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A Note on Linear Complementary Pairs of Group Codes

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Abstract

We give a short and elementary proof of the fact that for a linear complementary pair $(C, D)$, where $C$ and $D$ are 2-sided ideals in a group algebra, $D$ is uniquely determined by $C$ and the dual code $D^\perp$ is permutation equivalent to $C$. This includes earlier results of [3] and [6] on nD cyclic codes which have been proved by subtle and lengthy calculations in the space of polynomials.

Throughout this note let $K$ be a finite field. A pair $(C, D)$ of linear codes over $K$ of length $n$ is called a linear complementary pair (LCP) if $C \cap D = \{0\}$ and $C + D = K^n$, or equivalently if $C \oplus D = K^n$. In the special case that $D = C^\perp$ where the dual is taken with respect to the Euclidean inner product the code $C$ is referred to a linear complementary dual (LCD) code. LCD codes have first been considered by Massey in [7]. The nowadays interest of LCP codes aroused from the fact that they can be used in protection against side channel and fault injection attacks [1], [2], [4]. In this context the security of a linear complementary pair $(C, D)$ can be measured by the security parameter $\min\{d(C), d(D^\perp)\}$. Clearly, if $D = C^\perp$, then the security parameter for $(C, D)$ is $d(C)$.

Just recently, it has been shown in [3] that for linear complementary pairs $(C, D)$ the codes $C$ and $D^\perp$ are equivalent if $C$ and $D$ are both cyclic or 2D cyclic codes under the assumption that the characteristic of $K$ does not divide the length. In [6], this result has
been extended to the case that both $C$ and $D$ are $nD$ cyclic for $n \in \mathbb{N}$. In both papers the proof is rather complicated and formulated in the world of polynomials.

Recall that an $nD$ cyclic code is an ideal in the algebra

$$R_n = K[x_1, \ldots, x_n]/(x^{m_1} - 1, \ldots, x^{m_n} - 1),$$

and that $R_n$ is isomorphic to the group algebra $KG$ where $G = C_{m_1} \times \cdots \times C_{m_n}$ with cyclic groups $C_{m_i}$ of order $m_i$. Thus the above mentioned results are results on ideals in abelian group algebras.

A linear code $C$ is called a group code, or $G$-code, if $C$ is a right ideal in a group algebra

$$KG = \{a = \sum_{g \in G} a_g g \mid a_g \in K\}$$

where $G$ is a finite group. The vector space $KG \cong K^{[G]}$ with basis $g \in G$ serves as the ambient space and the weight function is defined by $\text{wt}(a) = |\{g \in G \mid a_g \neq 0\}|$ (which corresponds to the classical weight function via the isomorphism $KG \cong K^{[G]}$). Note that $KG$ carries a $K$-algebra structure via the multiplication in $G$. More precisely, if $a = \sum_{g \in G} a_g g$ and $b = \sum_{g \in G} b_g g$ are given, then

$$ab = \sum_{g \in G} (\sum_{h \in G} a_h b_{h^{-1}g})g.$$

In this sense $nD$ cyclic codes are group codes for abelian groups $G$ and vice versa since a finite abelian group is the direct product of cyclic groups.

There is a natural $K$-linear anti-algebra automorphism $^\ast : KG \rightarrow KG$ which is given by $g \mapsto g^{-1}$ for $g \in G$ (in the isomorphism $KG \cong K^{[G]}$, the automorphism $^\ast$ corresponds to a permutation of the coordinates). Thus we may associate to each $a = \sum_{g \in G} a_g g \in KG$ the adjoint $\hat{a} = \sum_{g \in G} a_g g^{-1}$ and call $a$ self-adjoint if $a = \hat{a}$.

In addition, the group algebra $KG$ carries a symmetric non-degenerate $G$-invariant bilinear form $\langle \cdot, \cdot \rangle$ which is defined by

$$\langle g, h \rangle = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{otherwise}. \end{cases}$$

Here $G$-invariance means that $\langle ag, bg \rangle = \langle a, b \rangle$ for all $a, b \in KG$ and all $g \in G$. Via the isomorphism $KG \cong K^{[G]}$, the above form corresponds to the usual Euclidean inner product. With respect to this form we may define the dual code $C^\perp$ of a group code $C \leq KG$ as usual and say that $C$ is self-dual if $C = C^\perp$. Note that for a group code $C$ the dual $C^\perp$ is a right ideal since for all $c \in C, c^\perp \in C^\perp$ and $g \in G$ we have

$$\langle c, c^\perp g \rangle = \langle cg^{-1}, c^\perp \rangle = 0.$$
Thus with $C$ the dual $C^\perp$ is a group code as well.

In [9] we classified completely group algebras which contain self-dual ideals. More precisely, a self-dual $G$-code exists over the field $K$ if and only if $|G|$ and the characteristic of $K$ are even. In [5] we investigated LCD group codes and characterized them via self-adjoint idempotents $e^2 = e = \hat{e}$ in the group algebra $KG$.

