sum_{i,j,k} \Gamma_{ij}^k(x) u^i w^j \partial_{x^k}$  and  $\Gamma_{ij}^k$  are the Christoffel symbols. There is a splitting

$$TTS^2 = \mathcal{H} \oplus \mathcal{V} \quad \mathcal{H} = \ker K, \quad \mathcal{V} = \ker d\pi$$

where both  $\mathcal{H}, \mathcal{V}$  are fiberwise two-dimensional. It follows that  $d\pi|_{\mathcal{H}}$  defines an isomorphism fiberwise. More precisely for all  $v, w \in T_x S^2$  there exists a unique  $w^{\text{hor}} \in \mathcal{H}_{(x,v)}$  satisfying  $d\pi_{(x,v)} \cdot w^{\text{hor}} = w$ , called the horizontal lift of  $w$ . The inverse mapping  $w \mapsto w^{\text{hor}}$  defines a linear isomorphism  $T_x S^2 \rightarrow \mathcal{H}_{(x,v)}$ . The vertical lift of  $w$  is defined as  $w^{\text{vert}} = \frac{d}{dt}|_{t=0}(v + tw)$ . The map  $w \mapsto w^{\text{vert}}$  is a linear isomorphism  $T_x S^2 \rightarrow \mathcal{V}_{(x,v)}$ .

The following facts are well-known, see [HS13] for details. The so-called Hilbert form  $\sum_{ij} g_{ij}(x) v^i dx^j$  restricts to a contact form  $\lambda_g$  on  $T_g^1 S^2$  whose Reeb flow coincides with the geodesic flow. For every  $(x, v) \in T_g^1 S^2$  denote by  $v^\perp \in T_x S^2$  the unique vector such that  $g(x)(v^\perp, v^\perp) = 1$ ,  $g(x)(v^\perp, v) = 0$ , and  $\{v, v^\perp\}$  is a positive basis. It follows that  $\{v^{\text{hor}}, (v^\perp)^{\text{vert}}, (v^\perp)^{\text{hor}}\}$  defines a basis of  $T_{(x,v)} T_g^1 S^2$ , where  $v^{\text{hor}}$  is the Reeb vector field at  $v$  and  $\{(v^\perp)^{\text{vert}}, (v^\perp)^{\text{hor}}\}$  is a  $d\lambda_g$ -symplectic frame of the contact structure  $\ker \lambda_g$ . In fact, we see from this that  $\ker \lambda_g$  is trivial as a (symplectic) vector bundle since it admits a global frame

$$\sigma_g : v \mapsto \{(v^\perp)^{\text{vert}}, (v^\perp)^{\text{hor}}\} \quad (71)$$

which we call the *geodesic frame*.

The Legendre transform  $\mathcal{L}_g : TS^2 \rightarrow T^* S^2$  is the vector bundle map defined by  $(x, v) \mapsto (x, p = g(x)(v, \cdot))$ . Let  $\tilde{\pi} : T^* S^2 \rightarrow S^2$  be the foot-point projection. The Hilbert form is pushed forward by  $\mathcal{L}_g$  to the tautological 1-form on  $T^* S^2$  defined by  $\zeta \in T_{(x,p)} T^* S^2 \mapsto p(d\tilde{\pi}_{(x,p)} \cdot \zeta) \in \mathbb{R}$ . Let  $F_g : T^* S^2 \rightarrow [0, +\infty)$  be defined by  $F_g(x, p) = \sqrt{g(x)(v, v)}$  where  $\mathcal{L}_g(x, v) = (x, p)$ . Then  $F_g$  restricts to a norm on the fibers of  $T^* S^2$ , and  $\mathcal{L}_g$  maps  $T_g^1 S^2$  to  $F_g^{-1}(1)$ . Denote by  $\Pi_g$  be the projection to  $F_g^{-1}(1)$  defined by  $p \mapsto p/F_g(x, p)$  on the complement of the zero section of  $T^* S^2$ . If  $h$  is another Riemannian metric on  $S^2$  then the map

$$\mathcal{L}_h^{-1} \circ \Pi_h \circ \mathcal{L}_g : T_g^1 S^2 \rightarrow T_h^1 S^2 \quad (72)$$

is diffeomorphism sending  $\ker \lambda_g$  to  $\ker \lambda_h$ , i.e. a contactomorphism. If  $g = h$  then this is the identity map.

We identify  $S^3 \simeq SU(2)$  by

$$(z, w) \simeq \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}, \quad |z|^2 + |w|^2 = 1,$$

and

$$S^2 \simeq \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in SU(2), \Re(z) = 0 \right\} \subset S^3. \quad (73)$$

Consider  $j, k \in SU(2)$  defined by

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Denote by  $g_0$  the round metric on  $S^2$ , given by  $g_0 = dy^2 + du^2 + dv^2$  along the 2-sphere (73) where  $z = x + iy$  and  $w = u + iv$ . There is a double covering map

$$D_0 : S^3 \rightarrow T_{g_0}^1 S^2 \quad Z \in SU(2) \simeq S^3 \mapsto (Z^{-1}jZ, -Z^{-1}kZ) \in T_{g_0}^1 S^2. \quad (74)$$

Following [HP08] one computes

$$D_0^* \lambda_{g_0} = 4\lambda_0 \quad (75)$$

where  $\lambda_0$  denotes the restriction to  $S^3$  of the 1-form

$$\frac{1}{4i}(\bar{z}dz - z d\bar{z} + \bar{w}dw - w d\bar{w}) = \frac{1}{2}(xdy - ydx + u dv - v du). \quad (76)$$

In other words, the geodesic flow of  $g_0$  lifts to the Hopf Reeb flow on  $S^3$  up to a constant time reparametrisation; to explain the factor 4 note that a Hopf fibre has Reeb flow period  $\pi$  with respect to  $\lambda_0$ , and is the lift of a great circle of length  $2\pi$  prescribed twice. Given any other metric  $g$  on  $S^2$  we get a double covering map

$$D_g = \mathcal{L}_g^{-1} \circ \Pi_g \circ \mathcal{L}_{g_0} \circ D_0 : S^3 \rightarrow T_g^1 S^2 \quad (77)$$

respecting contact structures, i.e.

$$D_g^* \lambda_g = f_g \lambda_0 \quad (78)$$

for some smooth  $f_g : S^3 \rightarrow \mathbb{R} \setminus \{0\}$ . The covering (77) is the universal covering and the group of deck transformations is  $\mathbb{Z}_2$  generated by the antipodal map. If  $g = g_0$  then  $f_{g_0} \equiv 4$  by (75). The following special case of a result from [HP08] relates Gaussian curvature to dynamical convexity; see also [HS13] for an alternative proof.

**Theorem 3.1** (Harris and Paternain). *If  $(S^2, g)$  is  $\delta$ -pinched with  $\delta > 1/4$ , then the Reeb flow of  $D_g^* \lambda_g$  is dynamically convex.*

*Remark 3.2.* In [HP08] one finds a version of the above theorem for Finsler metrics. Dynamical convexity is ensured once the flag curvatures are pinched by more than  $(r/(r+1))^2$  where  $r \geq 1$  is the reversibility parameter. In [HS13] one finds examples showing that this pinching condition is sharp for dynamical convexity.



From now on we denote

$$\begin{aligned}\phi_g^t & \text{ the geodesic flow on } T_g^1 S^2 \\ \phi^t & \text{ the lift of } \phi_g^t \text{ to } S^3 \text{ via } D_g\end{aligned}$$

so that the identity

$$\phi_g^t \circ D_g = D_g \circ \phi^t$$

holds. The generating vector field  $X$  of  $\phi^t$  is the Reeb vector field of the contact form  $f_g \lambda_0$ . Let  $\tilde{\sigma}_g$  be the lift of the frame  $\sigma_g$  (71) to  $S^3$  by  $D_g$ . Then  $\tilde{\sigma}_g$  is a  $d(f_g \lambda_0)$ -symplectic global frame of  $\xi = \ker \lambda_0$ , and as in (2) we get a well-defined global circle coordinate  $\Theta_{\tilde{\sigma}_g} : (\xi \setminus 0)/\mathbb{R}_+ \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ . The linearised flow  $D\phi^t$  on  $\xi$  induces a flow on  $(\xi \setminus 0)/\mathbb{R}_+$ , still denoted  $D\phi^t$ , whose generating vector field is denoted by  $\tilde{X}$ .

**Lemma 3.3.** *We have  $\min\{1, K_{\min}\} \leq i_{\tilde{X}} d\Theta_{\tilde{\sigma}_g} \leq \max\{1, K_{\max}\}$  where we denote by  $K_{\min}, K_{\max}$  the minimum and the maximum of the Gaussian curvature of  $(S^2, g)$ .*

*Proof.* Let  $p \in S^3$  and  $v_0 \in \xi_p$  be arbitrary. Then  $D_g(\phi^t(p)) = (d(t), \dot{d}(t))$  for some unit speed geodesic  $d(t)$  on  $(S^2, g)$ . With  $v(t) = D\phi^t \cdot v_0 \in \xi_{\phi^t(p)}$ , we have  $dD_g \cdot v(t) \in \ker \lambda_g$  and can use the splitting  $TTS^2 \simeq \mathcal{H} \oplus \mathcal{V}$  to decompose

$$dD_g \cdot v(t) = \left( \frac{DJ}{dt} \right)^{\text{vert}} + (J(t))^{\text{hor}}$$

into vertical and horizontal components, for some Jacobi field  $J(t)$  along  $d(t)$  everywhere perpendicular to  $\dot{d}(t)$ , see [HS13, Lemma 2.3]. We find a real-valued function  $b(t)$  such that  $J(t) = b(t)\dot{d}(t)^\perp$  and  $\frac{DJ}{dt} = b'(t)\dot{d}(t)^\perp$ . The equation for Jacobi fields becomes  $b''(t) = -K(t)b(t)$  where  $K(t)$  is the Gaussian curvature at  $d(t)$ . Using the frame  $\sigma_g$  (71) we can represent  $(dD_g)_{\phi^t(p)} \cdot v(t)$  as  $u(t) = (b'(t), b(t)) \in \mathbb{R}^2$  satisfying

$$u'(t) = J_0 S(t) u(t) \quad S(t) = \begin{pmatrix} 1 & 0 \\ 0 & K(t) \end{pmatrix} \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If we write  $u(t) = |u(t)|e^{i\theta(t)}$  then we get

$$\theta'(t) = \langle S(t)e^{i\theta(t)}, e^{i\theta(t)} \rangle$$

from where the desired conclusion follows.  $\square$

Let  $c : \mathbb{R}/L\mathbb{Z} \rightarrow (S^2, g)$  be a unit speed smooth immersion. It induces a smooth immersion  $s \in \mathbb{R}/L\mathbb{Z} \mapsto (c(s), \dot{c}(s)) \in T_g^1 S^2$  such that  $\lambda_g \cdot \frac{d}{ds}(c, \dot{c}) = g(c)(\dot{c}, \dot{c}) = 1$ . If  $c$  has no positive self-tangencies then  $(c, \dot{c})$  defines a knot on  $T_g^1 S^2$ . Note that  $\pi_1(T_{g_0}^1 S^2, \text{pt})$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ , and a generator can be taken as  $s \mapsto (c(s), \dot{c}(s))$  for an embedded unit speed loop  $s \mapsto c(s)$  in  $S^2$ .

**Lemma 3.4** ([HS13], subsection 3.1). *Let  $c : \mathbb{R}/L\mathbb{Z} \rightarrow S^2$  be a smooth unit speed embedding, and denote by  $\gamma_c : \mathbb{R}/2L\mathbb{Z} \rightarrow S^3$  a lift to  $S^3$  by  $D_g$  of the double cover of  $s \mapsto (c(s), \dot{c}(s))$ . Then  $\gamma_c$  is unknotted with self-linking number  $-1$ .*

Suppose now that  $c : \mathbb{R}/L\mathbb{Z} \rightarrow S^2$  is a unit speed embedding. Choose a lift  $\gamma_c : \mathbb{R}/2L\mathbb{Z} \rightarrow S^3$  of the double cover of  $s \mapsto (c(s), \dot{c}(s))$  to  $S^3$  by  $D_g$ . The Birkhoff annulus  $A_c \subset T_g^1 S^2$  associated to  $c$  is parametrised by

$$a : \mathbb{R}/L\mathbb{Z} \times [0, \pi] \rightarrow T_g^1 S^2, \quad a(s, \theta) = (c(s), \cos \theta \dot{c}(s) + \sin \theta \dot{c}(s)^\perp). \quad (79)$$

It follows from this formula that  $a$  admits a unique double lift to  $S^3$

$$\tilde{a} : \mathbb{R}/2L\mathbb{Z} \times [0, \pi] \rightarrow S^3 \quad (80)$$

fixed by requiring that  $D_g \circ \tilde{a}(s, \theta) = a(s \bmod L, \theta)$  holds identically, together with the boundary condition  $\tilde{a}(s, 0) = \gamma_c(s)$ . The image  $\tilde{A}_c$  of  $\tilde{a}$  is an embedded annulus in  $S^3$  and will be referred to as the *lifted Birkhoff annulus* associated to  $c$ . If  $A_c \subset T_g^1 S^2$  is the image of  $a$  then as sets we have  $\tilde{A}_c = D_g^{-1}(A_c)$  which holds because  $\tilde{A}_c$  is antipodal symmetric and satisfies  $D_g(\tilde{A}_c) = A_c$ . When  $\tilde{A}_c$  is oriented by  $ds \wedge d\theta$  the oriented boundary  $\partial \tilde{A}_c$  consists of  $\gamma_c$  together with a lift  $\hat{\gamma}_c : \mathbb{R}/2L\mathbb{Z} \rightarrow S^3$  of the double cover of  $s \mapsto (c(-s), -\dot{c}(-s))$ .

**Lemma 3.5.** *The identity*

$$\text{link}(\beta, \gamma_c) + \text{link}(\beta, \hat{\gamma}_c) = \text{int}(\beta, \tilde{A}_c)$$

*holds for every loop  $\beta$  on  $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ .*

*Proof.* Let  $D$  and  $\hat{D}$  be oriented disks spanned by  $\gamma_c$  and  $\hat{\gamma}_c$ , respectively, in such a way that the boundary orientations coincide with the flow orientation. Then  $C = D + \hat{D} - \tilde{A}_c$  is a 2-cycle on  $S^3$ . Thus  $0 = \text{int}(\beta, C)$  holds for every loop  $\beta$  on  $S^3$ . The conclusion follows.  $\square$

From now on we make the standing assumption that  $c$  is an embedded closed geodesic. We consider covering maps

$$\begin{aligned} P : \mathbb{R}/2L\mathbb{Z} \times [0, \pi] &\rightarrow \mathbb{R}/L\mathbb{Z} \times [0, \pi] & (s, \theta) &\mapsto (s \bmod L, \theta), \\ P_\infty : \mathbb{R} \times [0, \pi] &\rightarrow \mathbb{R}/2L\mathbb{Z} \times [0, \pi] & (s, \theta) &\mapsto (s \bmod 2L, \theta). \end{aligned} \quad (81)$$

We have an identity  $D_g \circ \tilde{a} = a \circ P$ . Let  $\Psi : A_c \rightarrow A_c$  be the return map, which exists by the result of Birkhoff [Bir66]. Note that, in principle, this return map would only be defined on the interior of  $A_c$  but then it can be extended smoothly up to the boundary by taking second conjugate points. We proceed assuming that the map has been extended in this manner. Then  $\tilde{A}_c$  is a global surface of section for the lifted flow on  $S^3$  and the return map  $\tilde{\Psi} : \tilde{A}_c \rightarrow \tilde{A}_c$  satisfies  $D_g \circ \tilde{\Psi} = \Psi \circ D_g$ . As above, this map exists and is smooth up on the closed annulus  $\tilde{A}_c$ .

Later we will use the geometry to choose an appropriate smooth lift

$$\bar{\Psi} : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R} \times [0, \pi] \quad (82)$$

of  $\tilde{\Psi}$ , i.e. a map that satisfies  $\tilde{a} \circ P_\infty \circ \bar{\Psi} = \tilde{\Psi} \circ \tilde{a} \circ P_\infty$ , and hence makes the following diagram commute

$$\begin{array}{ccccccc} \mathbb{R} \times [0, \pi] & \xrightarrow{P_\infty} & \mathbb{R}/2L\mathbb{Z} \times [0, \pi] & \xrightarrow{\tilde{a}} & \tilde{A}_c & \xrightarrow{D_g} & A_c \\ \downarrow \bar{\Psi} & & & & \downarrow \tilde{\Psi} & & \downarrow \Psi \\ \mathbb{R} \times [0, \pi] & \xrightarrow{P_\infty} & \mathbb{R}/2L\mathbb{Z} \times [0, \pi] & \xrightarrow{\tilde{a}} & \tilde{A}_c & \xrightarrow{D_g} & A_c \end{array}$$

### 3.2 Asymptotic estimates on linking and intersection numbers

Denote by  $\delta > 0$  the pinching factor. We may assume that

$$\delta = K_{\min} \leq K_{\max} = 1 \quad (83)$$

where  $K_{\min}, K_{\max}$  denote the minimum and the maximum of the Gaussian curvature. Recall that if  $\delta > 1/4$ , then  $\phi_g^t$  is dynamically convex ([HP08]), hence so is  $\phi^t$ . Lemma 3.4 and Theorem 1.10 imply that both  $\gamma_c$  and  $\hat{\gamma}_c$  span disk-like global surfaces of section. Denote by  $\hat{D}$  a  $\partial$ -strong disk-like surface of section spanned by  $\hat{\gamma}_c$ ; see Remark 1.11.

We write  $\Theta = \Theta_{\sigma_g}$  for simplicity. With  $u \in \xi$ ,  $u \neq 0$ , arbitrary we denote

$$\Delta\Theta(T, u) = \tilde{\Theta}(T, u) - \tilde{\Theta}(0, u)$$

where  $t \mapsto \tilde{\Theta}(t, u)$  is a continuous lift of  $t \mapsto \Theta(D\phi^t(u))$ . It does not depend on the choice of lift.

Let  $T \geq 0$ ,  $x \in S^3 \setminus \hat{\gamma}_c$ . Recall that  $k(T, x; \hat{D})$  is a loop obtained by closing the piece of trajectory  $\phi^{I(T, x; \hat{D})}(x)$  with a path  $\alpha$  in  $\hat{D} \setminus \hat{\gamma}_c$ . From now on we shall refer to  $\alpha$  as a closing path for  $(T, x; \hat{D})$ . Clearly,  $\text{link}(k(T, x; \hat{D}), \hat{\gamma}_c)$  does not depend on the choice of  $\alpha$ . Suppose further that  $x \in S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ . The number  $\text{int}(k(T, x; \hat{D}), \tilde{A}_c)$  might not be well-defined since  $\alpha$  could go through the (unique) intersection point of  $\gamma_c$  and  $\hat{D}$ . Even if it does not touch this point,  $\alpha$  can be chosen in such a way that  $\text{int}(k(T, x; \hat{D}), \tilde{A}_c)$  is any integer. Below we might write  $k_\alpha(T, x; \hat{D})$  if the dependence on  $\alpha$  needs to be made explicit.

**Lemma 3.6.** *There exists  $C > 0$  with the following property. For every point  $x \in S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$  and every  $T > 0$  there exists a closing path  $\alpha$  for  $(T, x; \hat{D})$  contained in  $\hat{D} \setminus (\gamma_c \cup \hat{\gamma}_c)$  such that*

$$|\text{int}(k_\alpha(T, x; \hat{D}), \tilde{A}_c) - \#\{t \in [0, T] \mid \phi^t(x) \in \tilde{A}_c\}| \leq C. \quad (84)$$

*Proof.* Since  $\hat{D}$  is  $\partial$ -strong we can invoke Lemma A.4 to slightly  $C^\infty$ -perturb  $\hat{D}$  to a new  $\partial$ -strong disk-like global surface of section  $\tilde{D}$  satisfying  $\partial\tilde{D} = \hat{\gamma}_c$  in relative generic position with respect to  $\tilde{A}_c$ .

Denote by  $K$  the closure of  $(\tilde{D} \setminus \partial \tilde{D}) \cap (\tilde{A}_c \setminus \partial \tilde{A}_c) = (\tilde{D} \cap \tilde{A}_c) \setminus (\hat{\gamma}_c \cup \gamma_c)$ . By Lemma A.6,  $K$  is a smooth compact 1-manifold,  $\partial K = K \cap (\hat{\gamma}_c \cup \gamma_c)$ , and at the boundary points of  $K$  the 1-manifolds  $\hat{\gamma}_c \cup \gamma_c$  and  $K$  are not tangent.

Recall the interval  $I(T, x; \hat{D})$  defined in (4). Denote by  $\hat{x}_0 \in \hat{D}$  and  $\hat{x}_1 \in \hat{D}$  the initial and end points of the trajectory  $\phi^{I(T, x; \hat{D})}(x)$ . We consider the small transfer map  $\psi : p \in \tilde{D} \mapsto \phi^{g(p)}(p) \in \hat{D}$  where  $g$  is the function in (e) Lemma A.4, see Remark A.8. Define  $\tilde{x}_0 = \psi^{-1}(\hat{x}_0)$ ,  $\tilde{x}_1 = \psi^{-1}(\hat{x}_1)$ . There is an induced interval  $\tilde{I}$  whose end points are close to those of  $I(T, x; \hat{D})$  such that  $\phi^{\tilde{I}}(x)$  is a piece of trajectory from  $\tilde{x}_0$  to  $\tilde{x}_1$ . The Lebesgue measure of the symmetric difference between  $\tilde{I}$  and  $I(T, x; \hat{D})$  is not larger than  $2\|g\|_{L^\infty} \ll 1$ .

