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► To cite this version:

Martino Borello, Javier de la Cruz, Wolfgang Willems. A note on linear complementary pairs of group codes. *Discrete Mathematics*, 2020, 343 (8), pp.111905. 10.1016/j.disc.2020.111905 . hal-03852292

HAL Id: hal-03852292

<https://hal.science/hal-03852292>

Submitted on 14 Nov 2022

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A Note on Linear Complementary Pairs of Group Codes

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Keywords. Group code, linear complementary pair (LCP)

MSC classification. 94B05, 94B99, 20C05

Abstract

We give a short and elementary proof of the fact that for a linear complementary pair (C, D) , where C and D are 2-sided ideals in a group algebra, D is uniquely determined by C and the dual code D^\perp is permutation equivalent to C . This includes earlier results of [3] and [6] on nD cyclic codes which have been proved by subtle and lengthy calculations in the space of polynomials.

Throughout this note let K be a finite field. A pair (C, D) of linear codes over K of length n is called a *linear complementary pair* (LCP) if $C \cap D = \{0\}$ and $C + D = K^n$, or equivalently if $C \oplus D = K^n$. In the special case that $D = C^\perp$ where the dual is taken with respect to the Euclidean inner product the code C is referred to a *linear complementary dual* (LCD) code. LCD codes have first been considered by Massey in [7]. The nowadays interest of LCP codes aroused from the fact that they can be used in protection against side channel and fault injection attacks [1], [2], [4]. In this context the security of a linear complementary pair (C, D) can be measured by the *security parameter* $\min\{d(C), d(D^\perp)\}$. Clearly, if $D = C^\perp$, then the security parameter for (C, D) is $d(C)$.

Just recently, it has been shown in [3] that for linear complementary pairs (C, D) the codes C and D^\perp are equivalent if C and D are both cyclic or 2D cyclic codes under the assumption that the characteristic of K does not divide the length. In [6], this result has

been extended to the case that both C and D are n D cyclic for $n \in \mathbb{N}$. In both papers the proof is rather complicated and formulated in the world of polynomials.

Recall that an n D cyclic code is an ideal in the algebra

$$R_n = K[x_1, \dots, x_n] / \langle x^{m_1} - 1, \dots, x^{m_n} - 1 \rangle,$$

and that R_n is isomorphic to the group algebra KG where $G = C_{m_1} \times \dots \times C_{m_n}$ with cyclic groups C_{m_i} of order m_i . Thus the above mentioned results are results on ideals in abelian group algebras.

A linear code C is called a *group code*, or G -code, if C is a right ideal in a group algebra

$$KG = \{a = \sum_{g \in G} a_g g \mid a_g \in K\}$$

where G is a finite group. The vector space $KG \cong K^{|G|}$ with basis $g \in G$ serves as the ambient space and the weight function is defined by $\text{wt}(a) = |\{g \in G \mid a_g \neq 0\}|$ (which corresponds to the classical weight function via the isomorphism $KG \cong K^{|G|}$). Note that KG carries a K -algebra structure via the multiplication in G . More precisely, if $a = \sum_{g \in G} a_g g$ and $b = \sum_{g \in G} b_g g$ are given, then

$$ab = \sum_{g \in G} \left(\sum_{h \in G} a_h b_{h^{-1}g} \right) g.$$

In this sense n D cyclic codes are group codes for abelian groups G and vice versa since a finite abelian group is the direct product of cyclic groups.

There is a natural K -linear anti-algebra automorphism $\hat{\cdot} : KG \rightarrow KG$ which is given by $g \mapsto g^{-1}$ for $g \in G$ (in the isomorphism $KG \cong K^{|G|}$, the automorphism $\hat{\cdot}$ corresponds to a permutation of the coordinates). Thus we may associate to each $a = \sum_{g \in G} a_g g \in KG$ the *adjoint* $\hat{a} = \sum_{g \in G} a_g g^{-1}$ and call a self-adjoint if $a = \hat{a}$.

In addition, the group algebra KG carries a symmetric non-degenerate G -invariant bilinear form $\langle \cdot, \cdot \rangle$ which is defined by

$$\langle g, h \rangle = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{otherwise.} \end{cases}$$

Here G -invariance means that $\langle ag, bg \rangle = \langle a, b \rangle$ for all $a, b \in KG$ and all $g \in G$. Via the isomorphism $KG \cong K^{|G|}$, the above form corresponds to the usual Euclidean inner product. With respect to this form we may define the dual code C^\perp of a group code $C \leq KG$ as usual and say that C is self-dual if $C = C^\perp$. Note that for a group code C the dual C^\perp is a right ideal since for all $c \in C$, $c^\perp \in C^\perp$ and $g \in G$ we have

$$\langle c, c^\perp g \rangle = \langle cg^{-1}, c^\perp \rangle = 0.$$

Thus with C the dual C^\perp is a group code as well.

