Annealed local limit theorem for Sinai's random walk in random environment

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ANNEALED LOCAL LIMIT THEOREM FOR SINAI'S RANDOM WALK IN RANDOM ENVIRONMENT

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ABSTRACT. We consider Sinai's random walk in random environment $(S_n)_{n\in\mathbb{N}}$. We prove a local limit theorem for $(S_n)_{n\in\mathbb{N}}$ under the annealed law $\mathbb{P}$. As a consequence, we get an equivalent for the annealed probability $\mathbb{P}(S_n = z_n)$ as n goes to infinity, when $z_n = O\big((\log n)^2\big)$. To this aim, we develop a path decomposition for the potential of Sinai's walk, that is, for some random walks with i.i.d. increments. The proof also relies on renewal theory, a coupling argument, a very careful analysis of the environments and trajectories of Sinai's walk satisfying $S_n = z_n$, and on precise estimates for random walks conditioned to stay positive or nonnegative.

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1. Introduction and statement of the main results

1.1. **Presentation of the model.** We consider a collection $\omega := (\omega_x)_{x\in\mathbb{Z}}$ of i.i.d. random variables, taking values in the interval $]0,1[$, with joint law P. A realization of ω is called an *environment*. A random walk $(S_k)_{k\in\mathbb{N}}$ in the environment ω is defined as follows. Conditionally on ω, $(S_k)_{k\in\mathbb{N}}$ is a Markov chain starting at $S_0 = 0$ and such that for every $k \in \mathbb{N} := \{0, 1, 2, \dots\}$, $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$,

$$P_\omega\big(S_{k+1} = y | S_k = x\big) = \begin{cases} \omega_x & \text{if } y = x+1, \\ 1-\omega_x & \text{if } y = x-1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We call P_ω the *quenched law*, and $S := (S_k)_k$ is a *random walk in random environment (RWRE)*. The *annealed law* is defined as follows:

$$\mathbb{P}[\cdot] := \int P_\omega[\cdot]\mathsf{P}(\mathrm{d}\omega).$$

Notice that $\mathbb{P}$ is not Markovian. The expectations with respect to $\mathbb{P}$, P_ω and P are denoted respectively by $\mathbb{E}$, E_ω and E.

One dimensional RWRE have many unusual properties, and have attracted much interest from mathematicians and physicists. For applications in physics and in biology, see e.g. Cocco et al. [17], Hughes [47] and more recently the introduction of Padash et al. [55]. Also, (one dimensional) RWRE are used to define or study some other mathematical models, see e.g. Kochler [49] (chapter 3) for random walks in oriented lattices with random environments, Zindy [66] for random walks in random environments with random scenery. Aurzada et al. [6] for branching processes in random environments, and Devulder [22] for branching random walks in random environments. We refer to Révész [56] and Zeitouni [65] for a general account on results on RWRE proved before 2005. For a statistical point of view, see e.g. Diel et al. [26] and references therein.

We assume that there exists $\varepsilon_0 \in]0, 1/2[$ such that

$$\mathsf{P}[\varepsilon_0 \leq \omega_0 \leq 1 - \varepsilon_0] = 1. \tag{2}$$

This classical condition is known as the *ellipticity* condition. We introduce $\rho_x := \frac{1-\omega_x}{\omega_x}$, $x \in \mathbb{Z}$. Solomon [62] proved that $(S_k)_k$ is recurrent for almost every environment ω if

$$\mathsf{E}[\log \rho_0] = 0, \tag{3}$$

and transient for almost every ω otherwise. Throughout the paper, log denotes the natural logarithm. We only consider the recurrent case (3) in the present paper. Also, in order to avoid the degenerate case of simple random walks, we assume that

$$\sigma := \big(\mathsf{E}\big[(\log \rho_0)^2\big]\big)^{1/2} > 0. \tag{4}$$

The asymptotic behaviour of S in the very delicate recurrent case was first analyzed in a celebrated paper of Sinai [61]. Indeed, Sinai [61] showed that under Hypotheses (2), (3) and (4), S_n is localized at time n, with large annealed probability, in the neighborhood of some random quantity $b'_{\log n}$, which depends only on the environment. More precisely, he proved that for every $\varepsilon > 0$,

$$\mathbb{P}\big[|S_n - b'_{\log n}| \leq \varepsilon (\log n)^2\big] \to_{n\to+\infty} 1.$$

He also proved that $\sigma^2 b'_{\log n}/(\log n)^2$ converges in law, as $n \to +\infty$, to some random variable b_∞, which is non degenerate and non gaussian. As a consequence, Sinai obtained the following convergence in law under the annealed law $\mathbb{P}$:

$$\frac{\sigma^2}{(\log n)^2} S_n \to_{n\to+\infty} b_\infty.$$

It was proved independently by Kesten [48] and Golosov [40] that $\mathsf{P}[b_\infty \in \mathrm{d}x] = \varphi_\infty(x)\mathrm{d}x$, where

$$\varphi_\infty(x) := \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(- \frac{(2k+1)^2\pi^2}{8} |x| \right), \qquad x \in \mathbb{R}. \tag{5}$$

This very slow movement of $(S_k)_{k\in\mathbb{N}}$, of order $(\log n)^2$ instead of $\sqrt{n}$ for simple random walks, is due to the presence of some traps which slow down the walk. Due to this result proved by Sinai, a random walk in random environment $(S_k)_{k\in\mathbb{N}}$ satisfying Hypotheses (2), (3) and (4) is often called a *Sinai walk*. Some other unusual properties of Sinai's walk are proved e.g. in Dembo et al. [20], Gantert et al. [35], [37], Hu et al. [45], [46] and Shi [58]. See also Shi [59] for a general account about Sinai's walk before 2001.

1.2. **Main results.** Throughout the paper, for sequences (d_n) and (m_n) with $m_n \neq 0$ for large n, we write $d_n \sim_{n\to+\infty} m_n$ if $d_n/m_n \to 1$ as $n \to +\infty$, $d_n = o(m_n)$ if $d_n/m_n \to 0$ as $n \to +\infty$, and $d_n = O(m_n)$ if $\limsup_{n\to+\infty} |d_n/m_n| < \infty$.

Our main result is the following local limit theorem for Sinai's walk $(S_n)_{n\in\mathbb{N}}$ under the annealed law $\mathbb{P}$:

Theorem 1.1. *Assume* (2), (3) *and* (4). *As* $n \to +\infty$,

$$\sup_{z\in(2\mathbb{Z}+n)} \left| \mathbb{P}\big(S_n = z\big) - \frac{2\sigma^2}{(\log n)^2} \varphi_\infty\left(\frac{\sigma^2 z}{(\log n)^2} \right) \right| = o\left(\frac{1}{(\log n)^2} \right),$$

where $2\mathbb{Z} + n$ *denotes the set of integers having the same parity as* n.

Notice that $S := (S_k)_{k\in\mathbb{N}}$ only makes ± 1 jumps and starts from 0 under $\mathbb{P}$, so $\mathbb{P}(S_n = z) = 0$ if n and z have different parity. Since $\varphi_\infty > 0$ and is continuous on $\mathbb{R}$, we get in particular:

Corollary 1.2. *Assume* (2), (3) *and* (4). *Let* $(z_n)_{n\in\mathbb{N}}$ *be a sequence of integers such that* $z_n = O\big((\log n)^2\big)$ *as* $n \to +\infty$, *and such that* z_n *and* n *have the same parity for every* $n \in \mathbb{N}$. *Then,*

$$\mathbb{P}\big(S_n = z_n\big) \sim_{n\to+\infty} \frac{2\sigma^2}{(\log n)^2}\,\varphi_\infty\bigg(\frac{\sigma^2 z_n}{(\log n)^2}\bigg).$$

Also $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \arctan(1) = \pi/4$, hence $\varphi_\infty(0) = 1/2$, so this leads to:

Corollary 1.3. *Assume* (2), (3) *and* (4). *We have,*

$$\mathbb{P}\big(S_{2n} = 0\big) \sim_{n\to+\infty} \frac{\sigma^2}{(\log n)^2},$$

and more generally $\mathbb{P}(S_{2n} = 2x) \sim_{n\to+\infty} \frac{\sigma^2}{(\log n)^2}$ *for every fixed* $x \in \mathbb{Z}$ *since* φ_∞ *is continuous on* $\mathbb{R}$. *Also, for every fixed* $x \in \mathbb{Z}$,

$$\mathbb{P}\big(S_{2n} = 2\big\lfloor (x/2)(\log n)^2\big\rfloor\big) \sim_{n\to+\infty} \frac{2\sigma^2\varphi_\infty(\sigma^2 x)}{(\log n)^2},$$

where for $y \in \mathbb{R}$, $\lfloor y\rfloor$ *denotes the integer part of* y.

In order to prove Theorem 1.1, we introduce in Section 2 (see (19)) a random quantity b_h, $h > 0$, depending only on the environment. It is defined differently from the localization point b'_h introduced by Sinai, but plays a similar role. Our b_h is defined in terms of left h-extrema, which are also introduced in Section 2 (see Definition 2.1). In order to prove our Theorem 1.1, we first prove a local limit theorem for b_h:

Theorem 1.4. *We have as* $h \to +\infty$,

$$\sup_{x\in\mathbb{Z}}\bigg|\mathsf{P}\big(b_h = x\big) - \frac{\sigma^2}{h^2}\varphi_\infty\bigg(\frac{\sigma^2 x}{h^2}\bigg)\bigg| = o\bigg(\frac{1}{h^2}\bigg).$$

Even though Theorem 1.4 looks, at first sight, very similar to Theorem 1.1, Theorem 1.1 is not a direct consequence of Theorem 1.4, because, loosely speaking, the event $\{S_n = z\}$ can be decomposed into a union of events $\{S_n = z\} \cap \{b_{\log n} = y\}$, and we will see that each one has a non-negligible probability for y "close" to z. Also, estimating the annealed probabilities of these events for y close to z, as well as proving that such probabilities are negligible for y "far" from z, is not immediate, since we have to decompose each of these events into many different cases, corresponding to different kinds of environments and trajectories.

The probability $\mathbb{P}(S_n = z_n)$ for Sinai's walk seems to have been first studied in a physics paper in 1985 by Nauenberg [53], by heuristic arguments in some particular cases and numerical simulations. However the function he obtained instead of our φ_∞ is $x \mapsto (C/2)\exp(-C|x|)$ for some $C > 0$, which is not correct. This function was also claimed in Nauenberg [53] to be the density of the limit law of $\frac{\sigma^2}{(\log n)^2}S_n$, and Kesten [48] already noticed that this is not the correct function, although $\varphi_\infty(x)$ is equivalent to some exponential as $x \to +\infty$.

There have been many papers dealing with local limit theorems for different models of random walks in random environments recently. For example, Dolgopyat and Goldsheid [27], [28], Leskela and Stenlund [52] and Berger et al. [8] prove local limit theorems for transient RWRE respectively on $\mathbb{Z}$ and on a strip, both in the diffusive regime, on $\mathbb{Z}$ with only 0 or 1 jumps, and for some ballistic multidimensional RWRE. See also Dolgopyat et al. [29] for diffusive recurrent RWRE on a strip, Takenami [63] for random walks on periodic environments, Chiarini et al. [16]

for some diffusions in random environment, and Andres et al. [5] for the random conductance model. We refer to the first two sections of Dolgopyat et al. [28] for a recent review of this subject. However, the previously cited papers consider transient or diffusive random walks or diffusions, whereas we consider Sinai's walk which is recurrent and subdiffusive. Also, we obtain probabilities of order $(\log n)^{-2}$ with a non gaussian limit law, instead of $n^{-1/2}$ with a gaussian limit law in their cases. Therefore, to the extent of our knowledge, our Theorem 1.1 is the first local limit theorem for (recurrent) subdiffusive RWRE.

Also, a similar local limit theorem for the quenched probability, replacing $\mathbb{P}$ by P_ω, does not hold. Indeed, $P_\omega(S_n = 0)$ almost surely takes very small values compared to $(\log n)^{-2}$ as $n \to +\infty$, since for $\eta \in]0,1[$, P-almost surely $P_\omega(S_n = 0) = O\big(\exp(-(\log n)^{1-\eta})\big)$ as $n \to +\infty$ (see Devulder et al. [24], last inequality of page 6). See also Gantert et al. ([34], Theorem 1.1) for previous results, Comets et al. ([19], Theorem 2.1 and Corollary 2.1) for estimates for a related model in continuous time, and Gantert et al. [36] for transient RWRE. So, contrarily to some of the previously cited papers on local limit theorems for RWRE, our annealed local limit theorem, Theorem 1.1, cannot be the consequence of a corresponding quenched local limit theorem.

We also mention that some estimates of $\mathbb{P}(S_n = z_n)$ when z_n is large, more precisely when $n = O(z_n)$, are given by Comets et al. [18]. For an overview of the vast literature about large deviations for RWRE, see e.g. Gantert et al. [38] and more recently Buraczewski et al. [13].

Finally, we think that the tools and technics developed in the present paper, in particular the ones of Section 2, will be useful for future research projects, including [23], which will study the rates of convergence in Sinai and Golosov localization theorems for Sinai's walk.

Acknowledgement: I am thankful to Yueyun Hu for asking, after a talk in a conference in Landela (France) in 2016, if I could give an estimate of $\mathbb{P}(S_{2n} = 0)$ as $n \to +\infty$, which made me aware that this question was still open. I also thank Françoise Pène for organizing this conference. Part of this work was done during a six months sabbatical "délégation CNRS".

1.3. **Organization of the proof and of the paper.** In Section 2, we recall the definition and use of the potential V. We also define left and right h-extrema for V, for $h > 0$. This allows us to introduce two path decompositions of the potential V, one with left h-extrema and one with right h-extrema. We can then define our localization point b_h. We describe the law of the potential V between two consecutive left (or right) h-extrema x_i and x_{i+1} when $0 \notin [x_i, x_{i+1}]$, which uses in particular the law of V or $-V$ conditioned to stay positive, or nonnegative, up to some hitting time (see Theorem 2.3). The law of V between the two left h-extrema surrounding 0 is given by a renewal theorem (see Theorem 2.5), and some independence is provided by Theorem 2.4. A first application of this renewal theorem is that we can give a simple formula for the law of b_h, that is, for $\mathsf{P}(b_h = x)$, $x \in \mathbb{Z}$ (in Lemma 2.6), which is an important tool in the proof of Theorem 1.4.

Section 3 is devoted to the proof of Theorem 1.4.

In Section 4, we first define an event $E_C^{(n)}(z)$, depending only on the environment and on z. On this event, we use a coupling argument, which helps us approximate the quenched probability $P_\omega(S_n = z)$ by $\widehat{\nu}_n(z)$, where $\widehat{\nu}_n$ is an invariant probability measure. This enables us to give an upper bound for the annealed probability that $S_n = z$ on $E_C^{(n)}(z)$ (see Proposition 4.8), giving the main contribution in the upper bound of Theorem 1.1. To this aim, loosely speaking, we express the expectation of $\widehat{\nu}_n(z)$ on each event $\{b_{\log n} = k + z\} \cap E_C^{(n)}(z)$ with quantities depending only on the laws of the potential V between consecutive left or right $(\log n)$-extrema; summing this over k makes appear, after some inequalities and computations using the tools

developed in Section 2, a formula equal to $\mathsf{P}(b_{\log n} = z)$ by Lemma 2.6. We conclude by applying Theorem 1.4.

In Section 5, we prove that the environments and trajectories such that $S_n = z$ which were not considered in Section 4 have a negligible annealed probability. This covers many different cases, which often combine conditions on both environments and trajectories of $(S_k)_k$. For example, z can be far from $b_{\log n}$, or the origin 0 can be very close to the maximum of the potential between two valleys (defined before (20)), or some of the valleys around the origin can have a height just slightly larger than $\log n$, or the central valley of height at least $\log n$ can include one or several subvalleys of height slightly less than $\log n$. The potentials for some of these cases are represented in Figures 6 page 43, 7 page 50, 8 page 59 and 9 page 61. In this section, we prove that all these cases, and some others, with $S_n = z$ have a negligible annealed probability (compared to $(\log n)^{-2}$). Combining this with the previous subsection, we get (uniformly on z) an upper bound of $\mathbb{P}(S_n = z)$, which completes the proof of the upper bound in Theorem 1.1. Even if this section mainly consider negligible events, it is maybe the most delicate of the paper.

Section 6 is devoted to the proof of the lower bound in Theorem 1.1, that is, we give (uniformly on z) a minoration of $\mathbb{P}(S_n = z)$. The proof is divided into three cases, depending on z being negative and far from 0, positive and far from 0, or z being close to 0. This uses results of all the other sections.

Finally, Section 7 is devoted to some important technical lemmas and their proofs. These lemmas mainly deal with the potential V, and with V conditioned to stay positive or nonnegative.

Outlines or sketches of proofs of several lemmas or theorems are also provided throughout the paper.

2. Potential, path decomposition and renewal theorem

2.1. **Definition and applications of the potential.** The *potential* $(V(x),\ x \in \mathbb{Z})$, which was first introduced by Sinai [61], is an important quantity which depends only on the environment ω. It is defined as follows:

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \frac{1-\omega_i}{\omega_i} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x+1}^{0} \log \frac{1-\omega_i}{\omega_i} & \text{if } x < 0. \end{cases} \tag{6}$$

We denote by P_ω^x the quenched probability for the RWRE $(S_k)_k$ starting at $x \in \mathbb{Z}$ instead of 0, and by E_ω^x the expectation with respect to P_ω^x. Also, let

$$\tau(y) := \inf\{k \geq 0 \ :\ S_k = y\}, \qquad \tau^*(y) := \inf\{k \geq 1 \ :\ S_k = y\}, \qquad\qquad y \in \mathbb{Z},$$

where by convention, $\inf \emptyset = +\infty$. In words, $\tau(y)$ (resp. $\tau^*(y)$) is the hitting time of (resp. return time to) the site y by the RWRE $(S_k)_k$. We also define for $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$,

$$\tau(x, y) := \inf\{k \in \mathbb{N} \ :\ S_{\tau(x)+k} = y\}.$$

We now recall some classical estimates, which explain why the potential is very useful. These formulas will be used throughout the paper. First, we have (see e.g. [65, (2.1.4)],

$$P_\omega^b[\tau(c) < \tau(a)] = \left(\sum_{j=a}^{b-1} e^{V(j)}\right)\left(\sum_{j=a}^{c-1} e^{V(j)}\right)^{-1}, \qquad a < b < c. \tag{7}$$

Furthermore (see e.g. [21] Lem. 2.2 coming from Zeitouni [65] p. 250), if $g < h < i$,

$$\mathsf{E}_\omega^h[\tau(g) \wedge \tau(i)] \leq \sum_{k=h}^{i-1} \sum_{\ell=g}^{k} \frac{\exp[V(k) - V(\ell)]}{\omega_\ell} \leq \varepsilon_0^{-1}(i-g)^2 \exp\left[\max_{g \leq \ell \leq k \leq i-1, k \geq h} (V(k) - V(\ell))\right], \tag{8}$$

where we used ellipticity (2) in the last inequality and with $x \wedge y := \min(x,y)$. For symmetry reasons, we also have

$$E^b_\omega[\tau(a) \wedge \tau(c)] \le \varepsilon_0^{-1}(c-a)^2 \exp\Big[\max_{a\le \ell\le k\le c-1,\ \ell\le b-1} \big(V(\ell)-V(k)\big)\Big], \qquad a<b<c\,. \tag{9}$$

Moreover, we have (see Golosov [39], Lemma 7, proved for a RWRE on $\mathbb{N}$ but still true for a RWRE on $\mathbb{Z}$),

$$P^b_\omega[\tau(c)<k] \le k \exp\left(\min_{\ell\in[b,c-1]} V(\ell) - V(c-1)\right), \qquad b<c\,. \tag{10}$$

Also by symmetry, we get (similarly as in Shi and Zindy [60], eq. (2.5) but with some slight differences for the values of ℓ)

$$P^b_\omega[\tau(a)<k] \;\le\; k \exp\left(\min_{\ell\in[a,b-1]} V(\ell) - V(a)\right), \qquad a<b\,. \tag{11}$$

Moreover, we have by Devulder et al. ([24], Lemma 4.10), if $a \ne b$,

$$\forall k\in\mathbb{N}, \qquad P^b_\omega[\tau(a)=k] \le P^b_\omega[\tau(a)<\tau^*(b)]. \tag{12}$$

Finally, we recall that, given ω, the Markov chain S is an electrical network where, for every $x\in\mathbb{Z}$, the conductance of the unoriented bond $(x,x+1)$ is $C_{(x,x+1)} = e^{-V(x)}$ (in the sense of Doyle and Snell [30]) (see also Levin et al. [51]). In particular, its reversible measure μ_ω (unique up to a multiplication by a constant) is given by

$$\mu_\omega(x) := e^{-V(x)} + e^{-V(x-1)}, \qquad z\in\mathbb{Z}, \tag{13}$$

where, for the sake of simplicity, we write $\mu_\omega(x)$ instead of $\mu_\omega(\{x\})$. For any process Y, we define

$$T_Y(A) \;:=\; \inf\{x\ge 0,\ Y(x)\in A\}, \qquad A\subset\mathbb{R}, \tag{14}$$

$$T^*_Y(A) \;:=\; \inf\{x> 0,\ Y(x)\in A\}, \qquad A\subset\mathbb{R}. \tag{15}$$

We sometimes write $T_Y(a) := T_Y([a,+\infty[)$ when $a>0$ and $T_Y(a) := T_Y(]-\infty,a])$ when $a<0$. Due to the ellipticity (2), we have

$$\forall x\in\mathbb{Z}, \qquad \big|V(x)-V(x-1)\big| \le \log\left(\frac{1-\varepsilon_0}{\varepsilon_0}\right) =: C_0. \tag{16}$$

In particular, thanks to (3) and (16), the following fact follows from the optimal stopping theorem applied to the martingale $(V(k),\ k\ge 0)$ at time $T_V([z,+\infty[) \wedge T_V(]-\infty,x])$:

$$\frac{y-x}{z-x+C_0} \le \mathsf{P}^y\big[T_V([z,+\infty[) < T_V(]-\infty,x])\big] \le \frac{y-x+C_0}{z-x+C_0}, \qquad x<y<z, \tag{17}$$

where P^y denotes the law of V starting from y instead of 0. Moreover, these inequalities remain valid if we replace $]-\infty,x]$ and/or $[z,+\infty[$ by the corresponding open interval $]-\infty,x[$ and/or $]z,+\infty[$. Also, there exist constants $c_1>0$ and $c_1^*>0$ such that (see e.g. Lemma 7.4),

$$\mathsf{P}[T_V(h) < T^*_V(\mathbb{R}_-)] \sim_{h\to+\infty} c_1^* h^{-1}, \quad \mathsf{P}[T_V(h) < T_V(\mathbb{R}^*_-)] \sim_{h\to+\infty} c_1 h^{-1}. \tag{18}$$

2.2. **Definition and properties of left and right h-extrema.** The point of view of h-extrema has been used recently in some papers for RWRE or diffusions in a random potential, either to prove localization results, see e.g. [3], [11], [24] and [33], or to use localization techniques, see e.g. [4], [15], [21] and [19] (where they are called e^h-stable points).

