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Abstract: In this paper, we develop a theory of stochastic Riesz spaces equipped with a stochastic topology that allows to define a general financial market model defined by a partial order. For such a model, we provide a construction of continuous-time portfolio processes from the discrete-time ones. We study the no-arbitrage condition AIP of the literature that states that the super-hedging prices of the non negative European claims are non negative. We show that this condition may be equivalent in discrete-time and in continuous-time and that the infimum super-hedging prices of a given payoff may also coincide in discrete-time and in continuous-time. At last, the construction of an upper linear stochastic integral is proposed in the setting of stochastic Riesz spaces.

Keywords and phrases: Stochastic Riesz spaces, Mathematical finance, Continuous-time financial market models, AIP no-arbitrage condition, Super-hedging prices, European claims, Stochastic integrals.

1. Introduction

The usual approach in mathematical finance is to define a financial market model as given by a stochastic basis $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P})$ where Ω is the set of all possible market states on some period $[0,T]$, $T > 0$ is the horizon date, $(\mathcal{F}_t)_{t\in[0,T]}$ is a filtration, i.e. an increasing sequence of σ-algebras and P is a probability measure. For each $t \in [0,T]$, $\mathcal{F}_t$ describes the available market information between time 0 and time t. Moreover, a stochastic price process $S = (S_t)_{t\in[0,T]}$, describing the prices of the risky assets composing the financial market, is exogenously given. At each time $t \in [0,T]$, S_t is supposed to be $\mathcal{F}_t$-measurable. This means that S_t is observable thanks to the information given by $\mathcal{F}_t$, i.e. at time t. Also, a strategy is defined as a stochastic process $\theta = (\theta_t)_{t\in[0,T]}$ describing the quantity that a financial agent holds. It is assumed that θ_t is $\mathcal{F}_t$-measurable, which means that the financial agent fixes the quantity θ_t at time t in terms of the available information $\mathcal{F}_t$ at time t. Therefore, the stochastic structure of the financial market models is fundamental as the financial decisions are based on the information. This gives rise to specific developments in the field of enlargement of filtration, see [5], where the natural question is how the additional information affects the financial market.

A very classical problem is the so-called super-hedging problem. The goal is to characterize the self-financing portfolio processes (investment portfolios), i.e. $V_t = \theta_t S_t$, $t \in [0,T]$, for models without transactions costs, such that the terminal value $V_T \geq h_T$ for some given payoff h_T. For the well known European Call option of strike $K > 0$, the payoff is $h_T = (S_T - K)^+$. Then, the initial value V_0 is interpreted as a price for the payoff h_T, as it allows to start an investment whose terminal value V_T is larger than or equal to the payoff h_T that the option contract seller must provide to the buyer at time T. The resolution of the super-hedging problem clearly depends on the stochastic structure of the financial market model. It also depends on the type of model, e.g. with transaction costs (we also say with friction) or without transaction costs (frictionless models).

The theory is very well developed for frictionless models. Some no-arbitrage conditions are imposed so that it is possible to characterize the super-hedging prices by the mean of dual elements. A no-arbitrage condition may be interpreted as a market equilibrium. The most famous one for discrete-time models is the no-arbitrage condition NA, see [[14], Section 2.1]. The Dalang-Morton-Willinger theorem [6] states that NA is equivalent to the existence of a martingale probability measures under which the discounted price process is a martingale. Then, a dual characterization of the super-hedging prices is deduced, see [[14], Theorem 2.1.11], using the martingale probability measures as dual elements. In continuous time, the theory is more sophisticated and several no-arbitrage conditions have been introduced as the No Free Lunch condition NFL, see [17] and [10], but also the NFLVR condition, see [9]. These conditions are equivalent in most of the cases to the existence of local martingale probability measures, which appear to be the dual elements to characterize the super-hedging prices. These no-arbitrage conditions implies that the price process is a semi-martingale and portfolio processes are stochastic integrals. This gave rise to a huge development of the stochastic calculus, see [15], since the pioneering works of Black, Scholes and Merton, see [3] and [22].

The theory for financial market models with friction is more recent. Most of the results are formulated for models with only proportional transaction costs, see [[14], Section 3]. Contrarily to the frictionless models, a portfolio process is expressed in physical values, i.e. it is a vector-valued financial position. A financial market model with friction may be simply defined by a stochastic basis and a partial order between the vector-valued financial positions, which is defined from a stochastic random set of all solvent financial positions, see [14], [21]. With proportional transaction costs, several no-arbitrage conditions exist which are mainly equivalent to the existence of dual elements, called Consistent Price Systems, see [13], that allow to get a dual characterization of the super-hedging prices as in the frictionless case, see [[14],Theorem 3.2.1, Theorem 3.3.3] and [8].

For more general transaction costs, in particular with fixed costs, the usual tools of the convex analysis are no more adapted and it is not possible to obtain a dual characterization of the super-hedging prices. A new approach is proposed in [20] where the financial market model is defined by a partial order. This gave rise to new problems in the field of random preorders with applications in finance as developed in [11], [12] and [19]. In this paper, we follow this philosophy by considering a general framework based on stochastic Riesz spaces that we introduce, see Section 2. Some stochastic topologies are introduced, see Section 3, in order to define continuous-time portfolio processes as limits of discrete-time portfolio processes with a financial meaning, see Section 4. Then, we study the recent no-arbitrage condition AIP of [4] and we show that this condition is equivalently satisfied in discrete and continuous time, see Section 5 and Section 6. At last, we propose in Section 7 the construction of a stochastic integral in the setting of the stochastic Riesz spaces that allows to consider a larger class of portfolio processes in continuous time.

2. Stochastic Riesz spaces

We consider two Riesz spaces $\mathcal{F}$ and $\mathcal{R}$, respectively endowed with partial orders that we denote by $\preceq$ and $\leq$ respectively. We refer to [24] for the theory of Riesz spaces. In the following, we suppose that $\mathcal{F}$ is a union indexed on time $t \in [0,T]$, $T > 0$, of some increasing Riesz subspaces, i.e. $\mathcal{F} = \cup_{t\in[0,T]}\mathcal{F}_t$ with $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ if $t_1 \leq t_2 \leq T$ and $\mathcal{F}_T = \mathcal{F}$. We say that the elements of $\mathcal{F}_t$ are the $\mathcal{F}_t$-measurable elements of $\mathcal{F}$, using here the usual vocabulary of stochastic finance. In finance, $\mathcal{F}_t$ can be interpreted as the set of all financial positions (which are random variables) at time $t \leq T$. In that case, the $\mathcal{F}_t$ measurability in understood in the usual sense with respect to a given filtration $(\mathcal{F}_t)_{t\in[0,T]}$ describing the available flow of information of the financial market.

Similarly, we suppose that $\mathcal{R} = \cup_{t\in[0,T]}\mathcal{R}_t$ with $\mathcal{R}_T = \mathcal{R}$ and the elements of the increasing Riesz subsets $\mathcal{R}_t$ are also called the $\mathcal{F}_t$-measurable [1] elements of $\mathcal{R}$ by analogy and an abuse of notation. We suppose that $\mathcal{R}_0$ is a vector space of dimension 1 so that, w.l.o.g., we assume that $\mathcal{R}_0 = \mathbf{R}$ is the real line. We introduce the notation $\mathcal{F}_t^+ := \{X_t \in \mathcal{F}_t : \ X_t \succeq 0\}$ to designate the $\mathcal{F}_t$-measurable elements of $\mathcal{F}_t$ and $\mathcal{R}_t^+ := \{X_t \in \mathcal{R}_t : \ X_t \geq 0\}$. Also $\mathcal{F}_T^+ = \mathcal{F}^+$ and $\mathcal{R}_T^+ = \mathcal{R}^+$.

The elements of $\mathcal{R}$ are interpreted as scalars. Precisely, we suppose that there exists a (left) product $(\alpha, F) \mapsto \alpha F$ between the elements $\alpha \in \mathcal{R}$ and $F \in \mathcal{F}$, i.e. $\alpha F \in \mathcal{F}$ makes sense for all $\alpha \in \mathcal{R}$ and $F \in \mathcal{F}$. We suppose the following usual properties:

Assumption A:

- A1) For any $\alpha_1, \alpha_2 \in \mathbf{R}$ and $F \in \mathcal{F}$, $(\alpha_1 + \alpha_2)F = \alpha_1 F + \alpha_2 F$.
- A2) For any $\alpha \in \mathbf{R}$ and $F_1, F_2 \in \mathcal{F}$, $\alpha(F_1 + F_2) = \alpha F_1 + \alpha F_2$.
- A3) For any $\alpha_t \in \mathcal{R}_t$ and $X_t \in \mathcal{F}_t$, we have $\alpha_t X_t \in \mathcal{F}_t$.
- A4) If $X, Y \in \mathcal{F}$ satisfies $X \preceq Y$, then $\alpha X \preceq \alpha Y$ for all $\alpha \in \mathcal{R}^+$.