In [9] we classified completely group algebras which contain self-dual ideals. More precisely, a self-dual G -code exists over the field K if and only if $|G|$ and the characteristic of K are even. In [5] we investigated LCD group codes and characterized them via self-adjoint idempotents $e^2 = e = \hat{e}$ in the group algebra KG .

In this short note we prove the following theorem which includes the above mentioned results of [3] and [6]. Observe that we require no assumption on the characteristic of the field K .

Theorem. Let G be a finite group. If $C \oplus D = KG$ where C and D are 2-sided ideals in KG , then D is uniquely determined by C and D^\perp is permutation equivalent to C . In particular $d(D^\perp) = d(C)$.

In order to prove the Theorem we state some elementary facts from representation theory.

Definition. If M is a right KG -module, then the dual vector space $M^* = \text{Hom}_K(M, K)$ becomes a right KG -module via

$$m(\alpha g) = (mg^{-1})\alpha$$

where $m \in M, \alpha \in M^*$ and $g \in G$. With this action M^* is called the *dual module* of M . Clearly, $\dim M^* = \dim M$.

Lemma A. (Okuyama-Tsushima, [8]) If $e = e^2 \in KG$, then $\hat{e}KG \cong eKG^*$. In particular, $\dim \hat{e}KG = \dim eKG$.

Proof: A short proof is given in Lemma 2.3 of [5]. □

Lemma B. If $D = eKG$ with $e^2 = e \in KG$, then $D^\perp = (1 - \hat{e})KG$.

Proof: First observe that $\hat{e}^2 = \hat{e}$ and that $\langle ab, c \rangle = \langle b, \hat{a}c \rangle$ for all $a, b, c \in KG$. Thus

$$\langle ea, (1 - \hat{e})b \rangle = \langle e^2a, (1 - \hat{e})b \rangle = \langle ea, \hat{e}(1 - \hat{e})b \rangle = 0$$

for all $a, b \in KG$. Hence, $(1 - \hat{e})KG \subseteq D^\perp$. As

$$\begin{aligned} \dim (1 - \hat{e})KG &= |G| - \dim \hat{e}KG \\ &= |G| - \dim eKG^* && \text{(by Lemma A)} \\ &= |G| - \dim eKG \\ &= \dim D^\perp \end{aligned}$$

we obtain $(1 - \hat{e})KG = D^\perp$. □

Proof of the Theorem: Since $C \oplus D = KG$ we may write $D = eKG$ and $C = (1-e)KG$ for a suitable central idempotent

$$e = e^2 \in Z(KG) = \{a \mid ab = ba \text{ for all } b \in KG\}$$

(see [5]). Lemma B says that $D^\perp = (1 - \hat{e})KG$. Via the map $\hat{\cdot}: KG \rightarrow KG$ the K -linear code $(1 - \hat{e})KG$ is permutation equivalent to the K -linear code $KG(1 - e)$. But $KG(1 - e) = (1 - e)KG = C$ since e is central, which completes the proof. \square

If C and D are only right ideals, then D is uniquely determined by C , but D^\perp , in general, is not necessarily permutation equivalent to C . It even may happen that $d(D^\perp) \neq d(C)$ as the next example shows.

Example. Let $K = \mathbb{F}_2$ and let

$$G = \langle a, b \mid a^7 = 1 = b^2, a^b = a^{-1} \rangle$$

be a dihedral group of order 14. If we put

$$e = 1 + a + a^2 + a^4 + b + a^2b + a^5b + a^6b,$$

then $e = e^2$. With MAGMA one easily computes $d((1 - e)KG) = 2$ and $d(KG(1 - e)) = 3$.

Now let $C = (1 - e)KG$ and $D = eKG$. By Lemma B, we have $D^\perp = (1 - \hat{e})KG$. Thus

$$d(D^\perp) = d((1 - \hat{e})KG) = d(KG(1 - e)) = 3,$$

but $d(C) = d((1 - e)KG) = 2$. We like to mention here that C and D are quasi-cyclic codes. \square

Remark. Let $|K| = q^2$. In this case we may consider the Hermitian inner product on KG which is defined by

$$\langle \sum_{g \in G} a_g g, \sum_{h \in G} b_h h \rangle = \sum_{g \in G} a_g b_g^q.$$

For $a = \sum_{g \in G} a_g g$ we put $a^{(q)} = \sum_{g \in G} a_g^q g$. With this notation we have

$$D^\perp = (1 - \widehat{e^{(q)}})KG$$

in Lemma B. Applying the anti-automorphism $\hat{\cdot}: KG \rightarrow KG$ we see that D^\perp is permutation equivalent to $KG(1 - e^{(q)})$. If in addition e is central, then $e^{(q)}$ is central. Thus D^\perp is permutation equivalent to $(1 - e^{(q)})KG = ((1 - e)KG)^{(q)}$.

It follow that

$$d(D^\perp) = d((1 - e)KG)^{(q)} = d((1 - e)KG) = d(C).$$

Thus, in the Hermitian case a linear complementary pair (C, D) of 2-sided group codes C and D also has security parameter $d(C)$.

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