However, these studies use h-extrema of a (maybe drifted) two-sided Brownian motion W, and sometimes transfer results about W to the potential V by Komlòs, Major and Tusnády strong approximation theorem [50]. This is not precise enough to prove our theorems, so we introduce and study variants of h-extrema directly for our potential V.

Let $h > 0$, and v be a function from $\mathbb{Z}$ to $\mathbb{R}$. Following Neveu and Pitman [54], we say that y is an *h-minimum* for v if there exist integers $\alpha < y < \beta$ such that $v(y) = \min_{[\alpha,\beta]} v$, $v(\alpha) \geq v(y)+h$ and $v(\beta) \geq v(y) + h$. We say that y is an *h-maximum* for v if it is an h-minimum for $-v$. In both cases, we say that y is an *h-extremum* for v.

One of the main differences with h-extrema of Brownian motion is that unfortunately, in the general case, h-maxima and h-minima for V do not necessarily alternate. For this reason, we introduce the following definitions (see Figure 1).

Definition 2.1. *Let $h > 0$ and v be a function from $\mathbb{Z}$ to $\mathbb{R}$. We say that $y \in \mathbb{Z}$ is a left h-minimum (resp. right h-minimum) for v if there exist $\alpha < y < \beta$ such that*

- $\min_{[\alpha,y-1]} v > v(y)$ *(resp $\geq$),*
- $\min_{[y+1,\beta]} v \geq v(y)$ *(resp. $>$),*
- $v(\alpha) \geq v(y) + h$,
- $v(\beta) \geq v(y) + h$.

We say that y is a left h-maximum (resp. right h-maximum) for v if it is a left h-minimum (resp. right h-minimum) for $-v$. In both cases, we say that y is a left h-extremum (resp. right h-extremum) for v.

With these definitions, left h-minima and left h-maxima for v alternate, and similarly right h-minima and right h-maxima for v alternate. The elementary proof is given in Lemma 7.7. Also, between two consecutive left h-maxima y_1 and y_2, more precisely in $[y_1, y_2[\cap\mathbb{Z}$, there are one or several h-minima, among which the smallest one is the only left h-minimum, which is y_1, and the largest one is the only right h-minimum, which we will not use in the present paper.

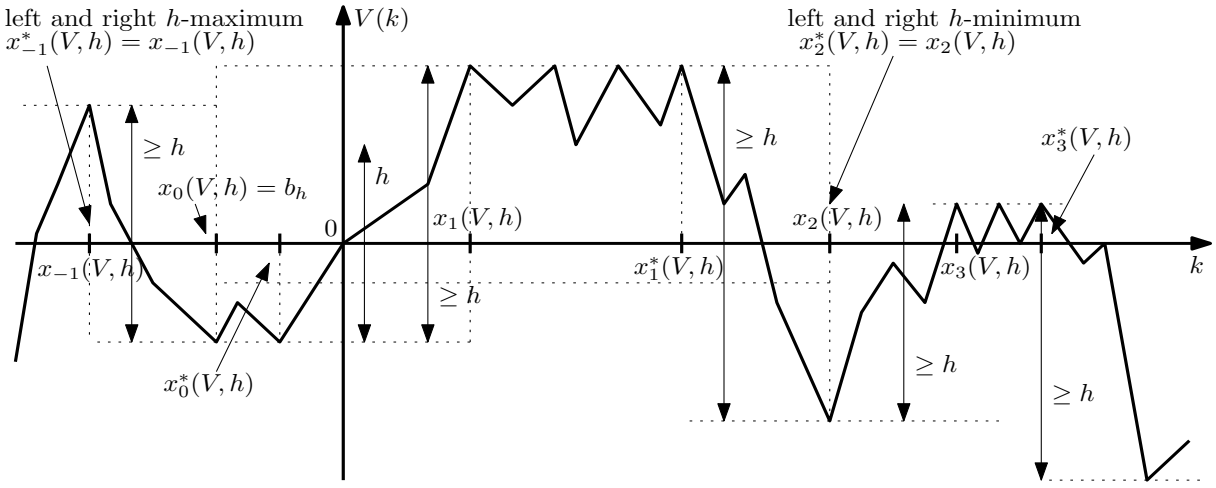

Figure 1. Schema of the potential V with left h-extrema $x_i(V,h)$ (defined before (19)) and right h-extrema $x^*_i(V,h)$ (defined before (46)).

Left and right h-extrema of V have the disadvantage of not being stopping times. However, we will see that they allow a very simple definition of the localization point b_h (see (19) below, which can be compared e.g. to (54)), that they have nice independence properties, that the properties of the law of trajectories of V between consecutive left or right h-extrema are convenient, and that we can use renewal theory, which enables for example to prove very useful formulas such as the law of b_h (see Lemma 2.6).

We now focus on left h-extrema. Let $\mathscr{V}$ be the set of functions v from $\mathbb{Z}$ to $\mathbb{R}$, such that $\liminf_{\pm\infty} v = -\infty$ and $\limsup_{\pm\infty} v = +\infty$. If $v \in \mathscr{V}$ and $h > 0$, then the set of left h-minima of v is unbounded from above and below, and so is the set of left h-maxima of v. Consequently, for $v \in \mathscr{V}$ for every $h > 0$, the set of left h-extrema of v can be denoted by $\{x_k(v,h),\ k \in \mathbb{Z}\}$, such that $k \mapsto x_k(v,h)$ is strictly increasing and $x_0(v,h) \le 0 < x_1(v,h)$. And also, $\lim_{k\to\pm\infty} x_k(v,h) = \pm\infty$. Notice that due to our hypotheses (3) and (4), $V \in \mathscr{V}$ almost surely.

Similarly as in the continuous case (see Cheliotis [14]), we can now define for $h > 0$,

$$b_h := \begin{cases} x_0(V,h) & \text{if } x_0(V,h) \text{ is a left } h\text{-minimum for } V, \\ x_1(V,h) & \text{otherwise.} \end{cases} \tag{19}$$

As already mentioned, the definition of the localization point b'_h given by Sinai [61] is not the same.

Similarly as in the continuous case for h-slopes, we introduce for each function $v \in \mathscr{V}$ and for each $i \in \mathbb{Z}$ and $h > 0$ the *left h-slope* $T_i(v,h) := (v(j) - v[x_i(v,h)],\ x_i(v,h) \le j \le x_{i+1}(v,h))$. Its *height* and its *excess height* are defined respectively as

$$H[T_i(v,h)] := \big|v[x_{i+1}(v,h)] - v[x_i(v,h)]\big| \ge h, \qquad e[T_i(v,h)] = H[T_i(v,h)] - h \ge 0.$$

If $x_i(v,h)$ is a left h-minimum (resp. maximum), then $T_i(v,h)$ is a nonnegative (resp. nonpositive) function, it is said to be an *upward slope* (resp. a *downward slope*) and its maximum (resp. minimum) is attained at $x_{i+1}(v,h)$, with $\sup_{[x_i(v,h),x_{i+1}(v,h)[} v < v[x_{i+1}(v,h)]$ (resp. $\inf_{[x_i(v,h),x_{i+1}(v,h)[} v > v[x_{i+1}(v,h)]$).

Similarly, if y_i and y_{i+1} are two consecutive right h-extrema of v, we say that $(v(j) - v(y_i),\ y_i \le j \le y_{i+1})$ is a *right h-slope* of v (see Subsection 2.6 for some properties of right h-slopes and extrema). More generally, we call a *slope* each $\mathrm{T}=(T(j),\ \alpha \le j \le \beta) \in \mathbb{R}^{\beta-\alpha+1}$, with $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}\cap]\alpha, +\infty[$, such that either $T(\alpha) = 0 = \min_{[\alpha,\beta]\cap\mathbb{Z}} T < \max_{[\alpha,\beta]\cap\mathbb{Z}} T = T(\beta)$ or $T(\beta) = \min_{[\alpha,\beta]\cap\mathbb{Z}} T < \max_{[\alpha,\beta]\cap\mathbb{Z}} T = T(\alpha) = 0$. Also, for each slope $T = (T(j),\ \alpha \le j \le \beta)$, we define its *length* $\ell(T) := \beta - \alpha$, its *height* $H(T) = |T(\beta) - T(\alpha)|$, and the *translated slope* $\theta(T) := (T(j+\alpha),\ 0 \le j \le \beta - \alpha)$.

We call *valleys of height at least h* of V the intervals $[x_i(V,h), x_{i+2}(V,h)]$, $i \in \mathbb{Z}$, such that $x_i(V,h)$ and $x_{i+2}(V,h)$ are (consecutive) left h-maxima. The *bottom* of such a valley is the left h-minimum $x_{i+1}(V,h)$. If its bottom is b_h, that is, if $b_h = x_{i+1}(V,h)$, then it is called the *central valley of height at least h* of V.

Knowing, for some $h > 0$, $\theta[T_i(V,h)]$ for each $i \in \mathbb{Z}^*$ and $(\theta[T_0(V,h)],\ x_0(V,h))$ allows us to reconstitute totally the process V since $V(0) = 0$. The two following subsections will provide their laws and independence properties.

2.3. **Definition and law of $\mathcal{T}^{\uparrow}_{V,h}$ and $\mathcal{T}^{\downarrow}_{V,h}$.** Let $h > 0$. We define by induction the following notation. Let $\tau_0^{(V)}(h) := 0$ and for $i \ge 0$ (see Figure 2),

$$\tau^{(V)}_{2i+1}(h) := \min\Big\{k \ge \tau^{(V)}_{2i}(h),\ V(k) - \min_{[\tau^{(V)}_{2i}(h),k]} V \ge h\Big\}, \tag{20}$$

$$m^{(V)}_{2i+1}(h) := \min\Big\{k \ge \tau^{(V)}_{2i}(h),\ V(k) = \min_{[\tau^{(V)}_{2i}(h),\tau^{(V)}_{2i+1}(h)]} V\Big\}, \tag{21}$$

$$\tau^{(V)}_{2i+2}(h) := \min\Big\{k \ge \tau^{(V)}_{2i+1}(h),\ \max_{[\tau^{(V)}_{2i+1}(h),k]} V - V(k) \ge h\Big\}, \tag{22}$$

$$m^{(V)}_{2i+2}(h) := \min\Big\{k \ge \tau^{(V)}_{2i+1}(h),\ V(k) = \max_{[\tau^{(V)}_{2i+1}(h),\tau^{(V)}_{2i+2}(h)]} V\Big\}. \tag{23}$$

Notice that that $\tau_i^{(V)}(h) < \infty$ P-a.s. for $i \geq 0$ since $V \in \mathscr{V}$ P-a.s. due to (3) and (4), and that the $\tau_i^{(V)}(h)$, $i \geq 0$, are stopping times for the natural filtration of $(V(\ell),\ \ell \geq 0)$.

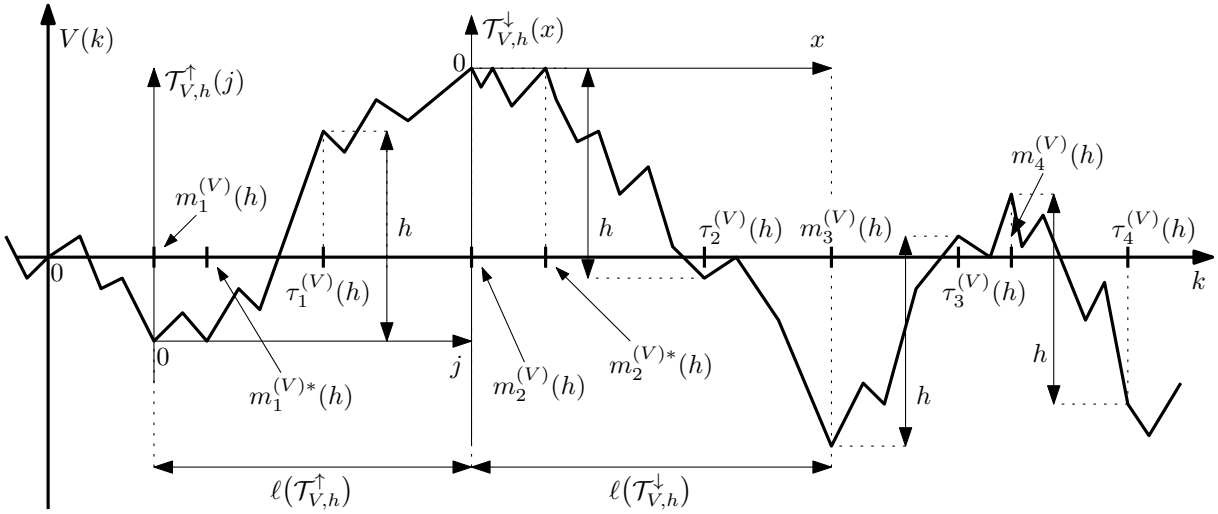

Figure 2. Schema of the potential V with the $\tau_i^{(V)}(h)$, $m_i^{(V)}(h)$, $\mathcal{T}^{\uparrow}_{V,h}$, $\mathcal{T}^{\downarrow}_{V,h}$ and $m_i^{(V)*}(h)$ (defined between (20) and (23), in Definition 2.2 and before (47)).

Let $\bigsqcup$ denote the disjoint union. Notice that, with a slight abuse of notation, each translated (left h-) slope $T = (T(0), T(1), \ldots, T(\ell(T)))$ belongs to $\mathbb{R}^{\ell(T)+1}$. So, we can consider our translated slopes (and $\mathcal{T}^{\uparrow}_{V,h}$ and $\mathcal{T}^{\downarrow}_{V,h}$ defined below) as random variables taking values into $\bigsqcup_{t\in\mathbb{N}^*}\mathbb{R}^t$, equipped with the σ-algebra $\{\bigsqcup_{t\in\mathbb{N}^*} A_t : \forall t \in \mathbb{N}^*,\ A_t \in \mathcal{B}(\mathbb{R}^t)\}$, where $\mathcal{B}(\mathbb{R}^t)$ is the Borel σ-algebra of $\mathbb{R}^t$. The following notations are useful to express the law of left h-slopes in the next subsection:

Definition 2.2. *Let $h > 0$. We introduce (see Figure 2),*

$$\begin{aligned}\mathcal{T}^{\uparrow}_{V,h} &:= \big(V\big[m_1^{(V)}(h)+x\big] - V\big[m_1^{(V)}(h)\big],\ 0 \leq x \leq m_2^{(V)}(h) - m_1^{(V)}(h)\big),\\ \mathcal{T}^{\downarrow}_{V,h} &:= \big(V\big[m_2^{(V)}(h)+x\big] - V\big[m_2^{(V)}(h)\big],\ 0 \leq x \leq m_3^{(V)}(h) - m_2^{(V)}(h)\big).\end{aligned}$$

In particular, $\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big) = m_2^{(V)}(h) - m_1^{(V)}(h)$ and $\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big) = m_3^{(V)}(h) - m_2^{(V)}(h)$. We sometimes write $\mathcal{T}^{\uparrow}_{V}$ and $\mathcal{T}^{\downarrow}_{V}$ instead of $\mathcal{T}^{\uparrow}_{V,h}$ and $\mathcal{T}^{\downarrow}_{V,h}$ to simplify the notation when no confusion is possible for the value of h. The laws of $\mathcal{T}^{\uparrow}_{V,h}$ and $\mathcal{T}^{\downarrow}_{V,h}$ are given in the following theorem.

Theorem 2.3. *Assume* (2), (3) *and* (4). *Let $h > 0$.*
(i) *The process $\mathcal{T}^{\uparrow}_{V,h}$ up to its first hitting time $T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[)$ of $[h,+\infty[$, that is, $\big(\mathcal{T}^{\uparrow}_{V,h}(k),\ 0 \leq k \leq T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[)\big)$, is equal in law to $\big(V(k),\ 0 \leq k \leq T_V([h,+\infty[)\big)$ conditioned on $\{T_V([h,+\infty[) < T_V(]-\infty,0])\}$. Moreover, it is independent of $\big(\mathcal{T}^{\uparrow}_{V,h}\big(T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[) + k\big) - \mathcal{T}^{\uparrow}_{V,h}\big(T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[)\big),\ 0 \leq k \leq \ell\big(\mathcal{T}^{\uparrow}_{V,h}\big) - T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[)\big)$, which has the same law as $(V(k),\ 0 \leq k \leq M_h^{\sharp})$, with $M_h^{\sharp} := \min\{k \in \mathbb{N},\ V(k) = \max_{[0,\tilde{\tau}_1(h)]} V\}$, where $\tilde{\tau}_1(h) := \min\{k \in \mathbb{N},\ \max_{[0,k]} V - V(k) \geq h\}$.*

(ii) *$\mathcal{T}^{\uparrow}_{-V,h} =_{law} -\mathcal{T}^{\downarrow}_{V,h}$ and $\mathcal{T}^{\downarrow}_{-V,h} =_{law} -\mathcal{T}^{\uparrow}_{V,h}$.*

(iii) *Also,* $\mathsf{E}\big(\ell(\mathcal{T}^{\uparrow}_{V,h})\big) < \infty$ *and* $\mathsf{E}\big(\ell(\mathcal{T}^{\downarrow}_{V,h})\big) < \infty$.

Before proving Theorem 2.3, we introduce some notation. For a slope $(T(i),\ 0 \le i \le \ell(T))$ (recall that $T(0) = 0$), we define the slope

$$\zeta(T) := \big(T[\ell(T) - i] - T[\ell(T)],\ 0 \le i \le \ell(T)\big), \tag{24}$$

with $\zeta \circ \zeta$ being identity (since $T(0) = 0$ when T is a slope).

Proof of Theorem 2.3: Let $h > 0$. Applying ([25], Proposition 5.2, **(ii)**), $\big(V\big[m_1^{(V)}(h) + x\big] - V\big[m_1^{(V)}(h)\big],\ 0 \le x \le \tau_1^{(V)}(h) - m_1^{(V)}(h)\big)$, is equal in law to $\big(V(k),\ 0 \le k \le T_V([h, +\infty[)\big)$ conditioned on $\{T_V([h, +\infty[) < T_V(]-\infty, 0[\}$, which proves the first part of **(i)**. The second one follows from the strong Markov property applied to $(V(k),\ k \ge 0)$ at stopping time $\tau_1^{(V)}(h)$, which is equal to $m_1^{(V)}(h) + T_{\mathcal{T}^{\uparrow}_{V,h}}([h, +\infty[)$.

We now prove some more general results, which will also be useful later. Due to Lemma 7.8, the $m_{2i+1}^{(V)}(h)$, $i \ge 1$, are left h-minima, the $m_{2i+2}^{(V)}(h)$, $i \ge 0$, are left h-maxima, and the $m_i^{(V)}(h)$, $i \ge 2$, are the only left h-extrema in $\big[\tau_1^{(V)}(h), +\infty\big[$. However, $m_1^{(V)}(h)$ is not necessarily a left h-minimum, depending on the values taken by $(V(k),\ k \le 0)$.

For $k \ge 1$, let $\widehat{\theta}^{(\ell)}_{k,h}(V) := \widehat{\theta}^{(\ell)}_{k,h} := \big(V\big(m_k^{(V)}(h) - x\big) - V\big(m_k^{(V)}(h)\big), 0 \le x \le m_k^{(V)}(h) - \tau_{k-1}^{(V)}(h)\big)$ and $\widehat{\theta}^{(r)}_{k,h}(V) := \widehat{\theta}^{(r)}_{k,h} := \big(V\big(m_k^{(V)}(h) + x\big) - V\big(m_k^{(V)}(h)\big),\ 0 \le x \le \tau_k^{(V)}(h) - m_k^{(V)}(h)\big)$. According to ([25], Proposition 5.2, **(i)**), the processes $\widehat{\theta}^{(\ell)}_{1,h}(V)$ and $\widehat{\theta}^{(r)}_{1,h}(V)$ are independent. Also, $\widehat{\theta}^{(\ell)}_{2,h}(V) = -\widehat{\theta}^{(\ell)}_{1,h}\big(V\big(\tau_1^{(V)}(h)\big) - V\big(\tau_1^{(V)}(h) + .\big)\big)$ and $\widehat{\theta}^{(r)}_{2,h}(V) = -\widehat{\theta}^{(r)}_{1,h}\big(V\big(\tau_1^{(V)}(h)\big) - V\big(\tau_1^{(V)}(h) + .\big)\big)$, so it follows from the previous result and from the strong Markov property applied at stopping time $\tau_1^{(V)}(h)$ that $\widehat{\theta}^{(\ell)}_{2,h}(V)$ and $\widehat{\theta}^{(r)}_{2,h}(V)$ are independent and more precisely that all the trajectories $\widehat{\theta}^{(\ell)}_{k,h}(V)$ and $\widehat{\theta}^{(r)}_{k,h}(V)$, $k \in \{1, 2\}$, are independent. Applying the same procedure by induction, with the strong Markov property applied successively at stopping times $\tau_k^{(V)}(h)$, $k \ge 1$, proves that all the trajectories $\widehat{\theta}^{(\ell)}_{k,h}$ and $\widehat{\theta}^{(r)}_{k,h}$, $k \ge 1$, are independent.

In what follows we will "glue" trajectories. For two trajectories $(f(i),\ a \le i \le b)$ and $(g(i),\ c \le i \le d)$, by gluing g to the right of f, we mean defining a new function $j : \{a, \dots, b + d - c\} \to \mathbb{R}$ such that

$$j(i) = \mathrm{Glue}(f, g)(i) := \begin{cases} f(i) & \text{if } a \le i \le b, \\ f(b) + g(i - b + c) - g(c) & \text{if } b \le i \le b + d - c. \end{cases} \tag{25}$$

Thanks to the previous paragraph, the trajectories

$$\widehat{\theta}^{(V)}_{k,h} := \big(V\big(x + m_k^{(V)}(h)\big) - V\big(m_k^{(V)}(h)\big),\ 0 \le x \le m_{k+1}^{(V)}(h) - m_k^{(V)}(h)\big), \qquad k \in \mathbb{N}^* \tag{26}$$

are independent, since the k-th one is obtained by gluing $\widehat{\theta}^{(r)}_{k,h}$ and, to its right, $\big(V\big(\tau_k^{(V)}(h) + x\big) - V\big(\tau_k^{(V)}(h)\big),\ 0 \le x \le m_{k+1}^{(V)}(h) - \tau_k^{(V)}(h)\big) = \zeta\big(\widehat{\theta}^{(\ell)}_{k+1,h}\big)$ (with ζ defined in (24)), that is, $\widehat{\theta}^{(V)}_{k,h} = \mathrm{Glue}\big[\widehat{\theta}^{(r)}_{k,h}, \zeta\big(\widehat{\theta}^{(\ell)}_{k+1,h}\big)\big]$.

Also by the strong Markov property applied at stopping time $\tau_{2k}^{(V)}(h)$, $\widehat{\theta}^{(\ell)}_{2k+i,h}(V) =_{law} \widehat{\theta}^{(\ell)}_{i,h}(V)$ and $\widehat{\theta}^{(r)}_{2k+i,h}(V) =_{law} \widehat{\theta}^{(r)}_{i,h}(V)$ for every $k \ge 1$ and $i \in \{1, 2\}$. Consequently, using the previous paragraph, $\widehat{\theta}^{(V)}_{2k+1,h} = \mathrm{Glue}\big[\widehat{\theta}^{(r)}_{2k+1,h}, \zeta\big(\widehat{\theta}^{(\ell)}_{2k+2,h}\big)\big] =_{law} \mathrm{Glue}\big[\widehat{\theta}^{(r)}_{1,h}, \zeta\big(\widehat{\theta}^{(\ell)}_{2,h}\big)\big] = \widehat{\theta}^{(V)}_{1,h} = \mathcal{T}^{\uparrow}_{V,h}$ and $\widehat{\theta}^{(V)}_{2k+2,h} = \mathrm{Glue}\big[\widehat{\theta}^{(r)}_{2k+2,h}, \zeta\big(\widehat{\theta}^{(\ell)}_{2k+3,h}\big)\big] =_{law} \mathrm{Glue}\big[\widehat{\theta}^{(r)}_{2,h}, \zeta\big(\widehat{\theta}^{(\ell)}_{3,h}\big)\big] = \widehat{\theta}^{(V)}_{2,h} = \mathcal{T}^{\downarrow}_{V,h}$ for every $k \in \mathbb{N}$.