At last, we define the set $\mathcal{I}_t$ of all $\mathcal{F}_t$-measurable components of $\mathcal{R}$ as the elements I_t of $\mathcal{R}_t^+$ such that $I_t \leq 1$ and $I_t \wedge (1 - I_t) = 0$. In the case where $\mathcal{R}_t$ is the set of all $\mathcal{F}_t$-measurable real-valued random variables, for a given σ-algebra $\mathcal{F}_t$, then $\mathcal{I}_t$ is the family of all indicator functions $I_t = 1_{F_t}$ defined as $1_{F_t}(\omega) = 1$ if $\omega \in F_t$ and $1_{F_t}(\omega) = 0$ if $\omega \notin F_t$ where $F_t \in \mathcal{F}_t$. Note that $0, 1 \in \mathcal{I}_t$.

[1] Instead of $\mathcal{R}_t$-measurable.

Definition 2.1. *Let $t \in [0,T]$. We say that a subset B of $\mathcal{F}$ (resp. $\mathcal{R}$) is $\mathcal{F}_t$ bounded from above if there exists $\gamma_t \in \mathcal{F}_t$ (resp. $\gamma_t \in \mathcal{R}_t$) such that $b \preceq \gamma_t$ (resp. $b \leq \gamma_t$) for all $b \in B$.*

Definition 2.2. *Let $t \in [0,T]$. We say that a subset B of $\mathcal{F}$ (resp. $\mathcal{R}$) is $\mathcal{F}_t$ bounded from below if there exists $\gamma_t \in \mathcal{F}_t$ (resp. $\gamma_t \in \mathcal{R}_t$) such that $b \succeq \gamma_t$ (resp. $b \geq \gamma_t$) for all $b \in B$.*

We also say that a subset is bounded if it is bounded from above and below. In the following, we suppose that $\mathcal{F}$ is Dedekind complete in the following sense:

Definition 2.3. *We say that $\mathcal{F}$ is Dedekind sup-complete if, for all $t \in [0,T]$, any $\mathcal{F}_t$ bounded from above subset B of $\mathcal{F}$ admits a supremum in $\mathcal{F}_t$ that we denote by* $\operatorname{ess\,sup}_{\mathcal{F}_t} B \in \mathcal{F}_t$.

Definition 2.4. *We say that $\mathcal{F}$ is Dedekind inf-complete if, for all $t \in [0,T]$, any $\mathcal{F}_t$ bounded from below subset B of $\mathcal{F}$ admits an infimum in $\mathcal{F}_t$ that we denote by* $\operatorname{ess\,inf}_{\mathcal{F}_t} B \in \mathcal{F}_t$.

In the case where $\mathcal{F} = L^0(\mathbf{R}, \mathcal{F}_T)$ and $\mathcal{F}_t = L^0(\mathbf{R}, \mathcal{F}_t)$, $t \in [0,T]$, are the families of $\mathcal{F}_t$-measurable variables with real values, for a given complete stochastic basis $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P})$, it is well known that $\mathcal{F}$ is both Dedekind sup and inf-complete, see [2] and [[14], Section 5.3.1].

Definition 2.5. *We say that $\mathcal{F}$ is Dedekind complete if it is both Dedekind sup-complete and Dedekind inf-complete.*

When $B = \{X\}$ is a singleton where $X \in \mathcal{F}$ is $\mathcal{F}_t$-bounded from below (resp. from above), we use the notation $\operatorname{ess\,inf}_{\mathcal{F}_t} X = \operatorname{ess\,inf}_{\mathcal{F}_t} B \in \mathcal{F}_t$ (resp. $\operatorname{ess\,sup}_{\mathcal{F}_t} X = \operatorname{ess\,sup}_{\mathcal{F}_t} B \in \mathcal{F}_t$) for $t \in [0,T]$. Note that, if X_t is $\mathcal{F}_t$-measurable, we have $\operatorname{ess\,sup}_{\mathcal{F}_t} X_t = \operatorname{ess\,inf}_{\mathcal{F}_t} X_t = X_t$.

Definition 2.6. *The stochastic Riesz space $\mathcal{F}$ is compactifiable if there exists a one to one correspondence c from $\mathcal{F}$ into $\mathcal{G}$ where $\mathcal{G} = \cup_{t\in[0,T]}\mathcal{G}_t$ is a stochastic Riesz space such that $c(\mathcal{F}_t) \subseteq \mathcal{G}_t$ for all $t \leq T$ and $\mathcal{G}$ is $\mathcal{G}_0$ bounded and Dedekind complete. Moreover, for any $f_1, f_1 \in \mathcal{F}$, $f_1 \preceq f_2$ if and only if we have $c(f_1) \leq c(f_2)$ where $\leq$ is the partial order of $\mathcal{G}$.*

The typical example in finance is when $\mathcal{F}_t = L^0(\mathbf{R}, \mathcal{F}_t)$, $t \in [0,T]$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is a filtration, and $c(x) = \arctan(x)$ so that $\mathcal{G}$ is $\mathcal{F}_0$ bounded by the deterministic random variable $\pi/2$. In that case, any element of $\mathcal{F}$ may be identified as an element of $\mathcal{G}$ and it is then possible to define $\operatorname{ess\,sup}_{\mathcal{F}_t} D$

and $\text{ess inf}_{\mathcal{F}_t} D$ for all subset D of $\mathcal{F}$ even if D is not $\mathcal{F}_t$ bounded. In particular, with $\mathcal{F}_t = L^0(\mathbf{R}, \mathcal{F}_t)$, $t \in [0, T]$, $\text{ess inf}_{\mathcal{F}_t} D$ may be infinite, see [[14], Section 5.3.1]. In the following, we shall suppose that $\mathcal{F}$ is compactifiable so that $\text{ess sup}_{\mathcal{F}_t} D$ and $\text{ess inf}_{\mathcal{F}_t} D$ always exit, at least in $\mathcal{G}$, but they are not necessary elements of $\mathcal{F}$. We easily deduce the following properties:

Lemma 2.7. *Let $t \in [0, T]$ and $X \in \mathcal{F}$. The following properties holds:*

1) If $X \succeq 0$, then $\text{ess sup}_{\mathcal{F}_t} X \succeq X \succeq \text{ess inf}_{\mathcal{F}_t} X \succeq 0$.
2) For any $D \subseteq \mathcal{F}$, $\text{ess sup}_{\mathcal{F}_t}(-D) = -\text{ess inf}_{\mathcal{F}_t} D$.
3) For any $D \subseteq \mathcal{F}$ and $t_1 \leq t_2 \leq T$, we have

$$\text{ess sup}_{\mathcal{F}_{t_1}} D = \text{ess sup}_{\mathcal{F}_{t_1}} \left(\text{ess sup}_{\mathcal{F}_{t_2}} D\right), \tag{2.1}$$

$$\text{ess inf}_{\mathcal{F}_{t_1}} D = \text{ess inf}_{\mathcal{F}_{t_1}} \left(\text{ess inf}_{\mathcal{F}_{t_2}} D\right). \tag{2.2}$$

4) If $D_t \subseteq \mathcal{F}_t$, then we have:

$$\text{ess sup}_{\mathcal{F}_t} D_t = \text{ess sup}_{\mathcal{F}_T} D_t, \quad \text{ess inf}_{\mathcal{F}_t} D_t = \text{ess inf}_{\mathcal{F}_T} D_t. \tag{2.3}$$

Proof. We show 3) as follows. Since $D \preceq \text{ess sup}_{\mathcal{F}_{t_1}} D$ and $\text{ess sup}_{\mathcal{F}_{t_1}} D$ is $\mathcal{F}_{t_1}$-measurable hence $\mathcal{F}_{t_2}$-measurable, we get that $\text{ess sup}_{\mathcal{F}_{t_2}} D \preceq \text{ess sup}_{\mathcal{F}_{t_1}} D$. It follows that $\text{ess sup}_{\mathcal{F}_{t_1}} \left(\text{ess sup}_{\mathcal{F}_{t_2}} D\right) \preceq \text{ess sup}_{\mathcal{F}_{t_1}} D$. Moreover, since we have $D \preceq \text{ess sup}_{\mathcal{F}_{t_2}} D$, we get that $\text{ess sup}_{\mathcal{F}_{t_1}} D \preceq \text{ess sup}_{\mathcal{F}_{t_1}} \left(\text{ess sup}_{\mathcal{F}_{t_2}} D\right)$. We then deduce (2.1) by antisymmetry. The same type of reasoning allows to show (2.2).

As we have seen above, if $D_t \subseteq \mathcal{F}_t$, $\text{ess sup}_{\mathcal{F}_t} D_t = \text{ess sup}_{\mathcal{F}_t} \left(\text{ess sup}_{\mathcal{F}_T} D_t\right)$. Moreover, $\text{ess sup}_{\mathcal{F}_T} D_t \preceq \text{ess sup}_{\mathcal{F}_t} D_t$ hence we deduce that

$$\text{ess sup}_{\mathcal{F}_t} \left(\text{ess sup}_{\mathcal{F}_T} D_t\right) \preceq \text{ess sup}_{\mathcal{F}_t} D_t.$$

We then deduce that 4) holds by antisymmetry. □

Example 2.8. We consider a Riesz space $\mathcal{F}$ which is Dedekind complete with weak order unit e and equipped with conditional expectations $\mathcal{T} = (\mathcal{T}_t)_{t \in [0,T]}$ on $\mathcal{F}$ which are strictly positive order continuous linear projections with $\mathcal{T}_t(e) = e$ and having a Dedekind complete range such that $\mathcal{T}_u \mathcal{T}_t = \mathcal{T}_t \mathcal{T}_u = \mathcal{T}_t$ for all $u \geq t$, see [16].