Assume  $\tilde{x}_0 \neq \tilde{x}_1$ . Consider a smooth immersion  $\alpha_0 : [0, 1] \rightarrow \tilde{D} \setminus (\hat{\gamma}_c \cup \gamma_c)$  from  $\tilde{x}_1 = \alpha_0(0)$  to  $\tilde{x}_0 = \alpha_0(1)$  that is transverse to  $K$ . Let  $\beta$  be a connected component of  $K$ , and suppose  $t_0 < t_1$  satisfy  $\alpha_0(t_0), \alpha_0(t_1) \in \beta$ . Consider the piecewise smooth arc  $\hat{\alpha}_0$  obtained from  $\alpha_0$  by replacing  $\alpha_0|_{[t_0, t_1]}$  with the arc in  $\beta$  from  $\alpha_0(t_0)$  to  $\alpha_0(t_1)$ . A further small perturbation of  $\hat{\alpha}_0$  creates a smooth immersed arc  $\alpha_1 : [0, 1] \rightarrow \tilde{D} \setminus (\hat{\gamma}_c \cup \gamma_c)$  from  $\tilde{x}_1$  to  $\tilde{x}_0$ , transverse to  $K$ , such that the number of intersection points of  $\alpha_1$  and any component of  $K$  is no larger than that of  $\alpha_0$ , and the number of intersection points of  $\alpha_1$  with  $\beta$  is strictly less than the number of intersection points of  $\alpha_0$  with  $\beta$ . Proceeding inductively in this way, after a finite number of steps we end up with an immersed arc  $\tilde{\alpha}$  in  $\tilde{D} \setminus (\hat{\gamma}_c \cup \gamma_c)$  from  $\tilde{x}_1$  to  $\tilde{x}_0$  which is transverse to  $K$  and intersects each connected component of  $K$  at most once. Consider the loop  $\tilde{k}$  obtained by concatenating  $\phi^{\tilde{I}}(x)$  with  $\tilde{\alpha}$ . Then

$$|\text{int}(\tilde{k}, \tilde{A}_c) - \#\{t \in \tilde{I} \mid \phi^t(x) \in \tilde{A}_c\}| \leq N + 2 \quad (85)$$

where  $N$  is the number of connected components of  $K$ . The image  $\alpha = \psi(\tilde{\alpha}) \subset \hat{D}$  is an arc from  $\hat{x}_1$  to  $\hat{x}_0$ . By construction  $k_\alpha(T, x; \hat{D})$  is homotopic to  $\tilde{k}$  on  $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ . Hence

$$\text{int}(\tilde{k}, \tilde{A}_c) = \text{int}(k_\alpha(T, x; \hat{D}), \tilde{A}_c) \quad (86)$$

Finally, consider  $\tau_{\max}(\hat{D})$  the supremum of the return time of  $\hat{D}$  and  $\tau_{\min}(\tilde{A}_c) > 0$  the infimum of the return time of  $\tilde{A}_c$ . Note that the Lebesgue measure of the symmetric difference of  $\tilde{I}$  and  $[0, T]$  is not larger than  $2(\tau_{\max}(\hat{D}) + \|g\|_{L^\infty})$ . Note also that for any interval  $J$  one estimates

$$\#\{t \in J \mid \phi^t(x) \in \tilde{A}_c\} \leq \frac{|J|}{\tau_{\min}(\tilde{A}_c)} + 2$$

Hence

$$\begin{aligned} & |\#\{t \in \tilde{I} \mid \phi^t(x) \in \tilde{A}_c\} - \#\{t \in [0, T] \mid \phi^t(x) \in \tilde{A}_c\}| \\ & \leq \frac{2(\tau_{\max}(\hat{D}) + \|g\|_{L^\infty})}{\tau_{\min}(\tilde{A}_c)} + 2 \end{aligned} \quad (87)$$

Combining (85), (86) and (87) we arrive at (84) with

$$C = N + \frac{2(\tau_{\max}(\hat{D}) + \|g\|_{L^\infty})}{\tau_{\min}(\tilde{A}_c)} + 4$$

as desired.

The case  $\tilde{x}_0 = \tilde{x}_1$  is left to the reader.  $\square$

Next we recall some definitions and facts about Riemannian geometry that will be useful all along this section; see [ABHS17].

**Definition 3.7.** A set  $D \subset S^2$  is said to be a geodesic polygon if it is the closure of an open disk bounded by a simple closed unit speed broken geodesic  $\gamma : \mathbb{R}/L\mathbb{Z} \rightarrow S^2$ . A corner of  $\gamma : \mathbb{R}/L\mathbb{Z} \rightarrow S^2$  is a point  $\gamma(t)$  such that  $\gamma'_+(t) \notin \mathbb{R}^+ \gamma'_-(t)$ , where  $\gamma'_\pm$  denote one-sided derivatives. The corners of  $\gamma$  are called vertices of  $D$ , and a side of  $D$  is a smooth geodesic arc contained in  $\partial D$  connecting two adjacent vertices.

For  $p \in S^2$  and  $u, v \in T_p S^2$  non-colinear vectors, consider the sets

$$\Delta(u, v) := \{su + tv \mid s, t \geq 0\}, \quad \Delta_r(u, v) := \{w \in \Delta(u, v) \mid |w| < r\}$$

for some  $r > 0$ . We denote by  $\text{inj}_p$  the injectivity radius at  $p$ , and  $\text{inj} = \inf_p \text{inj}_p$  the injectivity radius of  $g$ .

**Definition 3.8.** A geodesic polygon  $D \subset S^2$  is convex if for every corner  $p = \gamma(t)$  of  $\partial D$  we find  $0 < r < \text{inj}_p$  small enough such that  $D \cap B_r(p) = \exp(\Delta_r(-\gamma'_-(t), \gamma'_+(t)))$ .

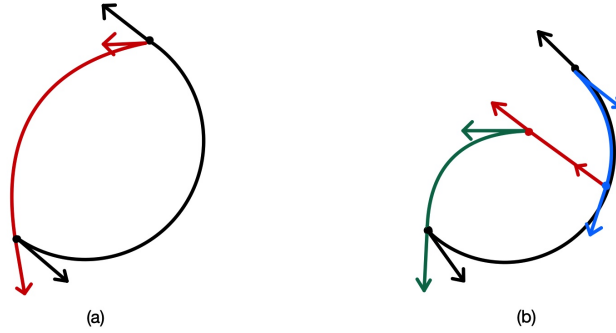


Figure 1: Examples of geodesic polygons, convex (a) and non convex (b).

**Theorem 3.9** (Corollary of Toponogov's Theorem [ABHS17, Theorem A.12]). *If  $D$  is a convex geodesic polygon in  $(S^2, g)$  then  $|\partial D| \leq 2\pi/\sqrt{\delta}$  where  $|\partial D|$  denotes the perimeter of  $D$ .*

**Theorem 3.10** ([Kli59], [Kli82, Theorem 2.6.9]). *The injectivity radius satisfies  $\text{inj} \geq \pi$ .*

In particular it follows from theorems 3.9 and 3.10 that if  $D \subset S^2$  is a convex geodesic polygon for the metric  $g$  then

$$2\pi \leq |\partial D| \leq \frac{2\pi}{\sqrt{\delta}}. \quad (88)$$

*Notation 3.11.* Let  $(x, v) \in T_g^1 S^2$  satisfy  $x \in c$ . If  $v$  is not tangent to  $c$  then define

$$\tau_+(x, v) := \min\{t > 0 \mid \pi \circ \phi_g^t(x, v) \in c\}. \quad (89)$$

That is,  $\tau_+(x, v)$  is the first positive time when the geodesic ray with initial conditions  $(x, v)$  meets again  $c$ . Since  $\delta > 1/4$ , by [ABHS17, Lemma 3.9] the geodesic arc  $\{\pi \circ \phi_g^t(x, v) \mid t \in [0, \tau_+(x, v)]\}$  is embedded, in particular  $x \neq \pi \circ \phi_g^{\tau_+(x, v)}(x, v)$ . If  $v$  is tangent to  $c$  then define  $\tau_+(x, v)$  as the time to the first conjugate point of  $(x, v)$ . Denote by  $\alpha_+(x, v)$  the path contained in  $c$  joining  $x$  to  $\pi \circ \phi_g^{\tau_+(x, v)}(x, v)$  following the orientation of  $c$ . See Figure 2.

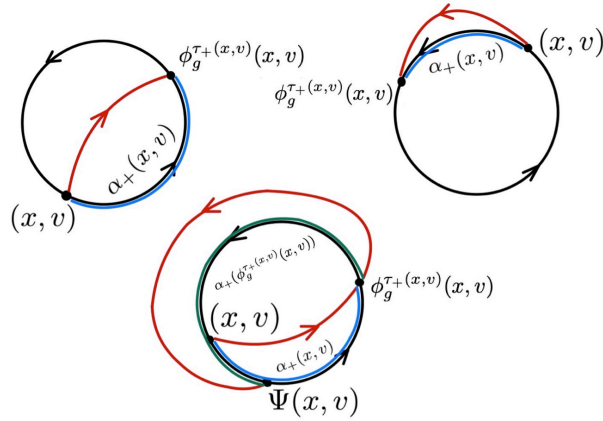


Figure 2: Examples of paths  $\alpha_+$ .

**Lemma 3.12.** Assume that  $\delta > 4/9$ . Let  $v$  be a unit vector based at  $x \in \gamma$ . The length of the arc obtained by concatenating  $\alpha_+(x, v)$  and  $\alpha_+(\phi_g^{\tau_+(x, v)}(x, v))$  lies in  $(2L/3, 3L/2)$ . In particular, for every  $(s, \theta) \in \mathbb{R}/L\mathbb{Z} \times [0, \pi]$  it holds

$$\pi \circ \Psi(a(s, \theta)) \neq c \left( s + \frac{L}{2} \right).$$

*Proof.* By [ABHS17, Lemma 3.9] the geodesic arc  $\phi_g^{[0, \tau_+(x, v)]}(x, v)$  is embedded, we denote by  $\ell$  its length. Thus the length  $\eta$  of the arc  $\alpha_+(x, v)$  satisfies  $\eta \in (0, L)$ . Similarly, the length  $\nu$  of  $\alpha_+(\phi_g^{\tau_+(x, v)}(x, v))$  satisfies  $\nu \in (0, L)$ . The length of the arc obtained by concatenating  $\alpha_+(x, v)$  and  $\alpha_+(\phi_g^{\tau_+(x, v)}(x, v))$  is then  $\eta + \nu$ .

Consider the convex geodesic polygon bounded by  $\phi_g^{[0, \tau_+(x, v)]}(x, v)$  and  $\alpha_+(x, v)$ . By (88) we have

$$2\pi \leq \eta + \ell \leq \frac{2\pi}{\sqrt{\delta}}. \quad (90)$$

Similarly, we consider the convex geodesic polygon whose boundary is made up of  $\phi_g^{[0, \tau_+(x, v)]}(x, v)$  and of  $c \setminus \alpha_+(x, v)$ . Observe that the length of  $c \setminus \alpha_+(a(s, \theta))$  is  $L - \eta \in (0, L)$ . Again by (88)

$$2\pi \leq L - \eta + \ell \leq \frac{2\pi}{\sqrt{\delta}}. \quad (91)$$

Thus, considering the difference between (90) and (91), it holds

$$\frac{L}{2} + \pi \left(1 - \frac{1}{\sqrt{\delta}}\right) \leq \eta \leq \frac{L}{2} + \pi \left(\frac{1}{\sqrt{\delta}} - 1\right).$$

By (88) applied at the geodesic loop  $c$ , we deduce that  $2\pi \leq L \leq 2\pi/\sqrt{\delta}$ . Hence

$$\frac{L}{2}\sqrt{\delta} \leq \eta \leq \frac{L}{2} \frac{1}{\sqrt{\delta}}. \quad (92)$$

Now, repeat then the same argument at the point  $(x', v') := \phi_g^{\tau_+(x, v)}(x, v)$ . Observe that

$$\phi_g^{\tau_+(x', v') + \tau_+(x, v)}(x, v) = \Psi(x, v).$$

As above one proves that

$$\frac{L}{2}\sqrt{\delta} \leq \nu \leq \frac{L}{2} \frac{1}{\sqrt{\delta}}. \quad (93)$$

Consequently,

$$L\sqrt{\delta} \leq \eta + \nu \leq \frac{L}{\sqrt{\delta}}.$$

Since  $\delta > 4/9$ , we deduce that

$$\frac{2}{3}L < \eta + \nu < \frac{3}{2}L \quad (94)$$

as desired.

Observe now that the distance along the geodesic  $c$  between  $c(s)$  and  $\pi \circ \Psi(a(s, \theta))$  is equal to  $\eta + \nu$  modulo  $L\mathbb{Z}$ . By (94) it cannot be equal to  $\frac{L}{2} \bmod L\mathbb{Z}$ .  $\square$

The proof of Lemma 3.12 actually also proves the following statement.

**Lemma 3.13.** *If  $\delta > 1/4$  then for every  $(x, v) \in T_g^1 S^2$  with  $x \in c$  it holds*

$$|\alpha_+(x, v)| \leq F(\delta) \quad (95)$$

where  $|\alpha_+(x, v)|$  is the length of  $\alpha_+(x, v)$  and  $F : (1/4, 1] \rightarrow [0, +\infty)$  is the function

$$F(\delta) := \frac{L}{2} + \pi \left(\frac{1}{\sqrt{\delta}} - 1\right). \quad (96)$$

The above estimates allow us to fix the special lift  $\bar{\Psi}$  (82).

**Corollary 3.14.** *If  $\delta > 4/9$  then there is a unique lift  $\bar{\Psi} : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R} \times [0, \pi]$  such that for every  $(S, \theta) \in \mathbb{R} \times [0, \pi]$*

$$p_1 \circ \bar{\Psi}(S, \theta) - S \in \left( \frac{2L}{3}, \frac{3L}{2} \right)$$

where  $p_1 : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R}$  is the projection on the first coordinate.

*Proof.* Consider the return map  $\Psi : A_c \rightarrow A_c$ . Let  $\bar{\Psi} : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R} \times [0, \pi]$  be any lift of  $\tilde{a}^{-1} \circ \tilde{\Psi} \circ \tilde{a}$ . The latter map is defined on  $\mathbb{R}/2L\mathbb{Z} \times [0, \pi]$ . Observe that  $\bar{\Psi}$  is also a lift of  $a^{-1} \circ \Psi \circ a$ . In particular,  $\bar{\Psi}$  is a diffeomorphism that preserves the area form  $\sin \theta \, ds \wedge d\theta$ , maps each boundary component into itself and satisfies

$$\bar{\Psi}(s + L, \theta) = \bar{\Psi}(s, \theta) + (L, 0).$$

Let  $(s, \theta) \in \mathbb{R}/L\mathbb{Z} \times [0, \pi]$ . The arc  $\alpha_+(a(s, \theta))$  has length  $\eta$ , and the arc  $\alpha_+(\phi_g^{\tau_+(a(s, \theta))}(a(s, \theta)))$  has length  $\nu$ , where we see  $\eta$  and  $\nu$  as functions of  $(s, \theta)$ . These functions are continuous. By Lemma 3.12 the length of the arc obtained by concatenating these two arcs is  $\eta + \nu \in (2L/3, 3L/2)$ , for every  $(s, \theta)$ . The endpoints of such arc are  $c(s)$  and  $c(p_1 \circ a^{-1} \circ \Psi \circ a(s, \theta))$ , where here we still denoted by  $p_1 : \mathbb{R}/L\mathbb{Z} \times [0, \pi] \rightarrow \mathbb{R}/L\mathbb{Z}$  the projection onto the first coordinate, with no fear of ambiguity. Thus, for every  $(S, \theta) \in \mathbb{R} \times [0, \pi]$

$$p_1 \circ \bar{\Psi}(S, \theta) - S = \eta + \nu + kL,$$

for some  $k \in \mathbb{Z}$ , where now  $\eta$  and  $\nu$  are seen as continuous functions of  $(S, \theta)$  that are  $L$ -periodic on  $S$ . To complete the proof we now simply need to consider the unique lift with  $k = 0$ .  $\square$

From Lemma 3.13 and Corollary 3.14, we also deduce the following

**Corollary 3.15.** *If  $\delta > 4/9$  then there exists a unique lift  $\bar{\Psi} : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R} \times [0, \pi]$  such that, for every  $(S, \theta) \in \mathbb{R} \times [0, \pi]$ , it holds*

$$L + 2\pi \left( 1 - \frac{1}{\sqrt{\delta}} \right) \leq p_1 \circ \bar{\Psi}(S, \theta) - S \leq L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right).$$

According to [HSW19, Appendix B], let  $(\Pi, \gamma_c \cup \hat{\gamma}_c)$  be the open book decomposition of  $S^3$  such that  $\tilde{A}_c$  is one of its pages. It holds that

$$H_1(S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)) \simeq H_1(\tilde{A}_c) \oplus \langle e \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$$

where  $e$  is a loop such that  $\Pi_* e$  is the positive generator of  $H_1(\mathbb{R}/\mathbb{Z})$ . In particular, we can choose  $e$  so that  $\text{link}(e, \gamma_c) = 0$  and  $\text{link}(e, \hat{\gamma}_c) = 1$ . Choose now a loop  $f$  lying in  $\tilde{A}_c \setminus (\gamma_c \cup \hat{\gamma}_c)$  which is a generator of  $H_1(\tilde{A}_c)$ . We can choose  $f$  so that  $\text{link}(f, \gamma_c) = 1$  and  $\text{link}(f, \hat{\gamma}_c) = -1$ . See Figure 3.



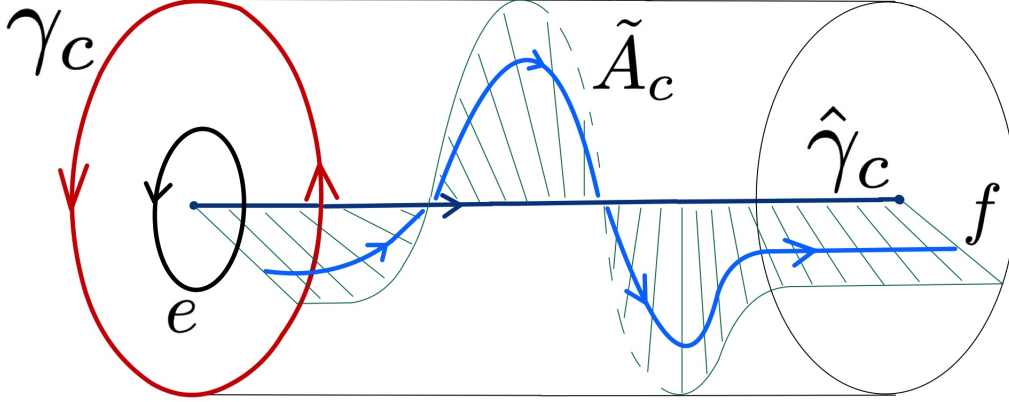


Figure 3: The chosen loops  $e$  and  $f$ .

**Lemma 3.16.** *The loop  $f$  is homotopic to the loop  $s \in [0, 2L] \mapsto \tilde{a} \circ P_\infty(-s, \theta_*)$  in  $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ , for any  $\theta_* \in (0, \pi)$ .*

*Proof.* The free homotopy classes of loops in  $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$  are determined by the linking numbers with  $\gamma_c$  and  $\hat{\gamma}_c$ . Note that  $\tilde{a} \circ P_\infty(-s, 0)$  is  $-\gamma_c$ , and that  $\tilde{a} \circ P_\infty(-s, \pi)$  is  $\hat{\gamma}_c$ , where  $s \in [0, 2L]$ . Hence the loop  $\beta : s \in [0, 2L] \mapsto \tilde{a} \circ P_\infty(-s, \theta_*)$  is homotopic to  $-\gamma_c$  in  $S^3 \setminus \hat{\gamma}_c$ , and is homotopic to  $\hat{\gamma}_c$  in  $S^3 \setminus \gamma_c$ . We conclude that  $\text{link}(\beta, \hat{\gamma}_c) = \text{link}(-\gamma_c, \hat{\gamma}_c) = -1$  and  $\text{link}(\beta, \gamma_c) = \text{link}(\hat{\gamma}_c, \gamma_c) = 1$ . Hence  $\beta$  and  $f$  are homotopic in  $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ .  $\square$

Let us introduce some notation needed for the statement of Lemma 3.17 below. Let  $T > 0$  and  $x \in S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ , and consider the loop  $k(T, x; \hat{D})$  in  $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ . There is a choice of closing path in  $\hat{D} \setminus \hat{\gamma}_c$  which is not made explicit in the notation. Assume that this closing path does not intersect  $\gamma_c$ , and denote

$$M(T, x) = \text{link}(k(T, x; \hat{D}), \hat{\gamma}_c), \quad N(T, x) = \text{link}(k(T, x; \hat{D}), \gamma_c). \quad (97)$$

Note that  $M$  depends only on  $(T, x)$  and that, in contrast,  $N$  highly depends on the choice of closing path; again we do not make this dependence explicit in the notation, for simplicity. By Lemma 3.5 we write in  $H_1(S^3 \setminus (\gamma_c \cup \hat{\gamma}_c))$

$$k(T, x; \hat{D}) = (M(T, x) + N(T, x))e + N(T, x)f. \quad (98)$$

Let  $\underline{t} > 0$  and  $\underline{x} \in \tilde{A}_c \setminus (\gamma_c \cup \hat{\gamma}_c)$  be defined by

$$\underline{t} = \min\{t \geq 0 \mid \phi^{t+\underline{t}^{\hat{D}}}(x) \in \tilde{A}_c\}, \quad \underline{x} = \phi^{t+\underline{t}^{\hat{D}}}(x). \quad (99)$$

Note that  $\underline{t}$  and  $\underline{x}$  depend only on  $x$  and  $\hat{D}$ .

**Lemma 3.17.** *There exists  $C \geq 0$  depending only on  $\hat{D}$  and  $\tilde{A}_c$  with the following significance. Let  $T > 0$  and  $x \in S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$  be arbitrary, and let  $(s_0, \theta_0) \in \mathbb{R} \times [0, \pi]$  satisfy  $\tilde{a} \circ P_\infty(s_0, \theta_0) = \underline{x}$ . Then there exists a closing path in  $\hat{D} \setminus \hat{\gamma}_c$  such that*

$$\left| \frac{p_1 \circ \bar{\Psi}^{M(T,x)+N(T,x)}(s_0, \theta_0) - s_0}{2L} - M(T, x) \right| \leq C$$

where  $p_1 : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R}$  is the projection onto the first coordinate.