Finally, by the strong Markov property applied at time $\tau_1^{(-V)}(h)$, $\big(\widehat{\theta}_{2,h}^{(\ell)}(-V), \widehat{\theta}_{2,h}^{(r)}(-V)\big)$ is equal in law to $\big(-\widehat{\theta}_{1,h}^{(\ell)}(V), -\widehat{\theta}_{1,h}^{(r)}(V)\big)$. Similarly, $\big(\widehat{\theta}_{3,h}^{(\ell)}(-V), \widehat{\theta}_{3,h}^{(r)}(-V)\big) =_{law} \big(-\widehat{\theta}_{2,h}^{(\ell)}(V), -\widehat{\theta}_{2,h}^{(r)}(V)\big)$. As a consequence, $\mathcal{T}_{-V,h}^{\downarrow} = \mathrm{Glue}\big[\widehat{\theta}_{2,h}^{(r)}(-V), \zeta\big(\widehat{\theta}_{3,h}^{(\ell)}(-V)\big)\big] =_{law} \mathrm{Glue}\big[-\widehat{\theta}_{1,h}^{(r)}(V), \zeta\big(-\widehat{\theta}_{2,h}^{(\ell)}(V)\big)\big]$ $= -\mathrm{Glue}\big[\widehat{\theta}_{1,h}^{(r)}(V), \zeta\big(\widehat{\theta}_{2,h}^{(\ell)}(V)\big)\big] = -\mathcal{T}_{V,h}^{\uparrow}$. Also, applying this to $-V$ instead of V gives $\mathcal{T}_{-V,h}^{\uparrow} =_{law} -\mathcal{T}_{V,h}^{\downarrow}$, which ends the proof of **(ii)**.

We now prove **(iii)**. Due to (3) and (4), there exist $a > 0$ such that $\mathsf{P}[V(1) \geq a] =: b > 0$. Let $d := \lfloor h/a \rfloor + 1$. Now, notice that $\tau_1^{(V)}(h) \leq d(N_d + 1)$, where $N_d := \min\{i \in \mathbb{N},\ \forall 0 \leq k \leq d,\ V(id+k) - V(id) \geq ak\}$. Hence, $\mathsf{E}\big(\tau_1^{(V)}(h)\big) \leq d(\mathsf{E}(N_d) + 1) < \infty$ since N_d is a geometric r.v. with parameter $\mathsf{P}[\forall 0 \leq k \leq d,\ V(k) \geq ak] \geq b^d > 0$. Using the strong Markov property, we get similarly $\mathsf{E}\big(\tau_2^{(V)}(h) - \tau_1^{(V)}(h)\big) < \infty$. Consequently, $\mathsf{E}\big(\ell\big(\mathcal{T}_{V,h}^{\uparrow}\big)\big) = \mathsf{E}\big(m_2^{(V)}(h) - m_1^{(V)}(h)\big) \leq \mathsf{E}\big(\tau_2^{(V)}(h)\big) < \infty$. Finally, applying this to $-V$, we get $\mathsf{E}\big[\ell\big(\mathcal{T}_{V,h}^{\downarrow}\big)\big] = \mathsf{E}\big[\ell\big(\mathcal{T}_{-V,h}^{\uparrow}\big)\big] < \infty$, since $\mathcal{T}_{-V,h}^{\uparrow} =_{law} -\mathcal{T}_{V,h}^{\downarrow}$ by **(ii)**. This proves **(iii)**. □

2.4. **Independence and law of translated left h-slopes via renewal theory.** Notice that the law of V may be nonsymmetric, so $\mathcal{T}_{V,h}^{\uparrow}$ and $-\mathcal{T}_{V,h}^{\downarrow} =_{law} \mathcal{T}_{-V,h}^{\uparrow}$ may have a different law, contrarily to what happens for Brownian motion (imagine for example that the jumps of V belong to $[-2,-1] \cup [4,5]$).

The following theorem is proved simultaneously as the next one. It says that for $h > 0$, roughly speaking, conditionally on the central left h-slope $T_0(V,h)$ being upward (or being downward), the translated left h-slopes $\theta[T_i(V,h)]$, $i \in \mathbb{Z}^*$, are independent and are independent of the (non translated) central left h-slope $T_0(V,h)$, and that the translated left h-slopes $\theta[T_i(V,h)]$, $i \in \mathbb{Z}^*$, have the same law as $\mathcal{T}_{V,h}^{\uparrow}$ (under P) for the upward ones (ie the ones with $i \in (2\mathbb{Z}) - \{0\}$ when $T_0(V,h)$ is upward, the ones for $i \in (2\mathbb{Z}+1)$ when $T_0(V,h)$ is downward) and the same law as $\mathcal{T}_{V,h}^{\downarrow}$ (under P) for the downward ones (the other ones).

We denote by $\mathscr{L}\big(\mathcal{T}_{V,h}^{\uparrow}\big)$ $\big($resp. $\mathscr{L}\big(\mathcal{T}_{V,h}^{\downarrow}\big)\big)$ the law of $\mathcal{T}_{V,h}^{\uparrow}$ $\big($resp. $\mathcal{T}_{V,h}^{\downarrow}\big)$ under P.

Theorem 2.4. *Let $h > 0$.* **(i)** *Conditionally on $\big\{V(x_1(V,h)) > V(x_0(V,h))\big\}$ (i.e. on the central left h-slope $T_0(V,h)$ being upward), the $\theta[T_{2i+1}(V,h)]$, $i \in \mathbb{Z}$ have the law $\mathscr{L}\big(\mathcal{T}_{V,h}^{\downarrow}\big)$ whereas the $\theta[T_{2i}(V,h)]$, $i \in \mathbb{Z}^*$ have the law $\mathscr{L}\big(\mathcal{T}_{V,h}^{\uparrow}\big)$, and $(\theta[T_0(V,h)], x_0(V,h), x_1(V,h))$, $\theta[T_i(V,h)]$, $i \in \mathbb{Z}^*$ are independent.*

(ii) *Conditionally on $\big\{V(x_1(V,h)) < V(x_0(V,h))\big\}$ (i.e. on the central left h-slope $T_0(V,h)$ being downward), the $\theta[T_{2i+1}(V,h)]$, $i \in \mathbb{Z}$ have the law $\mathscr{L}\big(\mathcal{T}_{V,h}^{\uparrow}\big)$, whereas the $\theta[T_{2i}(V,h)]$, $i \in \mathbb{Z}^*$ have the law $\mathscr{L}\big(\mathcal{T}_{V,h}^{\downarrow}\big)$, and $(\theta[T_0(V,h)], x_0(V,h), x_1(V,h))$, $\theta[T_i(V,h)]$, $i \in \mathbb{Z}^*$ are independent.*

However the law of the central left h-slope $T_0(V,h)$ is different. It is provided by the following renewal theorem.

Theorem 2.5. *Let $h > 0$, $\Delta_0 \subset \mathbb{Z}$ and $\Delta_1 \subset \mathbb{Z}$. For $A \in \{\bigsqcup_{t \in \mathbb{N}^*} A_t : \forall t \in \mathbb{N}^*,\ A_t \in \mathcal{B}(\mathbb{R}_+^t)\}$ (so that the only slopes in A are upward slopes), we have*

$$\begin{aligned} &\mathsf{P}[\theta(T_0(V,h)) \in A,\, x_0(V,h) \in \Delta_0,\, x_1(V,h) \in \Delta_1] \\ = \ &\frac{\mathsf{E}\big[\sharp\big\{0 \leq i < \ell(\mathcal{T}_{V,h}^{\uparrow}),\, (-i) \in \Delta_0,\, (\ell(\mathcal{T}_{V,h}^{\uparrow}) - i) \in \Delta_1\big\} \mathbf{1}_{\{\mathcal{T}_{V,h}^{\uparrow} \in A\}}\big]}{\mathsf{E}[\ell(\mathcal{T}_{V,h}^{\uparrow}) + \ell(\mathcal{T}_{V,h}^{\downarrow})]}. \end{aligned} \tag{27}$$

Moreover if $A \in \{\bigsqcup_{t\in\mathbb{N}^*} A_t \,:\, \forall t \in \mathbb{N}^*,\ A_t \in \mathcal{B}(\mathbb{R}_-^t)\}$ *(so that the only slopes in* A *are downward slopes), then*

$$\begin{aligned} & \mathsf{P}[\theta(T_0(V,h)) \in A,\, x_0(V,h) \in \Delta_0,\, x_1(V,h) \in \Delta_1] \\ = \ & \frac{\mathsf{E}\big[\sharp\big\{0 \le i < \ell(\mathcal{T}^{\downarrow}_{V,h}),\, (-i) \in \Delta_0,\, \big(\ell(\mathcal{T}^{\downarrow}_{V,h}) - i\big) \in \Delta_1\big\}\mathbf{1}_{\{\mathcal{T}^{\downarrow}_{V,h}\in A\}}\big]}{\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})\big]}. \end{aligned} \tag{28}$$

Finally, for all nonnegative function, $\varphi : \bigsqcup_{t\in\mathbb{N}^*} \mathbb{R}^t \to [0, +\infty[$*, measurable with respect to the* σ*-algebra* $\{\bigsqcup_{t\in\mathbb{N}^*} A_t \,:\, \forall t \in \mathbb{N}^*,\ A_t \in \mathcal{B}(\mathbb{R}^t)\}$*,*

$$\begin{aligned} & \mathsf{E}\Big[\varphi\big[\theta(T_0(V,h))\big]\mathbf{1}_{\{x_0(V,h)\in\Delta_0\}}\mathbf{1}_{\{x_1(V,h)\in\Delta_1\}}\Big] \\ = \ & \frac{\mathsf{E}\big[\sharp\big\{0 \le i < \ell(\mathcal{T}^{\uparrow}_{V,h}),\, (-i) \in \Delta_0,\, \big(\ell(\mathcal{T}^{\uparrow}_{V,h}) - i\big) \in \Delta_1\big\}\varphi(\mathcal{T}^{\uparrow}_{V,h})\big]}{\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})]} \\ & + \frac{\mathsf{E}\big[\sharp\big\{0 \le i < \ell(\mathcal{T}^{\downarrow}_{V,h}),\, (-i) \in \Delta_0,\, \big(\ell(\mathcal{T}^{\downarrow}_{V,h}) - i\big) \in \Delta_1\big\}\varphi(\mathcal{T}^{\downarrow}_{V,h})\big]}{\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})\big]}. \end{aligned} \tag{29}$$

Proof of Theorems 2.4 and 2.5: Let $h > 0$, $\Delta_0 \subset \mathbb{Z}$, $\Delta_1 \subset \mathbb{Z}$, $q \le 0 \le r$, and $B_i \in \{\bigsqcup_{t\in\mathbb{N}^*} A_t \,:\, \forall t \in \mathbb{N}^*,\ A_t \in \mathcal{B}(\mathbb{R}^t)\} =: \mathcal{G}$, for $q \le i \le r$. We first assume that $B_0 \in \{\bigsqcup_{t\in\mathbb{N}^*} A_t \,:\, \forall t \in \mathbb{N}^*,\ A_t \in \mathcal{B}(\mathbb{R}_+^t)\}$, so that B_0 contains only upward slopes.

For $t \in \mathbb{Z}$ and $v \in \mathscr{V}$, let, loosely speaking, $T_0(t, v, h)$ be the left h-slope around t for v, that is, the left h-slope whose of v domain contains t, and denote its domain as $[x_0(t, v, h), x_1(t, v, h)]$. More precisely and more generally, for $j \in \mathbb{Z}$, we define $T_j(t, v, h) = T_{i+j}(v, h)$ if and only if $x_i(v, h) \le t < x_{i+1}(v, h)$, and for this unique i, $x_j(t, v, h) := x_{i+j}(v, h)$ for $j \in \mathbb{Z}$ (recall that the notations x_{i+j}, T_{i+j} are defined before and after (19)). We also introduce $V_{-t}(k) := V(k - t) - V(-t)$ for $t \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Hence, for $t \in \mathbb{N}$,

$$\mathsf{P}\bigg(\big\{x_0(V,h) \in \Delta_0,\, x_1(V,h) \in \Delta_1\big\} \cap \bigcap_{i=q}^{r} \big\{\theta[T_i(V,h)] \in B_i\big\}\bigg) \tag{30}$$

$$\begin{aligned} = \ & \mathsf{P}\bigg(\big\{(x_0(t,V_{-t},h) - t) \in \Delta_0,\, (x_1(t,V_{-t},h) - t) \in \Delta_1\big\} \cap \bigcap_{i=q}^{r} \big\{\theta[T_i(t,V_{-t},h)] \in B_i\big\}\bigg) \\ = \ & \mathsf{P}[E_B(t)] \\ = \ & \mathsf{P}\big[E_B(t),\, m^{(V)}_{-q+3}(h) \le t\big] + \mathsf{P}\big[E_B(t),\, m^{(V)}_{-q+3}(h) > t\big], \end{aligned} \tag{31}$$

where $E_B(t) := \{(x_0(t, V, h)-t) \in \Delta_0,\, (x_1(t, V, h)-t) \in \Delta_1\} \cap \cap_{i=q}^{r} \big\{\theta[T_i(t, V, h)] \in B_i\big\}$, because $x_j(V, h) = x_j(t, V_{-t}, h) - t$ for $j \in \mathbb{Z}$, $\theta[T_i(V, h)] = \theta[T_i(t, V_{-t}, h)]$ for $i \in \mathbb{Z}$, and V_{-t} has the same law as V.

Let $(Y_k)_{k\in\mathbb{Z}}$ be a sequence of independent left h-slopes, such that $Y_{2k} =_{law} \mathcal{T}^{\uparrow}_{V,h}$ and $Y_{2k+1} =_{law} \mathcal{T}^{\downarrow}_{V,h}$ for every $k \in \mathbb{Z}$. We glue sequentially (see (25)) $Y_0, Y_1, \ldots, Y_k, \ldots$ to get a process $(Y(i),\ i \in \mathbb{N})$, starting from 0 (i.e. $Y(i) = Y_0(i)$ for $0 \le i \le \ell(Y_0)$). This process $(Y(i),\ i \in \mathbb{N})$ has the same law as $\big(V\big[m^{(V)}_1(h) + x\big] - V\big[m^{(V)}_1(h)\big],\ x \ge 0\big)$. Indeed, this last process can be obtained from gluing $\widehat{\theta}^{(V)}_{1,h}, \widehat{\theta}^{(V)}_{2,h}, \ldots, \widehat{\theta}^{(V)}_{k,h}, \ldots$ (see (26)), which are independent and such that $\widehat{\theta}^{(V)}_{k,h} =_{law} Y_{k-1}$, $k \in \mathbb{N}^*$ by definition of the Y_k and the law of the $\widehat{\theta}^{(V)}_{k,h}$ (see after (26)). We also glue sequentially the Y_k, $k < 0$ in the same way to the left of $(Y(i),\ i \in \mathbb{N})$, so that Y_k is followed by Y_{k+1}, $k \in \mathbb{Z}$. The resulting process is denoted by $(Y(i),\ i \in \mathbb{Z})$, with $Y(0) = 0$. Notice that

$x_0(Y,h)=0$, for $i\in\mathbb{N}^*$. We also have $x_i(Y,h)=\ell(Y_0)+\cdots+\ell(Y_{i-1})$, and for $i\in\mathbb{Z}_-^*$, we have $x_i(Y,h)=-\ell(Y_{-1})-\cdots-\ell(Y_i)$.

We can assume without loss of generality that $q\in(-2\mathbb{N}^*)$, so $T_q(V,h)$ is an upward slope when $\theta(T_0(V,h))\in B_0$. Using, in the second equality, the fact that $\big(V\big(x+m_3^{(V)}(h)\big)-V\big(m_3^{(V)}(h)\big),\ x\geq 0\big)$ has the same law as $(Y(x),\ x\geq 0)$ (see (26) and below), and is independent of $\big(V(x),\ x\leq m_3^{(V)}(h)\big)$ (see the paragraph before (25)), we have for $t\in\mathbb{N}$,

$$\begin{aligned}\mathsf{P}\big[E_B(t),\, m_{-q+3}^{(V)}(h)\leq t\big] &= \sum_{y=0}^{t}\mathsf{P}\big[E_B(t),\, m_3^{(V)}(h)=y,\, m_{-q+3}^{(V)}(h)\leq t\big]\\ &= \sum_{y=0}^{t}\mathsf{P}\big[m_3^{(V)}(h)=y\big]h_B(t-y)\end{aligned}\tag{32}$$

where $h_B(p):=\mathsf{P}\big[\widetilde{E}_B(p)\cap\{\ell(Y_0)+\cdots+\ell(Y_{-q-1})\leq p\}\big]$, with $\widetilde{E}_B(p):=\big\{\big(x_0(p,Y,h)-p\big)\in\Delta_0,\ \big(x_1(p,Y,h)-p\big)\in\Delta_1\big\}\cap\cap_{i=q}^r\big\{\theta\big[T_i(p,Y,h)\big]\in B_i\big\}$, $p\in\mathbb{N}$. Indeed, on $\big\{m_{-q+3}^{(V)}(h)\leq t\big\}$, we have $x_0(t,V,h)\geq m_{-q+3}^{(V)}(h)$ thanks to Lemma 7.8, thus $x_q(t,V,h)\geq m_3^{(V)}(h)$, so $E_B(t)$ depends only on $\big(V\big(x+m_3^{(V)}(h)\big)-V\big(m_3^{(V)}(h)\big),\ x\geq 0\big)=:(Y'(x),\ x\geq 0)$, with $x_i(t,V,h)=x_i(t-y,Y',h)+y$ and $\theta[T_i(t,V,h)]=\theta[T_i(t-y,Y',h)]$ for $i\geq q$ on $\big\{m_3^{(V)}(h)=y\big\}$ and $Y'=_{law}Y$.

We want to prove that $h_B(p)$ has a limit as $p\to+\infty$. For $p\in\mathbb{N}$, let

$$\begin{aligned}a'_p &:= \mathsf{P}\Big[\big\{(\ell(Y_0)+\cdots+\ell(Y_{-q-1})-p)\in\Delta_0,\ \big(\ell(Y_0)+\cdots+\ell(Y_{-q})-p\big)\in\Delta_1\big\}\\ &\qquad\cap\big\{0\leq p-\ell(Y_0)-\cdots-\ell(Y_{-q-1})<\ell(Y_{-q})\big\}\cap\bigcap_{i=q}^{r}\{Y_{i-q}\in B_i\}\Big].\end{aligned}$$

We have for $p\in\mathbb{N}$, since $q\in(-2\mathbb{N}^*)$,

$$\begin{aligned}&h_B(p)\\ =\ &\mathsf{P}\big[\widetilde{E}_B(p),\,\ell(Y_0)+\cdots+\ell(Y_{-q-1})\leq p<\ell(Y_0)+\cdots+\ell(Y_{-q-1})+\ell(Y_{-q})\big] &(33)\\ &+\mathsf{P}\big[\widetilde{E}_B(p),\,\ell(Y_0)+\cdots+\ell(Y_{-q})\leq p<\ell(Y_0)+\cdots+\ell(Y_{-q})+\ell(Y_{-q+1})\big] &(34)\\ &+\sum_{y=0}^{p}\mathsf{P}\big[\ell(Y_0)+\ell(Y_1)=y,\widetilde{E}_B(p),\,\ell(Y_2)+\cdots+\ell(Y_{-q+1})\leq p-y\big] &(35)\\ =\ &a'_p+0+\sum_{y=0}^{p}\mathsf{P}\big[\ell(Y_0)+\ell(Y_1)=y\big]\mathsf{P}\big[\widetilde{E}_B(p-y),\,\ell(Y_0)+\cdots+\ell(Y_{-q-1})\leq p-y\big]\\ =\ &a'_p+\sum_{y=0}^{p}\mathsf{P}\big[\ell(Y_0)+\ell(Y_1)=y\big]h_B(p-y).\end{aligned}$$

Indeed in the probability in line (33), the image by θ of the slope $T_0(p,Y,h)$ containing p is Y_{-q} and the 0 in the second equality comes from the fact that on the set inside the probability of line (34), $\theta\big[T_0(p,Y,h)\big]=Y_{-q+1}$ is a downward slope, whereas B_0 contains only upward slopes, and in the sets appearing in (35), there exists $j\geq 0$ such that $x_i(p-y,Y,h)\big]=\ell(Y_2)+\cdots+\ell(Y_{-q+1+j+i})$ and $\theta\big[T_i(p-y,Y,h)\big]=Y_{-q+2+j+i}$ for $i\geq q$ and (Y_0,Y_1) is independent of $(Y_{i+2},\ i\geq 0)$, which has the same law as $(Y_i,\ i\geq 0)$.