Definition 2.9. *A stochastic Riesz space is a pair of Riesz space $\mathcal{F} := (\mathcal{F}, \mathcal{R})$ as defined above, i.e. such that $\mathcal{F} = \cup_{t\in[0,T]}\mathcal{F}_t$ and $\mathcal{R} = \cup_{t\in[0,T]}\mathcal{R}_t$. Moreover, Assumption **A** is supposed to be satisfied and $\mathcal{F}$ is compactifiable so that the properties of Lemma 2.7 hold.*

We now suppose that $\mathcal{F}$ is equipped with a topology $\mathcal{O}$ which is compatible with the stochastic structure of $\mathcal{F}$ in the following sense:

Definition 2.10. *The topology $\mathcal{O}$ on the stochastic Riesz $\mathcal{F}$ is said compatible if the following properties hold at any time $t \in [0,T]$:*

1) *$\mathcal{F}_t^+$ is closed in $\mathcal{O}$.*
2) *If $(\alpha_t^n)_n \in \mathcal{F}_t^+$ converges to zero in $\mathcal{O}$ then, for any $(\beta_t^n)_n \in \mathcal{F}_t^+$, such that $\beta_t^n \preceq \alpha_t^n$, $(\beta_t^n)_n$ converges to zero.*
3) *If $(\alpha_t^n)_n \in \mathcal{F}_t^+$ converges to zero in $\mathcal{O}$ then, for any $\beta_t \in \mathbf{R}_t^+$, $\beta_t\alpha_t^n$ converges to zero in $\mathcal{O}$.*
4) *If $(\alpha_t^n)_n \in \mathcal{F}_t^+$ and $(\beta_t^n)_n \in \mathcal{F}_t^+$ converge to zero for $\mathcal{O}$ then $(\alpha_t^n + \beta_t^n)_n$ converges to zero for $\mathcal{O}$.*
5) *If $(\alpha_t^n)_n \in \mathcal{F}_t^+$ converges to zero in $\mathcal{O}$ then $(\alpha_t^n)_n$ is $\mathcal{F}_t$ bounded from above.*

Note that the topology of convergence in probability on $\mathcal{F} = L^0(\mathbf{R}, \mathcal{F})$ is compatible.

3. Stochastic topology on $\mathcal{F}$.

In the following, we consider a topology $\mathcal{O}$ which is compatible with the stochastic Riesz space $\mathcal{F}$.

Definition 3.1. *Let $t \in [0,T]$. We say that a sequence $(X^n)_n$ of $\mathcal{F}$ is $\mathcal{T}_t^+$-convergent to $X \in \mathcal{F}$, if $X \preceq X^n + \alpha_t^n$ for all $n \geq 1$, where $(\alpha_t^n)_n \in \mathcal{F}_t^+$ converges to zero in $\mathcal{O}$.*

The following result is trivial, as $\mathcal{O}$ is compatible, see Definition 2.10, 2) :

Lemma 3.2. *$(X^n)_{n\geq 1}$ $\mathcal{T}_t^+$-converges to X if and only if $\operatorname{ess\,sup}_{F_t}(X - X^n)^+$ converges to zero in $\mathcal{O}$, as $n \to \infty$.*

The $\mathcal{T}_t^+$-convergence we have defined above comes from a topology that we also denote by $\mathcal{T}_t^+$. Indeed, consider the set $\mathcal{O}_0$ of all neighborhoods of 0 in the topology $\mathcal{O}$. In particular, any finite intersection or arbitrary union of elements of $\mathcal{O}_0$ is still in $\mathcal{O}_0$. We then define a neighborhood base $\mathcal{B}_t(X)$ of any point X of $\mathcal{F}$ as follows:

$$\begin{aligned}\mathcal{B}_t(X): &= \{B_t(X,V_0):\ V_0 \in \mathcal{O}_0\},\\ B_t(X,V_0): &= \{Z \in \mathcal{F}; \text{ess sup}_{F_t}(X-Z)^+ \in V_0\}.\end{aligned}$$

A subset U of $\mathcal{F}$ is said to be an open set for the topology $\mathcal{T}_t^+$ if, for any $X \in U$, there exists $B_X \in \mathcal{B}_t(X)$ such that $X \in B_X \subset U$. We then easily verify that the collection of open sets U for $\mathcal{T}_t^+$, as previously defined, is a topology. Moreover, the convergence of Definition 3.1 coincides with the convergence induced by this topology.

Remark 3.3. *The topology $\mathcal{T}_t^+$ is not Hausdorff. For example, if $X, Y \in \mathcal{F}$ are such that $X \preceq Y$, then, $X - Y \preceq 0$ and $(X-Y)^+ = 0$. It follows that $\text{ess sup}_{F_t}(X-Y)^+ = 0 \in V_0$ for every $V_0 \in \mathcal{O}_0$ and we conclude that $Y \in B_t(X)$ for all $B_t(X) \in \mathcal{B}_t(X)$.*

Lemma 3.4. *If E is a closed set of $\mathcal{F}$ for $\mathcal{T}_t^+$, then E is a lower set, i.e. $E - \mathcal{F}^+ \subseteq E$.*

Proof. Indeed, consider $Z \preceq \gamma$ where $\gamma \in \mathcal{F}$. Then, $(Z-\gamma)^+ = 0$ hence the constant sequence $(\gamma_n = \gamma)_{n\geq 1}$ converges to Z and, finally, $Z \in \mathcal{F}$. □

Lemma 3.5. *If $(X^n)_{n\in\mathbb{N}}$ converges to X with respect to $\mathcal{T}_t^+$ and $(\tilde{X}^n)_{n\in\mathbb{N}}$ is another sequence such that $X^n \preceq \tilde{X}^n$, for all $n \in \mathbb{N}$, then $(\tilde{X}^n)_{n\in\mathbb{N}}$ converges to X with respect to $\mathcal{T}_t^+$.*

Proof. There exists a sequence $(\alpha_t^n)_n \in \mathcal{F}_t^+$ converging to zero in $\mathcal{O}$ such that $X \preceq X^n + \alpha_t^n \preceq \tilde{X}^n + \alpha_t^n$. The conclusion follows. □

Proposition 3.6. *Let $(X^n)_{n\in\mathbb{N}}$ and $(Y^n)_{n\in\mathbb{N}}$ be two sequences of elements of $\mathcal{F}$ which converge to X and Y respectively, with respect to $\mathcal{T}_t^+$. Then, the following convergences hold with respect to $\mathcal{T}_t^+$:*

1) *The sequence $(X^n + Y^n)_{n\in\mathbb{N}}$ converges to $X + Y$.*
2) *The sequence $(\alpha_t X^n)_{n\in\mathbb{N}}$ converges to $\alpha_t X$, for all $\alpha_t \in \mathcal{R}_t^+$.*
3) *The sequence $(\text{ess sup}_{\mathcal{F}_t}(X^n))_{n\in\mathbb{N}}$ converges to $\text{ess sup}_{\mathcal{F}_t}(X)$ with respect to $\mathcal{T}_t^+$.*

Proof. Observe that $X+Y \preceq X^n + \alpha_t^n + Y^n + \beta_t^n$ where $\alpha_t^n, \beta_t^n \in F_t^+$ converge to zero in $\mathcal{O}$. We then conclude by the statement 4) of Definition 2.10 that 1) holds. Similarly, $\alpha_t X \preceq \alpha_t X^n + \alpha_t \alpha_t^n$. We conclude by the statement

3) of Definition 2.10 that 2) holds. At last, $X = (X - X^n) + X^n$ hence $X \preceq \text{ess}\sup_{\mathcal{F}_t}(X - X^n) + \text{ess}\sup_{\mathcal{F}_t}(X^n)$. We deduce that

$$\text{ess}\sup_{\mathcal{F}_t}(X) \preceq \alpha_t^n + \text{ess}\sup_{\mathcal{F}_t}(X^n),$$

where $\alpha_t^n = \text{ess}\sup_{\mathcal{F}_t}(X - X^n)$ tends to 0 in $\mathcal{O}$ by Lemma 3.2. We then conclude that 3) holds. □

Proposition 3.7. *A sequence $(X^n)_{n\geq 1}$ of elements in $\mathcal{F}$ converges with respect to $\mathcal{T}_t^+$ if and only if $\text{ess}\inf_{\mathcal{F}_T}((X^n)_{n\geq 1})$ exists in $\mathcal{F}$ or, equivalently, the sequence $(X^n)_{n\geq 1}$ is $\mathcal{F}_T$-bounded from below. Moreover, under these equivalent conditions, $\text{ess}\inf_{\mathcal{F}_T}((X^n)_{n\geq 1})$ is a limit of $(X^n)_{n\geq 1}$ for $\mathcal{T}_t^+$.*