*Proof.* Denote  $p = \phi^{t_-^{\hat{D}}(x)}(x)$ . Consider

$$m(T, x) = \#\{t \in [0, T + t_+^{\hat{D}}(\phi^T(x)) - t_-^{\hat{D}}(x)] \mid \phi^t(p) \in \tilde{A}_c\}.$$

For each  $i = 0, \dots, m(T, x) - 1$  let  $t_i$  be defined by

$$\phi^{t_i}(p) \in \tilde{A}_c, \quad 0 \leq t_0 = \underline{t} < t_1 < \dots < t_{m(T,x)-1} \leq T + t_+^{\hat{D}}(\phi^T(x)) - t_-^{\hat{D}}(x).$$

If  $(s_0, \theta_0) \in \mathbb{R} \times [0, \pi]$  satisfies  $\tilde{a} \circ P_\infty(s_0, \theta_0) = \underline{x}$  and  $(s_i, \theta_i) := \bar{\Psi}^i(s_0, \theta_0)$  then we have

$$\tilde{a} \circ P_\infty(s_i, \theta_i) = \phi^{t_i}(p)$$

for every  $i = 0, \dots, m(T, x) - 1$ . Let  $k \in \mathbb{Z}$  be such that

$$k - 1 < \frac{s_{m(T,x)-1} - s_0}{2L} \leq k. \quad (100)$$

We introduce a path  $\varepsilon : [0, 1] \rightarrow \mathbb{R} \times [0, \pi]$  from  $\varepsilon(0) = (s_{m(T,x)-1}, \theta_{m(T,x)-1})$  to  $\varepsilon(1) = (s_0 + 2Lk, \theta_0)$  defined in the following way:

$$\varepsilon(\tau) = \begin{cases} (s_{m(T,x)-1} + 2\tau(s_0 + 2Lk - s_{m(T,x)-1}), \theta_{m(T,x)-1}) & 0 \leq \tau \leq 1/2, \\ (s_0 + 2Lk, \theta_{m(T,x)-1} + (2\tau - 1)(\theta_0 - \theta_{m(T,x)-1})) & 1/2 \leq \tau \leq 1. \end{cases}$$

See Figure 4.

For every  $i = 1, \dots, m(T, x) - 1$  construct a path  $\ell_i : [0, 1] \rightarrow \mathbb{R} \times [0, \pi]$  as follows. If  $i \leq m(T, x) - 2$  then

$$\ell_i(\tau) = \begin{cases} (s_i + 2\tau(s_{i-1} - s_i), \theta_i) & 0 \leq \tau \leq 1/2, \\ (s_{i-1}, \theta_i + (2\tau - 1)(\theta_{i-1} - \theta_i)) & 1/2 \leq \tau \leq 1. \end{cases}$$

The path  $\ell_{m(T,x)-1}$  is defined similarly, but following  $\varepsilon$  with a path from  $\varepsilon(1) = (s_0 + 2Lk, \theta_0)$  to  $(s_{m(T,x)-2}, \theta_{m(T,x)-2})$ , see Figure 4. Note that  $\ell_i$  connects  $(s_i, \theta_i)$  to  $(s_{i-1}, \theta_{i-1})$ .

Denote by  $\mathcal{L}$  the concatenation of the path  $\tau \mapsto \varepsilon(1 - \tau)$  with all the paths  $\ell_i$ ,  $i = 1, \dots, m(T, x) - 1$ . By Lemma 3.16, the path  $\tilde{a} \circ P_\infty(\mathcal{L})$  is a loop homotopic to  $kf$  in  $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ . Denote  $p = \phi^{t_-^{\hat{D}}(x)}(x)$ . For  $i = 1, \dots, m(T, x) - 2$  denote by  $\nu_i$  the loop

$$\nu_i = \phi^{[t_{i-1}, t_i]}(p) + \tilde{a} \circ P_\infty(\ell_i).$$

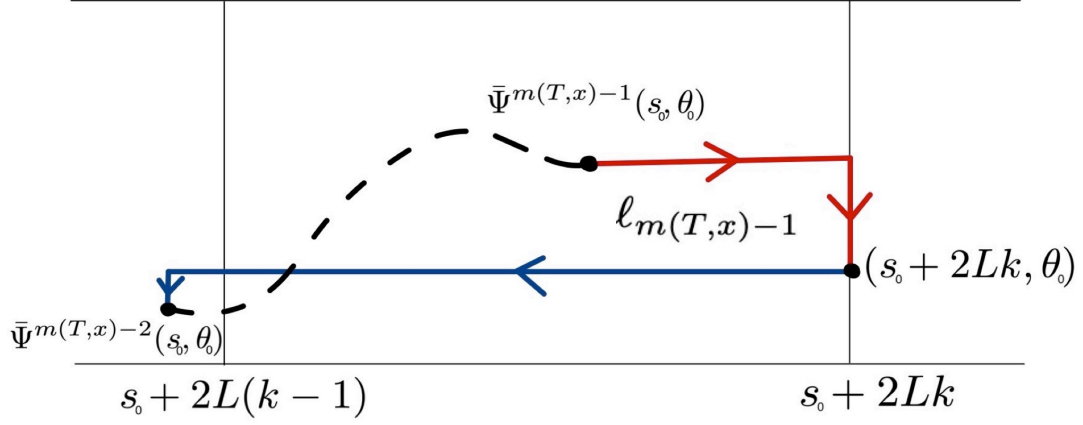


Figure 4: The path  $\ell_{m(T,x)-1}$ .

See Figure 5. Consider also

$$\nu_{m(T,x)-1} = \phi^{[t_{m(T,x)-2}, t_{m(T,x)-1}]}(p) + \tilde{a} \circ P_\infty(\ell_{m(T,x)-1}).$$

In  $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ , since  $\tilde{a} \circ P_\infty(\mathcal{L})$  is homotopic to  $kf$ , we have that

$$\phi^{[t_0, t_{m(T,x)-1}]}(p) + \tilde{a} \circ P_\infty(\varepsilon) + kf$$

is homologous to

$$\nu_1 + \cdots + \nu_{m(T,x)-1}.$$

This implies

$$\begin{aligned} & \text{link}(\phi^{[t_0, t_{m(T,x)-1}]}(p) + \tilde{a} \circ P_\infty(\varepsilon), \hat{\gamma}_c) - k \\ &= \text{link}(\phi^{[t_0, t_{m(T,x)-1}]}(p) + \tilde{a} \circ P_\infty(\varepsilon) + kf, \hat{\gamma}_c) \\ &= \sum_{i=1}^{m(T,x)-1} \text{link}(\nu_i, \hat{\gamma}_c). \end{aligned} \tag{101}$$

CLAIM 1. For every  $i = 1, \dots, m(T, x) - 1$  it holds  $\text{link}(\nu_i, \hat{\gamma}_c) = 0$ .

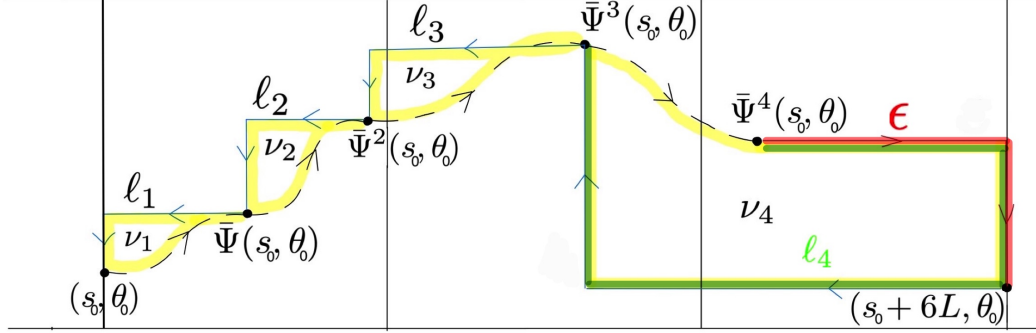


Figure 5: An example of the paths  $\ell_i, \nu_i$  and  $\epsilon$ .

*Proof of Claim 1.* The loop  $\nu_i \subset S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$  is made up of a trajectory of the lifted geodesic flow  $\phi^t$  and a path contained in the annulus  $\tilde{A}_c$ . Consider the loop  $D_g(\nu_i) \subset T_g^1 S^2$ , see Figure 6. This loop consists of the velocity vectors of a geodesic starting at (and transversely to)  $c$  up to the second hitting point with  $c$  (first hit with  $A_c$ ), and then of velocity vectors obtained by parallel transport back the initial point, and then of a vertical deformation. In particular,  $D_g(\nu_i)$  does not intersect the set  $\dot{c} \cup (-\dot{c}) = D_g(\gamma_c \cup \hat{\gamma}_c)$ . The path  $D_g(\nu_i)$  can be deformed in a continuous way into a loop  $\Gamma$  contained in  $\dot{c}$  as depicted in Figure 6: the initial velocity vector of  $\pi \circ D_g(\nu_i)$  becomes more and more positively tangent to  $c$ , the second hitting point converges to second conjugate point, and one parallel transport back by vectors positively tangent to  $c$ . Such a deformation can be realised within  $T_g^1 S^2 \setminus (-\dot{c})$ . Moreover, observe that the loop  $\Gamma \subset \dot{c}$  is homotopic to a point in  $\dot{c}$ . Thus,  $D_g(\nu_i)$  is contractible to a point in  $T_g^1 S^2 \setminus (-\dot{c})$ . By the homotopy lifting property, also the loop  $\nu_i$  is contractible to a point in  $S^3 \setminus \hat{\gamma}_c$ . Hence  $\text{link}(\nu_i, \hat{\gamma}_c) = 0$  and Claim 1 is proved.

From (101) and Claim 1 we deduce that

$$\text{link}(\phi^{[t_0, t_m(T, x) - 1]}(p) + \tilde{a} \circ P_\infty(\epsilon), \hat{\gamma}_c) = k. \quad (102)$$

CLAIM 2.  $\text{link}(\phi^{[t_0, t_m(T, x) - 1]}(p) + \tilde{a} \circ P_\infty(\epsilon), \hat{\gamma}_c) = M(T, x) + O(1)$ .

*Proof of Claim 2.* Recall that  $k(T, x; \hat{D})$  was defined as the concatenation of the piece of trajectory  $\phi^{[0, T + t_+^{\hat{D}}(\phi^T(x)) - t_-^{\hat{D}}(x)]}(p)$  with a path  $\alpha \subset \hat{D}$ . Hence

$$\begin{aligned} M(T, x) &= \text{link}(k(T, x; \hat{D}), \hat{\gamma}_c) \\ &= \#\{t \in [0, T + t_+^{\hat{D}}(\phi^T(x)) - t_-^{\hat{D}}(x)] \mid \phi^t(p) \in \hat{D}\} - 1. \end{aligned} \quad (103)$$

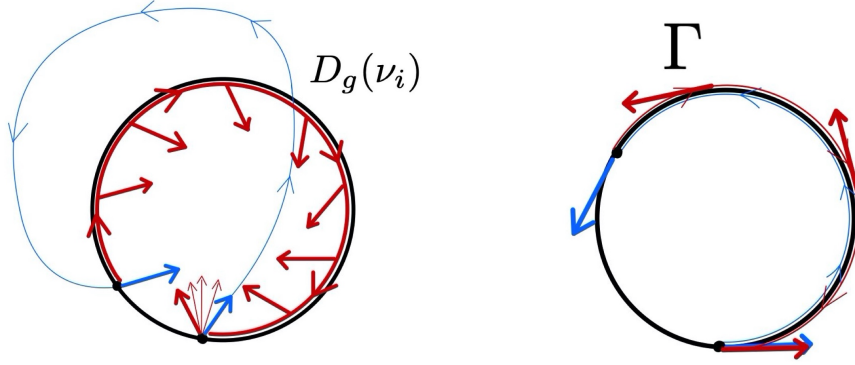


Figure 6: The projection through  $D_g$  of the loop  $\nu_i$  can be deformed into a loop contained in  $\dot{c}$ .

Now note that

$$\begin{aligned} & \text{link}(\phi^{[t_0, t_{m(T,x)}-1]}(p) + \tilde{a} \circ P_\infty(\varepsilon), \hat{\gamma}_c) \\ &= \#\{t \in [t_0, t_{m(T,x)}-1] \mid \phi^t(p) \in \hat{D}\} + Q \end{aligned} \quad (104)$$

where  $Q \in \mathbb{Z}$  satisfies

$$|Q| \leq \limsup_{\beta \rightarrow \tilde{a} \circ P_\infty(\varepsilon)} \text{int}(\beta, \hat{D}) + 2 \quad (105)$$

where the  $\limsup$  is taken with respect to smooth and  $C^0$ -small perturbations  $\beta$  of  $\tilde{a} \circ P_\infty(\varepsilon)$ , with endpoints fixed, that are transverse to  $\hat{D}$  (up to end points), as  $\beta$   $C^0$ -converges to  $\tilde{a} \circ P_\infty(\varepsilon)$ . Clearly this  $\limsup$  exists. Moreover, it is bounded from above by a constant independent of  $(T, x)$  since the horizontal translation of  $\epsilon$  in  $\mathbb{R} \times [0, \pi]$  is bounded by the constant  $2L$ , which is independent of  $(T, x)$ . It follows that

$$\begin{aligned} & |\text{link}(\phi^{[t_0, t_{m(T,x)}-1]}(p) + \tilde{a} \circ P_\infty(\varepsilon), \hat{\gamma}_c) - \text{link}(k(T, x; \hat{D}), \hat{\gamma}_c)| \\ & \leq \#\{t \in [0, t_0] \mid \phi^t(p) \in \hat{D}\} \\ & \quad + \#\{t \in (t_{m(T,x)}-1, T + t_+^{\hat{D}}(\phi^T(x)) - t_-^{\hat{D}}(x)) \mid \phi^t(p) \in \hat{D}\} \\ & \quad + \limsup_{\beta \rightarrow \tilde{a} \circ P_\infty(\varepsilon)} \text{int}(\beta, \hat{D}) + 2. \end{aligned} \quad (106)$$

Since  $\hat{D}$  is a strong global surface of section, we get estimates of the form

$$\#\{t \in [a, b] \mid \phi^t(y) \in \hat{D}\} \leq \frac{b-a}{\tau_{\min}} + 1$$

uniformly in  $y \in S^3 \setminus \hat{\gamma}_c$  and in  $[a, b] \subset \mathbb{R}$ . Here  $\tau_{\min} > 0$  denotes the infimum of the return time on  $\hat{D}$ . Finally note that the lengths of the intervals  $[0, t_0]$  and  $[t_{m(T,x)}-1, T + t_+^{\hat{D}}(\phi^T(x)) - t_-^{\hat{D}}(x)]$  are bounded from above by the supremum of the return times on  $\hat{D}$  and on  $\tilde{A}_c$ . Hence (106) is bounded from above by a constant

independent of  $(T, x)$ , and the proof of Claim 2 is complete.

**CLAIM 3.** There exists  $C \geq 0$  independent of  $(T, x)$  such that the inequality  $|m(T, x) - M(T, x) - N(T, x)| \leq C$  holds for some choice of closing path defining  $k(T, x; \hat{D})$ .

*Proof of Claim 3.* Note that  $\tilde{A}_c$  is a  $\partial$ -strong global surface of section. This is true by positivity of the curvature (along  $\gamma$ ). Hence, the return time back to (the interior of)  $\tilde{A}_c$  is bounded away from zero and bounded from above. We find  $\delta > 0$  depending only on  $\tilde{A}_c$  such that for every  $a < b$  and  $q \in S^3 \setminus \partial\tilde{A}_c$  we have  $\#\{t \in [a, b] \mid \phi^t(q) \in \tilde{A}_c\} \leq \delta(b - a)$ . Now note that  $\|t_{\pm}^{\hat{D}}\|_{L^\infty} < \infty$ . This is true because  $\hat{D}$  is  $\partial$ -strong. With  $L = \max\{\|t_-^{\hat{D}}\|_{L^\infty}, \|t_+^{\hat{D}}\|_{L^\infty}\}$  we get

$$\begin{aligned} 0 &\leq m(T, x) - \#\{t \in [0, T] \mid \phi^t(x) \in \tilde{A}_c\} \\ &\leq \#\{t \in [t_-^{\hat{D}}(x), 0] \mid \phi^t(x) \in \tilde{A}_c\} \\ &\quad + \#\{t \in [0, t_+^{\hat{D}}(\phi^T(x))] \mid \phi^{t+T}(x) \in \tilde{A}_c\} \\ &\leq 2L\delta. \end{aligned}$$

By Lemma 3.5 we have

$$M(T, x) + N(T, x) = \text{int}(k(T, x; \hat{D}), \tilde{A}_c),$$

and by Lemma 3.6 there exists  $C' \geq 0$  independent of  $(T, x)$  such that

$$|\#\{t \in [0, T] \mid \phi^t(x) \in \tilde{A}_c\} - \text{int}(k(T, x; \hat{D}), \tilde{A}_c)| \leq C'$$

holds for some particular choice of closing path. The proof is complete if we set  $C = C' + 2L\delta$ .

By Claim 2 and (102)

$$M(T, x) = k + O(1)$$

and (100) gives

$$k = \frac{p_1 \circ \bar{\Psi}^{m(T, x)-1}(s, \theta) - s}{2L} + O(1).$$

Thus

$$\left| M(T, x) - \frac{p_1 \circ \bar{\Psi}^{m(T, x)-1}(s, \theta) - s}{2L} \right| = O(1)$$

Up to now the constants  $O(1)$  are independent of  $(T, x)$  and also of choice of closing paths. By Claim 3 we find

$$|m(T, x) - M(T, x) - N(T, x)| = O(1)$$

where this last constant is still independent of  $(T, x)$ , although it depends on the particular choice of closing path. Hence

$$\left| \frac{p_1 \circ \bar{\Psi}^{m(T,x)-1}(s, \theta) - s}{2L} - \frac{p_1 \circ \bar{\Psi}^{M(T,x)+N(T,x)}(s, \theta) - s}{2L} \right| = O(1).$$

All of this implies that

$$\left| M(t, x) - \frac{p_1 \circ \bar{\Psi}^{M(T,x)+N(T,x)}(s, \theta) - s}{2L} \right| = O(1)$$

and the proof is complete.  $\square$

We now complete the proof of Theorem 1.13. Combining Corollary 3.15 with Lemma 3.17 we get

$$\begin{aligned} \frac{\text{int}(k(T, x; \hat{D}), \tilde{A}_c)}{\text{link}(k(T, x; \hat{D}), \hat{\gamma}_c)} &= \frac{M(T, x) + N(T, x)}{M(T, x)} \\ &= \frac{M(T, x) + N(T, x)}{\frac{p_1 \circ \bar{\Psi}^{M(T,x)+N(T,x)}(s, \theta) - s}{2L} + O(1)} \\ &\geq \frac{2L}{L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right)} + O\left( \frac{1}{M(T, x) + N(T, x)} \right) \end{aligned} \quad (107)$$

On the other hand by Lemma 3.3 and Lemma 3.6 we have an estimate

$$\frac{\Delta\Theta(T, u)}{\text{int}(k(T, x; \hat{D}), \tilde{A}_c)} \geq \frac{T\delta}{\frac{T}{\tau_{\min}} + O(1)} = \delta\tau_{\min} + O\left(\frac{1}{T}\right) \quad (108)$$

where  $\tau_{\min}$  denotes the infimum of the return time function of  $\tilde{A}_c$ . By Lemma 3.6 we can find some constant  $X > 0$  that is independent of the point  $x$  such that  $X^{-1}T \geq M(T, x) + N(T, x) \geq XT$ , which means that

$$O\left( \frac{1}{M(T, x) + N(T, x)} \right) = O\left( \frac{1}{T} \right) \quad \text{uniformly in } x \in S^3 \setminus (\gamma_c \cup \hat{\gamma}_c).$$

Plugging this together with (107) and (108) we finally arrive at

$$\frac{\Delta\Theta(T, u)}{\text{link}(k(T, x; \hat{D}), \hat{\gamma}_c)} \geq \delta\tau_{\min} \frac{2L}{L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right)} + O\left( \frac{1}{T} \right). \quad (109)$$

Let us consider an auxiliary constant  $\mu > 1$  and look for  $\delta < 1$  such that

$$\frac{2L}{L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right)} > \frac{\mu}{\delta} \quad \Leftrightarrow \quad L(2\delta - \mu) > 2\pi\mu \left( \frac{1}{\sqrt{\delta}} - 1 \right).$$

Since by (88) we have  $L \geq 2\pi$ , it suffices to ask for

$$\mu < 2\delta\sqrt{\delta}$$

Under this condition on  $\mu$ , by (109) we need

$$\tau_{\min}\mu > 2\pi$$

in order to get  $\kappa(\hat{\gamma}_c) > 2\pi$  and complete the proof. Toponogov's Theorem yields

$$\tau_{\min} \geq 2\pi \left(2 - \frac{1}{\sqrt{\delta}}\right)$$

hence we need

$$\mu \left(2 - \frac{1}{\sqrt{\delta}}\right) > 1.$$

Combining all of the above, we need to ask

$$2\delta\sqrt{\delta} > \mu > \frac{1}{2 - \frac{1}{\sqrt{\delta}}} = \frac{\sqrt{\delta}}{2\sqrt{\delta} - 1}. \quad (110)$$

Note that  $\frac{\sqrt{\delta}}{2\sqrt{\delta}-1} \geq 1$  as long as  $\frac{1}{4} < \delta \leq 1$ . Under this condition, a choice of  $\mu$  as in (110) is possible if

$$2\delta\sqrt{\delta} > \frac{\sqrt{\delta}}{2\sqrt{\delta}-1} \quad \Leftrightarrow \quad 2\delta(2\sqrt{\delta}-1) > 1. \quad (111)$$

Introducing the variable  $x = \sqrt{\delta}$  we get the inequality  $P(x) := 2x^2(2x-1) - 1 > 0$ . The polynomial  $P(x)$  has a unique real root  $x_*$  because it is negative at its critical points 0 and  $\frac{1}{3}$ . This root  $x_*$  lies in  $(\frac{1}{3}, 0.85)$  because  $P(0.85) > 0$ . Hence we get  $\delta_* = x_*^2 < 0.7225$  and (111) holds for all  $\delta \in (\delta_*, 1]$ , as claimed.