So, $h_B(p)$ is solution of the discrete time renewal equation $h_p=a'_p+\sum_{k=0}^{p}f_kh_{p-k}$, $p\in\mathbb{N}$, with $h_p=h_B(p)$ and $f_k=\mathsf{P}\big[\ell(Y_0)+\ell(Y_1)=k\big]$. Notice that $a'_p\geq 0$, $p\in\mathbb{N}$ and $\sum_{p=0}^{\infty}a'_p\leq$

$\mathsf{E}\big[\ell(Y_0)+\cdots+\ell(Y_{-q})\big]+a_0 \le (|q|+1)\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)+\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]+1<\infty$ by our Theorem 2.3 **(iii)**. So, Theorem 2.2 of Barbu and Limnios [7] with its notation $X_n=\ell(Y_{2n-2})+\ell(Y_{2n-1})>0$, $n\ge 1$ so that $f_k=\mathsf{P}[X_1=k]$ and $u_n:=\sum_{m=0}^{n}\mathsf{P}[X_1+\cdots+X_m=n]=\sum_{m=0}^{n}\mathsf{P}[\ell(Y_0)+\ell(Y_1)+\cdots+\ell(Y_{2m-2})+\ell(Y_{2m-1})=n]$ with $X_1+\cdots+X_0=0$ by convention, give us that this renewal equation has a unique solution, which is

$$h_B(p)=h_p=(u*a')_p=\sum_{k=0}^{p}u_{p-k}a'_k,\qquad p\in\mathbb{N}.$$

Let $n_1\in\mathbb{N}^*$ and $n_2\in\mathbb{N}^*$ be such that $\mathsf{P}[T_V(h)=n_1 \mid T_V(h)<T_V(\mathbb{R}^*_-)]=:c_2>0$ and $\mathsf{P}[\ell(Y_1)=n_2]=:c_3>0$ and let $c_4:=\mathsf{P}[T_V(-h)<T_V(]0,\infty[)]>0$ due to (3) and (4). Hence, using the law of $\mathcal{T}^{\uparrow}_{V,h}$ (see Theorem 2.3 **(i)**), $\mathsf{P}[\ell(Y_0)=n_1]=\mathsf{P}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)=n_1\big]\ge c_2c_4>0$. Also, $\mathsf{P}[\ell(Y_0)=n_1+1]\ge c_2\mathsf{P}[V(1)>0]c_4>0$. Thus, $\mathsf{P}[\ell(Y_0)+\ell(Y_1)=n_1+n_2]>0$ and $\mathsf{P}[\ell(Y_0)+\ell(Y_1)=n_1+n_2+1]>0$, and then the renewal chain $(X_1+\cdots+X_n)_n$ is aperiodic. It is also recurrent since $X_1<\infty$ a.s., e.g. because $\mathsf{E}(X_1)=\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,,h}\big)+\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]<\infty$ by Theorem 2.3 **(iii)**. So by the renewal theorem (see e.g. Barbu and Limnios [7], Theorem 2.6), we have $u_p\to_{p\to+\infty}1/\mathsf{E}(X_1)=1/\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)+\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]$. Moreover since this renewal chain is recurrent and aperiodic and $\sum_{p=0}^{\infty}|a'_p|<\infty$, we have by the key renewal theorem (see e.g. Barbu and Limnios [7], Theorem 2.7),

$$h_B(p)=h_p=\sum_{k=0}^{p}u_{p-k}a'_k\to_{p\to+\infty}\frac{1}{\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h})+\ell(\mathcal{T}^{\downarrow}_{V,h})]}\sum_{p=0}^{\infty}a'_p. \tag{36}$$

Also, let $A_{k_0,\ldots,k_{r-q}}:=\big\{p\in\mathbb{N},\ (k_0+\cdots+k_{-q-1}-p)\in\Delta_0,\ \big(k_0+\cdots+k_{-q}-p\big)\in\Delta_1\big\}\cap\big\{0\le p-k_0-\cdots-k_{-q-1}<k_{-q}\big\}$ for $(k_0,\ldots,k_{r-q})\in\mathbb{N}^{r-q+1}$. We have,

$$\begin{aligned}
&\sum_{p=0}^{\infty}a'_p\\
=\;&\sum_{p=0}^{\infty}\sum_{(k_0,\ldots,k_{r-q})\in\mathbb{N}^{r-q+1}}\mathsf{P}\Big[\bigcap_{j=0}^{r-q}\big\{\ell(Y_j)=k_j\big\}\cap\{Y_j\in B_{j+q}\}\cap\{p\in A_{k_0,\ldots,k_{r-q}}\}\Big]\\
=\;&\sum_{(k_0,\ldots,k_{r-q})\in\mathbb{N}^{r-q+1}}\Big(\prod_{j=0}^{r-q}\mathsf{P}\Big[\big\{\ell(Y_j)=k_j\big\}\cap\{Y_j\in B_{j+q}\}\Big]\Big)\sum_{p=0}^{\infty}\mathbf{1}_{A_{k_0,\ldots,k_{r-q}}}(p)\\
=\;&\sum_{(k_0,\ldots,k_{r-q})\in\mathbb{N}^{r-q+1}}\Big(\prod_{0\le j\le r-q,\,j\ne -q}\mathsf{P}\Big[\big\{\ell(Y_j)=k_j\big\}\cap\{Y_j\in B_{j+q}\}\Big]\Big)\\
&\times\mathsf{E}\Big(\mathbf{1}_{\{\ell(Y_{-q})=k_{-q},\ Y_{-q}\in B_0\}}\sum_{p=0}^{\infty}\mathbf{1}_{A_{k_0,\ldots,k_{r-q}}}(p)\Big)\\
=\;&\sum_{(k_0,\ldots,k_{r-q})\in\mathbb{N}^{r-q+1}}\Big(\prod_{0\le j\le r-q,\,j\ne -q}\mathsf{P}\Big[\big\{\ell(Y_j)=k_j\big\}\cap\{Y_j\in B_{j+q}\}\Big]\Big)\\
&\times\mathsf{E}\Big(\mathbf{1}_{\{\ell(Y_{-q})=k_{-q},\ Y_{-q}\in B_0\}}\sharp\{0\le m<\ell(Y_{-q}),(-m)\in\Delta_0,(\ell(Y_{-q})-m)\in\Delta_1\}\Big)\\
=\;&\Big(\prod_{0\le j\le r-q,\,j\ne -q}\mathsf{P}\Big[Y_j\in B_{j+q}\Big]\Big)
\end{aligned}$$

$$\times \mathsf{E}\Big(\mathbf{1}_{\{Y_{-q}\in B_0\}}\sharp\{0\le m<\ell(Y_{-q}),(-m)\in\Delta_0,(\ell(Y_{-q})-m)\in\Delta_1\}\Big). \tag{37}$$

Now, notice that by definition of $(Y_k)_{k\in\mathbb{Z}}$ and since $q\in(-2\mathbb{N}^*)$, the product in (37) is equal to

$$\Big(\prod_{q\le i\le r,\, i\neq 0,\, i\in(2\mathbb{Z})}\mathsf{P}\Big[\mathcal{T}^{\uparrow}_{V,h}\in B_i\Big]\Big)\times\Big(\prod_{q\le i\le r,\, i\in(2\mathbb{Z}+1)}\mathsf{P}\Big[\mathcal{T}^{\downarrow}_{V,h}\in B_i\Big]\Big). \tag{38}$$

The second probability in (31) is less than $\mathsf{P}\big[m^{(V)}_{-q+3}(h)>t\big]$ and then it goes to 0 as $t\to+\infty$ since $m^{(V)}_{-q+3}(h)<\tau^{(V)}_{-q+3}(h)<\infty$ a.s. since $V\in\mathscr{V}$ a.s.

Combining this with (32), letting $t\to+\infty$ and applying the dominated convergence theorem gives $(30)=\lim_{p\to+\infty}h_B(p)$ (since this limit exists by (36)). This, together with (36), (37), and (38) leads to

$$\begin{aligned}(30)=&\Big(\prod_{q\le i\le r,\, i\neq 0,\, i\in(2\mathbb{Z})}\mathsf{P}\Big[\mathcal{T}^{\uparrow}_{V,h}\in B_i\Big]\Big)\times\Big(\prod_{q\le i\le r,\, i\in(2\mathbb{Z}+1)}\mathsf{P}\Big[\mathcal{T}^{\downarrow}_{V,h}\in B_i\Big]\Big)\\ &\times\mathsf{E}\Big(\frac{\mathbf{1}_{\{Y_{-q}\in B_0\}}}{\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h})+\ell(\mathcal{T}^{\downarrow}_{V,h})]}\sharp\{0\le m<\ell(Y_{-q}),(-m)\in\Delta_0,(\ell(Y_{-q})-m)\in\Delta_1\}\Big).\end{aligned} \tag{39}$$

Moreover, taking (only here) all the B_i equal to $\bigsqcup_{t\in\mathbb{N}^*}\mathbb{R}^t$, except B_0 in (39), we get

$$\begin{aligned}&\mathsf{P}\big(x_0(V,h)\in\Delta_0,\, x_1(V,h)\in\Delta_1,\,\theta[T_0(V,h)]\in B_0\big)\\ &=\frac{\mathsf{E}\Big(\mathbf{1}_{\{\mathcal{T}^{\uparrow}_{V,h}\in B_0\}}\sharp\{0\le m<\ell(\mathcal{T}^{\uparrow}_{V,h}),(-m)\in\Delta_0,(\ell(\mathcal{T}^{\uparrow}_{V,h})-m)\in\Delta_1\}\Big)}{\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h})+\ell(\mathcal{T}^{\downarrow}_{V,h})]},\end{aligned} \tag{40}$$

since Y_{-q} has the same law as $\mathcal{T}^{\uparrow}_{V,h}$ because $q\in(2\mathbb{Z})$. This proves (27). Consequently, (39) becomes

$$\begin{aligned}(30)=&\Big(\prod_{q\le i\le r,\, i\neq 0,\, i\in(2\mathbb{Z})}\mathsf{P}\Big[\mathcal{T}^{\uparrow}_{V,h}\in B_i\Big]\Big)\times\Big(\prod_{q\le i\le r,\, i\in(2\mathbb{Z}+1)}\mathsf{P}\Big[\mathcal{T}^{\downarrow}_{V,h}\in B_i\Big]\Big)\\ &\times\mathsf{P}\big(\big\{x_0(V,h)\in\Delta_0,\, x_1(V,h)\in\Delta_1\big\}\cap\big\{\theta[T_0(V,h)]\in B_0\big\}\big).\end{aligned} \tag{41}$$

This proves Theorem 2.4 **(i)**.

We now prove (28) and Theorem 2.4 **(ii)**. We assume that $B_0\in\{\bigsqcup_{t\in\mathbb{N}^*}A_t:\forall t\in\mathbb{N}^*,\ A_t\in\mathcal{B}(\mathbb{R}^t_-)\}$, so that B_0 contains only downward slopes. Notice that $x_i(-V,h)=x_i(V,h)$ and $\theta[T_i(-V,h)]=-\theta[T_i(V,h)]$ for $i\in\mathbb{Z}$. Then, $-B_0=\{-f,\ f\in B_0\}\in\{\bigsqcup_{t\in\mathbb{N}^*}A_t:\forall t\in\mathbb{N}^*,\ A_t\in\mathcal{B}(\mathbb{R}^t_+)\}$, and for each $q\le i\le r$, $\theta(T_i(V,h))\in B_i$ iff $\theta(T_i(-V,h))\in(-B_i)$, for which we can apply (41) and (40) as follows. We get,

$$\begin{aligned}&\mathsf{P}\Big(\big\{x_0(V,h)\in\Delta_0,\, x_1(V,h)\in\Delta_1\big\}\cap\bigcap_{i=q}^{r}\big\{\theta[T_i(V,h)]\in B_i\big\}\Big) &(42)\\ =\ &\mathsf{P}\Big(\big\{x_0(-V,h)\in\Delta_0,\, x_1(-V,h)\in\Delta_1\big\}\cap\bigcap_{i=q}^{r}\big\{\theta[T_i(-V,h)]\in(-B_i)\big\}\Big)\\ =\ &\Big(\prod_{q\le i\le r,\, i\neq 0,\, i\in(2\mathbb{Z})}\mathsf{P}\Big[\mathcal{T}^{\uparrow}_{-V,h}\in(-B_i)\Big]\Big)\times\Big(\prod_{q\le i\le r,\, i\in(2\mathbb{Z}+1)}\mathsf{P}\Big[\mathcal{T}^{\downarrow}_{-V,h}\in(-B_i)\Big]\Big)\\ &\times\frac{\mathsf{E}\Big(\mathbf{1}_{\{\mathcal{T}^{\uparrow}_{-V,h}\in(-B_0)\}}\sharp\{0\le m<\ell(\mathcal{T}^{\uparrow}_{-V,h}),(-m)\in\Delta_0,(\ell(\mathcal{T}^{\uparrow}_{-V,h})-m)\in\Delta_1\}\Big)}{\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{-V,h})+\ell(\mathcal{T}^{\downarrow}_{-V,h})]},\end{aligned}$$

$$= \Big(\prod_{q\le i\le r,\, i\neq 0,\, i\in(2\mathbb{Z})} \mathsf{P}\Big[\mathcal{T}^{\downarrow}_{V,h}\in B_i\Big]\Big)\times\Big(\prod_{q\le i\le r,\, i\in(2\mathbb{Z}+1)} \mathsf{P}\Big[\mathcal{T}^{\uparrow}_{V,h}\in B_i\Big]\Big)$$
$$\times\frac{\mathsf{E}\Big(\mathbf{1}_{\{\mathcal{T}^{\downarrow}_{V,h}\in B_0\}}\sharp\{0\le m<\ell(\mathcal{T}^{\downarrow}_{V,h}),(-m)\in\Delta_0,(\ell(\mathcal{T}^{\downarrow}_{V,h})-m)\in\Delta_1\}\Big)}{\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h})+\ell(\mathcal{T}^{\downarrow}_{V,h})]},$$

since $\mathcal{T}^{\uparrow}_{-V,h}=_{law}-\mathcal{T}^{\downarrow}_{V,h}$ and $\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{-V,h}\big)+\ell\big(\mathcal{T}^{\downarrow}_{-V,h}\big)\big]=\mathsf{E}\big[\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)+\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)\big]$ by Theorem 2.3 **(ii)**. Taking all the B_i, $i\neq 0$, equal to $\bigsqcup_{t\in\mathbb{N}^*}\mathbb{R}^t$, this proves (28). This, in turn, proves that

$$(42)\quad = \Big(\prod_{q\le i\le r,\, i\neq 0,\, i\in(2\mathbb{Z})} \mathsf{P}\Big[\mathcal{T}^{\downarrow}_{V,h}\in B_i\Big]\Big)\times\Big(\prod_{q\le i\le r,\, i\in(2\mathbb{Z}+1)} \mathsf{P}\Big[\mathcal{T}^{\uparrow}_{V,h}\in B_i\Big]\Big)$$
$$\times\mathsf{P}\big(x_0(V,h)\in\Delta_0,\, x_1(V,h)\in\Delta_1,\, \theta[T_0(V,h)]\in B_0\big),$$

which proves Theorem 2.4 **(ii)**.

In order to prove (29), we first show that (29) is true for $\varphi=\mathbf{1}_A$ for any $A\in\{\bigsqcup_{t\in\mathbb{N}^*}A_t\,:\,\forall t\in\mathbb{N}^*,\ A_t\in\mathcal{B}(\mathbb{R}^t)\}=\mathcal{G}$. To this aim, let $A\in\mathcal{G}$. We introduce $\mathcal{S}_{\pm}:=\bigsqcup_{t\in\mathbb{N}^*}\mathbb{R}^t_{\pm}$. Applying (27) to $A\cap\mathcal{S}_+$ (resp. (28) to $A\cap\mathcal{S}_-$) proves (29) for $\varphi=\mathbf{1}_{A\cap\mathcal{S}_+}$ (resp. $\varphi=\mathbf{1}_{A\cap\mathcal{S}_-}$), since the second (resp. first) expectation in (29) is 0 when for $\varphi=\mathbf{1}_{A\cap\mathcal{S}_+}$ (resp. $\varphi=\mathbf{1}_{A\cap\mathcal{S}_-}$), because $\mathcal{T}^{\downarrow}_{V,h}\notin\mathcal{S}_+$ (resp. $\mathcal{T}^{\uparrow}_{V,h}\notin\mathcal{S}_-$). Also, (29) is true for $\varphi=\mathbf{1}_{A\cap(\mathcal{S}_+\cup\mathcal{S}_-)^c}$ since every term is equal to 0 in (29) in this case, since when $\theta[T_0(V,h)]$ is a downward (resp. upward) slope, it belongs to $\mathcal{S}_-$ (resp. $\mathcal{S}_+$) and $\mathcal{T}^{\downarrow}_{V,h}\in\mathcal{S}_-$ (resp. $\mathcal{T}^{\uparrow}_{V,h}\in\mathcal{S}_+$). Hence, adding (29) in the three previous cases proves that (29) is true for $\varphi=\mathbf{1}_A$, for every $A\in\mathcal{G}$.

Then by linearity, (29) is true for every simple function $\sum_{i=1}^p\alpha_i\mathbf{1}_{B_i}$ for $p\ge 1$, $\alpha_i\ge 0$ and $B_i\in\mathcal{G}$, $1\le i\le p$. Finally, (29) is true for any nonnegative $\mathcal{G}$-measurable function by the monotone convergence theorem, since every nonnegative $\mathcal{G}$-measurable function is the pointwise limit of a nondecreasing sequence of nonnegative simple $\mathcal{G}$-measurable functions. □

2.5. **A simple expression for $\mathsf{P}(b_h=x)$.** A first application of our renewal Theorem 2.5 is the following lemma, which contains key formulas to prove Theorem 1.4 and study the main contribution in Theorem 1.1 (see e.g. (129), (214) and (220)).

Lemma 2.6. *For $h>0$,*

$$\forall x\ge 0,\qquad \mathsf{P}(b_h=x) = \frac{\mathsf{P}\big[\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\ge x\big]}{\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)+\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]},\tag{43}$$

$$\forall x\le 0,\qquad \mathsf{P}(b_h=x) = \frac{\mathsf{P}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)>-x\big]}{\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)+\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]}.\tag{44}$$

Proof: Let $h>0$ and $x\in\mathbb{Z}$. If $x>0$, applying Theorem 2.5 eq. (28), with $A=\bigsqcup_{i=1}^{\infty}\mathbb{R}^i_-$, $\Delta_1=\{x\}$ and $\Delta_0=-\mathbb{N}$,

$$\mathsf{P}(b_h=x) = \mathsf{P}[b_h=x,\ x_1(V,h)=b_h>0]=\mathsf{P}\Big[x_1(V,h)=x,\ \theta(T_0(V,h))\in\bigsqcup_{i=1}^{\infty}\mathbb{R}^i_-\Big]$$
$$= \frac{\mathsf{E}\big(\sharp\big\{0\le i<\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big),\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)-i=x\big\}\big)}{\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)+\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]}=\frac{\mathsf{P}\big[\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\ge x\big]}{\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)+\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]}.$$

Similarly if $x \leq 0$, applying Theorem 2.5 eq. (27) with $A = \bigsqcup_{i=1}^{\infty} \mathbb{R}_+^i$, $\Delta_0 = \{x\}$ and $\Delta_1 = \mathbb{N}^*$,

$$\mathsf{P}(b_h = x) = \frac{\mathsf{E}\big(\sharp\big\{0 \leq i < \ell(\mathcal{T}^{\uparrow}_{V,h}), i = -x\big\}\big)}{\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})\big]} = \frac{\mathsf{P}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) > -x\big]}{\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})\big]}.$$

In particular,

$$\mathsf{P}(b_h = 0) = \frac{\mathsf{P}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) > 0\big]}{\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})\big]} = \frac{1}{\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})\big]} = \frac{\mathsf{P}\big[\ell(\mathcal{T}^{\downarrow}_{V,h}) \geq 0\big]}{\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})\big]}, \tag{45}$$

so both formulas of Lemma 2.6 are true for $x = 0$. □

2.6. **About right h-extrema and right h-slopes.** We have detailed in the previous subsections, for $h > 0$, a path decomposition of the potential V, which we cut into different trajectories, called left h-slopes, between random times which are the left h-extrema. We have also given the laws and independence properties of these left h-slopes, in particular in Theorems 2.3, 2.4 and 2.5.

We now focus on right h-extrema and provide a similar path decomposition of V with right h-slopes and right h-extrema. Similarly as for left h-minima, for $v \in \mathscr{V}$, for every $h > 0$, the set of right h-extrema of v can be denoted by $\{x_k^*(v,h),\ k \in \mathbb{Z}\}$, such that $k \mapsto x_k^*(v,h)$ is strictly increasing and $x_0^*(v,h) < 0 \leq x_1^*(v,h)$ (see Figure (1)), the first inequality being strict and second one being large, contrarily to inequalities for left h-extrema $x_i(v,h)$, $i \in \mathbb{Z}$, in order to get relation (46) below. Also, we prove below that the right h-extrema of v can be obtained from the left h-extrema of $v^-(.) := v_-(.) := v(-.)$ (and in particular, $V^-(.) := V_-(.) := V(-.)$; both notations V^- and V_- will be used throughout the paper, depending on which one is more convenient). More precisely, we have:

Lemma 2.7. *Let $v \in \mathscr{V}$. For $h > 0$,*

$$\forall i \in \mathbb{Z}, \qquad x_i^*(v,h) = -x_{1-i}(v^-,h). \tag{46}$$

Proof: Let $v \in \mathscr{V}$ and $h > 0$. First, notice that, applying Definition 2.1, $-x_j(v^-,h)$ is a right h-extremum for v for each $j \in \mathbb{Z}$, so $\{-x_j(v^-,h),\ j \in \mathbb{Z}\} \subset \{x_i^*(v,h),\ i \in \mathbb{Z}\}$. Similarly, for $i \in \mathbb{Z}$, $-x_i^*(v,h)$ is a left h-extremum for v^-, so $\{x_i^*(v,h),\ i \in \mathbb{Z}\} \subset \{-x_j(v^-,h),\ j \in \mathbb{Z}\}$, thus these two sets are equal. Moreover, $(x_i^*(v,h))_{i\in\mathbb{Z}}$ and $(-x_{-j}(v^-,h))_{j\in\mathbb{Z}}$ are two strictly increasing sequences, taking the same values, so there exists $k \in \mathbb{Z}$ such that $x_i^*(v,h) = -x_{k-i}(v^-,h)$ for every $i \in \mathbb{Z}$. Since $x_0^*(v,h) < 0 \leq x_1^*(v,h)$ and $-x_1(v^-,h) < 0 \leq -x_0(v^-,h)$, we have $k = 1$, which proves the lemma. □

Let $h > 0$. Similarly as for left h-extrema, for $v \in \mathscr{V}$, we introduce for each $i \in \mathbb{Z}$ the right h-*slope* $T_i^*(v,h) := (v(j) - v[x_i^*(V,h)],\ x_i^*(v,h) \leq j \leq x_{i+1}^*(v,h))$. If $x_i^*(v,h)$ is a right h-minimum (resp. maximum), then $\theta[T_i^*(v,h)]$ is strictly positive (resp. strictly negative) on $\{1,\dots,\ell(T_i^*(v,h))\}$. and its maximum (resp. minimum) is attained at $\ell(T_i^*(v,h))$. The notation with a star for x_i^* and T_i^* corresponds to this fact that the translated slopes $\theta[T_i^*(v,h)]$ are non-zero except at the origin.

Using the previous definition of $\tau_i^{(V)}(h)$ (see around (20)), we define for $i \geq 0$ (see Figure 2, in which $m_3^{(V)*}(h) = m_3^{(V)}(h)$ and $m_4^{(V)*}(h) = m_4^{(V)}(h)$),

$$\begin{aligned} m_{2i+1}^{(V)*}(h) &:= \max\Big\{k \in \big[\tau_{2i}^{(V)}(h), \tau_{2i+1}^{(V)}(h)\big] \cap \mathbb{N},\ V(k) = \min_{[\tau_{2i}^{(V)}(h), \tau_{2i+1}^{(V)}(h)]} V\Big\}, \\ m_{2i+2}^{(V)*}(h) &:= \max\Big\{k \in \big[\tau_{2i+1}^{(V)}(h), \tau_{2i+2}^{(V)}(h)\big] \cap \mathbb{N},\ V(k) = \max_{[\tau_{2i+1}^{(V)}(h), \tau_{2i+2}^{(V)}(h)]} V\Big\}. \end{aligned}$$

Also, similarly as in Definition 2.2, we introduce for $h>0$,

$$\mathcal{T}^{\uparrow *}_{V,h} := \big(V\big[m_1^{(V)*}(h)+x\big]-V\big[m_1^{(V)*}(h)\big],\ 0\le x\le m_2^{(V)*}(h)-m_1^{(V)*}(h)\big), \tag{47}$$

$$\mathcal{T}^{\downarrow *}_{V,h} := \big(V\big[m_2^{(V)*}(h)+x\big]-V\big[m_2^{(V)*}(h)\big],\ 0\le x\le m_3^{(V)*}(h)-m_2^{(V)*}(h)\big). \tag{48}$$

Recall T_V and T_V^* from (14) and (15). The following proposition is similar to ([25], Proposition 5.2) with $m_1^{(V)*}(h)$ instead of $m_1^{(V)}(h)$. The other main difference is that in **(ii)**, we condition by $\{T_V([h,+\infty[)<T_V^*(]-\infty,0])\}$, closed at 0, instead of $\{T_V([h,+\infty[)<T_V(]-\infty,0[)\}$. Since we did not find this lemma in the literature (in which our stopping time $\tau_1^{(V)}(h)$ is generally replaced by a deterministic time, see e.g. [9]), we give a detailed proof.