Proof. If the sequence $(X^n)_{n\geq 1}$ converges to X then $X \preceq X^n + \alpha_t^n$ where $\alpha_t^n \in \mathcal{F}_t^+$ converge to zero in $\mathcal{O}$. By the statement 5) of Definition 2.10, $(\alpha_t^n)_{n\geq 1}$ is bounded from above by some $\alpha_t \in \mathcal{F}_t^+$. Therefore, $X \preceq X^n + \alpha_t$ and $X^n \succeq X - \alpha_t$. This means that the sequence $(X^n)_{n\geq 1}$ is $\mathcal{F}_T$-bounded from below. As $\mathcal{F}$ is Dedekind complete, we then deduce that $\text{ess}\inf_{\mathcal{F}_T}((X^n)_{n\geq 1})$ exists in $\mathcal{F}$. Reciprocally, if $\text{ess}\inf_{\mathcal{F}_T}((X^n)_{n\geq 1})$ exists in $\mathcal{F}$, we deduce from $X^n \succeq \text{ess}\inf_{\mathcal{F}_T}((X^n)_{n\geq 1})$ that $(\text{ess}\inf_{\mathcal{F}_T}((X^n)_{n\geq 1}) - X^n)^+ = 0$. This implies that $\text{ess}\sup_{\mathcal{F}_t}(\text{ess}\inf_{\mathcal{F}_T}((X^n)_{n\geq 1}) - X^n)^+ = 0$ hence $(X_n)_{n\geq 1}$ converges to $\text{ess}\inf_{\mathcal{F}_T}((X^n)_{n\geq 1})$ with respect to $\mathcal{T}_t^+$. Therefore, the sequence $(X^n)_{n\geq 1}$ is convergent with respect to $\mathcal{T}_t^+$. □

Corollary 3.8. *A sequence $(X^n)_{n\geq 1}$ of elements in $\mathcal{F}$ is such that $(X^n)_{n\geq 1}$ and $(-X^n)_{n\geq 1}$ converge with respect to $\mathcal{T}_t^+$ if and only if $\text{ess}\sup_{\mathcal{F}_T}((|X^n|)_{n\geq 1})$ exists in $\mathcal{F}$.*

Proof. By Proposition 3.7, there exists $\alpha, \beta \in \mathcal{F}$ such that $X^n \succeq \beta$ and $-X^n \succeq -\alpha$ for all $n \geq 1$. We may replace β by $-\beta^-$ and α by α^+, hence we may suppose w.l.o.g. that $\alpha, \beta \succeq 0$ and $-\beta \preceq X^n \preceq \alpha$. Therefore, $(X^n)^+ \leq \alpha$ and $(X^n)^- \leq \beta$, for all $n \geq 1$. We then deduce that $|X^n| \preceq \alpha + \beta$, for all $n \geq 1$. As $\mathcal{F}$ is Dedekind complete, we conclude that $\text{ess}\sup_{\mathcal{F}_T}((|X^n|)_{n\geq 1})$ exists in $\mathcal{F}$. The reverse implication is immediate by Proposition 3.7. □

By the same type of reasoning, we deduce the following:

Lemma 3.9. *A sequence $(X^n)_{n\geq 1}$ of elements in $\mathcal{F}$ is such that $(X^n)_{n\geq 1}$ converges to X and $(-X^n)_{n\geq 1}$ converges to $-X$ with respect to $\mathcal{T}_t$ if and only if $\text{ess}\sup_{F_t}(|X - X^n|)$ converges to 0 for $\mathcal{O}$.*

In the following, we recall the classical definitions of limit infimum and supremum of a sequence $(X^n)_{n\geq 1}$ of $\mathcal{F}$, adapted to our Riesz structure:

$$\begin{aligned}\liminf_n X^n &:= \text{ess sup}_{\mathcal{F}_T}\{\text{ess inf}_{\mathcal{F}_T}\{X^k:\ k\geq n\}:\ n\geq 1\},\\ \limsup_n X^n &:= \text{ess inf}_{\mathcal{F}_T}\{\text{ess sup}_{\mathcal{F}_T}\{X^k:\ k\geq n\}:\ n\geq 1\}.\end{aligned}$$

We easily see that $\limsup_n X^n = -\liminf_n(-X^n)$ and, for any $X\in\mathcal{F}$, $\liminf_n(X+X^n) = X+\liminf_n X^n$ and $\limsup_n(X+X^n) = X+\limsup_n X^n$. Also, if $X^n \preceq Y^n$, for all $n\geq 1$, we have $\limsup_n X^n \leq \limsup_n Y^n$ and $\liminf_n X^n \leq \liminf_n Y^n$.

Definition 3.10. *We say that the topology $\mathcal{O}$ of the stochastic Riesz space $\mathcal{F}$ satisfies the* VLS *condition (Vanishing Limit Supremum) if, for any sequence $(X^n)_{n\geq 1}$ of $\mathcal{F}$ that converges to* 0 *in $\mathcal{O}$, there exists a subsequence $(X^{n_k})_{k\geq 1}$ such that* $\limsup_k X^{n_k} = 0$.

The property VLS is satisfied for the stochastic Riesz space $\mathcal{F} = L^0(\mathbf{R},\mathcal{F}_T)$ equipped with a complete stochastic basis $(\Omega,(\mathcal{F}_t)_{t\in[0,T]},\mathrm{P})$ and the topology $\mathcal{O}$ of convergence in topology. Indeed, it is possible to extract from any sequence that is convergent in probability a subsequence which converges almost surely.

Proposition 3.11. *Suppose that the topology satisfies the VLS condition. If a sequence $(X^n)_{n\geq 1}$ of elements in $\mathcal{F}$ converges to X, with respect to $\mathcal{T}_t^+$, then there exists subsequence $(X^{n_k})_{k\geq 1}$ such that*

$$X \preceq \liminf_k (X^{n_k}).$$

Proof. Recall that a sequence $(X^n)_{n\in\mathbb{N}}$ of elements in $\mathcal{F}$ converges to X with respect to $\mathcal{T}_t^+$ if and only if $\text{ess sup}_{\mathcal{F}_t}(X-X^n)^+$ converges to 0 in $\mathcal{O}$. By the VLS condition, there exists a subsequence $(X^{n_k})_{k\geq 1}$ such that $\limsup_k \text{ess sup}_{\mathcal{F}_t}(X-X^{n_k})^+ = 0$. As $X - X^{n_k} \preceq \text{ess sup}_{\mathcal{F}_t}(X-X^{n_k})^+$, we then deduce that the inequality $\liminf_k[X - \text{ess sup}_{\mathcal{F}_t}(X-X^{n_k})^+] \preceq \liminf_k(X^{n_k})$ holds. As

$$\begin{aligned}\liminf_k[X - \text{ess sup}_{\mathcal{F}_t}(X-X^{n_k})^+] &= X + \liminf_k[-\text{ess sup}_{\mathcal{F}_t}(X-X^{n_k})^+],\\ &= X - \limsup_k \text{ess sup}_{\mathcal{F}_t}(X-X^{n_k})^+,\\ &= X,\end{aligned}$$

the conclusion follows. □

For any converging sequence $X = (X^n)_{n\geq 1}$ of $\mathcal{F}$ with respect to $\mathcal{T}_t^+$, we denote by $\mathcal{L}_t(X)$ the set of all limits with respect to $\mathcal{T}_t^+$.

Example 3.12. Suppose that the VLS condition holds. Let $C \in \mathcal{F}$. Consider the sequence $X = (X^n)_{n\geq 1}$ such that $X^n = C$ for every $n \geq 1$. If $Z \in \mathcal{L}_t(X)$, $Z \leq C$, by Proposition 3.11. On the other hand, assume that $Z \in (-\infty, C]$. It follows that $0 \leq (Z - X^n)^+ \leq (C - X^n)^+ = 0$ hence $(X^n)_{n\geq 1}$ converges to Z in $\mathcal{T}_t^+$. We finally deduce that $\mathcal{L}_t(X) = (-\infty, C]$.

Definition 3.13. *A subset $E \subset \mathcal{F}$ is said to be $\mathcal{F}_t$-decomposable, $t \leq T$, if, for each $\xi, \eta \in E$ and each component I_t in $\mathcal{I}_t$, the element $I_t\xi + (1 - I_t)\eta$ belongs to E.*

Proposition 3.14. *Let $X = (X^n)_{n\geq 1}$ be a sequence in $\mathcal{F}$ that converges with respect to $\mathcal{T}_t^+$. The set of all possible limits $\mathcal{L}_t(X)$ is $\mathcal{F}_t$-decomposable.*

Proof. Let $\xi, \eta \in \mathcal{L}_t(X)$ and $I_t \in \mathcal{I}_t$. We have $\xi \preceq X^n + \alpha_t^n$ and $\eta \preceq X^n + \beta_t^n$, for all $n \geq 1$, where $\alpha_t^n, \beta_t^n \in \mathcal{F}_t^+$ converge to 0 in $\mathcal{O}$, as $n \to \infty$. By Assumption **A**, we deduce that the sequence $X = (X^n)_{n\geq 1}$ also satisfies $\gamma \preceq X^n + \gamma_t^n$ where $\gamma = I_t\xi + (1 - I_t)\eta$ and $\gamma_t^n = I_t\alpha_t^n + (1 - I_t)\beta_t^n$. By the statements 3) and 4) of Definition 2.10, we get that $\gamma_t^n \in \mathcal{R}_t^+$ converges to 0 in $\mathcal{O}$. Therefore, $\gamma \in \mathcal{L}_t(X)$ and the conclusion follows. □

