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# A Note on Linear Complementary Pairs of Group Codes

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## Abstract

We give a short and elementary proof of the fact that for a linear complementary pair  $(C, D)$ , where  $C$  and  $D$  are 2-sided ideals in a group algebra,  $D$  is uniquely determined by  $C$  and the dual code  $D^\perp$  is permutation equivalent to  $C$ . This includes earlier results of [3] and [6] on nD cyclic codes which have been proved by subtle and lengthy calculations in the space of polynomials.

Throughout this note let  $K$  be a finite field. A pair  $(C, D)$  of linear codes over  $K$  of length  $n$  is called a *linear complementary pair* (LCP) if  $C \cap D = \{0\}$  and  $C + D = K^n$ , or equivalently if  $C \oplus D = K^n$ . In the special case that  $D = C^\perp$  where the dual is taken with respect to the Euclidean inner product the code  $C$  is referred to a *linear complementary dual* (LCD) code. LCD codes have first been considered by Massey in [7]. The nowadays interest of LCP codes aroused from the fact that they can be used in protection against side channel and fault injection attacks [1], [2], [4]. In this context the security of a linear complementary pair  $(C, D)$  can be measured by the *security parameter*  $\min\{d(C), d(D^\perp)\}$ . Clearly, if  $D = C^\perp$ , then the security parameter for  $(C, D)$  is  $d(C)$ .

Just recently, it has been shown in [3] that for linear complementary pairs  $(C, D)$  the codes  $C$  and  $D^\perp$  are equivalent if  $C$  and  $D$  are both cyclic or 2D cyclic codes under the assumption that the characteristic of  $K$  does not divide the length. In [6], this result has

been extended to the case that both  $C$  and  $D$  are  $n$ D cyclic for  $n \in \mathbb{N}$ . In both papers the proof is rather complicated and formulated in the world of polynomials.

Recall that an  $n$ D cyclic code is an ideal in the algebra

$$R_n = K[x_1, \dots, x_n] / \langle x^{m_1} - 1, \dots, x^{m_n} - 1 \rangle,$$

and that  $R_n$  is isomorphic to the group algebra  $KG$  where  $G = C_{m_1} \times \dots \times C_{m_n}$  with cyclic groups  $C_{m_i}$  of order  $m_i$ . Thus the above mentioned results are results on ideals in abelian group algebras.

A linear code  $C$  is called a *group code*, or  $G$ -code, if  $C$  is a right ideal in a group algebra

$$KG = \left\{ a = \sum_{g \in G} a_g g \mid a_g \in K \right\}$$

where  $G$  is a finite group. The vector space  $KG \cong K^{|G|}$  with basis  $g \in G$  serves as the ambient space and the weight function is defined by  $\text{wt}(a) = |\{g \in G \mid a_g \neq 0\}|$  (which corresponds to the classical weight function via the isomorphism  $KG \cong K^{|G|}$ ). Note that  $KG$  carries a  $K$ -algebra structure via the multiplication in  $G$ . More precisely, if  $a = \sum_{g \in G} a_g g$  and  $b = \sum_{g \in G} b_g g$  are given, then

$$ab = \sum_{g \in G} \left( \sum_{h \in G} a_h b_{h^{-1}g} \right) g.$$

In this sense  $n$ D cyclic codes are group codes for abelian groups  $G$  and vice versa since a finite abelian group is the direct product of cyclic groups.

There is a natural  $K$ -linear anti-algebra automorphism  $\hat{\cdot} : KG \rightarrow KG$  which is given by  $g \mapsto g^{-1}$  for  $g \in G$  (in the isomorphism  $KG \cong K^{|G|}$ , the automorphism  $\hat{\cdot}$  corresponds to a permutation of the coordinates). Thus we may associate to each  $a = \sum_{g \in G} a_g g \in KG$  the *adjoint*  $\hat{a} = \sum_{g \in G} a_g g^{-1}$  and call  $a$  self-adjoint if  $a = \hat{a}$ .