Proposition 2.8. *Let $h>0$. Let V be a random walk given as in (6) by a sequence of partial sums of i.i.d. r.v. $\log\rho_i$, $i\in\mathbb{Z}$, such that $\mathsf{P}[\log\rho_0>0]>0$ and $\mathsf{P}[\log\rho_0<0]>0$ (this result does not require Hypotheses (2), (3) or (4)). If moreover $\liminf_{x\to+\infty}V(x)=-\infty$ a.s., then*
(i) *The processes $\big(V\big[m_1^{(V)*}(h)-k\big]-V\big[m_1^{(V)*}(h)\big],\ 0\le k\le m_1^{(V)*}(h)\big)$ and $\big(V\big[m_1^{(V)*}(h)+k\big]-V\big[m_1^{(V)*}(h)\big],\ 0\le k\le \tau_1^{(V)}(h)-m_1^{(V)*}(h)\big)$ are independent.*
(ii) *The process $\big(V\big[m_1^{(V)*}(h)+k\big]-V\big[m_1^{(V)*}(h)\big],\ 0\le k\le \tau_1^{(V)}(h)-m_1^{(V)*}(h)\big)$ is equal in law to $\big(V(k),\ 0\le k\le T_V([h,+\infty[)\big)$ conditioned on $\{T_V([h,+\infty[)<T_V^*(]-\infty,0])\}$.*

Proof: We fix $h>0$, and consider V satisfying the hypotheses. Let ψ_1 and ψ_2 be two nonnegative functions, $\bigsqcup_{t\in\mathbb{N}^*}\mathbb{R}^t\to[0,+\infty[$, measurable with respect to the σ-algebra $\{\bigsqcup_{t\in\mathbb{N}^*}A_t\ :\ \forall t\in\mathbb{N}^*,\ A_t\in\mathcal{B}(\mathbb{R}^t)\}$. To simplify the notation, we set $m_1^*:=m_1^{(V)*}(h)$ and $\tau_1^*:=\tau_1^{(V)}(h)$.

We now define by induction, e.g. as in Enriquez et al. [31] and [32], the weak descending ladder epochs for V as

$$e_0:=0,\qquad e_i:=\inf\{k>e_{i-1}\ :\ V(k)\le V(e_{i-1})\},\qquad i\ge 1, \tag{49}$$

with $e_i<\infty$ a.s. for each $i\ge1$ since $\liminf_{x\to+\infty}V(x)=-\infty$. In particular, the excursions $(V(k+e_i)-V(e_i),\ 0\le k\le e_{i+1}-e_i)$, $i\ge0$ are i.i.d. by the Strong Markov property. Also, the height H_i of the excursion $[e_i,e_{i+1}]$ is defined as

$$H_i:=\max_{e_i\le k\le e_{i+1}}[V(k)-V(e_i)],\qquad i\ge0. \tag{50}$$

Notice in particular that $m_1^*=e_L$, where $L:=\min\{\ell\ge0,\ H_\ell\ge h\}<\infty$ a.s. Hence, summing over the values of L, we get

$$\begin{aligned}
&\mathsf{E}\big[\psi_1\big(V(m_1^*-k)-V(m_1^*),\ 0\le k\le m_1^*\big)\psi_2\big(V(m_1^*+k)-V(m_1^*),\ 0\le k\le \tau_1^*-m_1^*\big)\big]\\
&=\sum_{\ell=0}^{\infty}\mathsf{E}\big[\psi_1\big(V\big(e_\ell-k\big)-V\big(e_\ell\big),\ 0\le k\le e_\ell\big)\mathbf{1}_{\cap_{i=0}^{\ell-1}\{H_i<h\}}\mathbf{1}_{\{H_\ell\ge h\}}\\
&\qquad\qquad\times\psi_2\big(V\big(e_\ell+k\big)-V\big(e_\ell\big),\ 0\le k\le T_{V(\cdot+e_\ell)-V(e_\ell)}([h,+\infty[)\big)\big]\\
&=\Pi_1\Pi_2,
\end{aligned}$$

due to the strong Markov property at stopping time e_ℓ, where, applying it again on the second equality,

$$\begin{aligned}
\Pi_1&:=\sum_{\ell=0}^{\infty}\mathsf{E}\big[\psi_1\big(V\big(e_\ell-k\big)-V\big(e_\ell\big),\ 0\le k\le e_\ell\big)\mathbf{1}_{\cap_{i=0}^{\ell-1}\{H_i<h\}}\big]\mathsf{P}[H_\ell\ge h]\\
&=\sum_{\ell=0}^{\infty}\mathsf{E}\big[\psi_1\big(V\big(e_\ell-k\big)-V\big(e_\ell\big),\ 0\le k\le e_\ell\big)\mathbf{1}_{\{L=\ell\}}\big]
\end{aligned}$$

$$= \mathsf{E}\big[\psi_1\big(V(m_1^*-k)-V(m_1^*),\ 0\le k\le m_1^*\big)\big]$$

and, since $\mathsf{P}[H_\ell \ge h] = \mathsf{P}[T_V([h,+\infty[) < T_V^*(]-\infty,0])]$,

$$\Pi_2 := \mathsf{E}\big[\psi_2\big(V(k),\ 0\le k\le T_V([h,+\infty[)) \mid T_V([h,+\infty[) < T_V^*(]-\infty,0])\big].$$

Since this is true for all ψ_1 and ψ_2, this proves the proposition. □

As a consequence, we get

Theorem 2.9. *Assume* (2), (3) *and* (4). *Let* $h>0$.
(i) *The process* $\mathcal{T}_{V,h}^{\uparrow *}$ *up to its first hitting time* $T_{\mathcal{T}_{V,h}^{\uparrow *}}([h,+\infty[)$ *of* $[h,+\infty[$, *that is,* $\big(\mathcal{T}_{V,h}^{\uparrow *}(k),\ 0\le k\le T_{\mathcal{T}_{V,h}^{\uparrow *}}([h,+\infty[)\big)$, *is equal in law to* $\big(V(k),\ 0\le k\le T_V([h,+\infty[)\big)$ *conditioned on* $\{T_V([h,+\infty[) < T_V^*(]-\infty,0])\}$. *Moreover, it is independent of* $\big(\mathcal{T}_{V,h}^{\uparrow *}\big(T_{\mathcal{T}_{V,h}^{\uparrow *}}([h,+\infty[)+k\big) - \mathcal{T}_{V,h}^{\uparrow *}\big(T_{\mathcal{T}_{V,h}^{\uparrow *}}([h,+\infty[)\big),\ 0\le k\le \ell\big(\mathcal{T}_{V,h}^{\uparrow *}\big) - T_{\mathcal{T}_{V,h}^{\uparrow *}}([h,+\infty[)\big)$, *which has the same law as* $\big(V(k),\ 0\le k\le \widetilde{M}_h^\sharp\big)$, *with* $\widetilde{M}_h^\sharp := \max\{0\le k\le \tilde\tau_1(h),\ V(k)=\max_{[0,\tilde\tau_1(h)]}V\}$, *where* $\tilde\tau_1(h) := \min\{k\ge 0,\ \max_{[0,k]}V - V(k)\ge h\}$.

(ii) $\mathcal{T}_{-V,h}^{\uparrow *} =_{law} -\mathcal{T}_{V,h}^{\downarrow *}$ *and* $\mathcal{T}_{-V,h}^{\downarrow *} =_{law} -\mathcal{T}_{V,h}^{\uparrow *}$.

(iii) *Also,* $\mathsf{E}\big(\ell\big(\mathcal{T}_{V,h}^{\uparrow *}\big)\big)<\infty$ *and* $\mathsf{E}\big(\ell\big(\mathcal{T}_{V,h}^{\downarrow *}\big)\big)<\infty$.

Proof: The proof of this theorem is the same as the proof of Theorem 2.3, with Proposition 2.8, $\mathcal{T}_{V,h}^{\uparrow *}$, $m_i^{(V)*}(h)$ and right extrema instead of ([25], Proposition 5.2), $\mathcal{T}_{V,h}^{\uparrow}$, $m_i^{(V)}(h)$ and left extrema respectively. □

The following lemma says that ζ, defined in (24), transforms translated left (resp. right) h-slopes for V into right (resp. left) ones for V^- (see Lemma 2.10 below), and upward ones into downward ones.

Lemma 2.10. *For* $i\in\mathbb{Z}$, $\zeta[\theta(T_i(V,h)))] = \theta[T_{-i}^*(V^-,h)]$.

Proof: Recall that $x_k^*(V,h) = -x_{1-k}(V^-,h)$ for $k\in\mathbb{Z}$ by Lemma 2.7. Hence for $i\in\mathbb{Z}$,

$$\begin{aligned}
&\zeta[\theta(T_i(V,h)))] = \zeta\big[\big(V[x_i(V,h)+j]-V[x_i(V,h)],\ 0\le j\le x_{i+1}(V,h)-x_i(V,h)\big)\big]\\
=\ & (V^-[-x_{i+1}(V,h)+j]-V^-[-x_{i+1}(V,h)],\ 0\le j\le x_{i+1}(V,h)-x_i(V,h))\\
=\ & \big(V^-[x_{-i}^*(V^-,h)+j]-V^-[x_{-i}^*(V^-,h)],\ 0\le j\le -x_{-i}^*(V^-,h))+x_{1-i}^*(V^-,h))\big)\\
=\ & \theta[T_{-i}^*(V^-,h)]. \qquad (51)
\end{aligned}$$

This proves the lemma.

As a consequence, we get the following result.

Theorem 2.11. *Theorems 2.4 and 2.5 remain valid if we replace "left" and each* $x_k(V,h)$, $T_k(V,h)$, $\mathcal{T}_{V,h}^{\uparrow}$ *and* $\mathcal{T}_{V,h}^{\downarrow}$ *respectively by "right",* $x_k^*(V,h)$, $T_k^*(V,h)$, $\mathcal{T}_{V,h}^{\uparrow *}$ *and* $\mathcal{T}_{V,h}^{\downarrow *}$, *and* $<$ *and* $\le$ *respectively by* $\le$ *and* $<$ *in Theorem 2.5.*

Proof: Indeed, their proofs remain valid if we make these replacements and also replace $m_k^{(V)}(h)$ by $m_k^{(V)*}(h)$, $k\in\mathbb{Z}$, and $T_j(t,V,h)$ by $T_j^*(t,V,h) = T_{i+j}^*(V,h)$ if $x_i^*(V,h) < t \le x_{i+1}^*(V,h)$, and for this unique i, $x_j^*(t,V,h) := x_{i+j}^*(V,h)$ for $j\in\mathbb{Z}$, and as a consequence, replace $<$ and $\le$ respectively by $\le$ and $<$ throughout the proof. □

The following proposition, combined with some other results such as Theorem 2.9, will be useful to obtain the law of V on the left of $x_i(V,h)$ (for $i \in \mathbb{Z}$) conditionally on $b_h \le 0$ or $b_h > 0$, in view of Theorems 2.4 and 2.5.

Proposition 2.12. *Let $h > 0$. Then,* **(i)** $\zeta\big(\mathcal{T}^{\uparrow}_{V,h}\big) =_{law} \mathcal{T}^{\downarrow *}_{V^-,h}$ *and* **(ii)** $\zeta\big(\mathcal{T}^{\downarrow}_{V,h}\big) =_{law} \mathcal{T}^{\uparrow *}_{V^-,h}$.

Proof: We denote by $\mathscr{L}\big(\mathcal{T}^{\downarrow *}_{V^-,h}\big)$ the law of $\mathcal{T}^{\downarrow *}_{V^-,h}$ under P. Conditionally on $\{V[x_0(V,h)] < V[x_1(V,h)]\} = \{V^-[x_1^*(V^-,h)] < V^-[x_0^*(V^-,h)]\}$ (thanks to (46)), $\theta[T_2(V,h)]$ has the law $\mathcal{L}\big(\mathcal{T}^{\uparrow}_{V,h}\big)$ by Theorem 2.4 **(i)**, whereas $\theta[T^*_{-2}(V^-,h)]$ has the law $\mathscr{L}\big(\mathcal{T}^{\downarrow *}_{V^-,h}\big)$ by the version of Theorem 2.4 **(ii)** with stars (see Theorem 2.11) applied to V^-. This and (51) prove our **(i)**. Applying the same arguments to $\theta[T_1(V,h)]$ and $\theta[T^*_{-1}(V^-,h)]$ proves **(ii)**. □

2.7. **Relation with another localization point.** In this subsection, we recall another way to define a localization point denoted by $b_h^{(K)}$, and we prove that $b_h^{(K]}$ is equal to b_h (defined in (19)) with large probability. The localization point $b_h^{(K)}$ is useful because we will apply the previous result of Kesten ([48], Thm 1.2) to the limit law of $b_h^{(K)}/h^2$ (in the proof of Theorem 1.4, see after (66)), whereas our b_h is convenient e.g. due to Lemma 2.6 and to the law of the potential near b_h (by Theorems 2.3, 2.4, and 2.5).

To this aim, we define for any process $(Z(k),\ k \ge 0)$, similarly as in Hu ([44] from eq. (2.1) to eq. (2.6)) but for processes indexed by $\mathbb{N}$,

$$\overline{Z}(t) := \sup_{0\le k\le t} Z(k), \qquad \underline{Z}(t) := \inf_{0\le k\le t} Z(k), \qquad Z^\sharp(t) := \sup_{0\le s\le t} \big(Z(s) - \underline{Z}(s)\big) \qquad t \ge 0,$$

$$d_Z(h) := \inf\{t \ge 0,\ Z^\sharp(t) \ge h\}, \qquad h > 0. \tag{52}$$

Also, with $V_-(k) := V(-k)$ for $k \ge 0$ as before, we introduce (see Figure 3)

$$\begin{aligned} b_V^+(h) &:= \inf\{0 \le u \le d_V(h),\ V(u) = \underline{V}(d_V(h))\}, \qquad h > 0, \\ b_V^-(h) &:= \sup\{0 \le u \le d_{V_-}(h),\ V_-(u) = \underline{V_-}(d_{V_-}(h))\}, \qquad h > 0. \end{aligned} \tag{53}$$

The sup instead of inf in the last line will be necessary so that in some cases, $-b_V^-(h)$ is a left h-minimum for V instead of a right one (as in Figure 3). Finally, we introduce

$$b_h^{(K)} := \begin{cases} b_V^+(h) & \text{if } \overline{V}[d_V(h)] < \overline{V_-}[d_{V_-}(h)], \\ -b_V^-(h) & \text{otherwise.} \end{cases} \tag{54}$$

Let $(W(x),\ x \in \mathbb{R})$ be a two-sided Brownian motion, and $W_- := (W(-x),\ x \ge 0)$. As in Hu ([44] eq. (2.6)), for $w = W$ or $w = \sigma W$, we define $b_h^{(K,w)}$ by the same formula as in (54), the previous notations of this Subsection 2.7 being the same, with V replaced by w, and the inf and sup being taken for real numbers instead of integers. As already stated by Hu ([44] after eq. (2.6), his $b(1)$ being a.s. equal to our $b_1^{(K,W)}$ since the sup in (53) is a.s. a inf when V is replaced by W), the density of $b_1^{(K,W)}$ is φ_∞, defined in (5). Indeed, it is easy to check that $b_1^{(K,W)}$ is a.s. equal to the r.v. L of Kesten ([48], as expressed in the statement of his Lemma 2.1), which has density φ_∞ by ([48], Thm 1.2).

For some choices of P, we have $\mathsf{P}\big[b_h \ne b_h^{(K)}\big] > 0$ for some $h > 0$. Indeed, for example, if $\mathsf{P}[V(1) = z] > 0$ for every $z \in \{-2,-1,0,1,2\}$, we have for $h \in \mathbb{N}^*$, with non zero probability, $V(-1) = V(0) = V(1) = 0$, with $V(k) = k-1$ for $1 \le k \le h+1$, $V(k) = |k|-1$ for $-h \le k \le -1$ and $V(-h-1) = h+1$, and so $b_h^{(K)} = b_V^+(h) = 0$ whereas $b_h = -1 \ne b_h^{(K)}$. However, we prove that $b_h^{(K)} = b_h$ with large probability. More precisely, we have:

Lemma 2.13. *There exists a constant $c_5 > 0$ such that, for large h,*

$$\mathsf{P}\big[b_h^{(K)} \neq b_h\big] \leq c_5 h^{-1}.$$

This lemma will be useful to prove Lemma 2.14 and Theorem 1.4. Moreover, we think it will also be necessary in a work in progress [23].

Proof of Lemma 2.13: Let $h > 0$. **First case:** we assume that

$$\max\Big[\overline{V}[b_V^+(h)], V[b_V^+(h)] + h\Big] \leq \overline{V_-}[d_{V_-}(h)] - 2C_0. \tag{55}$$

Let $c_-(h) := \sup\{k \leq 0,\ V(k) = \overline{V_-}[d_{V_-}(h)]\}$ (which may be $-d_{V_-}(h)$ or not). First, by

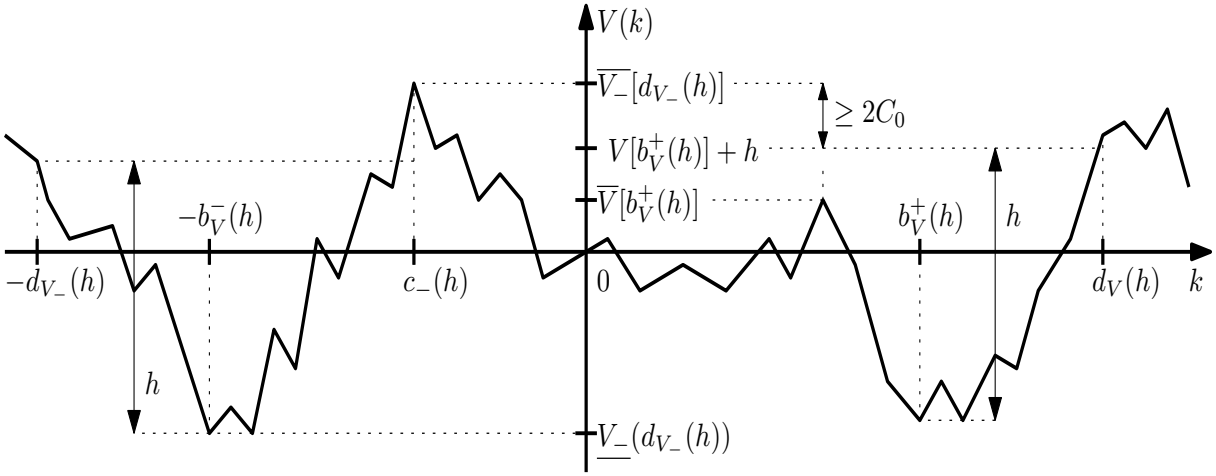

Figure 3. Schema of the potential V for the first case of the proof of Lemma 2.13 when $\overline{V}[b_V^+(h)] < V[b_V^+(h)] + h$.

definition of $b_V^+(h)$, we have $V[b_V^+(h)] = \min_{[b_V^+(h), d_V(h)]} V$. Also for the same reason, $V[b_V^+(h)] < \min_{[0, b_V^+(h)-1]} V$, with $\min \emptyset = +\infty$ by convention, and since $-d_{V_-}(h) \leq c_-(h) \leq 0$, we have $\min_{[c_-(h),0]} V \geq V(c_-(h)) - h - C_0 = \overline{V_-}[d_{V_-}(h)] - h - C_0 > V[b_V^+(h)]$ first by definition of $d_{V_-}(h)$ and ellipticity, then by definition of $c_-(h)$ followed by (55). So, $\min_{[c_-(h), b_V^+(h)-1]} V > V[b_V^+(h)]$. Moreover, by definition, $V[d_V(h)] \geq V[b_V^+(h)] + h$. Finally, $V[c_-(h)] = \overline{V_-}[d_{V_-}(h)] \geq V[b_V^+(h)] + h$ first by definition, then by (55). Consequently, $b_V^+(h)$ is a left h-minimum.

Assume that there exists a left h-extremum in $[0, b_V^+(h) - 1]$. Since $b_V^+(h)$ is a left h-minimum, and left h-maxima and minima for V alternate by Lemma 7.7, there would be at least one left h-maximum in this interval, which we denote by $\alpha \in [0, b_V^+(h)[$. Now, denote by γ the largest left h-minimum such that $\gamma < \alpha$, so that $[\gamma, \alpha]$ is (the support of) an upward left h-slope of V. In particular,

$$V(\gamma) \leq V(\alpha) - h \quad \text{and} \quad \gamma < \alpha. \tag{56}$$

Assume that $0 \leq \gamma$. So, $\gamma \in [0, \alpha]$, hence $V(\gamma) \geq \inf_{[0,\alpha]} V = \underline{V}(\alpha)$, and then using (56),

$$V^\sharp(\alpha) \geq V(\alpha) - \underline{V}(\alpha) \geq V(\alpha) - V(\gamma) \geq h.$$

By definition (52) of d_V, this would give $d_V(h) \leq \alpha$, which contradicts $\alpha < b_V^+(h) \leq d_V(h)$.

Hence we would have $\gamma < 0 \leq \alpha$. Using first the fact that $[\gamma, \alpha]$ is an upward left h-slope, then $\alpha \in [0, b_V^+(h)]$ and finally (55) would give

$$\sup_{[\gamma,\alpha]} V \leq V(\alpha) \leq \overline{V}[b_V^+(h)] < \overline{V_-}[d_{V_-}(h)] = V[c_-(h)].$$

So $c_-(h) \notin [\gamma, \alpha]$ and since $c_-(h) \leq 0 \leq \alpha$ by definition, this gives $c_-(h) < \gamma < 0$. Using ellipticity (16), then (55), we get

$$V[c_-(h)+1] \geq V[c_-(h)] - C_0 = \overline{V_-}[d_{V_-}(h)] - C_0 \geq \overline{V}[b_V^+(h)] + C_0.$$

Thus, using (56) in the second inequality,

$$V[c_-(h)+1] - V(\gamma) \geq \overline{V}[b_V^+(h)] + C_0 - V(\gamma) \geq \overline{V}[b_V^+(h)] + C_0 + h - V(\alpha) \geq C_0 + h,$$

since $\overline{V}[b_V^+(h)] \geq V(\alpha)$ because $\alpha \in [0, b_V^+(h)]$. So, $V_-(|c_-(h)|-1) - V_-(|\gamma|) > h$ with $0 < |\gamma| \leq |c_-(h)|-1$, which gives $d_{V_-}(h) \leq |c_-(h)| - 1 < |c_-(h)| \leq d_{V_-}(h)$, which is not possible.