Proposition 3.15. *For any $\mathcal{T}_t^+$-convergent sequence $X = (X^n)_n$ of $\mathcal{F}$, if $L_1, L_2 \in \mathcal{L}_t(X)$, then $L_1 \vee L_2 \in \mathcal{L}_t(X)$.*

Proof. We have $L_1 \preceq X^n + \alpha_t^n$ and $L_2 \preceq X^n + \beta_t^n$, for all $n \geq 1$, where $\alpha_t^n, \beta_t^n \in \mathcal{F}_t^+$ converge to 0 in $\mathcal{O}$, as $n \to \infty$. Therefore, $L_1 \preceq X^n + \gamma_t^n$ and $L_2 \preceq X^n + \gamma_t^n$ where $\gamma_t^n = \alpha_t^n + \beta_t^n$. It follows that $L_1 \vee L_2 \preceq X^n + \gamma_t^n$ for all $n \geq 1$. By the statement 4) of Definition 2.10, $\gamma_t^n \in \mathcal{F}_t^+$ converges to 0 in $\mathcal{O}$, as $n \to \infty$. We deduce that $L_1 \vee L_2 \in \mathcal{L}_t(X)$. □

Lemma 3.16. *Let $X = (X^n)_{n\geq 1}$ be a sequence of elements in $\mathcal{F}$ such that X and $-X$ are $\mathcal{T}_t^+$-convergent. Then $\xi = \operatorname{ess\,sup}_{\mathcal{F}_T}(\mathcal{L}_t(X))$ exists in $\mathcal{F}$.*

Proof. For every $Z \in \mathcal{L}_t(X)$, there exists a sequence $(\alpha_t^n)_{n\geq 1}$ such that we have $Z \preceq X^n + \alpha_t^n$ and $\alpha_t^n \in \mathcal{F}_t^+$ converge to 0 in $\mathcal{O}$, as $n \to \infty$. By the statement 5) of Definition 2.10, there exists $\alpha_t \in \mathcal{F}_t^+$ that dominates the sequence $(\alpha_t^n)_{n\geq 1}$. Moreover, the sequence $(X^n)_{n\geq 1}$ is dominated by $\operatorname{ess\,sup}_{\mathcal{F}_T}((|X^n|)_{n\geq 1}) \in \mathcal{F}$, see Corollary 3.8. Therefore, we deduce that

$Z \preceq \alpha_t + \text{ess sup}_{\mathcal{F}_T}((|X^n|)_{n\geq 1})$ for all $Z \in \mathcal{L}_t(X)$. This shows that $\mathcal{L}_t(X)$ is $\mathcal{F}_T$-bounded. As $\mathcal{F}$ is Dedekind complete, the conclusion follows. □

Consider the constant sequence $X = (X^n)_n$ of $\mathcal{F}$ such that $X^n = C \in \mathcal{F}$ for every $n \geq 1$. Then $\text{ess sup}_{\mathcal{F}_T}(\mathcal{L}_t(X)) = C \in \mathcal{L}_t(X)$.

Proposition 3.17. *Suppose that the* VLS *condition holds. Let $X = (X^n)_{n\geq 1}$ be a sequence of elements in $\mathcal{F}$ such that X and $-X$ are $\mathcal{T}_t^+$-convergent. Let us consider $\xi^{(t)} = \text{ess sup}_{\mathcal{F}_T}(\mathcal{L}_t(X)) \in \mathcal{F}$, which exists by Lemma 3.16. There exists a subsequence $(X^{n_k})_{k\geq 1}$ such that*

$$\liminf_n X^n \preceq \xi^{(t)} \preceq \liminf_k (X^{n_k}).$$

Proof. By Proposition 3.11, we have $\xi^{(t)} \preceq \liminf_k X^{n_k}$ for a subsequence $(X^{n_k})_{k\geq 1}$. We show that $\xi^{(t)} \succeq \liminf_n X^n$. For any $n \geq 1$, let us define the sequence $Z^n = \text{ess inf}_{\mathcal{F}_T}\{X^k : \ k \geq n\}$. By Corollary 3.8, $Z^n \in \mathcal{F}$ as $\mathcal{F}$ is Dedekind complete. Since $Z^n \preceq X^k$ for any $k \geq n$, we have $(Z^n - X^k)^+ = 0$ and we deduce that $Z^n \in \mathcal{L}_t(X)$. So $Z^n \leq \xi^{(t)}$, for all $n \geq 1$. We conclude that $\liminf_n X^n \leq \xi^{(t)}$. □

4. Financial market models derived from stochastic Riesz spaces

A financial market model is defined by a stochastic Riesz space $\mathcal{F} = (\mathcal{F}, \mathcal{R})$ as defined in Section 2. We suppose that the VLS condition is satisfied. Moreover, we consider a stochastic price process $(S_t)_{t\in[0,T]}$, $T > 0$, which is, by definition, a family of the time t elements S_t of a vector space $\mathcal{S}$. For each $t \in [0, T]$, S_t is interpreted as the collection of all prices at time t of the risky assets in the financial market. An investment strategy on the interval $[t, T]$, $t \in [0, T]$, is any stochastic process $(\theta_u)_{u\in[t,T]}$, which is, by definition, a family of the time $u \geq t$ elements θ_u of a vector space $\mathcal{Q}$. For each $u \in [t, T]$, θ_u is interpreted as the quantities invested at time u in the risky assets of the financial market. In finance, such strategies are supposed to satisfy some self-financing condition, see [18].

We suppose that there exists a left linear operator $\langle \theta, S\rangle$ between the elements $\theta \in \mathcal{Q}$ and $S \in \mathcal{S}$ with values in $\mathcal{F}$. For simplicity, we denote it by $\theta S = \langle \theta, S\rangle \in \mathcal{F}$ and we interpret it as a product between the quantities θ and the prices S.

Definition 4.1. *An investment strategy* $\theta = (\theta_u)_{u\in[t,T]}$*,* $t \in [0,T]$*, is said adapted to the stochastic structure of* $\mathcal{F}$ *(adapted in short) if the liquidation value at time* $u \geq t$*, given by* $V_u(\theta) = \theta_u S_u \in \mathcal{F}$ *satisfies* $V_u(\theta) \in \mathcal{F}_u$*, for all* $u \geq t$*.*

The definition above needs to be understood as follows. At time $u \geq t$, S_u is observable and the financial position θ_u is chosen by the manager in terms of the available information on the market. Then, $V_u(\theta)$ is the portfolio value, which is $\mathcal{F}_u$-measurable, i.e. observable at time u.

Example 4.2. In the classical theory of financial mathematics, the market is defined by a complete right-continuous stochastic basis $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathrm{P})$. Then, if the market is composed of $d \geq 1$ risky assets, $\mathcal{Q} = \mathcal{S} = L^0(\mathbf{R}^d, \mathcal{F}_T)$ is the set of all $\mathbf{R}^d$-valued $\mathcal{F}_T$-random variables. Moreover, it is supposed that $S_t^i > 0$, $i = 1, \cdots, d$, a.s., S_t is $\mathcal{F}_t$-measurable and any strategy θ_t is also $\mathcal{F}_t$-measurable. A discrete-time strategy θ satisfies by definition $\theta_t = \theta_{i-1}$ for any $t \in (t_{i-1}, t_i]$, $i = 1, \cdots, n$, where the $n+1$ discrete dates satisfy $t_0 = 0 < t_1 < \cdots < t_n = T$ and θ_{i-1} is $\mathcal{F}_{t_{i-1}}$-measurable (i.e. fixed at time t_{i-1}), for any $i = 1, \cdots, n$.

A discrete-time strategy as defined above is said self-financing if the liquidation value $V = V(\theta)$ given by Definition 4.1 satisfies $\Delta V_{t_i} = \theta_{i-1}\Delta S_{t_i}$ for all $i = 1, \cdots, n$, with the general notation $\Delta V_{t_i} = V_{t_i} - V_{t_{i-1}}$. This is the case for discrete-time financial models of the literature, provided that there is no transaction costs and the prices are discounted.

The value at time T of a discrete-time portfolio $V = V(\theta)$ starting from the initial capital V_0 at time 0 is then given by

$$V_T = x + \sum_{i=1}^{n} \theta_{i-1}\Delta S_{t_i}. \tag{4.4}$$

Of course, we may extend this definition to any strategy only defined on some interval $[t, T]$ with the initial capital V_t at time t.

The challenge in mathematical finance is to extend the concept of self-financing strategy, as presented in the example above, in continuous time. It is done by the mean of the Ito stochastic calculus, see [15], and more generally for semi-martingales price processes S, see [23]. In that case, a portfolio process is a stochastic integral in continuous-time, generalizing (4.4). Moreover, these stochastic integrals are limit in some sense (in L^2 with the Ito calculus) of discrete-time portfolio processes. Here, we propose to follow this general idea in the context of the stochastic Riesz space $(\mathcal{F}, \mathcal{R})$ that defines our financial model. This means that we consider elementary portfolio processes, that we call discrete-time portfolio processes as in the classical theory, and we aim to construct continuous-time portfolio processes as limits of such discrete-time portfolio processes.