## A Global surfaces of section in generic position

### A.1 Blow up construction

Let  $X$  be a smooth vector field on an oriented 3-manifold  $M$  with flow  $\phi^t$ . We need to consider the rank-2 vector bundle  $\xi = TM/\mathbb{R}X \rightarrow M$ , and the  $S^1$ -bundle  $\mathbb{P}_+\xi = (\xi \setminus 0)/\mathbb{R}_+ \rightarrow M$ . The linearised flow  $D\phi^t$  induces a flow on  $\xi$  still denoted by  $D\phi^t$ , and a flow on  $\mathbb{P}_+\xi$  again denoted  $D\phi^t$ . The generating vector field of  $D\phi^t$  on  $\mathbb{P}_+\xi$  is denoted  $\tilde{X}$ . Both  $\xi$  and  $\mathbb{P}_+\xi$  get oriented as bundles by the flow orientation and the orientation of  $M$ . Let  $\Gamma$  be a finite family of (non-constant) periodic orbits of  $\phi^t$ . Let  $\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow M$  be a periodic orbit in this family. The total space  $\mathbb{T}_\gamma$  of the (trivial)  $S^1$ -bundle  $\gamma(T\cdot)^*\mathbb{P}_+\xi \rightarrow \mathbb{R}/\mathbb{Z}$ , which we see as a submanifold of  $\mathbb{P}_+\xi$ ,



is a  $D\phi^t$ -invariant torus. As explained in the introduction, the dynamics of  $D\phi^t$  on  $\mathbb{T}_\gamma$  will be referred to as *linearised polar dynamics* along  $\gamma$ .

In [Fri82] Fried explains how to blow the orbits in  $\Gamma$  up to obtain a smooth 3-manifold with boundary where the restriction of  $\phi^t$  to  $M \setminus \Gamma$  can be extended as a smooth flow leaving the boundary invariant. Details are explained in the proof of the next lemma. Let us identify  $\Gamma$  with the corresponding link of periodic orbits. As a set the blown-up manifold is defined as

$$M_\Gamma = (M \setminus \Gamma) \sqcup \bigsqcup_{\gamma \in \Gamma} \mathbb{T}_\gamma$$

where the disjoint union is taken over the connected components  $\gamma$  of  $\Gamma$ .

**Lemma A.1.** *The following hold:*

- (a)  $M_\Gamma$  has the structure of a second countable, Hausdorff, smooth manifold with boundary such that  $\partial M_\Gamma = \bigsqcup_{\gamma \in \Gamma} \mathbb{T}_\gamma$  and  $M_\Gamma \setminus \partial M_\Gamma = M \setminus \Gamma$ . These identities respect the topology and differentiable structure previously defined on  $M \setminus \Gamma$  and on the  $\mathbb{T}_\gamma$ .
- (b) The flow  $\phi^t|_{M \setminus \Gamma}$  extends smoothly to a flow on  $M_\Gamma$  that agrees with  $D\phi^t$  on  $\partial M_\Gamma$ .

*Proof.* We sketch (a), details are left to the reader. Let  $\gamma$  be a connected component of  $\Gamma$ . Choose an orientation preserving embedding  $F : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \rightarrow N$ , where  $N$  is a compact neighbourhood of  $\gamma$ , satisfying  $F(\vartheta, 0) = \gamma(T\vartheta)$ . Equip  $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$  with coordinates  $(\vartheta, z = x + iy)$ . Consider  $\Phi : \mathbb{R}/\mathbb{Z} \times [0, 1] \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R} \times \mathbb{D}$  defined by  $\Phi(\vartheta, r, \theta) = (\vartheta, re^{i\theta})$ . Note that  $\Phi$  is smooth. Then  $(\vartheta, \theta)$  at  $r = 0$  define global coordinates on  $\mathbb{T}_\gamma$ , and  $(\vartheta, r, \theta) \in \mathbb{R}/\mathbb{Z} \times [0, 1] \times \mathbb{R}/2\pi\mathbb{Z}$  determine coordinates on  $(N \setminus \gamma) \sqcup \mathbb{T}_\gamma$  which we use to define the topology and the differentiable structure on a neighbourhood of  $\mathbb{T}_\gamma$  in  $M_\Gamma$ . One can repeat this process at every connected component  $\gamma$  of  $\Gamma$ .

For (b) we provide full details. We first claim that the vector field  $W = (F \circ \Phi)^* X$ , which is in principle only defined on  $\mathbb{R}/\mathbb{Z} \times (0, 1] \times \mathbb{R}/2\pi\mathbb{Z}$ , extends smoothly to  $\mathbb{R}/\mathbb{Z} \times [0, 1] \times \mathbb{R}/2\pi\mathbb{Z}$ . To prove this claim, write in components

$$F^* X = (X^\vartheta, Y = X^x + iX^y) \in \mathbb{R} \times \mathbb{C}$$

Note that  $X^\vartheta(\vartheta, 0) = T^{-1}$  and  $Y(\vartheta, 0) = 0$  for all  $\vartheta$ . Hence we can write

$$X^\vartheta(\vartheta, z) = T^{-1} + \Lambda_1(\vartheta, z)z \quad Y(\vartheta, z) = \Lambda_2(\vartheta, z)z$$

with smooth functions  $\Lambda_1, \Lambda_2$  satisfying

$$\Lambda_1(\vartheta, 0) = D_2 X^\vartheta(\vartheta, 0) \quad \Lambda_2(\vartheta, 0) = D_2 Y(\vartheta, 0)$$

Using the formula

$$D\Phi^{-1}(\vartheta, x + iy = re^{i\theta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -r^{-1} \sin \theta & r^{-1} \cos \theta \end{pmatrix}$$

we get

$$W = \left( T^{-1} + \Lambda_1(\vartheta, re^{i\theta})re^{i\theta}, \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \Lambda_2(\vartheta, re^{i\theta})e^{i\theta} \right) \quad (112)$$

The upshot is that (112) reveals that  $W$  has a smooth extension up to  $r = 0$ . Moreover, the vector field  $W$  on  $\{r = 0\} \simeq \mathbb{T}_\gamma$  is given by

$$W|_{\{r=0\}} = T^{-1}\partial_\vartheta + b(\vartheta, \theta)\partial_\theta$$

where

$$b(\vartheta, \theta) = \langle D_2Y(\vartheta, 0)e^{i\theta}, ie^{i\theta} \rangle$$

The linearised dynamics along  $\gamma$  is given by

$$\begin{pmatrix} \dot{\delta\vartheta}(t) \\ \dot{\delta z}(t) \end{pmatrix} = \begin{pmatrix} 0 & D_2X^\vartheta(t/T, 0) \\ 0 & D_2Y(t/T, 0) \end{pmatrix} \begin{pmatrix} \delta\vartheta(t) \\ \delta z(t) \end{pmatrix}$$

If one writes  $\delta z = |\delta z|e^{i\theta}$  then

$$\dot{\theta} = \operatorname{Re} \frac{D_2Y(t/T, 0)\delta z}{i\delta z} = \langle D_2Y(t/T, 0)e^{i\theta}, ie^{i\theta} \rangle = b(t/T, \theta)$$

Thus the dynamics of  $W|_{\{r=0\}}$  is precisely linearised polar dynamics along  $\gamma$ .  $\square$

## A.2 Global surfaces of section in generic position

We use the notation established above, and consider the case where  $\Gamma = \partial\Sigma$  is the boundary of some global surface of section  $\Sigma \hookrightarrow M$  for  $\phi^t$ . For each periodic orbit  $\gamma \subset \partial\Sigma$  consider

$$\nu^\Sigma\gamma = \{\mathbb{R}_+\nu \mid \nu \in T\Sigma|_\gamma \text{ is outward pointing}\}$$

Then  $\nu^\Sigma\gamma/\mathbb{R}X$  is the graph of a smooth section of  $\gamma(T\cdot)^*\mathbb{P}_+\xi$  and, as such, it defines a smooth submanifold of  $\mathbb{T}_\gamma$ .

**Definition A.2.** The global surface of section  $\Sigma$  is called  $\partial$ -strong if  $\nu^\Sigma\gamma/\mathbb{R}X$  is a global surface of section for  $D\phi^t$  on  $\mathbb{T}_\gamma$ , for every  $\gamma \subset \partial\Gamma$ .

**Definition A.3.** Two global surfaces of section  $\Sigma, \hat{\Sigma}$  for  $\phi^t$  are said to be in relative generic position if  $\Sigma \setminus \partial\Sigma$  intersects  $\hat{\Sigma} \setminus \partial\hat{\Sigma}$  transversely, and if for every connected component  $\gamma$  of  $\partial\hat{\Sigma} \cap \partial\Sigma$  the manifolds  $\nu^\Sigma\gamma$  and  $\nu^{\hat{\Sigma}}\gamma$  intersect transversely in  $\mathbb{T}_\gamma$ .

**Lemma A.4.** *Let  $\Sigma, \hat{\Sigma}$  be global surfaces of section for  $\phi^t$ . Assume that  $\hat{\Sigma}$  is  $\partial$ -strong. There exists an arbitrarily  $C^\infty$ -small perturbation of  $\hat{\Sigma}$  to a new embedded surface  $\tilde{\Sigma}$  such that the following hold:*

- (a)  $\partial\tilde{\Sigma} = \partial\hat{\Sigma}$ .
- (b)  $\tilde{\Sigma}$  is a  $\partial$ -strong global surface of section.
- (c) The infimum/supremum of the return time function on  $\tilde{\Sigma}$  is arbitrarily close to the infimum/supremum of the return time function on  $\hat{\Sigma}$ .
- (d)  $\Sigma$  and  $\tilde{\Sigma}$  are in relative generic position.
- (e) There exists a smooth function  $g : \tilde{\Sigma} \rightarrow \mathbb{R}$  that is  $C^\infty$ -close to zero, satisfying  $\phi^{g(p)}(p) \in \hat{\Sigma}$  for every  $p \in \tilde{\Sigma}$ .

*Remark A.5.* The function  $g$  is defined and smooth on  $\tilde{\Sigma}$  up to the boundary, and is  $C^\infty$ -small on the compact surface  $\tilde{\Sigma}$ .

**Lemma A.6.** *Let  $\Sigma, \hat{\Sigma}$  be global surfaces of section for  $\phi^t$ . Assume that they are in relative generic position. Then the closure<sup>3</sup>  $K$  of  $(\hat{\Sigma} \setminus \tilde{\Sigma}) \cap (\Sigma \setminus \partial\Sigma)$  is a smooth 1-dimensional manifold satisfying the following properties:*

- (a)  $\partial K = K \cap (\partial\Sigma \cup \partial\hat{\Sigma})$ .
- (b) If  $\gamma$  is a connected component of  $\partial\Sigma \cup \partial\hat{\Sigma}$  and  $p \in K \cap \gamma \subset \partial K$  then the identity  $T_p K \cap T_p \gamma = \{0\}$  holds.

*Proofs of lemmas A.4 and A.6.* If  $\partial\Sigma \cap \partial\hat{\Sigma} = \emptyset$  then the boundary of each of the global surfaces of section considered intersects the other in finitely many interior points. Lemma A.4 follows from a direct application of Sard's theorem, and Lemma A.6 is trivial in this case.

We proceed assuming that  $\partial\Sigma \cap \partial\hat{\Sigma} \neq \emptyset$ . This set is a finite collection of periodic orbits. Let the periodic orbit  $\gamma$  be a connected component of  $\partial\Sigma \cap \partial\hat{\Sigma}$ , and let  $F = F(\vartheta, z = x + iy)$  be an orientation preserving embedding of  $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$  onto a neighbourhood of  $\gamma$  as in the proof of Lemma A.1. By requirement, it satisfies  $\phi^t(F(0, 0)) = F(t/T, 0)$ .

Let  $(\rho, \alpha) \in (1 - \epsilon, 1] \times \mathbb{R}/\mathbb{Z}$  be smooth coordinates on a neighbourhood of  $\gamma$  in  $\Sigma$ . Hence, in the coordinates induced by  $F$ ,  $\Sigma$  is parametrised as an embedded strip near  $\gamma$

$$(\rho, \alpha) \mapsto (G_1(\rho, \alpha), G_2(\rho, \alpha)) \in \mathbb{R}/\mathbb{Z} \times \mathbb{D}$$

where  $G_1, G_2$  are smooth functions of  $(\rho, \alpha) \in (1 - \epsilon, 1] \times \mathbb{R}/\mathbb{Z}$ . We have  $G_2(1, \cdot) \equiv 0$  and  $G_2(\rho, \alpha) \neq 0$  if  $\rho < 1$ . Moreover  $\alpha \mapsto G_1(1, \alpha)$  defines a diffeomorphism of  $\mathbb{R}/\mathbb{Z}$ , and  $D_1 G_2(1, \alpha) \neq 0$  for all  $\alpha$ . It follows from this and the implicit function theorem

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<sup>3</sup>The closure is taken in  $\Sigma$ , or equivalently in  $\hat{\Sigma}$ , or equivalently in  $M$ .

that  $(\rho, \alpha) \mapsto (\rho, G_1(\rho, \alpha))$  defines a diffeomorphism between neighbourhoods of  $\{1\} \times \mathbb{R}/\mathbb{Z}$  in  $(1 - \epsilon, 1] \times \mathbb{R}/\mathbb{Z}$ . In other words, there is no loss of generality to assume that  $\alpha = \vartheta$ , i.e.  $\Sigma$  is parametrised as a strip  $(\rho, \vartheta) \mapsto (\vartheta, G(\rho, \vartheta))$  where the smooth function  $G : (1 - \epsilon, 1] \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{D}$  satisfies  $G(1, \cdot) \equiv 0$ ,  $D_1 G(1, \vartheta) \neq 0$  for all  $\vartheta$ . Hence one can write  $G(\rho, \vartheta) = (1 - \rho)G'(\rho, \vartheta)$  for some smooth function  $G' : (1 - \epsilon, 1] \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$  satisfying  $G'(1, \vartheta) = -D_1 G(1, \vartheta)$ . It follows that the argument  $\varphi(\rho, \vartheta) \in \mathbb{R}/2\pi\mathbb{Z}$  of  $G(\rho, \vartheta)$ , which is in principle only defined for  $\rho < 1$ , can be smoothly extended up to  $\rho = 1$  in such a way that  $\varphi(1, \vartheta)$  is the argument of  $-D_1 G(1, \vartheta)$ . If we denote  $r(\rho, \vartheta) = |G(\rho, \vartheta)|$  then the formula  $D_j r = \langle e^{i\varphi}, D_j G \rangle$  for the partial derivatives of  $r$  shows that  $r$  is smooth up to  $\rho = 1$ ; here  $D_1$  and  $D_2$  denote partial derivatives in  $\rho$  and  $\vartheta$ , respectively. Moreover, one computes

$$\begin{aligned} D_1 r(1, \vartheta) &= \langle e^{i\varphi(1, \vartheta)}, D_1 G(1, \vartheta) \rangle \\ &= \left\langle \frac{-D_1 G(1, \vartheta)}{|D_1 G(1, \vartheta)|}, D_1 G(1, \vartheta) \right\rangle \\ &= -|D_1 G(1, \vartheta)| < 0 \quad \forall \vartheta \end{aligned}$$

The implicit function theorem implies that we can use  $(r, \vartheta) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z}$  as coordinates of  $\Sigma$  near  $\gamma$ , and parametrise  $\Sigma$  smoothly as

$$(r, \vartheta) \mapsto (\vartheta, r e^{i\varphi(r, \vartheta)})$$

in terms of  $(r, \vartheta)$ . Repeating this analysis in a neighbourhood of  $\gamma$  in  $\hat{\Sigma}$ , we see that such a neighbourhood can be smoothly parametrised as

$$(r, \vartheta) \mapsto (\vartheta, r e^{i\hat{\varphi}(r, \vartheta)})$$

where  $(r, \vartheta) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z}$  are smooth coordinates of  $\hat{\Sigma}$  near  $\gamma$ , and  $\hat{\varphi}$  is smooth. The submanifolds  $\nu^\Sigma \gamma$  and  $\nu^{\hat{\Sigma}} \gamma$  are represented as the graphs of  $\vartheta \mapsto \varphi(1, \vartheta)$  and  $\vartheta \mapsto \hat{\varphi}(1, \vartheta)$ . This proves that after performing Fried's blow up construction along the connected components  $\gamma$  of  $\partial \hat{\Sigma} \cap \partial \Sigma$ , the surfaces  $\Sigma$  and  $\hat{\Sigma}$  induce embedded surfaces in the blown up 3-manifold meeting the boundary components  $\mathbb{T}_\gamma$  (corresponding to the various  $\gamma \subset \partial \hat{\Sigma} \cap \partial \Sigma$ ) transversely in embedded circles; the circles corresponding to  $\hat{\Sigma}$  are global surfaces of section for  $D\phi^t|_{\mathbb{T}_\gamma}$ . It follows from these local representations that a  $C^\infty$ -small perturbation of  $\hat{\Sigma}$  supported on an arbitrarily small neighbourhood of  $\gamma$ , keeping  $\gamma$  fixed as a boundary component of  $\hat{\Sigma}$ , will make  $\nu^{\hat{\Sigma}} \gamma$  transverse to  $\nu^\Sigma \gamma$ : this amounts to making the graph of  $\vartheta \mapsto \hat{\varphi}(1, \vartheta)$  transverse to the graph of  $\vartheta \mapsto \varphi(1, \vartheta)$ . Moreover, the infimum and supremum of return times to  $\hat{\Sigma}$  will only slightly change. It follows that for small  $r$  the graph of  $\vartheta \mapsto \varphi(r, \vartheta)$  is transverse to the graph of  $\vartheta \mapsto \hat{\varphi}(r, \vartheta)$ . We can repeat this process at every component  $\gamma$  of  $\partial \hat{\Sigma} \cap \partial \Sigma$ , and achieve transversality of the various pairs of sections  $\nu^{\hat{\Sigma}} \gamma$ ,  $\nu^\Sigma \gamma$  along the components  $\gamma$  of  $\partial \hat{\Sigma} \cap \partial \Sigma$ . It also follows that the interiors of  $\Sigma$  and  $\hat{\Sigma}$  intersect transversely near  $\partial \hat{\Sigma} \cap \partial \Sigma$ , but away from  $\partial \Sigma \cup \partial \hat{\Sigma}$  we achieve transversality by a further  $C^\infty$ -small perturbation supported in the interiors, using

Sard's theorem. Again the return time to  $\hat{\Sigma}$  can only change slightly. One does not need to perturb the surfaces near components of  $(\partial\Sigma \setminus \partial\hat{\Sigma}) \cup (\partial\hat{\Sigma} \setminus \partial\Sigma)$ . Properties (a)-(d) follow. Since the surfaces in the blown-up manifolds are  $C^\infty$ -close to each other the flow extends smoothly up to boundary, the function  $g$  as in (e) exists. The proof of Lemma A.4 is complete.

Let us now prove Lemma A.6. As we can see from the above, the intersection of the interiors of the two surfaces  $\Sigma$  and  $\hat{\Sigma}$ , which are assumed to be in generic relative position, cuts out near a connected component  $\gamma \subset \partial\Sigma \cap \partial\hat{\Sigma}$  a smooth 1-manifold intersecting  $\gamma$  transversely (relative to each surface). This is so since there the intersection is described as the equation  $\varphi = \hat{\varphi}$  which, by the implicit function theorem, determines  $\vartheta$  locally as a smooth function of  $r$ , up to  $r = 0$ . The assumed transversality of the intersection  $(\hat{\Sigma} \setminus \partial\hat{\Sigma}) \cap (\Sigma \setminus \partial\Sigma)$  guarantees that these graphs continue to the interiors as 1-manifolds. Hence the set  $K$  defined as the closure of  $(\hat{\Sigma} \setminus \partial\hat{\Sigma}) \cap (\Sigma \setminus \partial\Sigma)$  has the desired structure of a smooth 1-dimensional submanifold near connected components of  $\partial\Sigma \cap \partial\hat{\Sigma}$  and also on common interior points. The analysis near connected components of  $(\partial\Sigma \setminus \partial\hat{\Sigma}) \cup (\partial\hat{\Sigma} \setminus \partial\Sigma)$  is trivial.  $\square$

*Remark A.7.* In Lemma A.6 the boundary  $\partial K$  might contain points in  $\Sigma \setminus \partial\Sigma$ . Such points must also belong to  $\partial\hat{\Sigma}$ .

*Remark A.8.* There is a “small transfer map”  $\tilde{\Sigma} \rightarrow \hat{\Sigma}$  defined by  $p \mapsto \phi^{g(p)}(p)$  where  $g$  is given in (e) Lemma A.4. By transversality it restricts to a smooth diffeomorphism between interiors.

### A.3 Comparing linking numbers

Consider a disk-like global surface of section  $D$  of a flow on  $S^3$  spanned by the (unknotted) periodic orbit  $\gamma_0 = \partial D$ . Define for  $x \in S^3 \setminus \gamma_0$  the functions

$$t_-^D(x) = \sup\{t \leq 0 \mid \phi^t(x) \in D\} \quad t_+^D(x) = \inf\{t \geq 0 \mid \phi^t(x) \in D\} \quad (113)$$

Note that  $t_\pm^D$  are smooth on  $S^3 \setminus D$ , discontinuous and vanishing on points of  $D \setminus \gamma_0$ . For any  $x \in S^3 \setminus \gamma_0$  and  $T > 0$  define the interval

$$I(T, x; D) = [t_-^D(x), T + t_+^D(\phi^T(x))] \quad (114)$$

and denote by  $k(T, x; D)$  a loop obtained by concatenating to the piece of trajectory  $\phi^{I(T, x; D)}(x)$  a path in  $D \setminus \gamma_0$  from  $\phi^{T+t_+^D(\phi^T(x))}(x)$  to  $\phi^{t_-^D(x)}(x)$ . Note that

$$\begin{aligned} \text{link}(k(T, x; D), \gamma_0) &= \#\{t \in I(T, x; D) \mid \phi^t(x) \in D\} - 1 \\ &= \#\{t \in [0, T] \mid \phi^t(x) \in D\} + \ell - 1 \\ &\quad \text{for some } \ell \in \{0, 1, 2\} \end{aligned} \quad (115)$$

Now let  $\hat{D}$  be another global surface of section satisfying  $\partial\hat{D} = \gamma_0$ . Assume that  $\hat{D}$  and  $D$  are in generic relative position. Hence, by Lemma A.6, the closure  $K$  of

$(D \cap \hat{D}) \setminus \gamma_0$  in  $D$ , or equivalently in  $\hat{D}$  or  $S^3$ , is a 1-submanifold with boundary satisfying  $\partial K = K \cap \gamma_0$ . Moreover, at a point  $p \in \partial K$  the spaces  $T_p K$  and  $T_p \gamma_0$  are distinct. Let  $N$  denote the number of connected components of  $K$ .