In this short note we prove the following theorem which includes the above mentioned results of [3] and [6]. Observe that we require no assumption on the characteristic of the field $K$.

**Theorem.** Let $G$ be a finite group. If $C \oplus D = KG$ where $C$ and $D$ are 2-sided ideals in $KG$, then $D$ is uniquely determined by $C$ and $D^\perp$ is permutation equivalent to $C$. In particular $d(D^\perp) = d(C)$.

In order to prove the Theorem we state some elementary facts from representation theory.

**Definition.** If $M$ is a right $KG$-module, then the dual vector space $M^* = \text{Hom}_K(M, K)$ becomes a right $KG$-module via $$m(\alpha g) = (mg^{-1})\alpha$$
where $m \in M$, $\alpha \in M^*$ and $g \in G$. With this action $M^*$ is called the dual module of $M$. Clearly, $\dim M^* = \dim M$.

**Lemma A.** (Okuyama-Tsushima, [8]) If $e = e^2 \in KG$, then $\hat{e}KG \cong eKG^*$. In particular, $\dim \hat{e}KG = \dim eKG$.

**Proof:** A short proof is given in Lemma 2.3 of [5]. \hfill $\square$

**Lemma B.** If $D = eKG$ with $e^2 = e \in KG$, then $D^\perp = (1 - \hat{e})KG$. 

**Proof:** First observe that $\hat{e}^2 = \hat{e}$ and that $\langle ab, c \rangle = \langle b, \hat{ac} \rangle$ for all $a, b, c \in KG$. Thus
$$\langle ea, (1 - \hat{e})b \rangle = \langle e^2a, (1 - \hat{e})b \rangle = \langle ea, \hat{e}(1 - \hat{e})b \rangle = 0$$
for all $a, b \in KG$. Hence, $(1 - \hat{e})KG \subseteq D^\perp$. As
$$\dim (1 - \hat{e})KG = |G| - \dim \hat{e}KG$$
$$= |G| - \dim eKG^* \quad \text{(by Lemma A)}$$
$$= |G| - \dim eKG$$
$$= \dim D^\perp$$
we obtain $(1 - \hat{e})KG = D^\perp$. \hfill $\square$
Proof of the Theorem: Since $C \oplus D = KG$ we may write $D = eKG$ and $C = (1-e)KG$ for a suitable central idempotent $e = e^2 \in Z(KG) = \{a \mid ab = ba \text{ for all } b \in KG\}$ (see [5]). Lemma B says that $D^\perp = (1 - \hat{e})KG$. Via the map $^\ast : KG \rightarrow KG$ the $K$-linear code $(1 - \hat{e})KG$ is permutation equivalent to the $K$-linear code $KG(1 - e)$. But $KG(1 - e) = (1 - e)KG = C$ since $e$ is central, which completes the proof. \[\square\]

If $C$ and $D$ are only right ideals, then $D$ is uniquely determined by $C$, but $D^\perp$, in general, is not necessarily permutation equivalent to $C$. It even may happen that $d(D^\perp) \neq d(C)$ as the next example shows.

Example. Let $K = \mathbb{F}_2$ and let

$$G = \langle a, b \mid a^7 = 1 = b^2, a^b = a^{-1} \rangle$$

be a dihedral group of order 14. If we put

$$e = 1 + a + a^2 + a^4 + b + a^2b + a^5b + a^6b,$$

then $e = e^2$. With Magma one easily computes $d((1 - e)KG) = 2$ and $d(KG(1 - e)) = 3$.

Now let $C = (1 - e)KG$ and $D = eKG$. By Lemma B, we have $D^\perp = (1 - \hat{e})KG$. Thus $d(D^\perp) = d((1 - \hat{e})KG) = d(KG(1 - e)) = 3$,

but $d(C) = d((1 - e)KG) = 2$. We like to mention here that $C$ and $D$ are quasi-cyclic codes. \[\square\]

Remark. Let $|K| = q^2$. In this case we may consider the Hermitian inner product on $KG$ which is defined by

$$\langle \sum_{g \in G} a_g g, \sum_{h \in G} b_h h \rangle = \sum_{g \in G} a_g b_{g^{-1}}^q.$$ 

For $a = \sum_{g \in G} a_g g$ we put $a^{(q)} = \sum_{g \in G} a_g^q g$. With this notation we have

$$D^\perp = (1 - e^{(q)})KG$$

in Lemma B. Applying the anti-automorphism $^\ast : KG \rightarrow KG$ we see that $D^\perp$ is permutation equivalent to $KG(1 - e^{(q)})$. If in addition $e$ is central, then $e^{(q)}$ is central. Thus $D^\perp$ is permutation equivalent to $(1 - e^{(q)})KG = ((1 - e)KG)^{\langle q \rangle}$.

It follow that

$$d(D^\perp) = d((1 - e)KG)^{\langle q \rangle} = d((1 - e)KG) = d(C).$$

Thus, in the Hermitian case a linear complementary pair $(C, D)$ of 2-sided group codes $C$ and $D$ also has security parameter $d(C)$. 

4
References