In the following, we denote by $\mathcal{V}_{t,T}^d \subseteq \mathcal{F}_T$ the given set of all terminal values of discrete-time portfolios defined on the interval $[t, T]$, starting with the zero initial capital at time $t \in [0, T]$. We suppose that $\mathcal{V}_{t,T}^d$ is a convex set containing 0, which is $\mathcal{F}_t$-decomposable. The terminal value $W_{t,T}$ of an elementary portfolio defined on the interval $[t, T]$, starting with the initial capital V_t at time t, is then $W_{t,T} = V_t + V_{t,T}$ where $V_{t,T} \in \mathcal{V}_{t,T}^d$. By assumption, any $V_{t,T} \in \mathcal{V}_{t,T}^d$ is of the form $V_{t,T} = \theta_T S_T$ for some so-called discrete-time strategy $\theta = (\theta_u)_{u \in [t,T]}$ such that $V_{t,u} = \theta_u S_u \in \mathcal{F}_u$, for all $u \in [t, T]$. In the following, we shall also use the integral notation

$$V_{t,u}^{\theta} = \int_t^u \theta_r dS_r = \theta_u S_u \in \mathcal{F}_u, u \in [t, T], \tag{4.5}$$

which is derived from the self-financing property satisfied in the classical financial models of the literature. The set $\mathcal{V}_{t,T}^d$ may be interpreted as the portfolio terminal values we may obtain when following a piece-wise constant strategy, see the discrete-time models in [[14], Section 2].

5. Super-hedging prices and no-arbitrage condition AIP in discrete-time

In the following, we consider the financial model of Section 4 with the same notations and assumptions. In finance, a contingent claim or payoff is a terminal wealth $W_T \in \mathcal{F} = \mathcal{F}_T$ that must be delivered at some maturity date $T > 0$ to the holder of some financial contract. For instance, $W_T = (S_T - K)^+$ is the payoff of the so-called European Call option of underlying asset price S and strike $K > 0$. The European Call option contract corresponds to the possibility to buy the risky asset at price K instead of S_T. Clearly, it is interesting to exercice it if and only if $S_T \geq K$ so that the profit is $(S_T - K)^+$.

Definition 5.1. *A contingent claim $W_T \in \mathcal{F}$ is said to be super-hedgeable in discrete-time at time t if there exists a so-called super-hedging price $p_t \in \mathcal{F}_t$ and $V_{t,T} = V_{t,T}(\theta) \in \mathcal{V}_{t,T}^d$ such that $p_t + V_{t,T} \succeq W_T$.*

Note that the portfolio processes of our model take their values in $\mathcal{F}$ which is equipped with the partial order $\preceq$. For two terminal wealths $W_T^1, W_T^2 \in \mathcal{F}$, we may interpret the inequality $W_T^1 \preceq W_T^2$ as if W_T^2 is preferred to W_T^1. In that case, $\preceq$ designates the preferences of a given financial manager. There is also a second possible interpretation. In the models with transaction costs, see [[14], Section 3], $W_T^1 \preceq W_T^2$ if it is possible to change the financial position W_T^2 into W_T^1 by paying some transaction costs. We refer the readers to [11] and [12], for random partial orders especially designed for financial models with frictions. Here, the inequality $p_t + V_{t,T} \succeq W_T$ needs to be understood according to the second approach. The super-hedging price $p_t \in \mathcal{F}_t$ is the capital that the option contract seller asks at time t to the buyer so that he/she may initiate a portfolio strategy θ that generates the portfolio process $V_{t,\cdot} = V_{t,\cdot}(\theta)$ whose terminal value is sufficient (for the option seller) to deliver the payoff h_T to the buyer.

The set of all super-hedgeable claims from discrete-time strategies, with zero initial endowment at time t, is then given by

$$\mathcal{A}_{t,T}^d = \left\{ V_{t,T} - \varepsilon_T,\ V_{t,T} \in \mathcal{V}_{t,T}^d,\ \varepsilon_T \in \mathcal{F}_T^+ \right\}.$$

We denote by $\mathcal{P}_{t,T}^d(W_T)$ the set of all super-hedging prices $p_t \in \mathcal{F}_t$ at time $t \in [0,T]$ for the contingent claim W_T in discrete-time, as in Definition 5.1. The infimum super-hedging price is then defined as

$$\mathcal{P}_{t,T}^{d*}(W_T) := \operatorname{ess\,inf}_{F_t}(\mathcal{P}_t^d(W_T)),\ t \in [0,T].$$

Notice that we do not necessary have $\mathcal{P}_{t,T}^{d*}(W_T) \in \mathcal{F}_t$. In particular, $\mathcal{P}_{t,T}^{d*}(W_T)$ is not necessary a super-hedging price. We also adopt the following notations $\mathcal{P}_{t,T}^d := \mathcal{P}_{t,T}^d(0)$ and $\mathcal{P}_{t,T}^{d*} = \mathcal{P}_{t,T}^{d*}(0)$.

Proposition 5.2. *We have* $\mathcal{P}_{t,T}^d = \left(-\mathcal{A}_{t,T}^d\right) \cap \mathcal{F}_t$ *and*

$$\mathcal{P}_{t,T}^d = \left\{ \operatorname{ess\,sup}_{F_t}(-V_{t,T}) : V_{t,T} \in \mathcal{V}_{t,T}^d \right\} + \mathcal{F}_t^+.$$

Proof. It suffices to observe that p_t is a price for 0 if and only if there exists $V_{t,T} \in \mathcal{V}_{t,T}^d$ such that $p_t + V_{t,T} \succeq 0$, i.e, $p_t \succeq -V_{t,T}$ which is equivalent to $p_t \succeq \operatorname{ess\,sup}_{F_t}(-V_{t,T})$. In particular, $-p_t \preceq V_{t,T}$ hence $-p_t \in \mathcal{A}_{t,T}^d$. The conclusion follows. □

We introduce the notion of absence of instantaneous profit, as in [4], [1].

Definition 5.3 (AIP). *An instantaneous profit in discrete-time at time $t < T$ is the possibility to super-replicate the zero contingent claim from a discrete-time negative price, i.e. if there exists $p_{t,T} \in \mathcal{P}^d_{t,T} \cap \mathcal{F}^-_t$. On the contrary, we say that the Absence of Instantaneous Profit (*AIP *) holds if, for any $t \leq T$,*

$$\mathcal{P}^d_{t,T} \cap \mathcal{F}^-_t = \mathcal{A}^d_{t,T} \cap \mathcal{F}^+_t = \{0\}. \tag{5.6}$$

We obtain the following result.

Lemma 5.4. *The* AIP *condition holds in discrete-time if and only if, for any $t \leq T$ and for all $V_{t,T} \in \mathcal{V}^d_{t,T}$, we have* $\operatorname{ess\,inf}_{F_t}(V_{t,T}) \preceq 0$.

Proof. AIP holds at time t if and only if $\mathcal{P}^d_{t,T} \subset \mathcal{F}^+_t$. By Proposition 5.2, this is equivalent to $\operatorname{ess\,sup}_{F_t}(-V_{t,T}) \succeq 0$, i.e. $\operatorname{ess\,inf}_{F_t}(V_{t,T}) \preceq 0$, $\forall\, V_{t,T} \in \mathcal{V}^d_{t,T}$, by the statement 2) of Proposition 2.7. □

6. Super-hedging prices and no-arbitrage condition AIP in continuous-time

We still consider the financial model of Section 4 both with the definitions from Section 5 that we shall adapt immediately to the continuous-time setting we introduce. In the following, we denote by $\mathcal{V}^c_{t,T} \subseteq \mathcal{F}_T$ the set of all terminal values of continuous-time portfolios defined on the interval $[t,T]$, starting with the zero initial capital at time $t \in [0,T]$. By definition, $V^c_{t,T} \in \mathcal{V}^c_{t,T}$ if there exists a $\mathcal{T}^+_t$-convergent sequence $V^d_{t,T} = (V^{d,n}_{t,T})_{n\geq 1} \in \mathcal{V}^d_{t,T}$ such that $V^c_{t,T} \in \mathcal{L}_t(V^d_{t,T})$. We observe that $\mathcal{V}^d_{t,T} \subseteq \mathcal{V}^c_{t,T}$. To see it, it suffices to consider constant sequences constructed from discrete-time strategies.

The financial interpretation of a continuous-time portfolio $V^c_{t,T} \in \mathcal{V}^c_{t,T}$ is the following: It is possible to reach the value $V^c_{t,T}$ from above, in limit, from a sequence of discrete-time portfolios. This is a direct consequence of the definition of the $\mathcal{T}^+_t$-convergence.