**Lemma A.9.** *Suppose that the return time functions to  $D$  and to  $\hat{D}$  are bounded away from zero and bounded from above. Then there exists a constant  $C > 0$  depending only  $D$  and  $\hat{D}$  such that  $|\text{link}(k(T, x; D), \gamma_0) - \text{link}(k(T, x; \hat{D}), \gamma_0)| \leq C$  holds for all  $T > 0$  and all  $x \in S^3 \setminus \gamma_0$ .*

*Proof.* We start by proving that for every  $x \in S^3 \setminus \gamma_0$  and every  $T > 0$  there exists an open neighbourhood  $V \subset S^3 \setminus \gamma_0$  of  $x$  such that

$$\max_{\tilde{D}=D, \hat{D}} |\text{link}(k(T, x; \tilde{D}), \gamma_0) - \text{link}(k(T, x'; \tilde{D}), \gamma_0)| \leq 2 \quad \forall x' \in V. \quad (116)$$

It suffices to consider  $\tilde{D} = D$ . There exists a neighbourhood  $V \subset S^3 \setminus \gamma_0$  of  $x$  such that for every  $x' \in V$  the 1-cycles  $k(T, x; D)$  and  $k(T, x'; D)$  differ by a sum of two cycles  $\Gamma_1, \Gamma_2$  such that for  $i = 1, 2$  we have  $|\text{link}(\Gamma_i, \gamma_0)| \leq 1$ . Hence

$$|\text{link}(k(T, x; D), \gamma_0) - \text{link}(k(T, x'; D), \gamma_0)| \leq \sum_{i=1,2} |\text{link}(\Gamma_i, \gamma_0)| \leq 2.$$

By (116), in order to prove the lemma we can assume that  $x$  is such that

$$\left\{ \phi^{t_-^D(x)}(x), \phi^{T+t_+^D(\phi^T(x))}(x), \phi^{t_-^{\hat{D}}(x)}(x), \phi^{T+t_+^{\hat{D}}(\phi^T(x))}(x) \right\} \cap K = \emptyset.$$

Denote by  $\alpha$  the 1-chain consisting of the piece of trajectory from  $\phi^{t_-^{\hat{D}}(x)}(x)$  to  $\phi^{t_-^D(x)}(x)$ , oriented in favour (if  $t_-^{\hat{D}}(x) \leq t_-^D(x)$ ) or against (if  $t_-^{\hat{D}}(x) \geq t_-^D(x)$ ) the flow. Analogously, consider the 1-chain  $\beta$  consisting of the piece of flow trajectory from  $\phi^{T+t_+^{\hat{D}}(\phi^T(x))}(x)$  to  $\phi^{T+t_+^D(\phi^T(x))}(x)$ , oriented accordingly in favour or against the flow. Let  $c$  be the 1-chain in  $D \setminus \gamma_0$  and  $\hat{c}$  be the 1-chain in  $\hat{D} \setminus \gamma_0$  such that

$$k(T, x; D) = \phi^{I(T, x; D)}(x) + c \quad k(T, x; \hat{D}) = \phi^{I(T, x; \hat{D})}(x) + \hat{c}.$$

Then, as an identity of 1-cycles, we get

$$k(T, x; \hat{D}) = k(T, x; D) + \delta \quad \text{where} \quad \delta = \hat{c} + \alpha - c - \beta.$$

It follows that

$$\begin{aligned} \text{link}(k(T, x; \hat{D}), \gamma_0) &= \text{link}(k(T, x; D) + \delta, \gamma_0) \\ &= \text{link}(k(T, x; D), \gamma_0) + \text{int}(\delta, D) \end{aligned} \quad (117)$$

A  $C^\infty$ -small perturbation of  $\hat{c}$  keeping end points fixed will make  $\hat{c}$  transverse to  $K$ . It follows that

$$|\text{int}(\hat{c}, K)| \leq N + 2 \quad (118)$$

where we use here that every component of  $K$  is trivial in  $\pi_1(D, \partial D)$ . Note that  $\alpha$  and  $\beta$  are pieces of trajectories of time-lengths not larger than  $L := \max\{\tau_{\max}^D, \tau_{\max}^{\hat{D}}\}$ , where  $\tau_{\max}^D$  and  $\tau_{\max}^{\hat{D}}$  are the suprema of the return time functions to  $D$  and  $\hat{D}$  respectively. To see this for  $\alpha$ , consider the case  $t_-^{\hat{D}}(x) \leq t_-^D(x) \leq 0$ , the case  $t_-^D(x) \leq t_-^{\hat{D}}(x) \leq 0$  is analogous. Then the time-length of  $\alpha$  is

$$t_-^D(x) - t_-^{\hat{D}}(x) \leq -t_-^{\hat{D}}(x) \in [0, \tau_{\max}^{\hat{D}}]$$

For  $\beta$  one argues analogously. Clearly, all intersections of  $\alpha$  and  $\beta$  with  $D$  are transverse. From the estimate just proved, the number of such intersection points is bounded from above by  $L/\tau_{\min}^D$  where  $\tau_{\min}^D$  is the infimum of the return time function to  $D$ . Combining with (118) we get

$$|\text{int}(\delta, D)| \leq N + 2 + \frac{2 \max\{\tau_{\max}^D, \tau_{\max}^{\hat{D}}\}}{\tau_{\min}^D}.$$

Thus, the desired constant  $C$  in the statement of the lemma can be taken as

$$C = N + 6 + \frac{2 \max\{\tau_{\max}^D, \tau_{\max}^{\hat{D}}\}}{\tau_{\min}^D}.$$

□

From Lemma A.4 and Lemma A.9 we deduce the following corollary.

**Corollary A.10.** *Let  $\gamma_0$  be a periodic unknotted Reeb orbit whose self-linking number is  $-1$ . The quantity  $\kappa(\gamma_0)$  does not depend on the choice of the  $\partial$ -strong disk-like global surface of section spanned by  $\gamma_0$ .*

## A.4 Independence of the global frame

**Lemma A.11.** *Let  $\gamma_0 \subset S^3$  be a periodic unknotted Reeb orbit whose self-linking number is  $-1$ . The quantity  $\kappa(\gamma_0)$  does not depend on the choice of the global symplectic trivialisation  $\sigma$ .*

*Proof.* Let  $\Sigma$  be a  $\partial$ -strong disk-like global surface of section spanned by  $\gamma_0$ . Let  $\sigma_1, \sigma_2$  be two global symplectic trivialisations of the contact structure  $(\xi, d\lambda)$ . In particular,  $\sigma_1$  and  $\sigma_2$  are homotopic trivialisations. Denote then, for  $i = 1, 2$ ,

$$\kappa_i(\gamma_0) = \liminf_{T \rightarrow +\infty} \inf_{x, u} \frac{\tilde{\Theta}_{\sigma_i}(T, u) - \tilde{\Theta}_{\sigma_i}(0, u)}{\text{link}(k(T, x; \Sigma), \gamma_0)}.$$

As a first case, let us assume that  $\kappa_i(\gamma_0) \neq \pm\infty$  for  $i = 1, 2$ . Fix  $\epsilon > 0$ . For every  $T$  large enough it holds

$$\inf_{x, u} \left( \frac{\tilde{\Theta}_{\sigma_2}(T, u) - \tilde{\Theta}_{\sigma_2}(0, u)}{\text{link}(k(T, x; \Sigma), \gamma_0)} \right) + \epsilon \geq \kappa_2(\gamma_0).$$

Moreover, there exists a sequence  $(T_n)_{n \in \mathbb{N}}$ , satisfying  $T_n \rightarrow +\infty$  as  $n \rightarrow \infty$ , such that, for every  $n$ ,

$$\inf_{x,u} \left( \frac{\tilde{\Theta}_{\sigma_1}(T_n, u) - \tilde{\Theta}_{\sigma_1}(0, u)}{\text{link}(k(T_n, x; \Sigma), \gamma_0)} \right) - \epsilon \leq \kappa_1(\gamma_0).$$

Thus, for every  $T_n$  large enough, we have

$$\kappa_2(\gamma_0) - \kappa_1(\gamma_0) \leq \inf_{x,u} \left( \frac{\tilde{\Theta}_{\sigma_2}(T_n, u) - \tilde{\Theta}_{\sigma_2}(0, u)}{\text{link}(k(T_n, x; \Sigma), \gamma_0)} \right) - \inf_{x,u} \left( \frac{\tilde{\Theta}_{\sigma_1}(T_n, u) - \tilde{\Theta}_{\sigma_1}(0, u)}{\text{link}(k(T_n, x; \Sigma), \gamma_0)} \right) + 2\epsilon.$$

Let  $(x_n, u_n)$  be such that

$$\inf_{x,u} \left( \frac{\tilde{\Theta}_{\sigma_1}(T_n, u) - \tilde{\Theta}_{\sigma_1}(0, u)}{\text{link}(k(T_n, x; \Sigma), \gamma_0)} \right) \geq \frac{\tilde{\Theta}_{\sigma_1}(T_n, u_n) - \tilde{\Theta}_{\sigma_1}(0, u_n)}{\text{link}(k(T_n, x_n; \Sigma), \gamma_0)} - \epsilon.$$

Consequently, it holds

$$\kappa_2(\gamma_0) - \kappa_1(\gamma_0) \leq \frac{\tilde{\Theta}_{\sigma_2}(T_n, u_n) - \tilde{\Theta}_{\sigma_2}(0, u_n)}{\text{link}(k(T_n, x_n; \Sigma), \gamma_0)} - \frac{\tilde{\Theta}_{\sigma_1}(T_n, u_n) - \tilde{\Theta}_{\sigma_1}(0, u_n)}{\text{link}(k(T_n, x_n; \Sigma), \gamma_0)} + 3\epsilon$$

for  $n$  large enough. Observe that, for  $i = 1, 2$ ,

$$\tilde{\Theta}_{\sigma_i}(T_n, u_n) - \tilde{\Theta}_{\sigma_i}(0, u_n)$$

is the variation of the angle coordinate of the vector  $D\phi_t(x_n)u_n \in \xi_{\phi_t(x_n)}$ , between 0 and  $T_n$ , with respect to the frame  $\{X_1^i, X_2^i\}$ , where  $\sigma_i \circ X_1^i = (1, 0)$ ,  $\sigma_i \circ X_2^i = (0, 1)$ .

For  $i = 1, 2$ , let  $\theta_i : [0, T_n] \rightarrow \mathbb{R}$  be a continuous argument of  $t \mapsto D\phi_t(x_n)u_n$  with respect to the frame  $\{X_1^i, JX_1^i\}$ , where  $J$  is a fixed reference almost complex structure (compatible with  $d\lambda$ ). Then, by [Flo19a, Claim 1.1.1], it holds

$$\tilde{\Theta}_{\sigma_i}(T_n, \bar{u}_n) - \tilde{\Theta}_{\sigma_i}(0, \bar{u}_n) < \theta_i(T_n) - \theta_i(0) + 1. \quad (119)$$

For any given curve  $\gamma : [0, 1] \rightarrow S^3$ , denote as

$$\text{wind}_\gamma(\sigma_1, \sigma_2)$$

the variation of the oriented angle between  $X_1^1 = \sigma_1^{-1}(1, 0)$  and  $X_1^2 = \sigma_2^{-1}(1, 0)$  along the curve  $\gamma$ , with respect to the frame  $\{X_1^1, JX_1^1\}$ , where  $J$  is the aforementioned fixed almost complex structure.

Observe then that

$$(\theta_2(T_n) - \theta_2(0)) - (\theta_1(T_n) - \theta_1(0)) = \text{wind}_{(\phi_t(x_n))_{t \in [0, T_n]}}(\sigma_1, \sigma_2). \quad (120)$$



CLAIM. Fix a Riemannian metric  $g$  on  $S^3$ . Let  $\sigma_1, \sigma_2$  be two trivialisations of the contact plane. There exists a constant  $C > 0$  such that for every  $p, q \in S^3$

$$|\text{wind}_{\gamma_{(p,q)}}(\sigma_1, \sigma_2)| \leq C, \quad (121)$$

where  $\gamma_{(p,q)} : [0, 1] \rightarrow S^3$  is a path of the form  $t \in [0, 1] \mapsto \exp_p(tv) \in S^3$  for some  $v \in T_p S^3$  satisfying  $g_p(v, v) \leq \text{diam } S^3$ ,  $\exp_p(v) = q$ .

Since the two trivialisations are homotopic, considering the loop

$$\Gamma := (\phi_t(x_n))_{t \in [0, T_n]} + \gamma_{(\phi_{T_n}(x_n), x_n)},$$

it holds

$$0 = \text{wind}_\Gamma(\sigma_1, \sigma_2) = \text{wind}_{(\phi_t(x_n))_{t \in [0, T_n]}}(\sigma_1, \sigma_2) + \text{wind}_{\gamma_{(\phi_{T_n}(x_n), x_n)}}(\sigma_1, \sigma_2).$$

From the previous Claim, we deduce that

$$|\text{wind}_{(\phi_t(x_n))_{t \in [0, T_n]}}(\sigma_1, \sigma_2)| \leq C. \quad (122)$$

From (119), (120) and (122), it holds

$$\kappa_2(\gamma_0) - \kappa_1(\gamma_0) < \frac{C + 2}{\text{link}(k(T_n, x_n; \Sigma), \gamma_0)} + 3\epsilon.$$

Since  $\text{link}(k(T_n, x_n; \Sigma), \gamma_0) \rightarrow +\infty$  as  $n \rightarrow +\infty$ , we deduce that

$$\kappa_2(\gamma_0) - \kappa_1(\gamma_0) \leq 3\epsilon.$$

By exchanging the roles of  $\sigma_1$  and  $\sigma_2$ , and letting  $\epsilon \rightarrow 0$ , we conclude that

$$\kappa_1(\gamma_0) = \kappa_2(\gamma_0).$$

Assume now that  $\kappa_1(\gamma_0) = +\infty$  and  $\kappa_2(\gamma_0) < +\infty$ . Let  $M \in \mathbb{R}$  be such that  $M > 3\kappa_2(\gamma_0)$  and  $M \gg C$ , where  $C$  is the constant in (121). Then, there exists a sequence  $(T_n)_{n \in \mathbb{N}}$ , so that  $T_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ , and such that for every  $n$  it holds

$$\inf_{x,u} \left( \frac{\tilde{\Theta}_{\sigma_1}(T_n, u) - \tilde{\Theta}_{\sigma_1}(0, u)}{\text{link}(k(T_n, x; \Sigma), \gamma_0)} \right) > M > \frac{M}{2} > \inf_{x,u} \left( \frac{\tilde{\Theta}_{\sigma_2}(T_n, u) - \tilde{\Theta}_{\sigma_2}(0, u)}{\text{link}(k(T_n, x; \Sigma), \gamma_0)} \right).$$

Let  $(x_n, u_n)$  be such that

$$\left( \frac{\tilde{\Theta}_{\sigma_1}(T_n, u_n) - \tilde{\Theta}_{\sigma_1}(0, u_n)}{\text{link}(k(T_n, x_n; \Sigma), \gamma_0)} \right) > M > \frac{M}{2} > \left( \frac{\tilde{\Theta}_{\sigma_2}(T_n, u_n) - \tilde{\Theta}_{\sigma_2}(0, u_n)}{\text{link}(k(T_n, x_n; \Sigma), \gamma_0)} \right).$$

Consequently, it holds

$$\frac{(\tilde{\Theta}_{\sigma_1}(T_n, u_n) - \tilde{\Theta}_{\sigma_1}(0, u_n)) - (\tilde{\Theta}_{\sigma_2}(T_n, u_n) - \tilde{\Theta}_{\sigma_2}(0, u_n))}{\text{link}(k(T_n, x_n; \Sigma), \gamma_0)} > \frac{M}{2}.$$

As done in the previous case, from [Flo19a, Claim 1.1.1], (120) and (122), we conclude that

$$\frac{C+2}{\text{link}(k(T_n, x_n; \Sigma), \gamma_0)} > \frac{M}{2};$$

since  $\text{link}(k(T_n, x_n; \Sigma), \gamma_0) \rightarrow +\infty$  as  $n \rightarrow +\infty$  and since  $M \gg C$ , we obtain a contradiction. Thus, we deduce that if one between  $\kappa_1(\gamma_0)$  and  $\kappa_2(\gamma_0)$  is equal to  $+\infty$ , then also the other one has to be  $+\infty$ . An analogous argument allows us to deduce that if one between  $\kappa_1(\gamma_0)$  and  $\kappa_2(\gamma_0)$  is equal to  $-\infty$ , then also the other one has to be  $-\infty$ .  $\square$

## B Abundance of global surfaces of section

Here we prove Theorem 1.1. Let  $\gamma_1, \dots, \gamma_n$  be periodic orbits of the flow  $\phi^t$  of a smooth non-vanishing vector field  $X$  on a homology 3-sphere  $M$ . We see each  $\gamma_i$  as a knot oriented by  $X$ , and consider the oriented link  $L = \gamma_1 \cup \dots \cup \gamma_n$ .

Let  $\Sigma$  be any oriented Seifert surface satisfying  $\partial\Sigma = L$ , orientations included. Fix also oriented Seifert surfaces  $\Sigma_i$  satisfying  $\partial\Sigma_i = \gamma_i$ , orientations included. Note that  $\Sigma, \Sigma_i$  exist because  $M$  is a homology 3-sphere. Denote by  $\Sigma^* \in H^1(M \setminus L)$  the cohomology class dual to  $\Sigma$ , and by  $\Sigma_i^* \in H^1(M \setminus L)$  the restriction to  $M \setminus L$  of the cohomology classes in  $M \setminus \gamma_i$  dual to  $\Sigma_i$ . These cohomology classes exist with integer coefficients, but we only need to consider them with real coefficients.

Denote by  $\mathcal{P}(M \setminus L)$  the set of  $\phi^t$ -invariant Borel probability measures on  $M \setminus L$ . From [Hry20, Theorem 1.3] we get the following criterion for  $L$  to bind an open book decomposition with pages that are global surface of section:

- (I)  $\rho^{\Sigma^*}(\gamma_i) > 0$  holds for every  $i$ .
- (II)  $\mu \cdot \Sigma^* > 0$  holds for every  $\mu \in \mathcal{P}(M \setminus L)$ .

Hence, we are left with the task of checking these conditions under the assumption that  $\phi^t$  is right-handed.

We check (I). By definition, the right-handedness of  $\phi^t$  implies that any two periodic orbits of  $\phi^t$  (oriented by  $X$ ) are positively linked. It also implies that  $\rho^{\Sigma_i^*}(\gamma_i) > 0$ . Then (I) follows from a direct application of Corollary 2.4.

We now check (II). In [Hry20, section 3] one finds the construction of a compact smooth 3-manifold  $D_L$  with boundary, obtained by blowing the periodic orbits in  $L$  up. This construction is originally due to Fried [Fri82]. There is a natural inclusion

$$\iota : M \setminus L \rightarrow D_L$$

that defines a diffeomorphism  $M \setminus L \simeq D_L \setminus \partial D_L$ , and for each  $i$  there is a boundary torus associated to the end of  $M \setminus L$  near  $\gamma_i$ . The key property is that there is a smooth nowhere vanishing vector field  $\hat{X}$  on  $D_L$ , tangent to  $\partial D_L$ , such that  $\hat{X}|_{D_L \setminus \partial D_L} = \iota_*(X|_{M \setminus L})$ .

We denote the flow of  $\hat{X}$  by  $\hat{\phi}^t$ , and the set of  $\hat{\phi}^t$ -invariant Borel probability measures on  $D_L$  by  $\mathcal{P}(D_L)$ . Let  $\mathcal{P}_0(D_L)$  be the set of those  $\lambda \in \mathcal{P}(D_L)$  satisfying  $\lambda(\partial D_L) = 0$ . There is a bijection

$$\mathcal{P}(M \setminus L) \rightarrow \mathcal{P}_0(D_L), \quad \mu \mapsto \hat{\mu} := \iota_* \mu. \quad (123)$$

This follows from the formula  $\iota_* \mu(E) = \mu(\iota^{-1}(E))$  for every Borel set  $E \subset D_L$ . In [Hry20, section 2] the existence of a smooth closed 1-form  $\beta$  on  $M \setminus L$  representing the class  $\Sigma^*$  and satisfying  $\sup_{p \in M \setminus L} |i_X \beta(p)| < \infty$  is established. Likewise, for every  $i$  there is a smooth closed 1-form  $\beta_i$  on  $M \setminus \gamma_i$  representing the class  $\Sigma_i^*$  and satisfying  $\sup_{p \in M \setminus \gamma_i} |i_X \beta_i(p)| < \infty$ . Let  $\hat{\beta}, \hat{\beta}_i$  be the 1-forms on  $D_L \setminus \partial D_L = M \setminus L$  induced by  $\beta, \beta_i$  respectively. Hence  $i_{\hat{X}} \hat{\beta} \circ \iota = i_X \beta$  and  $i_{\hat{X}} \hat{\beta}_i \circ \iota = i_X \beta_i$  on  $M \setminus L$ . Since  $\lambda(\partial D_L) = 0$ , for every  $\lambda$  in  $\mathcal{P}_0(D_L)$ , we conclude that

$$i_{\hat{X}} \hat{\beta}, i_{\hat{X}} \hat{\beta}_1, \dots, i_{\hat{X}} \hat{\beta}_n \in L^\infty(D_L, \lambda), \quad \forall \lambda \in \mathcal{P}_0(D_L).$$

We see from [Hry20, section 2] that for every  $\mu \in \mathcal{P}(M \setminus L)$

$$\begin{aligned} \mu \cdot \Sigma^* &= \int_{M \setminus L} i_X \beta \, d\mu = \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta} \, d\hat{\mu} \\ \mu \cdot \Sigma_i^* &= \int_{M \setminus L} i_X \beta_i \, d\mu = \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta}_i \, d\hat{\mu} \end{aligned} \quad (124)$$

holds.