Hence there is no left h-extremum in $[0, b_V^+(h) - 1]$. Since $b_V^+(h)$ is a left h-minimum, this gives $x_1(V,h) = b_V^+(h)$ if $b_V^+(h) \neq 0$ and $x_0(V,h) = b_V^+(h)$ if $b_V^+(h) = 0$, and by definition (19) of b_h, it follows that $b_h = b_V^+(h)$. Since $\overline{V}[d_V(h)] \leq \max\left[\overline{V}[b_V^+(h)], \max_{[b_V^+(h), d_V(h)]} V\right] \leq \max\left[\overline{V}[b_V^+(h)], V[b_V^+(h)]+h+C_0\right] < \overline{V_-}[d_{V_-}(h)]$ by ellipticity and (55), we also have $b_h^{(K)} = b_V^+(h)$ by (54). Hence, $b_h = b_h^{(K)}$ when (55) holds.

Second case: we assume that

$$\max\left[\overline{V_-}[b_V^-(h)], V_-[b_V^-(h)] + h\right] \leq \overline{V}[d_V(h)] - 2C_0. \tag{57}$$

This case is nearly the symmetric of the previous one, the only asymmetry being the sup in (53) (which is necessary for $-b_V^-(h)$ to be a left h-minimum instead of a right one). So we prove similarly as in the first case that $b_h = -b_V^-(h) = b_h^{(K)}$ when (57) holds.

Third step: Consequently, if $b_h \neq b_h^{(K)}$ then neither (55) nor (57) hold, and so

$$\begin{aligned} \overline{V_-}[d_{V_-}(h)] - 2C_0 &< \max\left[\overline{V}[b_V^+(h)], V[b_V^+(h)] + h\right] \leq \overline{V}[d_V(h)] \\ &< \max\left[\overline{V_-}[b_V^-(h)], V_-[b_V^-(h)] + h\right] + 2C_0 \leq \overline{V_-}[d_{V_-}(h)] + 2C_0, \end{aligned}$$

where we first used the negation of (55), then the definitions of $d_V(h)$ and $b_V^+(h)$, then the negation of (57) and finally the definitions of $d_{V_-}(h)$ and $b_V^-(h)$. In view of these inequalities, we define

$$\begin{aligned} E_1 &:= \left\{-2C_0 < \max\left[\overline{V}[b_V^+(h)], V[b_V^+(h)] + h\right] - \overline{V_-}[d_{V_-}(h)] < 2C_0\right\}, \\ E_2 &:= \left\{V[b_V^+(h)] + h < \overline{V}[b_V^+(h)]\right\}, \end{aligned}$$

so that $\mathsf{P}\left[b_h \neq b_h^{(K)}\right] \leq \mathsf{P}[E_1]$.

First, notice that on $E_1 \cap E_2$, writing here $\beta := \overline{V_-}[d_{V_-}(h)]$ to simplify the notation, we have $\beta - 2C_0 < \overline{V}[b_V^+(h)] < \beta + 2C_0$, and so $V[b_V^+(h)] < \beta + 2C_0 - h$ thanks to E_2. Hence, $T_V([\beta - 2C_0, +\infty[) \leq b_V^+(h)$ and $V[. + T_V([\beta - 2C_0, +\infty[)]$ hits $V[b_V^+(h)] \in]-\infty, \beta + 2C_0 - h]$ before $[\beta + 2C_0, +\infty[$. Thus, since V_- is independent of $(V(x),\, x \geq 0)$, the strong Markov property, and then (17) lead to, if $h > 4C_0$,

$$\begin{aligned} & \mathsf{P}[E_1 \cap E_2 \mid V_-] \\ \leq\ & \mathsf{E}\big(\mathsf{P}^{T_V([y-2C_0, +\infty[)}[T_V(]-\infty, y + 2C_0 - h]) < T_V([y + 2C_0, +\infty[)]_{|y=\beta} \mid V_-\big) \\ \leq\ & 5C_0(h + C_0)^{-1}. \end{aligned}$$

Consequently, $\mathsf{P}[E_1 \cap E_2] \leq 6C_0 h^{-1}$ for large h.

Similarly, notice that on $E_1 \cap E_2^c$, once more with the notation $\beta := \overline{V_-}[d_{V_-}(h)]$, we have $\beta - 2C_0 < V[b_V^+(h)] + h < \beta + 2C_0$. So, $T_V(]-\infty, \beta + 2C_0 - h]) \leq b_V^+(h)$. Also, $\min_{[0, d_V(h)]} V = V[b_V^+(h)] > \beta - 2C_0 - h$ and $V[d_V(h)] \geq V[b_V^+(h)] + h > \beta - 2C_0$, thus $V[. + T_V(]-\infty, \beta + 2C_0 - h])$

hits $[\beta-2C_0,+\infty[$ before $]-\infty,\beta-2C_0-h]$. Hence as previously, since V_- is independent of $(V(x),\ x\geq 0)$, by the strong Markov property, and then by (17), if $h>4C_0$,

$$\begin{aligned}&\mathsf{P}[E_1\cap E_2^c\mid V_-]\\ \leq\ &\mathsf{E}\big(\mathsf{P}^{T_V(]-\infty,y+2C_0-h])}[T_V([y-2C_0,+\infty[)<T_V(]-\infty,y-2C_0-h])]_{y=\beta}\mid V_-\big)\\ \leq\ &5C_0(h+C_0)^{-1}.\end{aligned}$$

Consequently, $\mathsf{P}[E_1\cap E_2^c]\leq 6C_0h^{-1}$ for large h. Finally, $\mathsf{P}\big[b_h\neq b_h^{(K)}\big]\leq\mathsf{P}[E_1]\leq 12C_0h^{-1}$ for large h, which proves the lemma. □

Lemma 2.14. *There exists a constant $c_6>0$ such that*

$$\mathsf{P}\big[b_h>0\big]\to_{h\to+\infty}1/2,\qquad \mathsf{P}\big[b_h=0\big]\sim_{h\to+\infty}c_6h^{-2}.$$

Proof: For the equivalent, observe that by (19), $b_h=0$ if and only if 0 is a left h-minimum for V, that is if and only if V and $V(-.)=:V_-(.)$ hit $[h,+\infty[$ before going back to $]-\infty,0]$ for V_-, and before hitting $]-\infty,0[$ for $(V(k),\ k\geq 0)$. So by independence of $(V(k),\ k\geq 0)$ and V_- and (18) (or (235)),

$$\begin{aligned}\mathsf{P}\big[b_h=0\big]&=\mathsf{P}\big[T_{V_-}([h,+\infty[)<T^*_{V_-}(]-\infty,0])\big]\mathsf{P}\big[T_V([h,+\infty[)<T_V(]-\infty,0[)\big]\\ &\sim_{h\to+\infty}c_6h^{-2}\end{aligned}\tag{58}$$

with $c_6>0$ being the product of c_1^* (for the law of V_-) and of c_1 (for the law of V) with the notation of (18) (and (235)). This proves the second claim in Lemma 2.14. Notice that this constant c_6 depends on the law of ω_0, that is, c_6 depends on P.

For the first limit of the lemma, notice that $\mathsf{P}[b_h>0]=\mathsf{P}\big[b_h^{(K)}>0\big]+O(1/h)$ as $h\to+\infty$ by Lemma 2.13, so we just have to prove that $\mathsf{P}\big[b_h^{(K)}>0\big]\to_{h\to+\infty}1/2$. We now consider a two sided Brownian motion $(W(x),\ x\in\mathbb{R})$, and consider $W_-(x):=W(-x)$ for $x\geq 0$, and define $\overline{W}$, $\overline{W_-}$, d_W, d_{W_-}, as explained after (54). By (54), we have for $h>0$,

$$\begin{aligned}\mathsf{P}\big[b_h^{(K)}>0\big]&=\mathsf{P}\big[\overline{V}[d_V(h)]<\overline{V_-}[d_{V_-}(h)],\ b_V^+(h)\neq 0\big]\\ &=\mathsf{P}\big[\overline{V}[d_V(h)]<\overline{V_-}[d_{V_-}(h)]\big]+O(1/h)\end{aligned}\tag{59}$$

since $\mathsf{P}\big[b_V^+(h)=0\big]=\mathsf{P}\big[T_V([h,+\infty[)<T_V(]-\infty,0[)\big]=O(1/h)$ as $h\to+\infty$ similarly as in (58). By the theorem of Donsker, the limit of the probability in (59) as $h\to+\infty$ is $\mathsf{P}\big[\overline{\sigma W}[d_{\sigma W}(1)]<\overline{\sigma W_-}[d_{\sigma W_-}(1)]\big]$, which is 1/2 by symmetry and because $\mathsf{P}\big[\overline{\sigma W}[d_{\sigma W}(1)]=\overline{\sigma W_-}[d_{\sigma W_-}(1)]\big]=0$ since the r.v. $\overline{\sigma W}[d_{\sigma W}(1)]$ and $\overline{\sigma W_-}[d_{\sigma W_-}(1)]$ are independent and have a density (by Hu [44] Lemma 2.1 and by scaling). Hence $\mathsf{P}\big[b_h^{(K)}>0\big]\to_{h\to+\infty}1/2$ and so $\mathsf{P}\big[b_h>0\big]\to_{h\to+\infty}1/2$. □

Lemma 2.15. *There exists a constant $c_7:=(2c_6)^{-1}>0$ such that*

$$\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h})]\sim_{h\to+\infty}\mathsf{E}[\ell(\mathcal{T}^{\downarrow}_{V,h})]\sim_{h\to+\infty}c_7h^2.$$

Proof: Applying (45), and Theorem 2.5, using (27) with $A=\mathcal{S}_+=\bigsqcup_{t=1}^{\infty}\mathbb{R}_+^t$, $\Delta_1=\mathbb{N}^*$ and $\Delta_0=-\mathbb{N}$, we have, since $b_h\leq 0$ if and only if $\theta(T_0(V,h))\in\mathcal{S}_+$ by (19),

$$\mathsf{P}(b_h=0)=\frac{1}{\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h})+\ell(\mathcal{T}^{\downarrow}_{V,h})]},\qquad \mathsf{P}(b_h\leq 0)=\frac{\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h})\big]}{\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h})+\ell(\mathcal{T}^{\downarrow}_{V,h})]}.$$

Consequently, $\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h})\big]=\frac{\mathsf{P}(b_h\leq 0)}{\mathsf{P}(b_h=0)}\sim_{h\to+\infty}h^2/(2c_6)$ by Lemma 2.14. Similarly, we obtain $\mathsf{E}\big[\ell(\mathcal{T}^{\downarrow}_{V,h})\big]=\frac{\mathsf{P}(b_h>0)}{\mathsf{P}(b_h=0)}\sim_{h\to+\infty}h^2/(2c_6)$, which proves the lemma. □

2.8. **An inequality for the excess height of left h-slopes.**

Lemma 2.16. *There exists a constant $c_8 > 0$ such that, for large h,*

$$\forall i \in \mathbb{Z},\, \forall C_0 < \Delta < h, \qquad \mathsf{P}\Big(e[T_i(V,h)] \le \Delta | b_h \le 0\Big) \le c_8 \frac{\Delta}{h}. \tag{60}$$

This remains true if $b_h \le 0$ is replaced by $b_h > 0$.

Proof: Let $h > 0$ and $C_0 < \Delta < h$. Applying Theorem 2.4 **(i)** since $\{V(x_1(V,h)) > V(x_0(V,h))\} = \{b_h \le 0\}$, then Theorem 2.3 **(i)**, and then (17), we have for $i \neq 0$, since $C_0 < \Delta < h$,

$$\begin{aligned}\mathsf{P}\big(e[T_{2i}(V,h)] \le \Delta | b_h \le 0\big) &= \mathsf{P}\Big(H[\theta(T_{2i}(V,h))] - h \le \Delta | b_h \le 0\Big) = \mathsf{P}\Big(H\big(\mathcal{T}^{\uparrow}_{V,h}\big) - h \le \Delta\Big) \\ &\le \mathsf{P}\Big(T_V(-h+\Delta) \le \tilde{\tau}_1(h) < T_V(]\Delta, +\infty[)\Big) \le \frac{\Delta + C_0}{h + C_0} \le \frac{2\Delta}{h}. \end{aligned} \tag{61}$$

Similarly, applying Theorem 2.4 **(i)**, then $\mathcal{T}^{\downarrow}_{V,h} =_{law} -\mathcal{T}^{\uparrow}_{-V,h}$ by Theorem 2.3 **(ii)**,

$$\mathsf{P}\Big(e[T_{2i+1}(V,h)] \le \Delta | b_h \le 0\Big) = \mathsf{P}\Big(H\big(\mathcal{T}^{\downarrow}_{V,h}\big) - h \le \Delta\Big) = \mathsf{P}\Big(H\big(\mathcal{T}^{\uparrow}_{-V,h}\big) - h \le \Delta\Big) \le \frac{2\Delta}{h}$$

similarly as before for $i \in \mathbb{Z}$ and $C_0 < \Delta < h$. This proves (60) for $i \neq 0$.

The proof is similar when conditioning by $b_h > 0$, applying Theorem 2.4 **(ii)** instead of **(i)**.

We now consider the case $i = 0$. We have, by Theorem 2.5 eq. (27) applied with $\Delta_0 = \Delta_1 = \mathbb{Z}$,

$$\begin{aligned}&\mathsf{P}\Big(e[T_0(V,h)] \le \Delta | V(x_1(V,h)) > V(x_0(V,h))\Big) \\ = \;& \frac{\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h}) \mathbf{1}_{\{H(\mathcal{T}^{\uparrow}_{V,h}) - h \le \Delta\}}\big]}{\mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})] \mathsf{P}\big[V(x_1(V,h)) > V(x_0(V,h))\big]}. \end{aligned} \tag{62}$$

Notice by Theorem 2.3 **(i)** and since $H(\mathcal{T}^{\uparrow}_{V,h}) = \mathcal{T}^{\uparrow}_{V,h}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)\big]$, $T_{\mathcal{T}^{\uparrow}_{V,h}}([h, +\infty[) \le \ell(\mathcal{T}^{\uparrow}_{V,h})$ and $\Delta - h < 0$, and finally by (17),

$$\begin{aligned}&\mathsf{E}\big[T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[) \mathbf{1}_{\{H(\mathcal{T}^{\uparrow}_{V,h}) - h \le \Delta\}}\big] \\ \le\;& \mathsf{E}\big[T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[) \mathbf{1}_{\{H(\mathcal{T}^{\uparrow}_{V,h}) - T^{\uparrow}_{V,h}(T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[)) \le \Delta\}}\big] \\ =\;& \mathsf{E}\big[T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[)\big] \mathsf{P}\big[H(\mathcal{T}^{\uparrow}_{V,h}) - T^{\uparrow}_{V,h}(T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[)) \le \Delta\big] \\ \le\;& \mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)\big] \mathsf{P}\big(T_V(-h+\Delta) < T_V(]\Delta, +\infty[)\big) \\ \le\;& \mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)\big] 2\Delta h^{-1}. \end{aligned} \tag{63}$$

Finally, once more by Theorem 2.3 **(i)** with its notation,

$$\begin{aligned}\mathsf{E}\big[\big(\ell(\mathcal{T}^{\uparrow}_{V,h}) - T_{\mathcal{T}^{\uparrow}_{V,h}}([h,+\infty[)\big) \mathbf{1}_{\{H(\mathcal{T}^{\uparrow}_{V,h}) - h \le \Delta\}}\big] &\le \mathsf{E}\big[M_h^{\sharp} \mathbf{1}_{\{\tilde{\tau}_1(h) < T_V(]\Delta,+\infty[)\}}\big] \\ &\le \mathsf{E}\big[\tilde{\tau}_1(h) \mathbf{1}_{\{\tilde{\tau}_1(h) < T_V(]\Delta,+\infty[)\}}\big]. \end{aligned} \tag{64}$$

Notice that $\widehat{X}_k := (V(k))^2 - \sigma^2 k$, $k \in \mathbb{N}$ is a martingale for the filtration $\mathcal{F}_{V,k} := \sigma(V(1), \ldots, V(k))$, $k \in \mathbb{N}$. Moreover, the stopping time $\tilde{\tau}_1(h) \wedge T_V(]\Delta, +\infty[)$ has finite expectation, since $\mathsf{E}[\tilde{\tau}_1(h) \wedge T_V(]\Delta, +\infty[)] \le \mathsf{E}[\tilde{\tau}_1(h)] = \mathsf{E}\big[\tau_2^{(V)}(h) - \tau_1^{(V)}(h)\big] \le \mathsf{E}[\ell(\mathcal{T}^{\uparrow}_{V,h}) + \ell(\mathcal{T}^{\downarrow}_{V,h})] < \infty$ by (20), (22), Definition 2.2 (see also Figure 2) and Theorem 2.3 **(iii)**. Also, for every $k \in \mathbb{N}$,

$$\mathsf{E}\big[\big|\widehat{X}_{k+1} - \widehat{X}_k\big| \mid \mathcal{F}_{V,k}\big] = \mathsf{E}\big[\big|(V(k+1))^2 - (V(k))^2 - \sigma^2\big| \mid \mathcal{F}_{V,k}\big] \le 2C_0(\Delta + h + C_0) + \sigma^2$$

a.s. on $\{k < \widetilde{\tau}_1(h) \wedge T_V(]\Delta, +\infty[)\}$, since $V(k)$ and $V(k+1)$ belong to $[-h - C_0, \Delta + C_0]$ on this event and $|V(k+1) - V(k)| \leq C_0$. Hence by the optimal stopping time theorem (see e.g. [41], (9) p. 492), we have $\mathsf{E}\big[\widehat{X}_{\widetilde{\tau}_1(h) \wedge T_V(]\Delta,+\infty[)}\big] = \mathsf{E}\big[\widehat{X}_0\big] = 0$. This gives

$$\mathsf{E}\big[\big([V(\widetilde{\tau}_1(h))]^2 - \sigma^2 \widetilde{\tau}_1(h)\big)\mathbf{1}_{\{\widetilde{\tau}_1(h) < T_V(]\Delta,+\infty[)\}}\big] + \mathsf{E}\big[\widehat{X}_{T_V(]\Delta,+\infty[)}\mathbf{1}_{\{\widetilde{\tau}_1(h) > T_V(]\Delta,+\infty[)\}}\big] = 0,$$

since $\widetilde{\tau}_1(h) \neq T_V(]\Delta, +\infty[)$ a.s. Consequently, using $\widehat{X}_k \leq (V(k))^2$ and ellipticity (16),

$$\begin{aligned}
& \sigma^2 \mathsf{E}\big[\widetilde{\tau}_1(h)\mathbf{1}_{\{\widetilde{\tau}_1(h) < T_V(]\Delta,+\infty[)\}}\big] \\
= \;& \mathsf{E}\big[[V(\widetilde{\tau}_1(h))]^2 \mathbf{1}_{\{\widetilde{\tau}_1(h) < T_V(]\Delta,+\infty[)\}}\big] + \mathsf{E}\big[\widehat{X}_{T_V(]\Delta,+\infty[)}\mathbf{1}_{\{\widetilde{\tau}_1(h) > T_V(]\Delta,+\infty[)\}}\big] \\
\leq \;& (h + C_0)^2 \mathsf{P}[T_V(-h + \Delta) < T_V(]\Delta, +\infty[)] + (\Delta + C_0)^2 \\
\leq \;& (h + C_0)^2 2\Delta h^{-1} + (\Delta + C_0)^2,
\end{aligned}$$

as before since $C_0 < \Delta < h$. This and (64) give for large h for every $\Delta \in]C_0, h[$,

$$\mathsf{E}\big[\big(\ell(\mathcal{T}^{\uparrow}_{V,h}) - T_{\mathcal{T}^{\uparrow}_{V,h}}([h, +\infty[)\big)\mathbf{1}_{\{H(\mathcal{T}^{\uparrow}_{V,h}) - h \leq \Delta\}}\big] \leq \sigma^{-2}(3\Delta h + 3\Delta h).$$

This together with (63) gives

$$\mathsf{E}\big[\ell(\mathcal{T}^{\uparrow}_{V,h})\mathbf{1}_{\{H(\mathcal{T}^{\uparrow}_{V,h}) - h \leq \Delta\}}\big] \leq 6\sigma^{-2}\Delta h + \mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)\big] 2\Delta h^{-1}. \tag{65}$$

Moreover, $\mathsf{P}\big[V(x_1(V,h)) > V(x_0(V,h))\big] = \mathsf{P}(b_h \leq 0) \to 1/2$ as $h \to +\infty$ by Lemma 2.14, so (62), (65) and Lemma 2.15 give for large h for every $\Delta \in]C_0, h[$,

$$\mathsf{P}\Big(e[T_0(V,h)] \leq \Delta | b_h \leq 0\Big) \leq \frac{6\sigma^{-2}\Delta h}{2c_7 h^2 .1/3} + 5\frac{\Delta}{h} \leq c_8 \frac{\Delta}{h}$$

with $c_8 := 9\sigma^{-2}/c_7 + 5$. The proof is similar if we replace $b_h \leq 0$ by $b_h > 0$, using Theorem 2.5 eq. (28) instead of eq. (27) and since $\mathcal{T}^{\downarrow}_{V,h} =_{law} -\mathcal{T}^{\uparrow}_{-V,h}$ by Theorem 2.3 **(ii)**. This proves (60) in the case $i = 0$, which ends the proof of the lemma. □

3. Proof of Theorem 1.4

The proof relies mainly on the expression of $\mathsf{P}(b_h = x)$ provided by Lemma 2.6, the monotonicity of $x \mapsto \mathsf{P}(b_h = x)$ on $\mathbb{N}$ and $-\mathbb{N}$ due to Lemma 2.6, the uniform continuity of φ_∞, Donsker's theorem, Kesten [48]'s result and some estimates on the laws of left h-slopes. The proof is divided into three steps, depending on whether x is far from 0, close to 0, or in between.

Proof of Theorem 1.4: Let $0 < \varepsilon < 1/2$.

First step: Notice that by Lemma 2.6 and Markov inequality, for $h > 0$,

$$\forall x > 0, \qquad \mathsf{P}(b_h = x) = \frac{\mathsf{P}\big[\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big) \geq x\big]}{\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big) + \ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]} \leq \frac{\mathsf{E}\big[\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]}{x\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big) + \ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]} \leq \frac{1}{x}$$

and similarly $\mathsf{P}(b_h = x) \leq \frac{1}{|x|}$ for all $x < 0$. Moreover, $\lim_{x \to \pm\infty} \varphi_\infty(x) = 0$, so we can fix some $A > 0$ such that, for every $h > 0$, for all $x \in \mathbb{Z}$ such that $|x| > Ah^2$, we have

$$\begin{aligned}
\left|\mathsf{P}\big(b_h = x\big) - \frac{\sigma^2}{h^2}\varphi_\infty\left(\frac{\sigma^2 x}{h^2}\right)\right| \;&\leq\; \mathsf{P}\big(b_h = x\big) + \frac{\sigma^2}{h^2}\varphi_\infty\left(\frac{\sigma^2 x}{h^2}\right) \leq \frac{1}{|x|} + \frac{\sigma^2}{h^2}\sup_{|y| \geq A\sigma^2}\varphi_\infty(y) \\
&\leq\; \frac{1}{h^2}\left(\frac{1}{A} + \sigma^2 \sup_{|y| \geq A\sigma^2}\varphi_\infty(y)\right) \leq \frac{\varepsilon}{h^2}.
\end{aligned} \tag{66}$$

Second step: By Donsker's theorem, $b^{(K)}_h/h^2$ converges in law as $h \to +\infty$ under P to $b^{(K,\sigma W)}_1$ (defined after (54)), which has the same law as $\sigma^{-2} b^{(K,W)}_1$ by scaling. Also, the law of $b^{(K,W)}_1$ is

$\varphi_\infty(x)\mathrm{d}x$ by Kesten [48] as explained after our (54). Also, $\mathsf{P}\big[b_h \neq b_h^{(K)}\big] \to_{h\to+\infty} 0$ by Lemma 2.13, so $\sigma^2 b_h/h^2$ converges in law under P to $\varphi_\infty(x)\mathrm{d}x$ as $h \to +\infty$.