Definitions 5.1 and 5.3 are easily adapted to the continuous-time case if we replace $\mathcal{V}^d_{t,T}$ by $\mathcal{V}^c_{t,T}$. Then, we observe that Proposition 5.2 is still valid in continuous time as well as Lemma 5.4. We now propose to show that the AIP conditions, respectively in discrete and continuous time, are equivalent:

Theorem 6.1. *The* AIP *conditions are equivalent in discrete and continuous time.*

Proof. It suffices to prove that AIP holds in continuous time if it holds in discrete time. By Lemma 5.4, we have $\operatorname{ess\,inf}_{F_t}(V_{t,T}) \preceq 0$ for all $V_{t,T} \in \mathcal{V}^d_{t,T}$. We have to show the same for $V^c_{t,T} \in \mathcal{V}^c_{t,T}$. By definition, $V^c_{t,T} \preceq V^{d,n}_{t,T} + \alpha^n_t$ for all $n \geq 1$, where $\alpha^n_t \in \mathcal{F}^+_t$ converges to 0 in $\mathcal{O}$ and $V^{d,n}_{t,T} \in \mathcal{V}^d_{t,T}$. As α^n_t is $\mathcal{F}_t$-measurable, we deduce that $\operatorname{ess\,inf}_{F_t} V^c_{t,T} \preceq \operatorname{ess\,inf}_{F_t} V^{d,n}_{t,T} + \alpha^n_t \preceq \alpha^n_t$. As, we suppose that the VLS condition holds, we have $\operatorname{ess\,inf}_{F_t} V^c_{t,T} \preceq \alpha^{n_k}_t$ by AIP, for a subsequence $(\alpha^{n_k}_t)_{k\geq 1}$ such that $\limsup_k \alpha^{n_k}_t = 0$. By (2.3), we know that $\operatorname{ess\,inf}_{F_t} V^c_{t,T} = \operatorname{ess\,inf}_{F_T} \operatorname{ess\,inf}_{F_t} V^c_{t,T}$. So, $\operatorname{ess\,inf}_{F_t} V^c_{t,T} \preceq \limsup_k \alpha^{n_k}_t$ and we conclude that $\operatorname{ess\,inf}_{F_t} V^c_{t,T} \preceq 0$. The conclusion follows. □

We now compare the super-hedging prices in continuous and discrete time. To do so, for any payoff $W_T \in \mathcal{F}$, we use the notations $\mathcal{P}^c_{t,T}(W_T)$ and $\mathcal{P}^{c*}_{t,T}(W_T)$ for the continuous-time super-hedging prices, analogs of the discrete-time prices $\mathcal{P}^d_{t,T}(W_T)$ and $\mathcal{P}^{d*}_{t,T}(W_T)$.

Theorem 6.2. *Let $W_T \in \mathcal{F}$ be a payoff. For any $t \in [0,T]$, we have $\mathcal{P}^d_{t,T}(W_T) \subseteq \mathcal{P}^c_{t,T}(W_T)$ and $\mathcal{P}^{c*}_{t,T}(W_T) = \mathcal{P}^{d*}_{t,T}(W_T)$.*

Proof. Since $\mathcal{V}^d_{t,T} \subseteq \mathcal{V}^c_{t,T}$ for $t \in [0,T]$, $\mathcal{P}^d_{t,T}(W_T) \subseteq \mathcal{P}^c_{t,T}(W_T)$ for any payoff $W_T \in \mathcal{F}$ and, finally, $\mathcal{P}^{c*}_{t,T}(W_T) \preceq \mathcal{P}^{d*}_{t,T}(W_T)$. Moreover, if $p^c_t \in \mathcal{P}^c_{t,T}(W_T)$, then $p^c_t + V^c_{t,T} \succeq W_T$ for some $V^c_{t,T} \in \mathcal{V}^c_{t,T}$. By definition, $V^c_{t,T} \preceq V^{d,n}_{t,T} + \alpha^n_t$ where the sequence $(V^{d,n}_{t,T})_{n\geq 1}$ belong to $\mathcal{V}^d_{t,T}$ and $\alpha^n_t \in \mathcal{F}^+_t$ tends to 0 in $\mathcal{O}$, as $n \to \infty$. We deduce that $p^c_t + V^{d,n}_{t,T} + \alpha^n_t \succeq W_T$ so that $p^c_t + \alpha^n_t \in \mathcal{P}^d_{t,T}(W_T)$ for all $n \geq 1$. Therefore, $p^c_t + \alpha^n_t \succeq \mathcal{P}^{d*}_{t,T}(W_T)$. By the VLS condition, we may assume w.l.o.g. that $\limsup_n \alpha^n_t = 0$ so that we finally conclude that $p^c_t \succeq \mathcal{P}^{d*}_{t,T}(W_T)$. As p^c_t is arbitrarily chosen, we conclude that $\mathcal{P}^{c*}_{t,T}(W_T) \succeq \mathcal{P}^{d*}_{t,T}(W_T)$ and the conclusion follows. □

7. Stochastic integrals in continuous-time

We consider the financial model as considered in Section 6. Recall that each $V^d_{t,T} \in \mathcal{V}^d_{t,T}$ may be seen as a stochastic integral as in (4.5), i.e. generated by a discrete-time strategy θ on $[t,T]$. In this section, we provide a way to extend such a stochastic integral in continuous time on $[t,T]$.

Definition 7.1. *Let $t \in [0,T]$. A continuous-time strategy θ^c on $[t,T]$ is a sequence $\theta^c = (\theta^n)_{n\geq 1}$ of discrete-time strategies θ^n, $n \geq 1$, such that the sequence $V^d_{t,T}(\theta^n) = \int_t^T \theta^n_u dS_u$, $n \geq 1$, is $\mathcal{T}^+_t$-convergent. We then define the* *stochastic integral of* θ^c *with respect to* S *as:*

$$\int_t^T \theta_u^c dS_u := \text{ess sup}_{F_T} \mathcal{L}_t(\theta^c), \quad \mathcal{L}_t(\theta^c) := \mathcal{L}_t\left(\left(\int_t^T \theta_u^n dS_u\right)_{n\geq 1}\right) \tag{7.7}$$

The continuous-time stochastic integral $\int_t^T \theta_u^c dS_u$ above may be also interpreted as a continuous-time portfolio terminal value. In that case, we define the set of all attainable claims as the family $\bar{\mathcal{V}}_{t,T}^c$ of all $\bar{V}_{t,T}^c \in \mathcal{F}$ such that $\bar{V}_{t,T}^c \preceq \int_t^T \theta_u^c dS_u$ for some continuous-time strategy θ^c on $[t,T]$. We easily see that $\mathcal{V}_{t,T}^c \subseteq \bar{\mathcal{V}}_{t,T}^c$ so that we have enlarged the set of all possible continuous-time portfolio processes.

Note that, in the classical setting of stochastic calculus, see [23] or [15], a continuous-time stochastic integral $I = \int_0^T \theta_t dS_t$ is the limit of discrete-time integrals $I^n = \int_0^T \theta_t^n dS^t$ with respect to the topology of convergence in probability and $-I^n$ converges also to $-I$. By Proposition 3.17, we then deduce that $I = \int_0^T \theta_t dS^t$ coincides with the integral as constructed in Definition 7.1. This means that our stochastic integral is an extension of the usual one.

In the following, we suppose that the family of discrete-time strategies generating the discrete-time portfolios $\mathcal{V}_{t,T}^d$ is a convex cone. By Proposition 3.6, we deduce that the family of continuous-time strategies generating the continuous-time portfolios $\bar{\mathcal{V}}_{t,T}^c$ is also a convex cone. Moreover, we suppose that the conditional operator $\text{ess sup}_{\mathcal{F}_T}$ satisfy the following properties:

Condition **H**: If $\alpha \in \mathcal{R}^+$, then $\text{ess sup}_{\mathcal{F}_T}(\alpha D) = \alpha \,\text{ess sup}_{\mathcal{F}_T}(D)$ for any $D \subseteq \mathcal{F}$.

In Proposition 8.1 of Appendix, we give some conditions on $(\mathcal{F}, \mathcal{R})$ such that Condition **H** holds.

Definition 7.2. *Let* $t \in [0,T]$*. The integral operator* $\theta \mapsto \int_t^T \theta_u dS_u$ *is* $\mathcal{F}_t$ *upper linear in discrete time (resp. in continuous time) if*

$$\int_t^T (\theta_u^1 + \alpha_t \theta_u^2) dS_u \succeq \int_t^T \theta_u^1 dS_u + \alpha_t \int_t^T \theta_u^2 dS_u, \tag{7.8}$$

for any discrete-time (resp. continuous-time) strategies θ^1, θ^2 *and* $\alpha_t \in \mathcal{R}_t^+$*.*

Similarly, we may define the lower linearity by replacing $\succeq$ by $\preceq$ in the inequality above. Then, an operator is $\mathcal{F}_t$ linear if it is both lower and upper linear. Note that the inequality (7.8) means that the portfolio terminal value $\int_t^T (\theta_u^1 + \alpha_t \theta_u^2) dS_u$ is greater than the combination of the two isolated portfolio terminal values $\int_t^T \theta_u^i 1 dS_u$, $i = 1, 2$. This is an interesting property in finance, meaning that we improve the terminal wealth by diversification. Note that the strict equality is generally not satisfied in finance if there are transaction costs.

We observe that an $\mathcal{F}_t$ upper linear operator is strictly $\mathcal{F}_t$ positively homogeneous in the sense that the inequality (7.8) is an equality for $\theta^1 = 0$, any θ^2, and any $\alpha_t \in \mathcal{R}_t^+$ which is invertible in $\mathcal{R}_t^+$, i.e. $\alpha_t \alpha_t^{-1} = \alpha_t^{-1} \alpha_t = 1$ for some $\alpha_t^{-1} \in \mathcal{R}_t^+$.