Let  $\mu \in \mathcal{P}(M \setminus L)$  be arbitrary. Let  $\mathcal{E} \subset \mathcal{P}(D_L)$  be the set of ergodic measures. By Choquet's theorem there is a Borel probability measure  $\nu_{\hat{\mu}}$  on  $\mathcal{E}$  such that the following holds. Let  $f : U \rightarrow \mathbb{R}$  be a bounded continuous function defined on some open set  $U \subset D_L$ , and consider the Borel measurable function

$$\mathcal{E} \rightarrow \mathbb{R}, \quad \mu_e \mapsto \int_U f \, d\mu_e.$$

Then

$$\int_U f \, d\hat{\mu} = \int_{\mathcal{E}} \left( \int_U f \, d\mu_e \right) d\nu_{\hat{\mu}}(\mu_e). \quad (125)$$

Denote  $\mathcal{E}_L = \mathcal{E} \setminus \mathcal{P}_0(D_L)$ , i.e.  $\mu_e \in \mathcal{E}_L \Leftrightarrow \mu_e(\partial D_L) = 1$ . We claim that:

- (i)  $\nu_{\hat{\mu}}(\mathcal{E}_L) = 0$  for every  $\hat{\mu}$  in the image of the map (123).
- (ii)  $\int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta} \, d\mu_e > 0$  for every  $\mu_e \in \mathcal{E} \setminus \mathcal{E}_L$ .

To prove (i) note that  $\mathbf{1}_{\mathcal{P}_0(D_L)}(\mu_e) = \mu_e(D_L \setminus \partial D_L) \, \forall \mu_e \in \mathcal{E}$  where we denote by  $\mathbf{1}$  the characteristic function. This and (125) imply that

$$1 = \hat{\mu}(D_L \setminus \partial D_L) = \int_{\mathcal{E}} \mu_e(D_L \setminus \partial D_L) d\nu_{\hat{\mu}}(\mu_e) = \int_{\mathcal{E}} \mathbf{1}_{\mathcal{P}_0(D_L)} \, d\nu_{\hat{\mu}} = \nu_{\hat{\mu}}(\mathcal{P}_0(D_L) \cap \mathcal{E})$$

and hence  $\nu_{\hat{\mu}}(\mathcal{E}_L) = 1 - \nu_{\hat{\mu}}(\mathcal{P}_0(D_L) \cap \mathcal{E}) = 0$ .

Let us now prove (ii). Fix any  $\mu_e \in \mathcal{E} \setminus \mathcal{E}_L \subset \mathcal{P}_0(D_L)$ . Then  $\mu_e(D_L \setminus \partial D_L) = 1$ . From Birkhoff's ergodic theorem and Poincaré's recurrence combined we get a Borel set  $E_{\mu_e} \subset D_L \setminus \partial D_L$  consisting of recurrent points of  $\hat{\phi}^t$  such that

- $\mu_e(E_{\mu_e}) = 1$ .
- For every  $\hat{p} \in E_{\mu_e}$  and every  $i$ , if  $p := \iota^{-1}(\hat{p})$  then

$$\begin{aligned} \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta} \, d\mu_e &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{\hat{\phi}^{[0, T]}(\hat{p})} \hat{\beta} = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{\phi^{[0, T]}(p)} \beta, \\ \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta}_i \, d\mu_e &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{\hat{\phi}^{[0, T]}(\hat{p})} \hat{\beta}_i = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{\phi^{[0, T]}(p)} \beta_i. \end{aligned}$$

Note that  $p$  is a recurrent point of  $\phi^t$ . If  $T_n \rightarrow +\infty$  satisfies  $\phi^{T_n}(p) \rightarrow p$  and  $\alpha_n$  is a short geodesic arc<sup>4</sup> from  $\phi^{T_n}(p)$  to  $p$ , and if we denote by  $k(T_n, p)$  the loop obtained by concatenating  $\alpha_n$  to  $\phi^{[0, T_n]}(p)$ , then

$$\begin{aligned} \left| \text{int}(k(T_n, p), \Sigma) - \int_{\phi^{[0, T_n]}(p)} \beta \right| &\leq \|\beta\|_{L^\infty(B)} \text{Length}(\alpha_n) \\ \left| \text{link}(k(T_n, p), \gamma_i) - \int_{\phi^{[0, T_n]}(p)} \beta_i \right| &\leq \|\beta_i\|_{L^\infty(B)} \text{Length}(\alpha_n) \end{aligned}$$

where  $B \subset M \setminus L$  is a compact neighborhood of  $p$ . Hence for every  $p \in \iota^{-1}(E)$  and every  $i$

$$\begin{aligned} \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta} \, d\mu_e &= \lim_{n \rightarrow \infty} \frac{1}{T_n} \text{int}(k(T_n, p), \Sigma), \\ \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta}_i \, d\mu_e &= \lim_{n \rightarrow \infty} \frac{1}{T_n} \text{link}(k(T_n, p), \gamma_i). \end{aligned} \tag{126}$$

Since  $H_2(M, \mathbb{Z})$  vanishes, the 2-cycle  $\Sigma - \Sigma_1 - \dots - \Sigma_n$  is a boundary and we get

$$\text{int}(c, \Sigma) = \sum_{i=1}^n \text{int}(c, \Sigma_i) = \sum_{i=1}^n \text{link}(c, \gamma_i)$$

for every closed loop  $c$  in  $M \setminus L$ . It follows that

$$\int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta} \, d\mu_e = \sum_{i=1}^n \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta}_i \, d\mu_e \quad \forall \mu_e \in \mathcal{E} \setminus \mathcal{E}_L = \mathcal{P}_0(D_L) \cap \mathcal{E}. \tag{127}$$

Now we use the assumption of right-handedness to prove that

$$\int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta}_i \, d\mu_e > 0 \quad \forall \mu_e \in \mathcal{E} \setminus \mathcal{E}_L, \forall i. \tag{128}$$

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<sup>4</sup>Fix an auxiliary Riemannian metric on the background.

In fact, consider any  $\mu_e \in \mathcal{E} \setminus \mathcal{E}_L$  and the  $\phi^t$ -invariant Borel probability measure  $\tilde{\mu}_e$  on  $M$  uniquely determined by  $\iota_*(\tilde{\mu}_e|_{M \setminus L}) = \mu_e$ . In other words,  $\tilde{\mu}_e(A) = \mu_e(\iota(A \setminus L))$  for every Borel set  $A \subset M$ . If  $A \subset M$  is a  $\phi^t$ -invariant Borel set then  $\iota(A \setminus L)$  is a  $\hat{\phi}^t$ -invariant Borel set in  $D_L$  and  $\tilde{\mu}_e(A) = \mu_e(\iota(A \setminus L)) \in \{0, 1\}$  since  $\mu_e$  is ergodic. This shows that  $\tilde{\mu}_e$  is ergodic. For each  $i$  consider the  $\phi^t$ -invariant ergodic measure

$$\mu_{\gamma_i} = \frac{(\gamma_i)_* \text{Leb}}{\tau_i}$$

on  $M$ , where  $\tau_i > 0$  is the primitive  $\gamma_i$  which is seen as a map  $\gamma_i : \mathbb{R}/\tau_i\mathbb{Z} \rightarrow M$ , and  $\text{Leb}$  denotes Lebesgue measure on  $\mathbb{R}/\tau_i\mathbb{Z}$ . By definition  $\mu_{\gamma_i}$  is supported in  $\gamma_i \subset L$  and  $\tilde{\mu}_e(L) = 0$ . Hence for each  $i$  the pair of ergodic measures  $\tilde{\mu}_e, \mu_{\gamma_i}$  falls into case A explained in subsection 2.2. Note that  $\mu_{\gamma_i} \times \tilde{\mu}_e(\gamma_i \times \iota^{-1}(E_{\mu_e})) = 1$ . For  $\mu_{\gamma_i} \times \tilde{\mu}_e$ -almost all points  $(x, p) \in \gamma_i \times \iota^{-1}(E_{\mu_e})$ , we can consider any sequence  $T_j \rightarrow +\infty$  satisfying  $\phi^{T_j}(p) \rightarrow p$ , and estimate using (126)

$$\begin{aligned} \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta}_i d\mu_e &= \lim_{j \rightarrow \infty} \frac{1}{T_j} \text{link}(\gamma_i, k(T_j, p)) \\ &= \tau_i \lim_{j \rightarrow \infty} \frac{1}{j\tau_i T_j} \text{link}(k(j\tau_i, x), k(T_j, p)) > 0. \end{aligned}$$

The last inequality follows from the definition of right-handedness and the assumption that  $\phi^t$  is a right-handed flow. We have proved (128). From (i), (124), (125), (127) and (128) one can estimate

$$\begin{aligned} \mu \cdot \Sigma^* &= \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta} d\hat{\mu} = \int_{\mathcal{E}} \left( \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta} d\mu_e \right) d\nu_{\hat{\mu}}(\mu_e) \\ &= \int_{\mathcal{E} \setminus \mathcal{E}_L} \left( \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta} d\mu_e \right) d\nu_{\hat{\mu}}(\mu_e) \\ &= \sum_{i=1}^n \int_{\mathcal{E} \setminus \mathcal{E}_L} \left( \int_{D_L \setminus \partial D_L} i_{\hat{X}} \hat{\beta}_i d\mu_e \right) d\nu_{\hat{\mu}}(\mu_e) > 0. \end{aligned}$$

The proof of (II) is complete.

A direct application of [Hry20, Theorem 1.3] provides the desired global surface of section with boundary equal to  $L$ . The proof of Theorem 1.1 is complete.

## C Blown-up global coordinates

In this appendix we prove Proposition 2.8. The proof is divided into several steps.

### Step 1: A $C^1$ global surface of section from a finite energy plane

**Definition C.1.** Let  $\gamma$  be a periodic Reeb orbit of a contact form  $\lambda$  on a 3-dimensional manifold  $M$ , of primitive period  $T > 0$ . We may think it as a map

$\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow M$ . The contact structure  $\xi = \ker \lambda$  is oriented by  $d\lambda$ , and  $M$  by  $\lambda \wedge d\lambda$ . A *Martinet tube* around  $\gamma$  is, by definition, a diffeomorphism

$$F : \mathbb{R}/\mathbb{Z} \times \text{int}(\mathbb{D}) \subset \mathbb{R}/\mathbb{Z} \times \mathbb{C} \rightarrow U \quad (129)$$

onto an open neighbourhood  $U$  of  $\gamma$  such that

$$(MT1) \quad F(t, 0) = \gamma(Tt).$$

$$(MT2) \quad \text{There exists a smooth function } f : \mathbb{R}/\mathbb{Z} \times \text{int}(\mathbb{D}) \rightarrow (0, +\infty) \text{ such that } f(\vartheta, 0) \equiv T, df(\vartheta, 0) \equiv 0, \text{ and } F^*\lambda = f(\vartheta, x + iy)(d\vartheta + xdy).$$

The following lemma is proved in [HWZ96].

**Lemma C.2.** *Martinet tubes always exist. Moreover, the vector fields  $\partial_x, \partial_y$  along  $\gamma$  can be arranged to be any  $d\lambda$ -positive frame of  $\xi$  along  $\gamma$  given a priori.*

Let us consider the so-called *asymptotic operator*  $A$  associated to  $(\gamma_0, \lambda, J)$ , where  $J : \xi \rightarrow \xi$  is any  $d\lambda$ -compatible complex structure on  $\xi$ . It is an unbounded operator on the space of  $L^2$ -sections of the bundle  $\gamma_0(T_0 \cdot)^*\xi \rightarrow \mathbb{R}/\mathbb{Z}$ . It is defined by

$$A\eta = -J(\nabla_t \eta - T_0 \nabla_\eta X) \quad (130)$$

where  $\nabla$  is any symmetric connection on  $M$ , and  $\nabla_t$  denotes the associated covariant differentiation along  $t \mapsto \gamma_0(T_0 t)$ . It turns out that  $A$  does not depend on the choice of  $\nabla$ , as we shall see below.

To get a more concrete representation of  $A$  we use a Martinet tube  $F$  around  $\gamma_0$  to represent all objects with respect to the induced coordinates  $(\vartheta, x + iy)$ . Let  $\nabla$  be any symmetric connection on  $M$ . If we pull back  $\nabla$  by  $F$  then we get a symmetric connection on  $\mathbb{R}/\mathbb{Z} \times \text{int}(\mathbb{D})$  still denoted by  $\nabla$ . It can be explicitly written in these local coordinates as  $\nabla_v Y = DYv + B(v, Y)$  where  $B$  is a smooth map defined on  $U$  with values on the space of symmetric bilinear maps  $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . In view of the form of  $F^*\lambda$  we have that, along  $\gamma_0(T_0 t) \simeq (t, 0)$ ,  $\xi$  is represented as  $0 \times \mathbb{C}$  and  $d\lambda$  on  $\xi_{\gamma_0(T_0 t)}$  is represented as  $T_0 dx \wedge dy$ , for every  $t$ . If a section  $\eta(t)$  of  $\gamma_0(T_0 \cdot)^*\xi$  is represented as  $t \in \mathbb{R}/\mathbb{Z} \mapsto (0, v(t)) \in \{0\} \times \mathbb{C}$  then  $\nabla_t \eta$  is represented by  $(0, \dot{v}) + B(t, 0)((1, 0), (0, v))$ . The operator  $\eta \mapsto \nabla_t \eta - T_0 \nabla_\eta X$  gets represented as

$$\begin{aligned} (0, v) &\mapsto (0, \dot{v}) + B(t, 0)((1, 0), (0, v)) - T_0 DX(t, 0)(0, v) - T_0 B(t, 0)((0, v), X(t, 0)) \\ &= (0, \dot{v}) - T_0 DX(t, 0)(0, v). \end{aligned}$$

Here it was used that  $X(\vartheta, 0) = (T_0^{-1}, 0) = T_0^{-1} \partial_\vartheta$  and that  $B$  is symmetric. Now we need to compute  $DX(\vartheta, 0)$ . Since  $\lambda$  is pulled back to  $f(d\vartheta + xdy)$  by  $F$  we get that  $X$  is written locally as  $X = X^\vartheta \partial_\vartheta + X^x \partial_x + X^y \partial_y$  where

$$X^\vartheta = \frac{f + x f_x}{f^2} \quad X^x = \frac{f_y - x f_\vartheta}{f^2} \quad X^y = -\frac{f_x}{f^2} \quad (131)$$

Using that  $f(\vartheta, 0) = T_0$  and  $df(\vartheta, 0) = 0$  one can compute

$$DX(\vartheta, 0) = \frac{1}{T_0^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_{xy}(\vartheta, 0) & f_{yy}(\vartheta, 0) \\ 0 & -f_{xx}(\vartheta, 0) & -f_{yx}(\vartheta, 0) \end{pmatrix} = \frac{1}{T_0} \begin{pmatrix} 0 & 0 \\ 0 & -J_0 S(\vartheta) \end{pmatrix}$$

where the first matrix ( $3 \times 3$ ) is expressed in the basis  $\{\partial_\vartheta, \partial_x, \partial_y\}$ , the second matrix is written in blocks, and

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S(\vartheta) = \frac{1}{T_0} \begin{pmatrix} f_{xx}(\vartheta, 0) & f_{yx}(\vartheta, 0) \\ f_{xy}(\vartheta, 0) & f_{yy}(\vartheta, 0) \end{pmatrix}.$$

If the matrix-valued function

$$J(\vartheta, x + iy) = \begin{pmatrix} J_{11}(\vartheta, x + iy) & J_{12}(\vartheta, x + iy) \\ J_{21}(\vartheta, x + iy) & J_{22}(\vartheta, x + iy) \end{pmatrix} \quad (132)$$

represents the complex structure  $J$  on  $\xi$  with respect to the (conformally symplectic) frame  $\{\partial_x, \partial_y - x\partial_\vartheta\}$ , then with the above formulas together we represent  $A$  as

$$v(t) \mapsto -J(t, 0)(\dot{v}(t) + J_0 S(t)v(t)). \quad (133)$$

This is a symmetric operator with the respect to the Hilbert bundle structure induced by the inner-product

$$\langle v, w \rangle = \int_{\mathbb{R}/\mathbb{Z}} \langle J_0 v(t), J(t, 0)w(t) \rangle dt.$$

From (133) it follows that the spectrum  $\sigma(A)$  is real, discrete, accumulates only at  $\pm\infty$ , see [HWZ95]. We still denote by  $A$  this local representation, with no fear of ambiguity.

It is known from [HWZ99] that dynamical convexity of a contact form  $\lambda$  on  $S^3$  forces  $\xi$  to be a tight contact structure. The main result from [Hry14] gives a global surface of section spanned by any unknotted periodic Reeb orbit  $\gamma_0$  with self-linking number  $-1$ . Let us choose a  $d\lambda$ -compatible complex structure  $J$  on  $\xi$ . This means that  $d\lambda(\cdot, J\cdot)$  defines a metric on the vector bundle  $\xi$ . Then we can follow Hofer [Hof93] and consider an  $\mathbb{R}$ -invariant almost complex structure  $\tilde{J}$  on  $\mathbb{R} \times S^3$  defined by

$$\tilde{J} : \partial_a \mapsto X \quad \tilde{J}|_\xi = J \quad (134)$$

where  $X$  denotes the Reeb vector field of  $\lambda$ . Here we see  $X$  and  $\xi$  as  $\mathbb{R}$ -invariant objects in  $\mathbb{R} \times S^3$ . The interior of the global surface of section produced by [Hry14] is the projection to  $S^3$  of a smooth embedding

$$\tilde{u} : \mathbb{C} \rightarrow \mathbb{R} \times S^3 \quad (135)$$

solving the Cauchy-Riemann equations

$$d\tilde{u} + \tilde{J}(\tilde{u}) \circ d\tilde{u} \circ i = 0$$

asymptotic to  $\gamma_0 : \mathbb{R}/T_0\mathbb{Z} \rightarrow S^3$  in the following sense. If we write in components  $\tilde{u}(s, t) = (a(s, t), u(s, t))$  with respect to exponential polar coordinates

$$(s, t) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \mapsto e^{2\pi(s+it)} \in \mathbb{C} \setminus \{0\}$$

then  $a(s, t) \rightarrow +\infty$  uniformly in  $t \in \mathbb{R}/\mathbb{Z}$  as  $s \rightarrow +\infty$  and

$$\lim_{s \rightarrow +\infty} u(s, t + t_0) = \gamma_0(T_0 t) \quad C^\infty(\mathbb{R}/\mathbb{Z}, S^3) \quad (136)$$

for some  $t_0$ .

**Lemma C.3.** *There exists an embedded closed disk  $\mathcal{D} \hookrightarrow S^3$  of class  $C^1$  such that  $\partial\mathcal{D} = \gamma_0$  and  $\mathcal{D} \setminus \gamma_0 = u(\mathbb{C})$ .*

The above lemma is probably well-known to specialists, but we prove it here since we need to understand well the behaviour of  $\mathcal{D}$  near  $\gamma_0$ .

*Proof.* Fix a choice  $F$  of Martinet tube around  $\gamma_0$ . In view of (136) we can represent the map  $u(s, t)$  in the  $(\vartheta, x + iy)$ -coordinates as

$$F^{-1} \circ u : (s, t) \mapsto (\vartheta(s, t), z(s, t) = x(s, t) + iy(s, t)) \quad (s \geq s_0) \quad (137)$$

for some  $s_0 \gg 1$  large enough. In [HWZ98, section 8] one finds the proof of a very precise asymptotic formula for the plane (135) which we will now describe. There exist  $b > 0$ ,  $a_0 \in \mathbb{R}$  and  $t_0 \in \mathbb{R}/\mathbb{Z}$  such that

$$\begin{aligned} \lim_{s \rightarrow +\infty} \sup_{t \in \mathbb{R}/\mathbb{Z}} e^{bs} |D^\alpha[a(s, t) - T_0 s - a_0]| &= 0 \\ \lim_{s \rightarrow +\infty} \sup_{t \in \mathbb{R}/\mathbb{Z}} e^{bs} |D^\alpha[\vartheta(s, t) - t - t_0]| &= 0 \end{aligned} \quad (138)$$

holds for all  $D^\alpha = \partial_s^{\alpha_1} \partial_t^{\alpha_2}$ . Moreover there is an eigenvalue  $\mu_0 < 0$  of  $A$ , an eigenvector  $e(t) \in \ker(A - \mu_0 I)$  and smooth functions  $\mu : [s_0, +\infty) \rightarrow \mathbb{R}$  and  $R : [s_0, +\infty) \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$  such that

$$z(s, t) = e^{\int_{s_0}^s \mu(\tau) d\tau} (e(t) + R(s, t)) \quad (139)$$

$$\begin{aligned} \lim_{s \rightarrow +\infty} |D^j[\mu(s) - \mu_0]| &= 0 \quad \forall j \geq 0 \\ \lim_{s \rightarrow +\infty} \sup_{t \in \mathbb{R}/\mathbb{Z}} |D^\alpha R(s, t)| &= 0 \quad \forall D^\alpha = \partial_s^{\alpha_1} \partial_t^{\alpha_2} \end{aligned} \quad (140)$$

hold.

*Remark C.4.* It should be noted that, in general, such asymptotic behaviour is only guaranteed when the asymptotic limit  $\gamma_0$  is non-degenerate. But in [HWZ98, section 8] Hofer, Wysocki and Zehnder obtain this formula for the special pseudo-holomorphic plane  $\tilde{u}$  without any non-degeneracy assumptions. The same asymptotic behaviour is proved for the pseudo-holomorphic planes from [Hry14].