In addition, the group algebra  $KG$  carries a symmetric non-degenerate  $G$ -invariant bilinear form  $\langle \cdot, \cdot \rangle$  which is defined by

$$\langle g, h \rangle = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{otherwise.} \end{cases}$$

Here  $G$ -invariance means that  $\langle ag, bg \rangle = \langle a, b \rangle$  for all  $a, b \in KG$  and all  $g \in G$ . Via the isomorphism  $KG \cong K^{|G|}$ , the above form corresponds to the usual Euclidean inner product. With respect to this form we may define the dual code  $C^\perp$  of a group code  $C \leq KG$  as usual and say that  $C$  is self-dual if  $C = C^\perp$ . Note that for a group code  $C$  the dual  $C^\perp$  is a right ideal since for all  $c \in C$ ,  $c^\perp \in C^\perp$  and  $g \in G$  we have

$$\langle c, c^\perp g \rangle = \langle cg^{-1}, c^\perp \rangle = 0.$$

Thus with  $C$  the dual  $C^\perp$  is a group code as well.

In [9] we classified completely group algebras which contain self-dual ideals. More precisely, a self-dual  $G$ -code exists over the field  $K$  if and only if  $|G|$  and the characteristic of  $K$  are even. In [5] we investigated LCD group codes and characterized them via self-adjoint idempotents  $e^2 = e = \hat{e}$  in the group algebra  $KG$ .

In this short note we prove the following theorem which includes the above mentioned results of [3] and [6]. Observe that we require no assumption on the characteristic of the field  $K$ .

**Theorem.** Let  $G$  be a finite group. If  $C \oplus D = KG$  where  $C$  and  $D$  are 2-sided ideals in  $KG$ , then  $D$  is uniquely determined by  $C$  and  $D^\perp$  is permutation equivalent to  $C$ . In particular  $d(D^\perp) = d(C)$ .

In order to prove the Theorem we state some elementary facts from representation theory.

**Definition.** If  $M$  is a right  $KG$ -module, then the dual vector space  $M^* = \text{Hom}_K(M, K)$  becomes a right  $KG$ -module via

$$m(\alpha g) = (mg^{-1})\alpha$$

where  $m \in M, \alpha \in M^*$  and  $g \in G$ . With this action  $M^*$  is called the *dual module* of  $M$ . Clearly,  $\dim M^* = \dim M$ .

**Lemma A.** (Okuyama-Tsushima, [8]) If  $e = e^2 \in KG$ , then  $\hat{e}KG \cong eKG^*$ . In particular,  $\dim \hat{e}KG = \dim eKG$ .

**Proof:** A short proof is given in Lemma 2.3 of [5]. □

**Lemma B.** If  $D = eKG$  with  $e^2 = e \in KG$ , then  $D^\perp = (1 - \hat{e})KG$ .

**Proof:** First observe that  $\hat{e}^2 = \hat{e}$  and that  $\langle ab, c \rangle = \langle b, \hat{a}c \rangle$  for all  $a, b, c \in KG$ . Thus

$$\langle ea, (1 - \hat{e})b \rangle = \langle e^2a, (1 - \hat{e})b \rangle = \langle ea, \hat{e}(1 - \hat{e})b \rangle = 0$$

for all  $a, b \in KG$ . Hence,  $(1 - \hat{e})KG \subseteq D^\perp$ . As

$$\begin{aligned} \dim (1 - \hat{e})KG &= |G| - \dim \hat{e}KG \\ &= |G| - \dim eKG^* && \text{(by Lemma A)} \\ &= |G| - \dim eKG \\ &= \dim D^\perp \end{aligned}$$

we obtain  $(1 - \hat{e})KG = D^\perp$ . □

**Proof of the Theorem:** Since  $C \oplus D = KG$  we may write  $D = eKG$  and  $C = (1-e)KG$  for a suitable central idempotent

$$e = e^2 \in Z(KG) = \{a \mid ab = ba \text{ for all } b \in KG\}$$

(see [5]). Lemma B says that  $D^\perp = (1 - \hat{e})KG$ . Via the map  $\hat{\cdot} : KG \rightarrow KG$  the  $K$ -linear code  $(1 - \hat{e})KG$  is permutation equivalent to the  $K$ -linear code  $KG(1 - e)$ . But  $KG(1 - e) = (1 - e)KG = C$  since  $e$  is central, which completes the proof.  $\square$

If  $C$  and  $D$  are only right ideals, then  $D$  is uniquely determined by  $C$ , but  $D^\perp$ , in general, is not necessarily permutation equivalent to  $C$ . It even may happen that  $d(D^\perp) \neq d(C)$  as the next example shows.