Proposition 7.3. *Suppose that Condition* **H** *holds. If the integral operator* $\theta \mapsto \int_t^T \theta_u dS_u$ *is* $\mathcal{F}_t$ *upper linear in discrete time, then it is* $\mathcal{F}_t$ *upper-linear in continuous time.*

Proof. Let $\theta^{c,i} = (\theta^{n,i})_{n \geq 1}$, $i = 1, 2$, be two continuous-time strategies where $\theta^{n,i}$ are discrete-time strategies. By definition, $(V_{t,T}^d(\theta^{ni}))_{n \geq 1}$, $i = 1, 2$, are $\mathcal{T}_t^+$-convergent sequences. Note that $(V_{t,T}^d(\alpha_t \theta^{n2}))_{n \geq 1} = (\alpha_t V_{t,T}^d(\theta^{n2}))_{n \geq 1}$ is also $\mathcal{T}_t^+$-convergent by Proposition 3.6 when $\alpha_t \in \mathcal{R}_t^+$. Using the the upper linearity assumption for the discrete-time strategies and the statements 3) and 4) of Definition 2.10, we get that $\mathcal{L}_t(\theta^{c,1}) + \alpha_t \mathcal{L}_t(\theta^{c,2}) \subseteq \mathcal{L}_t(\theta^{c,1} + \alpha_t \theta^{c,2})$. It follows that $\xi^1 + \alpha_t \xi^2 \leq \int_t^T (\theta_u^{c,1} + \alpha_t \theta_u^{c,2}) \, dS_u$ for any $\xi^i \in \mathcal{L}_t(\theta^{c,i})$, $i = 1, 2$. Using Condition **H**, and considering the essential supremum knowing $\mathcal{F}_T$ over all ξ^1 and ξ^2, successively, we then deduce that

$$\int_t^T \theta_u^{c,1} dS_u + \alpha_t \int_t^T \theta_u^{c,2} dS_u \leq \int_t^T \left(\theta_u^{c,1} + \alpha_t \theta_u^{c,2}\right) dS_u.$$

The conclusion follows. □

Through the stochastic integrals as defined in Definition 7.1, we have enlarged the class of continuous-time portfolio processes. Some problems are open: no-arbitrage characterizations, super-hedging problems, as it is done in the usual setting of stochastic finance.

8. Appendix

The property $\alpha = (\alpha \wedge 1)(\alpha \vee 1)$ that we require in the following proposition holds in the setting of the Riesz spaces which are f-algebras, as shown in De Pagter's thesis, see [7].

Proposition 8.1. *Suppose that there exists an inner product $\alpha\beta \in \mathcal{R}^+$ between elements of $\alpha, \beta \in \mathcal{R}^+$ and suppose that, any $\alpha \in \mathcal{R}^+$ such that $\alpha \succeq 1$ is invertible in $\alpha \in \mathcal{R}^+$. Moreover, suppose that any $\alpha \in \mathcal{R}^+$ satisfies $\alpha = (\alpha \wedge 1)(\alpha \vee 1)$. Then, for any family $D \subseteq \mathcal{F}$, we have*

$$\text{ess sup}_{F_T}(\alpha D) = \alpha \,\text{ess sup}_{F_T}(D).$$

Proof. Since $D \preceq \text{ess sup}_{F_T}(D)$, we get that $\alpha D \preceq \alpha \,\text{ess sup}_{F_T}(D)$ hence

$$\text{ess sup}_{F_T}(\alpha D) \preceq \alpha \,\text{ess sup}_{F_T}(D). \tag{8.9}$$

<u>1rst case: $\alpha \preceq 1$</u>

By (8.9), we also have

$$\text{ess sup}_{F_T}((1-\alpha)D) \preceq (1-\alpha)\,\text{ess sup}_{F_T}(D). \tag{8.10}$$

Adding this inequality (8.10) to (8.9), we get that

$$\text{ess sup}_{F_T}(\alpha D) + \text{ess sup}_{F_T}((1-\alpha)D) \preceq \text{ess sup}_{F_T}(D).$$

As $\text{ess sup}_{F_T}(\alpha D) + \text{ess sup}_{F_T}((1-\alpha)D) \succeq \alpha d + (1-\alpha)d$ for all $d \in D$, we finally deduce that $D \preceq \text{ess sup}_{F_T}(\alpha D) + \text{ess sup}_{F_T}((1-\alpha)D)$ hence

$$\text{ess sup}_{F_T}(D) \preceq \text{ess sup}_{F_T}(\alpha D) + \text{ess sup}_{F_T}((1-\alpha)D) \preceq \text{ess sup}_{F_T}(D). \tag{8.11}$$

By antisymmetry, we get that

$$\text{ess sup}_{F_T}(\alpha D) + \text{ess sup}_{F_T}((1-\alpha)D) = \text{ess sup}_{F_T}(D).$$

Using (8.10) and (8.9), we deduce that necessarily (8.10) and (8.9) are equalities.

<u>2nd case: $\alpha \succeq 1$</u>

By assumption, α is invertible hence (8.9) may be applied to α^{-1} and α so that we deduce that (8.9) is an equality.

<u>General case</u>

By assumption, $\alpha = (\alpha \wedge 1)(\alpha \vee 1)$. Therefore, applying successively the first and the second case, we deduce that

$$\begin{aligned}\text{ess sup}_{F_T}(\alpha D) &= \text{ess sup}_{F_T}((\alpha \wedge 1)(\alpha \vee 1)D) = (\alpha \wedge 1)\,\text{ess sup}_{F_T}((\alpha \vee 1)D)\\ &= (\alpha \wedge 1)(\alpha \vee 1)\,\text{ess sup}_{F_T} D = \alpha \,\text{ess sup}_{F_T} D.\end{aligned}$$

The conclusion follows. □

References

[1] Baptiste J., Carassus L. and Lépinette E. Pricing without martingale measure, 2018. Preprint, https://hal.archives-ouvertes.fr/hal-01774150.

[2] Barron E.N., Cardaliaguet P. and Jensen R. Conditional essential suprema with applications. Applied Mathematics and Optimization, 48, 229-253, 2003.

[3] Black F. and Scholes M. The pricing of options and corporate liabilities. Journal of Political Economy, 81, 3, 637-659, 1973.

[4] Carassus L. and Lépinette E. Pricing without no-arbitrage condition in discrete-time. Journal of Mathematical Analysis and Applications, 505, 1, 125441, 2022.

[5] Choulli T. and Deng J. Structure condition under initial enlargement of filtration. Science China Mathematics, 60, 301-316, 2017.

[6] Dalang R., Morton A. and Willinger W. Equivalent martingale measures and no-arbitrage in stochastic securities market models. Stochastic and Stochastic Reports, 29, 185-201, 1990.

[7] De Pagter B. f-Algebras and orthomorphisms, Ph.D. Thesis, University of Leiden, 1981.

[8] De Vallière D., Kabanov Y. and Denis E. Hedging of American options under transaction costs. Finance and Stochastics 13 , 1, 105-119, 2009.

[9] Delbaen F. and Schachermayer W. A general version of the fundamental theorem of asset pricing. Mathematische Annalen, 300, 463-520, 1994.

[10] Harrison M. and Kreps D. Martingales and stochastic integrals in multiperiod security markets. Journal of Economical Theory, 20, 381-408, 1979.

[11] Kabanov Y. and Lépinette E. Essential supremum and essential maximum with respect to random preference relations. Journal of Mathematical Economics, 49, 6, 488-495, 2013.

[12] Kabanov Y. and Lépinette E. Essential supremum with respect to a random partial order. Journal of Mathematical Economics, 49 , 6, 478-487, 2013.

[13] Kabanov Y. and Lépinette E. Consistent price systems and arbitrage opportunities of the second kind in models with transaction costs. Finance and Stochastics. 16, 1, 135-154, 2011.

[14] Kabanov Y. and Safarian M. Markets with transaction costs. Mathe-

matical theory. Springer-Verlag Berlin Heidelberg, 2009.

[15] Karatzas I. and Shreve S.E. Brownian motion and stochastic calculus. Graduate Texts in Mathematics, 2nd Ed., Springer, 1998.

[16] Kuo W.C., Labuschagne Coenraad C.A. and Watson B.A. Discrete-time stochastic processes on Riesz spaces. Indag. Mathem., N.S., 15, 3, 435-451, 2004.

[17] Kreps D.M. Arbitrage and equilibrium in economies with infinitely many commodities. Journal of Mathematical Economics, 8, 15-35, 1981.

[18] Lamberton D. et Lapeyre B. Introduction to stochastic calculus applied to finance, 2nd Edition. Chapman and Hall, CRC Press, 2007. Version PDF http://damsteam.free.fr/calculsto/intro-calcul-sto.pdf.

[19] Lépinette E. On supremal and maximal sets with respect to random partial orders. "Set Optimization - State of the Art and Applications in Finance." Ed. A. Hamel, Springer, 151, 275-291, 2015.

[20] Lépinette E. and Tran T.Q. Arbitrage theory for non convex financial market models. Stochastic Processes and Applications, 127, 10, 3331-3353, 2017.

[21] Lépinette E. and Tran T.Q. General financial market model defined by a liquidation value process. Stochastics, 88, 3, 437-459, 2016.

[22] Merton R.C. Optimum consumption and portfolio rules in a continuous time model. Journal of Economic Theory, 3, 4, 373-433, 1971.

[23] Protter P. Stochastic integration and differential equations. Applications of Mathematics. Springer Berlin Heidelberg, 1990.

[24] Zaanen A.C. Introduction to Operator Theory in Riesz Spaces. Springer-Verlag Berlin Heidelberg, 1997.