Perhaps after translating in the coordinate  $t$  and increasing  $s_0$  we may assume that  $t_0 = 0$ ,  $s \geq s_0 \Rightarrow \mu(s) \leq \mu_0/2 < 0$ , and the map  $t \mapsto \vartheta(s, t)$  defines an orientation preserving self-diffeomorphism of  $\mathbb{R}/\mathbb{Z}$  for all  $s \geq s_0$ . This follows from (138). Hence

$$\psi(s, t) = (s, \vartheta(s, t))$$

defines a self-diffeomorphism of  $[s_0, +\infty) \times \mathbb{R}/\mathbb{Z}$ . Again by (138) with  $t_0 = 0$  we have

$$\lim_{s \rightarrow +\infty} \sup_{t \in \mathbb{R}/\mathbb{Z}} e^{bs} |D^\alpha [D\psi(s, t) - I]| = 0, \quad \forall D^\alpha = \partial_s^{\alpha_1} \partial_t^{\alpha_2}.$$

For the inverse mapping  $\psi^{-1}(s, \vartheta) = (s, t(s, \vartheta))$  we must also have

$$\lim_{s \rightarrow +\infty} \sup_{\vartheta \in \mathbb{R}/\mathbb{Z}} e^{bs} |D^\alpha [D\psi^{-1}(s, \vartheta) - I]| = 0 \quad \forall D^\alpha = \partial_s^{\alpha_1} \partial_\vartheta^{\alpha_2}$$

from where it follows that

$$\lim_{s \rightarrow +\infty} \sup_{\vartheta \in \mathbb{R}/\mathbb{Z}} |D^\alpha [e(\vartheta) - e(t(s, \vartheta))]| = 0, \quad \forall D^\alpha = \partial_s^{\alpha_1} \partial_\vartheta^{\alpha_2}.$$

Hence in view of (139) and (140) we can write

$$z \circ \psi^{-1}(s, \vartheta) = e^{\int_{s_0}^s \mu(\tau) d\tau} (e(\vartheta) + \hat{R}(s, \vartheta))$$

where  $\hat{R}(s, \vartheta) = e(t(s, \vartheta)) - e(\vartheta) + R(\psi^{-1}(s, \vartheta))$  satisfies

$$\lim_{s \rightarrow +\infty} \sup_{\vartheta \in \mathbb{R}/\mathbb{Z}} |D^\alpha \hat{R}(s, \vartheta)| = 0, \quad \forall D^\alpha = \partial_s^{\alpha_1} \partial_\vartheta^{\alpha_2}.$$

Let us denote by  $G : [s_0, +\infty) \rightarrow (0, +\infty)$  the function

$$G(s) = e^{\int_{s_0}^s \mu(\tau) d\tau}$$

It satisfies  $G(s_0) = 1$ ,  $G'(s) = \mu(s)G(s) < 0$ , and  $G(s) \rightarrow 0$  as  $s \rightarrow +\infty$ . Consider a parameter  $\rho \geq 0$  and a solution  $s(\rho)$  of the initial value problem

$$s'(\rho)G'(s(\rho)) = -1, \quad s(0) = s_0.$$

Then  $G(s(\rho)) = 1 - \rho$ , from where it follows that the maximal forward existence interval of  $s(\rho)$  is  $[0, 1)$  and that  $s(\rho) \rightarrow +\infty$  as  $\rho \rightarrow 1$ . In particular the map  $G(s(\rho))$  extends smoothly to  $[0, 1]$  as the function  $\rho \mapsto 1 - \rho$ . We claim that

$$\epsilon(\rho, \vartheta) := (1 - \rho)\hat{R}(s(\rho), \vartheta) \tag{141}$$

extends to a  $C^1$  function on  $[0, 1] \times \mathbb{R}/\mathbb{Z}$  whose 1-jet vanishes at  $\{1\} \times \mathbb{R}/\mathbb{Z}$ . To see this, first note that the function and the partial derivative in  $\vartheta$  clearly converge to

0 uniformly in  $\vartheta$  as  $\rho \rightarrow 1$ . For the partial derivative in  $\rho$  we have

$$\begin{aligned}
& \left| \partial_\rho [(1 - \rho)\hat{R}(s(\rho), \vartheta)] \right| \\
&= \left| -\hat{R}(s(\rho), \vartheta) + (1 - \rho)\partial_s \hat{R}(s(\rho), \vartheta)s'(\rho) \right| \\
&= \left| -\hat{R}(s(\rho), \vartheta) + \partial_s \hat{R}(s(\rho), \vartheta)s'(\rho) \frac{G'(s(\rho))}{\mu(s)} \right| \\
&= \left| -\hat{R}(s(\rho), \vartheta) - \frac{1}{\mu(s)} \partial_s \hat{R}(s(\rho), \vartheta) \right| \\
&\leq C \left( |\hat{R}(s(\rho), \vartheta)| + |\partial_s \hat{R}(s(\rho), \vartheta)| \right) \rightarrow 0 \text{ as } \rho \rightarrow 1
\end{aligned} \tag{142}$$

as claimed. Denote

$$\Gamma(\rho, \vartheta) = z \circ \psi^{-1}(s(\rho), \vartheta) = (1 - \rho)e(\vartheta) + \epsilon(\rho, \vartheta) \tag{143}$$

where  $\epsilon(\rho, \vartheta)$  is defined in (141). Above we proved that  $\epsilon(\rho, \vartheta)$  defines a  $C^1$ -function up to  $\rho = 1$  whose 1-jet vanishes on  $\{\rho = 1\}$ . Hence the map  $\Gamma$ , which was defined on  $[0, 1) \times \mathbb{R}/\mathbb{Z}$ , admits a  $C^1$  extension to  $[0, 1] \times \mathbb{R}/\mathbb{Z}$ . We still denote this extension by  $\Gamma$ , it satisfies

$$\Gamma(1, \vartheta) = 0, \quad D_1 \Gamma(1, \vartheta) = -e(\vartheta), \quad \forall \vartheta \in \mathbb{R}/\mathbb{Z}. \tag{144}$$

As consequence, the map

$$(\rho, \vartheta) \mapsto (\vartheta, \Gamma(\rho, \vartheta)) \tag{145}$$

extends the map  $F^{-1} \circ u \circ \psi^{-1}(s(\rho), \vartheta)$ , and defines an embedding. Since  $u$  is an embedding into  $S^3 \setminus \gamma_0$  we can glue the embedding  $u|_{\{|w| \leq e^{2\pi s_0}\}}$  with the embedded strip (145) to produce the  $C^1$  embedded disk  $\mathcal{D}$  as desired.  $\square$

The following lemma provides additional information about the second order derivatives of  $\Gamma$ ; it will be crucial later on.

**Lemma C.5.** *The partial derivatives  $D_{22}\Gamma$ ,  $D_{12}\Gamma = D_{21}\Gamma$  of the map (143) extend continuously to  $[0, 1] \times \mathbb{R}/\mathbb{Z}$  as*

$$D_{22}\Gamma(1, \vartheta) = 0, \quad D_{21}\Gamma(1, \vartheta) = D_{12}\Gamma(1, \vartheta) = -e'(\vartheta).$$

*The partial derivative  $D_{11}\Gamma(\rho, \vartheta)$  satisfies*

$$\lim_{\rho \rightarrow 1^-} (1 - \rho) \sup_{\vartheta \in \mathbb{R}/\mathbb{Z}} |D_{11}\Gamma(\rho, \vartheta)| = 0. \tag{146}$$

*Proof.* Note that

$$D_2[\Gamma(\rho, \vartheta) - (1 - \rho)e(\vartheta)] = D_2[(1 - \rho)\hat{R}(s(\rho), \vartheta)] = (1 - \rho)D_2\hat{R}(s(\rho), \vartheta)$$

hence

$$\begin{aligned} D_{22}[\Gamma(\rho, \vartheta) - (1 - \rho)e(\vartheta)] &= (1 - \rho)D_{22}\hat{R}(s(\rho), \vartheta), \\ D_{21}[\Gamma(\rho, \vartheta) - (1 - \rho)e(\vartheta)] &= (1 - \rho)s'(\rho)D_{21}\hat{R}(s(\rho), \vartheta) - D_2\hat{R}(s(\rho), \vartheta). \end{aligned} \quad (147)$$

But note that  $s'(\rho)(1 - \rho) = -\mu(s(\rho))^{-1}$  is bounded as  $\rho \rightarrow 1^-$ . Hence, in view of (140), the partial derivatives (147) converge to zero uniformly in  $\vartheta$  as  $\rho \rightarrow 1^-$ . For the second assertion we compute

$$\begin{aligned} D_{11}\Gamma(\rho, \vartheta) &= D_{11}[(1 - \rho)\hat{R}(s(\rho), \vartheta)] \\ &= D_1[-\hat{R}(s(\rho), \vartheta) + (1 - \rho)s'(\rho)D_1\hat{R}(s(\rho), \vartheta)] \\ &= D_1\left[-\hat{R}(s(\rho), \vartheta) - \frac{1}{\mu(s(\rho))}D_1\hat{R}(s(\rho), \vartheta)\right] \\ &= -s'(\rho)D_1\hat{R}(s(\rho), \vartheta) + \frac{\mu'(s(\rho))}{\mu(s(\rho))^2}s'(\rho)D_1\hat{R}(s(\rho), \vartheta) \\ &\quad - \frac{1}{\mu(s(\rho))}s'(\rho)D_{11}\hat{R}(s(\rho), \vartheta) \end{aligned} \quad (148)$$

and hence

$$\begin{aligned} (1 - \rho)D_{11}\Gamma(\rho, \vartheta) &= (1 - \rho)s' \left[ -D_1\hat{R}(s, \vartheta) + \frac{\mu'(s)}{\mu(s)^2}D_1\hat{R}(s, \vartheta) - \frac{1}{\mu(s)}D_{11}\hat{R}(s, \vartheta) \right] \\ &= -\frac{1}{\mu(s)} \left[ -D_1\hat{R}(s, \vartheta) + \frac{\mu'(s)}{\mu(s)^2}D_1\hat{R}(s, \vartheta) - \frac{1}{\mu(s)}D_{11}\hat{R}(s, \vartheta) \right] \end{aligned} \quad (149)$$

Using that  $\mu(s)$  is bounded and bounded away from zero, that  $\mu'(s)$  is bounded and that the partial derivatives of  $\hat{R}$  converge to zero as  $s \rightarrow +\infty$ , we conclude that the function in (149) converges to zero uniformly in  $\vartheta$  as  $\rho \rightarrow 1^-$ .  $\square$

## Step 2: Regularising the disk $\mathcal{D}$ to a $\partial$ -strong $C^\infty$ global surface of section

From now on we assume that the frame  $\partial_x, \partial_y$  along  $\mathbb{R}/\mathbb{Z} \times \{0\}$  corresponds to a  $J$ -complex frame along  $\gamma_0$ , which means that

$$J(\vartheta, 0) = J_0 \quad \forall \vartheta \in \mathbb{R}/\mathbb{Z} \quad (150)$$

This is not a necessary assumption, but it simplifies our calculations. In complex notation, the asymptotic operator (133) assumes the simplified form  $-i\partial_t + S(t)$ , and is symmetric with respect to the standard  $L^2$  inner-product.

**Lemma C.6.** *Let  $\mathcal{D}$  be the  $C^1$  disk provided by Lemma C.3. There exists a  $C^\infty$  family of embedded  $C^\infty$  disks  $\mathcal{D}_\delta$ ,  $0 < \delta \ll 1$ , satisfying*

(i)  $\mathcal{D}_\delta$  is a  $\partial$ -strong global surface of section satisfying  $\partial\mathcal{D}_\delta = \gamma_0$ .

(ii)  $\mathcal{D}_\delta \rightarrow \mathcal{D}$  in  $C^1$  as  $\delta \rightarrow 0$ .

*Proof.* A neighbourhood of  $\partial\mathcal{D}$  in  $\mathcal{D}$  is a strip parametrised with coordinates  $(\rho, \vartheta)$  by the map (145) where  $\Gamma(\rho, \vartheta)$  is the map (143). Consider the smooth map

$$\Phi : \mathbb{R}/\mathbb{Z} \times [0, 1) \times \mathbb{R}/2\pi \rightarrow \mathbb{R}/\mathbb{Z} \times \text{int}(\mathbb{D}) \quad \Phi(\vartheta, r, \theta) = (\vartheta, re^{i\theta}). \quad (151)$$

Direct calculations found in Appendix A, see also [Hry20, section 3], show that  $\Phi^*X$  extends smoothly to a vector field  $W$  on  $\mathbb{R}/\mathbb{Z} \times [0, 1) \times \mathbb{R}/2\pi$  satisfying

$$W(\vartheta, 0, \theta) = T_0^{-1}\partial_\vartheta + b(\vartheta, \theta)\partial_\theta \quad (152)$$

for some smooth function  $b : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}$ . In the following we need a precise description of the function  $b(\vartheta, \theta)$  appearing in (152). Denote

$$Y(\vartheta, x + iy) = X^x(\vartheta, x + iy) + iX^y(\vartheta, x + iy) : \mathbb{R}/\mathbb{Z} \times \text{int}(\mathbb{D}) \rightarrow \mathbb{C}$$

and write

$$X(\vartheta, z) = (T_0^{-1} + \Lambda_1(\vartheta, z)z, \Lambda_2(\vartheta, z)z)$$

where  $z = x + iy$  and

$$\Lambda_1(\vartheta, 0) = 0 \quad \Lambda_2(\vartheta, 0) = D_2Y(\vartheta, 0) = -T_0^{-1}J_0S(\vartheta)$$

Here we used that  $D_2X^\vartheta(\vartheta, 0) = 0$ , which follows from (131) and the properties of  $f$ . In the basis  $\{\partial_\vartheta, \partial_x, \partial_y\}$  we can write

$$D\Phi^{-1}(\vartheta, r \cos \theta + ir \sin \theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -r^{-1} \sin \theta & r^{-1} \cos \theta \end{pmatrix}$$

Hence

$$W(\vartheta, r, \theta) = \left( T_0^{-1} + \Lambda_1(\vartheta, re^{i\theta})re^{i\theta}, \begin{bmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \Lambda_2(\vartheta, re^{i\theta})e^{i\theta} \right) \quad (153)$$

showing that  $W$  extends smoothly all the way to  $r = 0$  and satisfies (152) with

$$b(\vartheta, \theta) = \langle -T_0^{-1}J_0S(\vartheta)e^{i\theta}, ie^{i\theta} \rangle = -T_0^{-1} \langle S(\vartheta)e^{i\theta}, e^{i\theta} \rangle \quad (154)$$

A neighbourhood of  $\partial\mathcal{D} = \gamma_0$  in  $\mathcal{D}$  is  $C^1$ -parametrised as a the strip (145) where  $\partial\mathcal{D}$  corresponds to  $\rho = 1$ . Denote

$$\Phi^{-1}(\vartheta, \Gamma(\rho, \vartheta)) = (\vartheta, \Gamma^r(\rho, \vartheta), \Gamma^\theta(\rho, \vartheta)) \quad (155)$$

Note that  $\Gamma^\theta$  is, at this moment, only defined for  $\rho < 1$ , but now we argue to prove that it extends continuously up to  $\rho = 1$ . We have  $\Gamma = \Gamma^r e^{i\Gamma^\theta}$ . Since  $\Gamma$  is  $C^1$

$$\lim_{\rho \rightarrow 1^-} \frac{\Gamma(\rho, \vartheta)}{1 - \rho} = \lim_{\rho \rightarrow 1^-} -\frac{\Gamma(\rho, \vartheta) - \Gamma(1, \vartheta)}{\rho - 1} = -D_1\Gamma(1, \vartheta) = e(\vartheta)$$

uniformly in  $\vartheta$ . Since  $e(\vartheta)$  never vanishes we conclude that  $\Gamma^\theta$  extends continuously up to  $\rho = 1$ , and if we still denote this extension by  $\Gamma^\theta$  then

$$e(\vartheta) = |e(\vartheta)|e^{i\Gamma^\theta(1,\vartheta)}. \quad (156)$$

This defines an extension of (155) to a  $C^0$  embedded strip in  $\mathbb{R}/\mathbb{Z} \times [0, 1] \times \mathbb{R}/2\pi\mathbb{Z}$ . Fix a smooth bump function  $\chi : \mathbb{R} \rightarrow [0, 1]$  satisfying

- $\chi(\rho) = 1$  if  $\rho \leq -1$ ,  $\chi(\rho) = 0$  if  $\rho \geq 0$ .
- $\chi'(\rho) \leq 0 \ \forall \rho$ ,  $\text{supp}(\chi') \subset (-1, 0)$ ,  $\|\chi'\|_{L^\infty} \leq 2$ .

With  $\delta > 0$  consider

$$\chi_\delta : \mathbb{R} \rightarrow [0, 1], \quad \chi_\delta(\rho) = \chi(\delta^{-1}(\rho - 1)).$$

We have

$$\text{supp}(\chi'_\delta) \subset (1 - \delta, 1) \quad \|\chi'_\delta\|_{L^\infty} \leq \frac{2}{\delta} \quad (157)$$

and

$$|(1 - \rho)\chi'_\delta(\rho)| = \left| \frac{1 - \rho}{\delta} \delta \chi'_\delta(\rho) \right| \leq 2 \quad \forall \rho \ \forall \delta \in (0, 1] \quad (158)$$

since  $\chi'_\delta(\rho) \neq 0 \Rightarrow \rho > 1 - \delta \Rightarrow \frac{1 - \rho}{\delta} < 1$ . One can write

$$\begin{aligned} \Gamma(\rho, \vartheta) &= (1 - \rho)e(\vartheta) + \epsilon(\rho, \vartheta) \\ \Gamma_\delta(\rho, \vartheta) &= (1 - \rho)e(\vartheta) + \epsilon_\delta(\rho, \vartheta) \end{aligned} \quad (159)$$

with

$$\epsilon(\rho, \vartheta) = \Gamma(\rho, \vartheta) - (1 - \rho)e(\vartheta), \quad \epsilon_\delta(\rho, \vartheta) = \chi_\delta(\rho)\epsilon(\rho, \vartheta). \quad (160)$$

The map  $\epsilon$  was already defined in (141). It follows that  $\Gamma_\delta(\rho, \vartheta) \neq 0$  for all  $\rho < 1$  if  $\delta > 0$  is small enough. Lemma C.5 together with (157)-(158) gives

$$\begin{aligned} \lim_{\rho \rightarrow 1^-} \sup_{\delta \in (0, 1]} \frac{\|\epsilon_\delta(\rho, \cdot)\|_{L^\infty(\mathbb{R}/\mathbb{Z})} + \|D_2 \epsilon_\delta(\rho, \cdot)\|_{L^\infty(\mathbb{R}/\mathbb{Z})}}{1 - \rho} &= 0 \\ \lim_{\rho \rightarrow 1^-} \sup_{\delta \in (0, 1]} \|D_1 \epsilon_\delta(\rho, \cdot)\|_{L^\infty(\mathbb{R}/\mathbb{Z})} &= 0 \end{aligned} \quad (161)$$

Analogously as before define  $\Gamma_\delta^r, \Gamma_\delta^\theta$  by  $\Gamma_\delta = \Gamma_\delta^r e^{i\Gamma_\delta^\theta}$ . Then

$$\Phi^{-1}(\vartheta, \Gamma_\delta(\rho, \vartheta)) = (\vartheta, \Gamma_\delta^r(\rho, \vartheta), \Gamma_\delta^\theta(\rho, \vartheta)) \quad (162)$$

The strip (155) is not even  $C^1$  but (162) is  $C^\infty$  since  $\Gamma_\delta(\rho, \vartheta) = (1 - \rho)e(\vartheta)$  for  $\rho$  close enough to 1. Moreover  $\Gamma_\delta^r(1, \vartheta) = \Gamma^r(1, \vartheta) = 0$  for all  $\delta > 0$ . From (161) we

conclude that

$$\begin{aligned}
\lim_{\rho \rightarrow 1^-} \sup_{\delta \in (0,1]} \left\| \frac{\Gamma_\delta(\rho, \cdot)}{(1-\rho)e(\cdot)} - 1 \right\|_{L^\infty(\mathbb{R}/\mathbb{Z})} &= 0 \\
\lim_{\rho \rightarrow 1^-} \sup_{\delta \in (0,1]} \left\| \frac{D_2\Gamma_\delta(\rho, \cdot)}{(1-\rho)} - e'(\cdot) \right\|_{L^\infty(\mathbb{R}/\mathbb{Z})} &= 0 \\
\lim_{\rho \rightarrow 1^-} \sup_{\delta \in (0,1]} \left\| \frac{D_1\Gamma_\delta(\rho, \cdot)}{e(\cdot)} + 1 \right\|_{L^\infty(\mathbb{R}/\mathbb{Z})} &= 0
\end{aligned} \tag{163}$$

which can then be used to obtain

$$\begin{aligned}
\lim_{\rho \rightarrow 1^-} \sup_{\delta \in (0,1]} \| D_1\Gamma_\delta^r(\rho, \cdot) + |e(\cdot)| \|_{L^\infty(\mathbb{R}/\mathbb{Z})} &= 0 \\
\limsup_{\rho \rightarrow 1^-} \sup_{\delta \in (0,1]} \|(1-\rho)D_1\Gamma_\delta^\theta(\rho, \cdot)\|_{L^\infty(\mathbb{R}/\mathbb{Z})} &= 0 \\
\lim_{\rho \rightarrow 1^-} \sup_{\delta \in (0,1]} \left\| D_2\Gamma_\delta^\theta(\rho, \cdot) - \operatorname{Im} \frac{e'(\cdot)}{e(\cdot)} \right\|_{L^\infty(\mathbb{R}/\mathbb{Z})} &= 0 \\
\lim_{\rho \rightarrow 1^-} \sup_{\delta \in (0,1]} \| D_2\Gamma_\delta^r(\rho, \cdot) D_1\Gamma_\delta^\theta(\rho, \cdot) \|_{L^\infty(\mathbb{R}/\mathbb{Z})} &= 0
\end{aligned} \tag{164}$$

The first three limits in (164) are straightforward consequences of (163) and are left to the reader. Here we only spell out the details for the last limit.

$$\begin{aligned}
&D_2\Gamma_\delta^r D_1\Gamma_\delta^\theta \\
&= |\Gamma_\delta| \operatorname{Re} \left( \frac{D_2\Gamma_\delta}{\Gamma_\delta} \right) \operatorname{Im} \left( \frac{D_1\Gamma_\delta}{\Gamma_\delta} \right) = \operatorname{Re} \left( \frac{D_2\Gamma_\delta}{\Gamma_\delta} \right) \operatorname{Im} \left( \frac{D_1\Gamma_\delta}{\Gamma_\delta/|\Gamma_\delta|} \right) \\
&= \operatorname{Re} \left( \frac{D_2\Gamma_\delta}{1-\rho} \frac{(1-\rho)e(\vartheta)}{\Gamma_\delta} \frac{1}{e(\vartheta)} \right) \operatorname{Im} \left( \frac{D_1\Gamma_\delta}{e(\vartheta)} \frac{(1-\rho)e(\vartheta)}{\Gamma_\delta} \frac{|\Gamma_\delta|}{(1-\rho)|e(\vartheta)|} |e(\vartheta)| \right) \\
&\rightarrow \operatorname{Re} \left( \frac{e'(\vartheta)}{e(\vartheta)} \right) \operatorname{Im}(-|e(\vartheta)|) = 0
\end{aligned} \tag{165}$$

where the convergence is uniform in  $(\delta, \vartheta) \in (0, 1] \times \mathbb{R}/\mathbb{Z}$  in view of (163).