Since φ_∞ is continuous on $\mathbb{R}$ and $\lim_{x\to\pm\infty}\varphi_\infty(x) = 0$, φ_∞ is uniformly continuous on $\mathbb{R}$. Hence, there exists $\eta > 0$ such that

$$\forall x \in \mathbb{R},\ \forall y \in \mathbb{R}, \qquad |x-y| \le \eta \Rightarrow |\varphi_\infty(x) - \varphi_\infty(y)| < \varepsilon, \tag{67}$$

and we can choose $\eta > 0$ small enough so that $5\eta\sigma^{-2} \le A$, $5c_7^{-1}\exp[-90^{-1}\eta^{-1}] \le \varepsilon\sigma^2$ and $3\sqrt{5\eta} \le 1$, where $c_7 > 0$ is a constant introduced in Lemma 2.15. We can now fix $N_0 \in \mathbb{N}$ such that $[-A, A] \subset [-N_0\eta\sigma^{-2}, N_0\eta\sigma^{-2}]$. Since $\sigma^2 b_h/h^2$ converges in law under P to $\varphi_\infty(x)\mathrm{d}x$ as $h \to +\infty$, for all $j \in \{-N_0-3, \dots, N_0+3\}$,

$$\mathsf{P}\big(\sigma^2 b_h/h^2 \in [j\eta, (j+1)\eta[\big) \to_{h\to+\infty} \int_{j\eta}^{(j+1)\eta} \varphi_\infty(u)\mathrm{d}u.$$

Hence there exists $h_0 > 0$ such that $\eta\sigma^{-2}h_0^2 > 2$, $1 \le [(1-\varepsilon)^{-1} - 1]\eta\sigma^{-2}h_0^2$, $1 \le [1-(1+\varepsilon)^{-1}]\eta\sigma^{-2}h_0^2$ and

$$\forall h \ge h_0, \forall j \in \{-N_0-3, \dots, N_0+3\}, \quad \left|\mathsf{P}\left(\frac{\sigma^2 b_h}{h^2} \in [j\eta, (j+1)\eta[\right) - \int_{j\eta}^{(j+1)\eta} \varphi_\infty(u)\mathrm{d}u\right| \le \eta\varepsilon.$$

This, combined with (67), gives for all $j \in \{-N_0-3, \dots, N_0+3\}$,

$$\forall h \ge h_0, \quad \eta[\varphi_\infty(j\eta) - \varepsilon] - \eta\varepsilon \le \mathsf{P}\big(\sigma^2 b_h/h^2 \in [j\eta, (j+1)\eta[\big) \le \eta[\varphi_\infty(j\eta) + \varepsilon] + \eta\varepsilon. \tag{68}$$

We consider $h \ge h_0$. Due to Lemma 2.6, $x \mapsto \mathsf{P}(b_h = x)$ is nonincreasing on $\mathbb{N}$, and nondecreasing on $-\mathbb{N}$. Hence, for $0 \le j \le N_0 + 3$,

$$\begin{aligned}\mathsf{P}\big(\sigma^2 b_h/h^2 \in [j\eta, (j+1)\eta[\big) &= \sum_{i \in [j\eta\sigma^{-2}h^2, (j+1)\eta\sigma^{-2}h^2[\cap\mathbb{N}} \mathsf{P}(b_h = i)\\ &\le (1-\varepsilon)^{-1}\eta\sigma^{-2}h^2\mathsf{P}\big(b_h = \lfloor j\eta\sigma^{-2}h^2\rfloor\big),\end{aligned}$$

due to the second inequality defining h_0. This and (68) give for such j,

$$\begin{aligned}\mathsf{P}\big(b_h = \lfloor j\eta\sigma^{-2}h^2\rfloor\big) &\ge \frac{\eta[\varphi_\infty(j\eta) - \varepsilon] - \eta\varepsilon}{(1-\varepsilon)^{-1}\eta\sigma^{-2}h^2} = \sigma^2[\varphi_\infty(j\eta) - 2\varepsilon]h^{-2}(1-\varepsilon)\\ &\ge \sigma^2[\varphi_\infty(j\eta) - 3\varepsilon]h^{-2},\end{aligned} \tag{69}$$

since $\varphi_\infty(u) \in [0, 2/\pi]$ for all $u \in \mathbb{R}$. Similarly for such j,

$$\mathsf{P}\big(\sigma^2 b_h/h^2 \in [j\eta, (j+1)\eta[\big) \ge \big[\eta\sigma^{-2}h^2 - 1\big]\mathsf{P}\big(b_h = \lfloor (j+1)\eta\sigma^{-2}h^2\rfloor\big).$$

This and (68) give, using the third inequality in the definition of h_0,

$$\begin{aligned}\mathsf{P}\big(b_h = \lfloor (j+1)\eta\sigma^{-2}h^2\rfloor\big) &\le \frac{\eta[\varphi_\infty(j\eta) + \varepsilon] + \eta\varepsilon}{(1+\varepsilon)^{-1}\eta\sigma^{-2}h^2} = (1+\varepsilon)\sigma^2[\varphi_\infty(j\eta) + 2\varepsilon]h^{-2}\\ &\le \sigma^2[\varphi_\infty(j\eta) + 4\varepsilon]h^{-2},\end{aligned} \tag{70}$$

since $0 < \varepsilon < 1/2$ and $\varphi_\infty(u) \in [0, 2/\pi]$ for all $u \in \mathbb{R}$.

Now, let $j \in \{2, \dots, N_0\}$ and $x \in \mathbb{N}$ such that $j\eta\sigma^{-2}h^2 \le x < (j+1)\eta\sigma^{-2}h^2$. We have since $\mathsf{P}(b_h = .)$ is nonincreasing on $\mathbb{N}$ and $x \le \lfloor (j+1)\eta\sigma^{-2}h^2\rfloor$, then by (69) and finally by (67),

$$\begin{aligned}\mathsf{P}(b_h = x) &\ge \mathsf{P}(b_h = \lfloor (j+1)\eta\sigma^{-2}h^2\rfloor) \ge \sigma^2[\varphi_\infty((j+1)\eta) - 3\varepsilon]h^{-2}\\ &\ge \sigma^2\big[\varphi_\infty\big(x\sigma^2h^{-2}\big) - 4\varepsilon\big]h^{-2}.\end{aligned}$$

Similarly, using (70) applied to $j-1 \ge 1$ instead of (69), followed by (67),

$$\begin{aligned}\mathsf{P}(b_h = x) &\le \mathsf{P}(b_h = \lfloor j\eta\sigma^{-2}h^2\rfloor)\\ &\le \sigma^2[\varphi_\infty((j-1)\eta) + 4\varepsilon]h^{-2} \le \sigma^2\big[\varphi_\infty\big(x\sigma^2h^{-2}\big) + 6\varepsilon\big]h^{-2}.\end{aligned}$$

Since this is true for all $h \geq h_0$, every $j \in \{2, \dots, N_0\}$ and for every $x \in \mathbb{N}$ such that $j\eta\sigma^{-2}h^2 \leq x < (j+1)\eta\sigma^{-2}h^2$ for such j, and $A \leq N_0\eta\sigma^{-2}$, this gives

$$\forall h \geq h_0, \qquad \max_{x \in [2\eta\sigma^{-2}h^2, Ah^2] \cap \mathbb{Z}} \big|\mathsf{P}(b_h = x) - \varphi_\infty\big(x\sigma^2h^{-2}\big)\sigma^2h^{-2}\big| \leq 6\varepsilon\sigma^2h^{-2}. \tag{71}$$

We get similarly

$$\forall h \geq h_0, \qquad \max_{x \in [-Ah^2, -2\eta\sigma^{-2}h^2] \cap \mathbb{Z}} \big|\mathsf{P}(b_h = x) - \varphi_\infty\big(x\sigma^2h^{-2}\big)\sigma^2h^{-2}\big| \leq 6\varepsilon\sigma^2h^{-2}. \tag{72}$$

Third step: Now, for $-5\eta\sigma^{-2}h^2 \leq x \leq 0$, we have by (44) and (45),

$$\big|\mathsf{P}(b_h = x) - \mathsf{P}(b_h = 0)\big| = \frac{\mathsf{P}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big) \leq -x\big]}{\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big) + \ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big]} \leq \frac{\mathsf{P}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big) \leq 5\eta\sigma^{-2}h^2\big]}{c_7h^2} \tag{73}$$

(uniformly) for all $-5\eta\sigma^{-2}h^2 \leq x \leq 0$ for large h, since $\mathsf{E}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big)\big] \sim_{h\to+\infty} \mathsf{E}\big[\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big)\big] \sim_{h\to+\infty} c_7h^2$ by Lemma 2.15.

We know from Theorem 2.3 **(i)** that up to its first hitting time of $[h, +\infty[$, $\mathcal{T}^{\uparrow}_{V,h}$ has the same law as $(V(k),\ 0 \leq k \leq T_V([h, +\infty[)$ conditioned by $\{T_V([h, +\infty[) < T_V(]-\infty, 0[)\}$. Thus for $\alpha > 0$, applying the strong Markov property in the last equality, and ellipticity (16) in the last line (for h large enough so that $C_0 < h/6$),

$$\begin{aligned}
&\mathsf{P}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,h}\big) \leq \alpha h^2\big] \leq \mathsf{P}\big[T_{\mathcal{T}^{\uparrow}_{V,h}}([h, +\infty[) - T_{\mathcal{T}^{\uparrow}_{V,h}}([h/2, +\infty[) \leq \alpha h^2\big] \\
= \ & \frac{\mathsf{P}\big[T_V([h, +\infty[) - T_V([h/2, +\infty[) \leq \alpha h^2, T_V([h, +\infty[) < T_V(]-\infty, 0[)\big]}{\mathsf{P}\big[T_V([h, +\infty[) < T_V(]-\infty, 0[)\big]} \\
= \ & \frac{\mathsf{E}\big[\mathbf{1}_{\{T_V([h/2,+\infty[) < T_V(]-\infty,0[)\}} \mathsf{P}^{V(T_V([h/2,+\infty[))}\big[T_V([h, +\infty[) \leq (\alpha h^2) \wedge T_V(]-\infty, 0[)\big]\big]}{\mathsf{P}\big[T_V([h, +\infty[) < T_V(]-\infty, 0[)\big]} \\
\leq \ & \frac{\mathsf{P}\big[T_V([h/2, +\infty[) < T_V(]-\infty, 0[)\big]}{\mathsf{P}\big[T_V([h, +\infty[) < T_V(]-\infty, 0[)\big]} \mathsf{P}\big[T_V([h/3, +\infty[) \leq \alpha h^2\big].
\end{aligned}$$

Using $\mathsf{P}\big[T_V([h, +\infty[) < T_V(]-\infty, 0[)\big] \sim_{h\to+\infty} c_1h^{-1}$ (see (18)) and Donsker's theorem, the last line is equivalent, as $h \to +\infty$, to

$$\begin{aligned}
2\mathsf{P}\big[T_{\sigma W}([1/3, +\infty[)) \leq \alpha\big] &= 2\mathsf{P}\Big[\sup_{[0,\alpha]}(\sigma W) \geq 1/3\Big] = 2\mathsf{P}\big[\sigma|W(\alpha)| \geq 1/3\big] \\
&= 2\mathsf{P}\big[|W(1)| \geq (3\sigma\sqrt{\alpha})^{-1}\big] \leq 4\exp[-(3\sigma\sqrt{\alpha})^{-2}/2]
\end{aligned}$$

if $3\sigma\sqrt{\alpha} \leq 1$, where $(W(x),\ x \in \mathbb{R})$ is a two-sided Brownian motion as before. Since $3\sqrt{5\eta} \leq 1$, this and (73) give for large h,

$$\max_{-5\eta\sigma^{-2}h^2 \leq x \leq 0} \big|\mathsf{P}(b_h = x) - \mathsf{P}(b_h = 0)\big| \leq 5c_7^{-1}h^{-2}\exp\big[-\big(3\sqrt{5\eta}\big)^{-2}/2\big] \leq \varepsilon\sigma^2h^{-2}$$

by the second inequality after (67). Since we have a similar result for $0 \leq x \leq 5\eta\sigma^{-2}h^2$, using (43) instead of (44) and e.g. $\ell\big(\mathcal{T}^{\downarrow}_{V,h}\big) =_{law} \ell\big(\mathcal{T}^{\uparrow}_{-V,h}\big)$ (see Theorem 2.3 **(ii)**), there exists $h_1 > h_0$ such that

$$\forall h \geq h_1, \qquad \max_{-5\eta\sigma^{-2}h^2 \leq x \leq 5\eta\sigma^{-2}h^2} \big|\mathsf{P}(b_h = x) - \mathsf{P}(b_h = 0)\big| \leq \varepsilon\sigma^2h^{-2}. \tag{74}$$

We already know, from (69), that $\forall h \geq h_1 \geq h_0, \mathsf{P}(b_h = 0) \geq \sigma^2[\varphi_\infty(0) - 3\varepsilon]h^{-2}$. Moreover, using (74), (71) and then (67),

$$\begin{aligned}
\mathsf{P}(b_h = 0) &\leq \mathsf{P}(b_h = \lfloor 4\eta\sigma^{-2}h^2\rfloor) + \varepsilon\sigma^2h^{-2} \leq [\phi_\infty(\lfloor 4\eta\sigma^{-2}h^2\rfloor\sigma^2h^{-2}) + 7\varepsilon]\sigma^2h^{-2} \\
&\leq [\phi_\infty(0) + 11\varepsilon]\sigma^2h^{-2}
\end{aligned}$$

for $h \geq h_1$. So,

$$\forall h \geq h_1, \qquad \big|\mathsf{P}(b_h = 0) - \sigma^2 \varphi_\infty(0) h^{-2}\big| \leq 11\varepsilon\sigma^2 h^{-2}. \tag{75}$$

Finally, once more by (67), $\big|\varphi_\infty\big(x\sigma^2 h^{-2}\big) - \varphi_\infty(0)\big| \leq 5\varepsilon$ for $x \in \mathbb{Z}$ such that $|x| \leq 5\eta\sigma^{-2}h^2$. This, combined with (74) and (75) and the triangular inequality yields to

$$\forall h \geq h_1, \qquad \max_{x \in [-5\eta\sigma^{-2}h^2, 5\eta\sigma^{-2}h^2] \cap \mathbb{Z}} \big|\mathsf{P}(b_h = x) - \varphi_\infty\big(x\sigma^2 h^{-2}\big)\sigma^2 h^{-2}\big| \leq 17\varepsilon\sigma^2 h^{-2}. \tag{76}$$

This, together with (71) and (72) leads to

$$\forall h \geq h_1, \qquad \max_{x \in [-Ah^2, Ah^2] \cap \mathbb{Z}} \big|\mathsf{P}(b_h = x) - \varphi_\infty\big(x\sigma^2 h^{-2}\big)\sigma^2 h^{-2}\big| \leq 17\varepsilon\sigma^2 h^{-2}. \tag{77}$$

This, combined with (66), proves Theorem 1.4. □

4. Coupling argument when $b_{\log n}$ is close to z

In this section, we use a coupling argument, in order to approximate the quenched probability $P_\omega[S_n = z]$ by the invariant probability measure at z of a RWRE reflected inside the central valley of the potential. In order to make this approximation, we require some conditions, mainly for the environment.

4.1. An inequality related to hitting times of $(S_k)_k$.

Before dealing with the coupling argument, we prove a useful inequality about hitting times. This lemma is in the same spirit as ([24], Lemma 4.7), but is more general. We will use this lemma with different values of ξ_1. See Figure 4 for the schema of the potential V under the hypotheses of this lemma.

Lemma 4.1. *Assume* (2). *Let $\xi_1 > 0$, $\xi_2 > 0$ and $\alpha > 0$. There exists $\widehat{h}_2 = \widehat{h}_2(\xi_1, \xi_2) > 1$ such that, for almost every environment ω, for every $a < b < c$ and $h \geq \widehat{h}_2$ such that* **(i)** *$V(b) = \max_{[a,c]} V$,* **(ii)** *$\max_{b \leq \ell \leq k \leq c-1} \big(V(k) - V(\ell)\big) \leq h - \xi_1 \log h$,* **(iii)** *$\max_{a \leq \ell \leq k \leq b-1} \big(V(\ell) - V(k)\big) \leq h - \xi_1 \log h$ and* **(iv)** *$|c - a| \leq 2h^\alpha$, and for every $a \leq x \leq c$, we have*

$$P_\omega^x\big[\tau(a) \wedge \tau(c) \geq \xi_2 e^h\big] \leq 24\xi_2^{-1}\varepsilon_0^{-2} h^{2\alpha - \xi_1 + 8} + 4\varepsilon_0^{-1} h^{\alpha - 8}, \tag{78}$$

and is, in particular, uniformly less than h^{-4} for all $h \geq \widehat{h}_2$ if $\alpha = 3$ and $\xi_1 > 19$.

Proof: We cannot apply directly (8) or (9) to $\mathsf{E}_\omega[\tau(a) \wedge \tau(c)]$, because the $\max(\dots)$ which appear in these inequalities can be much too large, since they can be respectively nearly as large as $V(b) - V(a)$ or $V(b) - V(c)$, which can be much larger than our h. Consider $\widehat{h}_2 > 1$ such that $h - (\xi_1 - 8)\log h > 0$ for every $h \geq \widehat{h}_2$. We fix $h \geq \widehat{h}_2$, and assume that the hypotheses of the lemma are satisfied for this h. We define (see Figure 4), with $x \vee y := \max(x, y)$,

$$\begin{aligned} A^- &:= a \vee \big(\max\{y \leq b,\ V(b) - V(y) \geq h - (\xi_1 - 8)\log h\}\big), \\ A^+ &:= c \wedge \big(\min\{y \geq b,\ V(b) - V(y) \geq h - (\xi_1 - 8)\log h\}\big). \end{aligned}$$

First case: we assume that $a \leq x \leq A^-$. We start with the sub-case $a < x \leq A^-$, which implies that $A^- = \max\{\dots\} \neq a$ in the definition of A^-. Then, by Markov inequality, (9) and Hypotheses **(iii)** and **(iv)**,

$$\begin{aligned} P_\omega^x[\tau(a) \wedge \tau(b) \geq \xi_2 e^h/2] &\leq 2\xi_2^{-1} e^{-h} \varepsilon_0^{-1} (b-a)^2 \exp\Big[\max_{a \leq \ell \leq k \leq b-1} \big(V(\ell) - V(k)\big)\Big] \\ &\leq 8\xi_2^{-1} e^{-h} \varepsilon_0^{-1} h^{2\alpha} \exp(h - \xi_1 \log h) = 8\xi_2^{-1}\varepsilon_0^{-1} h^{2\alpha - \xi_1}. \end{aligned} \tag{79}$$

Also, notice that since $a < A^- < b$, using Hypothesis **(iii)**,

$$\max_{[a, A^-]} V \leq V(A^-) + \max_{a \leq \ell \leq k \leq b-1} \big(V(\ell) - V(k)\big)$$

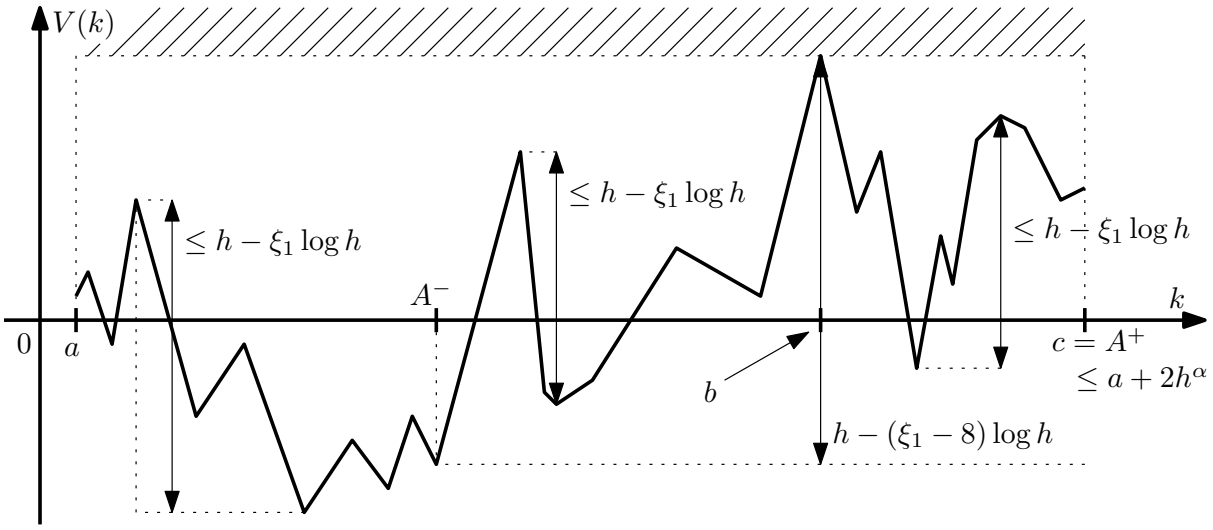

Figure 4. Schema of the potential V for Lemma 4.1 between a and c when $c = A^+$.

$$\leq \quad V(b) - (h - (\xi_1 - 8)\log h) + (h - \xi_1 \log h) \leq V(b) - 8\log h. \tag{80}$$

Hence using (7), $a < x \leq A^- < b$, then Hypothesis **(iv)**, ellipticity (2) and (80),

$$P_\omega^x[\tau(b) < \tau(a)] \quad \leq \quad (x-a)\exp[\max_{[a,x-1]} V - V(b-1)] \leq 2h^\alpha \varepsilon_0^{-1} h^{-8}.$$

Consequently, this and (79) lead to

$$\begin{aligned} & P_\omega^x[\tau(a) \wedge \tau(c) \geq \xi_2 e^h/2] \\ \leq \quad & P_\omega^x[\tau(b) < \tau(a)] + P_\omega^x[\tau(a) \wedge \tau(c) \geq \xi_2 e^h/2, \tau(a) < \tau(b) < \tau(c)] \\ \leq \quad & 2\varepsilon_0^{-1} h^{\alpha-8} + P_\omega^x[\tau(a) \wedge \tau(b) \geq \xi_2 e^h/2] \leq 2\varepsilon_0^{-1} h^{\alpha-8} + 8\xi_2^{-1}\varepsilon_0^{-1} h^{2\alpha-\xi_1}. \end{aligned} \tag{81}$$

This remains true if $x = a$, whether $a = A^-$ or $a \neq A^-$, and so for every $a \leq x \leq A^-$. This already proves (78) in this case.