**Example.** Let  $K = \mathbb{F}_2$  and let

$$G = \langle a, b \mid a^7 = 1 = b^2, a^b = a^{-1} \rangle$$

be a dihedral group of order 14. If we put

$$e = 1 + a + a^2 + a^4 + b + a^2b + a^5b + a^6b,$$

then  $e = e^2$ . With MAGMA one easily computes  $d((1 - e)KG) = 2$  and  $d(KG(1 - e)) = 3$ .

Now let  $C = (1 - e)KG$  and  $D = eKG$ . By Lemma B, we have  $D^\perp = (1 - \hat{e})KG$ . Thus

$$d(D^\perp) = d((1 - \hat{e})KG) = d(KG(1 - e)) = 3,$$

but  $d(C) = d((1 - e)KG) = 2$ . We like to mention here that  $C$  and  $D$  are quasi-cyclic codes.  $\square$

**Remark.** Let  $|K| = q^2$ . In this case we may consider the Hermitian inner product on  $KG$  which is defined by

$$\left\langle \sum_{g \in G} a_g g, \sum_{h \in G} b_h h \right\rangle = \sum_{g \in G} a_g b_g^q.$$

For  $a = \sum_{g \in G} a_g g$  we put  $a^{(q)} = \sum_{g \in G} a_g^q g$ . With this notation we have

$$D^\perp = (1 - \widehat{e^{(q)}})KG$$

in Lemma B. Applying the anti-automorphism  $\hat{\cdot} : KG \rightarrow KG$  we see that  $D^\perp$  is permutation equivalent to  $KG(1 - e^{(q)})$ . If in addition  $e$  is central, then  $e^{(q)}$  is central. Thus  $D^\perp$  is permutation equivalent to  $(1 - e^{(q)})KG = ((1 - e)KG)^{(q)}$ .

It follows that

$$d(D^\perp) = d((1 - e)KG)^{(q)} = d((1 - e)KG) = d(C).$$

Thus, in the Hermitian case a linear complementary pair  $(C, D)$  of 2-sided group codes  $C$  and  $D$  also has security parameter  $d(C)$ .

## References

- [1] S. BHASIN, J.-L. DANGER, S. GUILLEY, Z. NAJIM AND X.T. NGO, “Linear complementary dual code improvement to strengthen encoded circuit against hardware Trojan horses.” In *Proc. IEEE Int. Symp. Hardware Oriented Secur. Trust (HOST)* 2015, pp. 82-87.
- [2] J. BRINGER, C. CARLET, H. CABANNE, S. GUILLEY AND H. MAGHREBI, “Orthogonal direct sum making: A smartcard friendly computation paradigm in a code, with builtin protection against side-channel and fault attacks.” in *Proc. WIST*, Springer 2014, pp. 40-56.
- [3] C. CARLET, C. GÜNERI, F. ÖZBUDAK, B. ÖZKAYA AND P. SOLÉ, “On linear complementary pairs of codes.” *IEEE Trans. Inform. Theory*, vol. 64, pp. 6583-6589, 2018.
- [4] C. CARLET and S. GUILLEY, “Complementary Dual Codes for Counter-measures to Side-Channel Attacks. In “Coding Theory and Applications.” Eds. R. Pinto, P. Rocha Malonek and P. Vettorey, *CIM Series in Math. Sciences*. vol. 3, pp. 97-105, Springer 2015.
- [5] J. DE LA CRUZ AND W. WILLEMS, “ On group codes with complementary duals.” *Des. Codes Cryptogr.*, vol. 86, pp. 2065-2073, 2018.
- [6] C. GÜNERI, B. ÖZKAYA AND S. SAYICI, “On linear complementary pair of nD cyclic codes.” *IEEE Commun. Lett.*, vol. 22, pp. 2404-2406, 2018.
- [7] J.L. MASSEY, “Linear codes with complementary duals.” *Discrete Math.*, vol. 106/107, 337-342, 1992.
- [8] T. OKUYAMA AND Y. TSUSHIMA, “On a conjecture of P. Landrock.” *J. of Algebra*, vol. 104, pp. 203-208, 1986.
- [9] W. WILLEMS, “A note on self-dual group codes.” *IEEE Trans. Inf. Theory*, vol. 48, pp. 3107-3109, 2002.