Denote the components of  $W$  by

$$W = (W^\vartheta, W^r, W^\theta) = W^\vartheta \partial_\vartheta + W^r \partial_r + W^\theta \partial_\theta$$

Consider

$$\begin{aligned}
K_\delta(\rho, \vartheta) &= \det \begin{pmatrix} 1 & 0 & W^\vartheta \\ D_2\Gamma_\delta^r & D_1\Gamma_\delta^r & W^r \\ D_2\Gamma_\delta^\theta & D_1\Gamma_\delta^\theta & W^\theta \end{pmatrix} \\
&= D_1\Gamma_\delta^r W^\theta - D_1\Gamma_\delta^\theta W^r + W^\vartheta (D_2\Gamma_\delta^r D_1\Gamma_\delta^\theta - D_1\Gamma_\delta^r D_2\Gamma_\delta^\theta)
\end{aligned} \tag{166}$$

where  $W^\vartheta, W^r, W^\theta$  are evaluated at (162). Consider also the function

$$K(\vartheta) = |e(\vartheta)| \left( -b(\vartheta, \Gamma^\theta(1, \vartheta)) + T_0^{-1} \operatorname{Im} \left( \frac{e'(\vartheta)}{e(\vartheta)} \right) \right) \tag{167}$$

It follows from (163) and (164), together with the fact that  $W$  is smooth and satisfies  $W^r|_{\{r=0\}} = 0$ , that

$$\lim_{\rho \rightarrow 1^-} \sup_{\delta \in (0,1]} \|K_\delta(\rho, \cdot) - K(\cdot)\|_{L^\infty(\mathbb{R}/\mathbb{Z})} = 0 \quad (168)$$

Here it was used that  $W^r/(1-\rho)$  is bounded as  $\rho \rightarrow 1^-$ , uniformly in  $\delta$ .

We now claim that

$$K(\vartheta) < 0 \quad \forall \vartheta \in \mathbb{R}/\mathbb{Z} \quad (169)$$

Note that  $\Gamma^\theta(1, \vartheta)$  is the argument of  $e(\vartheta)$ , hence by (154)

$$b(\vartheta, \Gamma^\theta(1, \vartheta)) = -T_0^{-1} \frac{\langle S(\vartheta)e(\vartheta), e(\vartheta) \rangle}{|e(\vartheta)|^2} \quad (170)$$

By (150) the ODE solved by  $e(\vartheta)$  is written in complex notation as

$$-ie'(\vartheta) + S(\vartheta)e(\vartheta) = \mu_0 e(\vartheta)$$

from where we conclude that

$$\operatorname{Im} \left( \frac{e'(\vartheta)}{e(\vartheta)} \right) = \mu_0 - \frac{\langle S(\vartheta)e(\vartheta), e(\vartheta) \rangle}{|e(\vartheta)|^2} \quad (171)$$

from where it follows that  $K(\vartheta) = T_0^{-1} \mu_0 |e(\vartheta)| < 0$  thus proving (169).

We can find  $\eta > 0$  and  $\delta_\eta \in (0, 1)$  such that

$$0 < \delta < \delta_\eta \quad \Rightarrow \quad |K_\delta(\rho, \vartheta)| \geq \eta \quad \forall (\rho, \vartheta) \in (1 - \delta, 1] \times \mathbb{R}/\mathbb{Z}. \quad (172)$$

In fact, by (169) we can fix  $0 < 2\eta < \inf_\vartheta |K(\vartheta)|$ . By (168) there is  $\rho_\eta \in (0, 1)$  such that

$$\rho_\eta < \rho \leq 1 \quad \Rightarrow \quad \sup_{\delta \in (0,1], \vartheta \in \mathbb{R}/\mathbb{Z}} |K_\delta(\rho, \vartheta) - K(\vartheta)| \leq \eta$$

Now take  $0 < \delta_\eta < 1 - \rho_\eta$ . Hence  $0 < \delta < \delta_\eta$  and  $\rho > 1 - \delta$  implies  $\rho > \rho_\eta$  and

$$|K_\delta(\rho, \vartheta)| \geq |K(\vartheta)| - |K_\delta(\rho, \vartheta) - K(\vartheta)| \geq 2\eta - \eta = \eta$$

for all  $\vartheta$ .

Define  $\mathcal{D}_\delta$  by replacing the strip (155) by the strip (162). Hence  $\mathcal{D}_\delta$  is smooth and differs from  $\mathcal{D}$  only on a  $\delta$ -small neighbourhood of  $\gamma_0$  and  $\mathcal{D}_\delta \rightarrow \mathcal{D}$  in  $C^1$  as  $\delta \rightarrow 0$ . It follows from (172) that  $\mathcal{D}_\delta$  is transverse to  $X$  for all  $\delta$  small enough. The proof of Lemma C.6 is now complete since the graph of the argument of  $e(\vartheta)$ , i.e. the graph of  $\vartheta \mapsto \theta = \Gamma^\theta(1, \vartheta)$ , is a global surface of section of the linearised dynamics on the torus  $r = 0$ .  $\square$

### Step 3: Projecting vector fields onto $\mathcal{D}_\delta \setminus \gamma_0$

Fix  $\delta$  small. Near  $\gamma_0$  we can find a non-vanishing section  $Y_0$  of  $\xi$  that on points of  $\mathcal{D}_\delta$  near  $\gamma_0$  satisfies  $\mathbb{R}Y_0 = T\mathcal{D}_\delta \cap \xi$ . Using the coordinates  $(\vartheta, x + iy)$  in the Martinet tube near  $\gamma_0$ , we can write

$$Y_0 = a_0 \partial_x + b_0(\partial_y - x \partial_\vartheta)$$

where the  $\mathbb{C}$ -valued function  $a_0 + ib_0$  does not vanish. Consider the vector field

$$Y_1 = a_1 \partial_x + b_1(\partial_y - x \partial_\vartheta) \quad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \cos(2\pi\vartheta) & \sin(2\pi\vartheta) \\ -\sin(2\pi\vartheta) & \cos(2\pi\vartheta) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}. \quad (173)$$

Since  $\gamma_0$  has self-linking number  $-1$  and  $Y_1$  winds  $-1$  with respect to  $Y_0$  along  $\gamma_0$ , there exists a smooth extension of  $Y_1$  as a non-vanishing section of  $\xi \rightarrow S^3$ .

By Lemma C.6 we know that  $\mathcal{D}_\delta$  is a global surface of section spanned by  $\gamma_0$  for small enough  $\delta$ , and  $\mathcal{D}_\delta \rightarrow \mathcal{D}$  in  $C^1$  as  $\delta \rightarrow 0$ . From now on we will denote by  $\dot{\mathcal{D}}_\delta = \mathcal{D}_\delta \setminus \partial\mathcal{D}_\delta = \mathcal{D}_\delta \setminus \gamma_0$  the interior of  $\mathcal{D}_\delta$ . The projection along the Reeb direction is a smooth vector bundle map

$$P : TS^3|_{\dot{\mathcal{D}}_\delta} \rightarrow T\dot{\mathcal{D}}_\delta$$

characterised by

$$P^2 = P, \quad \ker P = \mathbb{R}X.$$

Note that  $P$  becomes singular on the boundary since  $X$  is tangent to  $\partial\mathcal{D}_\delta = \gamma_0$ . Note also that  $P$  defines a vector bundle isomorphism between  $\xi|_{\dot{\mathcal{D}}_\delta}$  and  $T\dot{\mathcal{D}}_\delta$ , this follows from the transversality between the Reeb vector field and  $\dot{\mathcal{D}}_\delta$ .

Let  $A_\delta \subset \mathcal{D}_\delta$  be a small compact neighborhood of  $\gamma_0 = \partial\mathcal{D}_\delta$  in  $\mathcal{D}_\delta$ , small enough so that it is contained in the domain of definition of  $Y_0$ . Denote  $\dot{A}_\delta = A_\delta \setminus \gamma_0$ . Note that  $A_\delta$  is equipped with polar coordinates  $(\rho, \vartheta)$  when parametrised as the strip  $(\rho, \vartheta) \mapsto (\vartheta, \Gamma_\delta(\rho, \vartheta))$ . Here  $\Gamma_\delta$  is the map (159). There is no loss of generality to assume that  $A_\delta$  corresponds to  $\{(\rho, \vartheta) \in [1 - \epsilon, 1] \times \mathbb{R}/\mathbb{Z}\}$  for  $\epsilon > 0$  small enough. Hence  $\dot{A}_\delta$  corresponds to  $\{(\rho, \vartheta) \in [1 - \epsilon, 1) \times \mathbb{R}/\mathbb{Z}\}$ . In particular the local slice disk  $\{\vartheta \equiv 0 \pmod{\mathbb{Z}}\}$  intersects  $A_\delta$  in a smooth arc  $\eta$  transverse to  $\partial A_\delta$  defining a generator of  $H_1(A_\delta, \partial A_\delta)$ . In the coordinates  $(\rho, \vartheta)$ ,  $\eta$  corresponds to  $\{\rho \in [1 - \epsilon, 1], \vartheta \equiv 0 \pmod{\mathbb{Z}}\}$ .

By (173),  $Y_0$  and  $Y_1$  are positively collinear only at  $\{\vartheta \equiv 0 \pmod{\mathbb{Z}}\}$ . Hence on  $\dot{A}_\delta$  the vector fields  $P(Y_1)$  and  $Y_0 = P(Y_0)$  are positively collinear only at  $\eta \setminus \gamma_0$ . The universal covering of  $A_\delta$  can be given coordinates  $(\rho, \tilde{\vartheta}) \in [1 - \epsilon, 1] \times \mathbb{R}$ , where  $\vartheta = \tilde{\vartheta} \pmod{\mathbb{Z}}$ . Hence the universal covering of  $\dot{A}_\delta$  is  $[1 - \epsilon, 1) \times \mathbb{R}$ . The disk  $\mathcal{D}_\delta$  can be equipped with global coordinates  $u + iv \in \mathbb{D}$  such that  $u + iv = \rho e^{i2\pi\vartheta}$  near the boundary. We can write

$$\begin{cases} Y_0 = R_0(\cos \varphi_0 \partial_u + \sin \varphi_0 \partial_v) & \text{on } A_\delta \\ P(Y_1) = R_1(\cos \varphi_1 \partial_u + \sin \varphi_1 \partial_v) & \text{on } \dot{\mathcal{D}}_\delta \end{cases}$$



in polar coordinates, where

$$\varphi_0 : [1 - \epsilon, 1] \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/2\pi\mathbb{Z} \quad \varphi_1 : \dot{\mathcal{D}}_\delta \rightarrow \mathbb{R}/2\pi\mathbb{Z}$$

are smooth. Choose smooth lifts

$$\tilde{\varphi}_0 : [1 - \epsilon, 1] \times \mathbb{R} \rightarrow \mathbb{R} \quad \text{and} \quad \tilde{\varphi}_1 : \dot{\mathcal{D}}_\delta \rightarrow \mathbb{R}$$

of  $\varphi_0$  and  $\varphi_1$ , respectively. We will also write  $\tilde{\varphi}_1(\rho, \tilde{\vartheta})$  to denote the corresponding lift of the restriction of  $\varphi_1$  to  $\dot{A}_\delta$ , with no fear of ambiguity. Note that

$$\begin{cases} \tilde{\varphi}_0(\rho, \tilde{\vartheta} + 1) = \tilde{\varphi}_0(\rho, \tilde{\vartheta}) + 2\pi, \\ \tilde{\varphi}_1(\rho, \tilde{\vartheta} + 1) = \tilde{\varphi}_1(\rho, \tilde{\vartheta}). \end{cases} \quad (174)$$

Note also that  $\tilde{\varphi}_0$  is bounded on any compact subset of  $[1 - \epsilon, 1] \times \mathbb{R}$  since  $Y_0$  is smooth on  $A_\delta$ . As observed before,

$$\tilde{\varphi}_1(\rho, \tilde{\vartheta}) - \tilde{\varphi}_0(\rho, \tilde{\vartheta}) \in 2\pi\mathbb{Z} \quad \Leftrightarrow \quad \tilde{\vartheta} \in \mathbb{Z}. \quad (175)$$

The lifts of  $\eta$  divide the universal covering of  $\dot{A}_\delta$  in fundamental domains. It follows from (175) that on each such fundamental domain  $[1 - \epsilon, 1] \times [k, k + 1]$  the function  $\tilde{\varphi}_1 - \tilde{\varphi}_0$  takes values on  $[2\pi(m - 1), 2\pi m]$  for some  $m \in \mathbb{Z}$ , in particular it is bounded there. Hence the function  $\tilde{\varphi}_1 = \tilde{\varphi}_0 + \tilde{\varphi}_1 - \tilde{\varphi}_0$  is bounded on each fundamental domain  $[1 - \epsilon, 1] \times [k, k + 1]$ . In view of (174) we can conclude that  $\tilde{\varphi}_1$  is bounded on  $\dot{A}_\delta$ . By compactness of the closure of  $\dot{\mathcal{D}}_\delta \setminus \dot{A}_\delta$ , we get that  $\tilde{\varphi}_1 \in L^\infty(\dot{\mathcal{D}}_\delta)$ .

This shows that the desired vector field  $Z$  satisfying property (e) in Proposition 2.8 can be taken as  $Z = Y_1$ .

#### Step 4: Completing the proof

In *Step 1* we fixed a Martinet tube  $F : \mathbb{R}/\mathbb{Z} \times \text{int}(\mathbb{D}) \rightarrow U$  onto a neighbourhood  $U$  of  $\gamma_0$  as in Lemma C.2. Consider the space

$$M = (S^3 \setminus \gamma_0 \sqcup \mathbb{R}/\mathbb{Z} \times [0, 1] \times \mathbb{R}/2\pi\mathbb{Z}) / \sim \quad (176)$$

where a point  $F(\vartheta, re^{i\theta}) \in U \setminus \gamma_0$  is identified with  $(\vartheta, r, \theta) \in \mathbb{R}/\mathbb{Z} \times (0, 1) \times \mathbb{R}/2\pi\mathbb{Z}$ . One gets a differentiable structure on  $M$  which makes  $M$  diffeomorphic to  $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$  in such a way that  $re^{i2\pi\vartheta}$  are polar coordinates near the boundary of the  $\mathbb{D}$ -factor. Starting from a special finite-energy plane  $\tilde{u}$  obtained from [HWZ98], or from [Hry14], we showed in *Step 2* that there exists a smooth embedded disk  $\mathcal{D}_\delta \subset S^3$  satisfying  $\partial\mathcal{D}_\delta = \gamma_0$ , and such that  $\dot{\mathcal{D}}_\delta = \mathcal{D}_\delta \setminus \partial\mathcal{D}_\delta$  is the interior of a smoothly embedded meridional disk  $D \subset M$  intersecting  $\partial M$  cleanly. This claim follows from (162) since  $\Gamma_\delta(\rho, \vartheta) = (1 - \rho)e(\vartheta)$  when  $\rho$  is close to 1; here  $e(\vartheta)$  is the eigenvector of the asymptotic operator (130) governing the asymptotic behaviour of  $\tilde{u}$  near  $\infty$ . Moreover, we proved that  $X|_{S^3 \setminus \gamma_0}$  extends to a smooth vector field

$W$  on  $M$  represented as (152) in the coordinates  $(\vartheta, \theta)$  on  $\partial M$ . In particular,  $W$  is tangent to  $\partial M$ . Finally, in *Step 2* we showed with (168)-(169) that  $D$  is transverse to  $W$  up to the boundary. From (152) we see that  $D \cap \partial M = \partial D$  is a global surface of section for the flow of  $W$  on  $\partial M$ . Since  $D \setminus \partial M = \mathring{D}_\delta$  is also a global surface of section for the flow of  $W$  on  $M \setminus \partial M$ , one gets that  $D$  is a global section for the flow of  $W$  on  $M$ . Using the flow of  $W$  to deform  $D$  one constructs a smooth foliation  $\{D_t\}_{t \in \mathbb{R}/\mathbb{Z}}$  of  $M$  by embedded disks in such that  $D = D_0$  and all  $D_t$  are transverse to  $W$ . It follows that all  $D_t$  are global surfaces of section for the flow of  $W$  on  $M$  since  $D = D_0$  is. Hence we get a diffeomorphism

$$M \simeq \mathbb{R}/\mathbb{Z} \times \mathbb{D} \qquad D_t \simeq \{t\} \times \mathbb{D}$$

Properties (a)-(d) follow from this construction.

We fix coordinates  $u + iv$  on  $D_0 \simeq \{0\} \times \mathbb{D}$ . In *Step 3* we proved that there is a non-vanishing vector field  $Z$  on  $S^3$  tangent to the contact structure  $\xi = \ker \lambda$  that projects to  $\mathring{D}_\delta \simeq D_0 \setminus \partial D_0 \simeq \{0\} \times \mathring{\mathbb{D}}$  along the direction of  $X \simeq W$  to a non-vanishing vector field on  $\mathring{\mathbb{D}}$  for which any continuous choice of argument (in the coordinates  $u + iv$ ) defines a function in  $L^\infty(\mathring{\mathbb{D}})$ . This is property (e).

## References

- [ABHS17] A. Abbondandolo, B. Bramham, U. Hryniewicz, and P. A. S. Salomão, *A systolic inequality for geodesic flows on the two-sphere*, Math. Ann. **367** (2017), no. 1-2, 701–753.
- [Arn86] V. I. Arnold, *The asymptotic Hopf invariant and its applications*, Selecta Math. Soviet. **5** (1986), no. 4, 327–345. Selected translations.
- [Bir13] G. D. Birkhoff, *Proof of Poincaré’s geometric theorem*, Trans. Amer. Math. Soc. **14** (1913), no. 1, 14–22.
- [Bir66] ———, *Dynamical systems*, With an addendum by Jurgen Moser. American Mathematical Society Colloquium Publications, Vol. IX, American Mathematical Society, Providence, R.I., 1966.
- [Deh17] P. Dehornoy, *Which geodesic flows are left-handed?*, Groups Geom. Dyn. **11** (2017), no. 4, 1347–1376.
- [DR20] P. Dehornoy and A. Rechtman, *Vector Fields and Genus in Dimension 3*, International Mathematics Research Notices (2020).
- [Flo19a] A. Florio, *Asymptotic maslov indices*, 2019. PhD thesis, Avignon Université.
- [Flo19b] ———, *Torsion and linking number for a surface diffeomorphism*, Math. Z. **292** (2019), no. 1-2, 231–265.
- [Fri82] D. Fried, *The geometry of cross sections to flows*, Topology **21** (1982), no. 4, 353–371.
- [Ghy09] E. Ghys, *Right-handed vector fields & the Lorenz attractor*, Jpn. J. Math. **4** (2009), no. 1, 47–61.
- [Gro00] M. Gromov, *Three remarks on geodesic dynamics and fundamental group*, Enseign. Math. (2) **46** (2000), no. 3-4, 391–402.

- [GRS06] C. Grotta-Ragazzo and P. A. S. Salomão, *The Conley-Zehnder index and the saddle-center equilibrium*, Journal of Differential Equations **220** (2006), no. 1, 259–278.
- [Hof93] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. **114** (1993), no. 3, 515–563.
- [HP08] A. Harris and G. P. Paternain, *Dynamically convex Finsler metrics and J-holomorphic embedding of asymptotic cylinders*, Ann. Global Anal. Geom. **34** (2008), no. 2, 115–134.
- [Hry14] U. Hryniewicz, *Systems of global surfaces of section for dynamically convex Reeb flows on the 3-sphere*, J. Symplectic Geom. **12** (2014), no. 4, 791–862.
- [Hry20] ———, *A note on Schwartzman-Fried-Sullivan theory, with an application*, J. Fixed Point Theory Appl. **22** (2020), no. 1, Paper No. 25, 20.
- [HS13] U. Hryniewicz and P. A. S. Salomão, *Global properties of tight Reeb flows with applications to Finsler geodesic flows on  $S^2$* , Math. Proc. Cambridge Philos. Soc. **154** (2013), no. 1, 1–27.
- [HSW19] U. Hryniewicz, P. A. S. Salomão, and K. Wysocki, *Genus zero global surfaces of section for reeb flows and a result of birkhoff* (2019). arXiv:1912.01078v2.
- [HWZ95] H. Hofer, K. Wysocki, and E. Zehnder, *Properties of pseudo-holomorphic curves in symplectisations. II. Embedding controls and algebraic invariants*, Geom. Funct. Anal. **5** (1995), no. 2, 270–328.
- [HWZ96] ———, *Properties of pseudoholomorphic curves in symplectisations. I. Asymptotics*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), no. 3, 337–379. MR1395676
- [HWZ98] ———, *The dynamics on three-dimensional strictly convex energy surfaces*, Ann. of Math. (2) **148** (1998), no. 1, 197–289.
- [HWZ99] ———, *A characterization of the tight 3-sphere. II*, Comm. Pure Appl. Math. **52** (1999), no. 9, 1139–1177.
- [Kli59] W. Klingenberg, *Contributions to Riemannian geometry in the large*, Ann. of Math. (2) **69** (1959), 654–666.
- [Kli82] ———, *Riemannian geometry*, de Gruyter Studies in Mathematics, vol. 1, Walter de Gruyter & Co., Berlin-New York, 1982.
- [Poi12] H. Poincaré, *Sur un théorème de géométrie*, Rend. Circ. Mat. Palermo **33** (1912), 375–407.

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