Second case: we now assume that $A^+ \leq x \leq c$. This case is similar as the first one, so we get by symmetry, using **(ii)** instead of **(iii)** and (8) instead of (9),

$$P_\omega^x[\tau(a) \wedge \tau(c) \geq \xi_2 e^h/2] \leq 2\varepsilon_0^{-1} h^{\alpha-8} + 8\xi_2^{-1}\varepsilon_0^{-1} h^{2\alpha-\xi_1}. \tag{82}$$

This already proves (78) in the this case.

Third case: We now assume that $A^- < x < A^+$. Using Markov inequality, (8) and Hypothesis **(iv)** and $a \leq A^- < A^+ \leq c \leq a + 2h^\alpha$ in the first line, then $\max_{[A^-,A^+]} V = V(b)$ (due to Hypothesis **(i)** and $b \in [A^-, A^+] \subset [a, c]$) and $\min_{[A^-,A^+]} V \geq V(b) - (h - (\xi_1 - 8)\log h) - \log \varepsilon_0^{-1}$ (by definition of $A^\pm$ and ellipticity (16)), we have

$$\begin{aligned} P_\omega^x[\tau(A^-) \wedge \tau(A^+) \geq \xi_2 e^h/2] \quad & \leq \quad 2\xi_2^{-1} e^{-h} \varepsilon_0^{-1} (2h^\alpha)^2 \exp\Big[\max_{[A^-,A^+]} V - \min_{[A^-,A^+]} V\Big] \\ & \leq \quad 8\xi_2^{-1} e^{-h} \varepsilon_0^{-1} h^{2\alpha} \exp\Big[h - (\xi_1 - 8)\log h + \log \varepsilon_0^{-1}\Big] \\ & = \quad 8\xi_2^{-1} \varepsilon_0^{-2} h^{2\alpha - \xi_1 + 8}. \end{aligned}$$

Consequently, we have by the strong Markov property applied at time $\tau(A^-) \wedge \tau(A^+)$,

$$\begin{aligned} P_\omega^x\big[\tau(a) \wedge \tau(c) \geq \xi_2 e^h\big] \quad \leq \quad & P_\omega^x\big[\tau(A^-) \wedge \tau(A^+) \geq \xi_2 e^h/2\big] + P_\omega^{A^-}\big[\tau(a) \wedge \tau(c) \geq \xi_2 e^h/2\big] \\ & + P_\omega^{A^+}\big[\tau(a) \wedge \tau(c) \geq \xi_2 e^h/2\big] \end{aligned}$$

$$\begin{aligned} &\leq 8\xi_2^{-1}\varepsilon_0^{-2}h^{2\alpha-\xi_1+8} + 2\big(2\varepsilon_0^{-1}h^{\alpha-8} + 8\xi_2^{-1}\varepsilon_0^{-1}h^{2\alpha-\xi_1}\big) \\ &\leq 24\xi_2^{-1}\varepsilon_0^{-2}h^{2\alpha-\xi_1+8} + 4\varepsilon_0^{-1}h^{\alpha-8} \end{aligned}$$

by (81) and (82) applied respectively at A^- and A^+. This proves (78) in this third case, so (78) is proved in every case for every h larger than some constant $\widehat{h}_2 > 1$. Finally, when $\alpha = 3$ and $\xi_1 > 19$, we have $24\xi_2^{-1}\varepsilon_0^{-2}h^{2\alpha-\xi_1+8} + 4\varepsilon_0^{-1}h^{\alpha-8} \leq (24\xi_2^{-1}\varepsilon_0^{-2} + 4\varepsilon_0^{-1})h^{-5}$ which is $o(h^{-4})$ as $h \to +\infty$, so, up to a change of $\widehat{h}_2$, the right hand side of (78) is less than h^{-4} for all $h > \widehat{h}_2$, which ends the proof of the lemma. □

4.2. **Some events useful for the coupling argument.** In order to evaluate the probability $\mathbb{P}(S_n = z)$, we decompose the event $\{S_n = z\}$ into smaller ones, and to this aim we introduce some conditions on the environment ω. First, we fix $C_1 > 20$, $C_2 > 9$, and $\delta_1 \in]0, 2/3[$. For $n \geq 3$, we introduce

$$h_n := \log n - C_1 \log_2 n, \qquad \widetilde{h}_n := h_n - C_1 \log_2 n, \qquad \Gamma_n := \big\lfloor (\log n)^{4/3+\delta_1} \big\rfloor,$$

where for $x > 1$, $\log_2 x := \log\log x$. We also fix an integer $n_3 \geq 3$ such that, for all $n \geq n_3$, $\log_2 n > C_0 + 1$, $\log n > \max\big[2\varepsilon_0^{-1}, \widehat{h}_2(2C_1, 1/10), \widehat{h}_2(C_1, 1/10), \widehat{h}_2(2C_1, 1), p_5\big]$, $h_n - C_1 \log_2 n > \max\{3C_0 + 10\log_2 n, (\log n)/2 + (2C_1 + C_2 + 2)\log_2 n\}$, $(\log_2 n)^6 \leq \log n$, $n \geq (\log n)^{C_1+4}$ and $\Gamma_n \geq p_4$, with p_4 and p_5 defined in Proposition 7.3 and $\widehat{h}_2$ in Lemma 4.1. We also define for $n \geq n_3$ and $z \in \mathbb{Z}$,

$$\begin{aligned} E_-^{(n)} &:= \{b_{\log n} \leq 0\} = \{b_{\log n} = x_0(V, \log n)\}, \qquad (83) \\ E_+^{(n)} &:= \{b_{\log n} > 0\} = \{b_{\log n} = x_1(V, \log n)\} = \big(E_-^{(n)}\big)^c, \\ E_3^{(n)} &:= \cap_{i=-10}^{10}\big\{H[T_i(V, h_n - C_1\log_2 n)] \geq \log n + C_2 \log_2 n\big\}, \\ E_4^{(n)}(z) &:= \{V(z) - V(b_{\log n}) \geq 5\log_2 n\} \\ &\quad \cup\Big(E_-^{(n)} \cap \Big\{\max_{[b_{\log n}, 0]} V < V[x_1(V, \log n)] - 9\log_2 n\Big\}\Big) \\ &\quad \cup\Big(E_+^{(n)} \cap \Big\{\max_{[0, b_{\log n}]} V < V[x_0(V, \log n)] - 9\log_2 n\Big\}\Big), \\ E_5^{(n)} &:= \big\{-(\log n)^{2+\delta_1} \leq x_{-12}(V, \log n) \leq x_{12}(V, \log n) \leq (\log n)^{2+\delta_1}\big\}, \\ E_6^{(n)} &:= \big\{\max\{V(b_{\log n} + i) - V(b_{\log n}), |i| \leq \Gamma_n\} < \log n\big\}, \\ E_7^{(n)}(z) &:= \{|b_{\log n} - z| \leq \Gamma_n\}. \qquad (84) \end{aligned}$$

Finally, let

$$E_C^{(n)}(z) := E_3^{(n)} \cap E_4^{(n)}(z) \cap E_5^{(n)} \cap E_6^{(n)} \cap E_7^{(n)}(z). \qquad (85)$$

Remark 4.2. *For $\omega \in E_3^{(n)}$, for every $-9 \leq i \leq 10$, $H(T_{i-1}(V, h_n - C_1\log_2 n)) \geq \log n$ and $H(T_i(V, h_n - C_1\log_2 n)) \geq \log n$, so $x_i(V, h_n - C_1\log_2 n)$ is also a left $(\log n)$-extremum. So, $x_i(V, \log n) = x_i(V, h_n - C_1\log_2 n)$ for every $-9 \leq i \leq 10$, and as a consequence, $H[T_i(V, \log n)] = H[T_i(V, h_n - C_1\log_2 n)]$ for every $-9 \leq i \leq 9$.*

The previous events depend only on the environment ω and on z. They are useful for the coupling argument used in this section. More precisely, we saw in Remark 4.2 that $E_3^{(n)}$ ensures that $x_i(V, h_n - C_1\log_2 n) = x_i(V, \log n)$ for $|i| \leq 9$, and as a consequence, there is no subvalley of height slightly less than $\log n$ in the $(\log n)$-central valley (defined after (87)), so $(S_k)_k$ is not trapped a long time in such subvalleys, which helps $(S_k)_k$ to go quickly to $b_{\log n}$ with large quenched probability.

Also, $E_4^{(n)}(z)$ is useful to prove a technical lemma, Lemma 4.6. $E_5^{(n)}$ says that the $|x_i(V, \log n)|$ are quite small, which will often be useful in applying inequalities such as (7), ..., (11) to prove that some events are negligible. Finally, $E_6^{(n)}$ and $E_7^{(n)}(z)$ will imply in particular that z is inside the $(\log n)$-central valley (see (107)).

We will use, in the proof of Theorem 1.1, left h-extrema of V for three different values of h. In particular, left $(\log n)$-extrema are useful to define $b_{\log n}$, left $\widetilde{h}_n$-extrema are useful e.g. to use $E_3^{(n)}$ as explained previously, and the proof of Lemma 5.9 uses left h-extrema with two different values strictly less than $\log n$, which are h_n and $\widetilde{h}_n$; left h_n-extrema are used in Lemma 5.11 (in view of (164) and Lemma 5.10), whereas left $\widetilde{h}_n$-extrema are also used in Lemma 5.13 and in the proof of the lower bound of Theorem 1.1 (see Section 6).

In the rest of the paper, the n_i, $3 \le i \le 19$, denote some integers with $n_i \le n_{i+1}$ for $3 \le i \le 18$, which are useful to get the uniformity in Theorem 1.1 (n_3 being defined before (83)).

4.3. **Definition of the coupling.** We fix an integer $n \ge n_3$, $z \in \mathbb{Z}$, and an environment $\omega \in E_C^{(n)}(z)$. In all the remaining of Section 4, we set $x_i := x_i(V, \log n)$, $i \in \mathbb{Z}$ (defined before (19)), to simplify the notation. Notice that, since $\omega \in E_3^{(n)}$, $x_i = x_i(V, \log n) = x_i(V, h_n - C_1 \log_2 n)$ for every $-9 \le i \le 10$ by Remark 4.2. We also introduce

$$\widehat{b}(n) := 2\lfloor b_{\log n}/2 \rfloor + \mathbf{1}_{2\mathbb{N}+1}(n), \tag{86}$$

which belongs to $\{b_{\log n} - 1,\, b_{\log n},\, b_{\log n} + 1\}$ and has the same parity as n. We define

$$M^- := \begin{cases} x_{-1} & \text{if } b_{\log n} \le 0, \\ x_0 & \text{if } b_{\log n} > 0, \end{cases} \qquad M^+ := \begin{cases} x_1 & \text{if } b_{\log n} \le 0, \\ x_2 & \text{if } b_{\log n} > 0. \end{cases} \tag{87}$$

Since $b_{\log n} = x_0$ when $b_{\log n} \le 0$ and $b_{\log n} = x_1$ when $b_{\log n} > 0$, M^- and M^+ are the two left $(\log n)$-maxima surrounding $b_{\log n}$, respectively on its left and on its right. For this reason, $[M^-, M^+]$ is called the $(\log n)$-*central valley* (see Figure 5); also $0 \in [M^-, M^+]$.

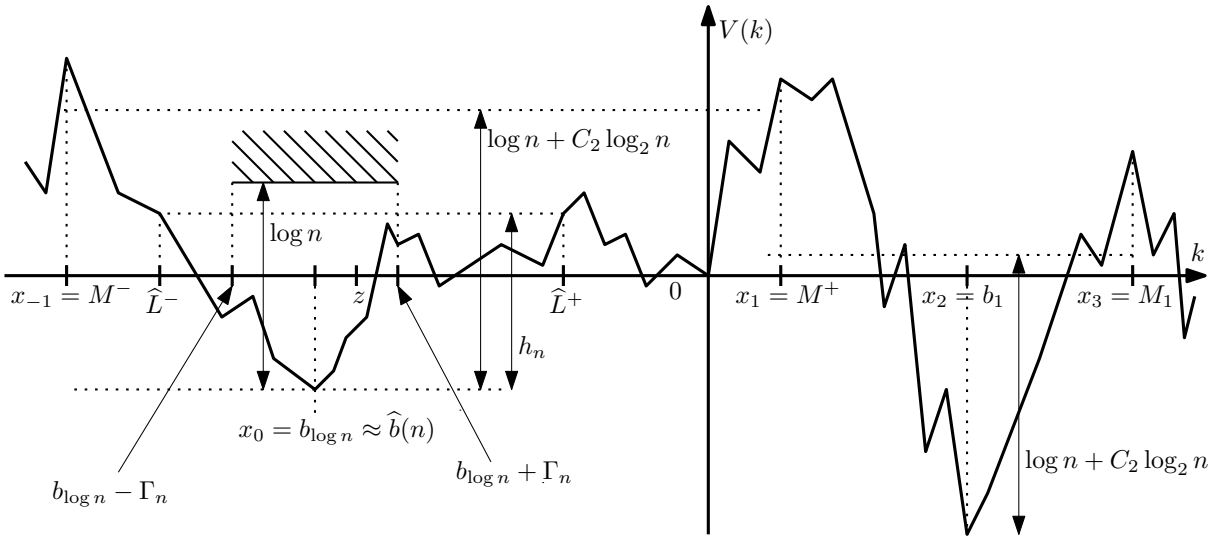

Figure 5. Schema of the potential V for $\omega \in E_C^{(n)}(z)$ in the case $b_{\log n} \le 0$.

Similarly as in Brox [12] and Andreoletti et al. [3] for diffusions in a random environment, and as in Devulder et al. [24] and [25] for RWRE, but with some adaptations, we use a coupling

between $S = (S_k)_k$ $\big($under $P_\omega^{\widehat{b}(n)}\big)$ and a reflected RWRE $\widehat{S}$ defined below. To this aim, we define, for fixed n, $\big(\widehat{\omega}_x\big)_{x\in\mathbb{Z}}$ as follows:

$$\widehat{\omega}_{M^-} := 1, \qquad \widehat{\omega}_x := \omega_x \text{ if } x \notin \{M^-, M^+\}, \qquad \widehat{\omega}_{M^+} := 0.$$

We can now introduce, for fixed ω and n, a random walk $\widehat{S} := \big(\widehat{S}_k\big)_{k\in\mathbb{N}}$ in the environment $\widehat{\omega} := \big(\widehat{\omega}_x\big)_{M^-\le x\le M^+}$, starting from $y \in [M^-, M^+]$, and denote its law by $P_{\widehat{\omega}}^y$. So, $\widehat{S}$ satisfies (1) with ω and S replaced respectively by $\widehat{\omega}$ and $\widehat{S}$. In words, $\widehat{S}$ is a random walk in the environment ω, starting from $y \in [M^-, M^+]$, and reflected at M^- and M^+. We also define the measure $\widehat{\mu}_n$ on $\mathbb{Z}$ by

$$\begin{aligned} \widehat{\mu}_n(M^-) &:= e^{-V(M^-)}, \qquad \widehat{\mu}_n(M^+) := e^{-V(M^+-1)}, \\ \widehat{\mu}_n(x) &:= e^{-V(x)} + e^{-V(x-1)}, \quad M^- < x < M^+, \end{aligned} \tag{88}$$

and $\widehat{\mu}_n(x) := 0$ for $x \notin [M^-, M^+]$ (where $\widehat{\mu}_n(x)$ denotes $\widehat{\mu}_n(\{x\})$ for simplicity).

Observe that for fixed n and ω, $\widehat{\mu}_n(.)/\widehat{\mu}_n(\mathbb{Z})$ is an invariant probability measure for $\widehat{S}$.

Consequently, similarly as in ([25] eq. (55)), for every fixed n and ω, the measure $\widehat{\nu}_n$ defined by

$$\widehat{\nu}(x) := \widehat{\nu}_n(x) := \begin{cases} \widehat{\mu}_n(x)\mathbf{1}_{2\mathbb{Z}}(x)/\widehat{\mu}_n(2\mathbb{Z}) & \text{if } n \in (2\mathbb{N}), \\ \widehat{\mu}_n(x)\mathbf{1}_{2\mathbb{Z}+1}(x)/\widehat{\mu}_n(2\mathbb{Z}+1) & \text{if } n \in (2\mathbb{N}+1), \end{cases} \qquad x \in \mathbb{Z}, \tag{89}$$

is an invariant probability measure for $\big(\widehat{S}_{2k}\big)_{k\in\mathbb{N}}$. This means that $P_{\widehat{\omega}}^{\widehat{\nu}}\big(\widehat{S}_{2k} = x\big) = \widehat{\nu}(x)$ for all $x \in \mathbb{Z}$ and $k \in \mathbb{N}$, where $P_{\widehat{\omega}}^{\widehat{\nu}}(.) := \sum_{y\in\mathbb{Z}} \widehat{\nu}(y) P_{\widehat{\omega}}^y(.)$. Observe that $\widehat{\omega}$, $\widehat{S}$, $\widehat{\mu}_n$, $\widehat{\nu}_n$ and some other notation of this subsection defined below, depend on M^- and M^+ and so on n and ω, but we often do not write the subscript n in the following to simplify the notation.

We now have all the ingredients to build, for fixed n and ω, our coupling Q_ω of S and $\widehat{S}$ as follows and similarly as in ([25] around eq. (56)):

$$Q_\omega\big(\widehat{S} \in .\big) = P_{\widehat{\omega}}^{\widehat{\nu}}\big(\widehat{S} \in .\big), \qquad Q_\omega\big(S \in .\big) = P_\omega^{\widehat{b}(n)}\big(S \in .\big), \tag{90}$$

so that under Q_ω, the two Markov chains $\widehat{S}$ and S move independently until

$$\tau_{\widehat{S}=S} := \inf\big\{\ell \ge 0,\ \widehat{S}_\ell = S_\ell\big\},$$

which is their first meeting time, then $\widehat{S}_k = S_k$ for all $\tau_{\widehat{S}=S} \le k < \tau_{\text{exit}}$, where

$$\tau_{\text{exit}} := \inf\big\{\ell > \tau_{\widehat{S}=S},\ S_\ell \notin [M^-, M^+]\big\}$$

is the first exit time of S from the central valley $[M^-, M^+]$ after the meeting time $\tau_{\widehat{S}=S}$, and then $\widehat{S}$ and S move independently again after τ_{exit}.

4.4. **Approximation of the quenched probability measure.** The next step is to prove that, under Q_ω, $\widehat{S}$ and S meet quickly, and more precisely that $\tau_{\widehat{S}=S} \le n/10$ with large probability. For this purpose, we define, for $n \ge n_3$, in view of $E_3^{(n)}$,

$$\widehat{L}^- := \max\{k \le b_{\log n},\ V(k) - V(b_{\log n}) \ge h_n\}, \tag{91}$$

$$\widehat{L}^+ := \min\{k \ge b_{\log n},\ V(k) - V(b_{\log n}) \ge h_n\}. \tag{92}$$

Loosely speaking, $\widehat{L}^-$ and $\widehat{L}^+$ are useful because $V\big(\widehat{L}^\pm\big) - V(b_{\log n})$ is approximatively h_n and then is quite lower than $\log n$, so $\widehat{L}^-$ and $\widehat{L}^+$ will be hit quickly by S under Q_ω (see Lemma 4.3 below), but $V\big(\widehat{L}^\pm\big) - V(b_{\log n})$ is also chosen quite large because the invariant measure $\widehat{\nu}$ outside of $\big[\widehat{L}^-, \widehat{L}^+\big]$ needs to be small (see Lemma 4.4). We introduce the notation $u \vee v := \max(u, v)$. We prove the three following lemmas, which are uniform on z since they do not depend on z.

Lemma 4.3. *We have, with $\tau(.)$ denoting the hitting times by S as before,*

$$\forall n \geq n_3,\ \forall \omega \in E_3^{(n)} \cap E_5^{(n)}, \qquad Q_\omega\big[\tau\big(\widehat{L}^-\big) \vee \tau\big(\widehat{L}^+\big) > n/10\big] \leq (\log n)^{-3}.$$

Proof: Assume that $n \geq n_3$ and $\omega \in E_3^{(n)} \cap E_5^{(n)}$. Since $V(M^\pm) - V(b_{\log n}) \geq \log n + C_2 \log_2 n > h_n + C_0 \geq V\big(\widehat{L}^+\big) - V(b_{\log n})$ by $E_3^{(n)}$ (see also Remark 4.2) and using ellipticity (16), and since $h_n > 0$ by definition of n_3, we have $b_{\log n} < \widehat{L}^+ < M^+$. Moreover,

$$\max_{M^- \leq \ell \leq k \leq \widehat{L}^+, k \geq \widehat{b}(n)} [V(k) - V(\ell)] \leq \max_{[\widehat{b}(n), \widehat{L}^+]} V - \min_{[M^-, \widehat{L}^+]} V \leq V\big(\widehat{L}^+\big) - V\big(b_{\log n}\big) \leq h_n + \log(\varepsilon_0^{-1})$$

by ellipticity, i.e. by (16), and because $[M^-, M^+]$ is the $(\log n)$-central valley, its bottom being $b_{\log n}$. Consequently, using (8) and Markov's inequality since $M^- < \widehat{b}(n) < \widehat{L}^+$ because $V\big(M^-\big) > V\big(\widehat{L}^+\big) \geq V\big(b_{\log n}\big) + h_n \geq V\big(b_{\log n}\big) + 3C_0 > V\big(\widehat{b}(n)\big)$, then $\big[M^-, \widehat{L}^+\big] \subset [x_{-1}, x_2[\subset [-(\log n)^3, (\log n)^3[$ because $\omega \in E_5^{(n)}$ and $\delta_1 \in]0, 2/3[$, this leads to

$$\begin{aligned} P_\omega^{\widehat{b}(n)}\big[\tau(M^-) \wedge \tau\big(\widehat{L}^+\big) > n/10\big] &\leq 10 n^{-1} \varepsilon_0^{-1} (2(\log n)^3)^2 \varepsilon_0^{-1} e^{h_n} \\ &= 40 \varepsilon_0^{-2} (\log n)^{6 - C_1} \leq (\log n)^{-3}/4, \end{aligned}$$

since $n \geq n_3$ and $C_1 > 20$. Moreover, applying (7), then $\omega \in E_5^{(n)}$ and the definition of $\widehat{L}^+$ and finally using $V(M^\pm) - V(b_{\log n}) \geq \log n + C_2 \log_2 n$ on $E_3^{(n)}$ as before,

$$\begin{aligned} P_\omega^{\widehat{b}(n)}\big[\tau(M^-) < \tau\big(\widehat{L}^+\big)\big] &\leq \big[\widehat{L}^+ - \widehat{b}(n)\big] \exp\Big[\max_{[\widehat{b}(n), \widehat{L}^+ - 1]} V - V(M^-)\Big] \\ &\leq 2(\log n)^3 \exp\big[V(b_{\log n}) + h_n - (V(b_{\log n}) + \log n)\big] \\ &\leq 2(\log n)^{3 - C_1} \leq (\log n)^{-3}/4, \end{aligned}$$

since $n \geq n_3$ and $C_1 > 20$. As a consequence, using (90),

$$\begin{aligned} Q_\omega\big[\tau\big(\widehat{L}^+\big) > n/10\big] &= P_\omega^{\widehat{b}(n)}\big[\tau\big(\widehat{L}^+\big) > n/10\big] \\ &\leq P_\omega^{\widehat{b}(n)}\big[\tau(M^-) < \tau\big(\widehat{L}^+\big)\big] + P_\omega^{\widehat{b}(n)}\big[\tau(M^-) \wedge \tau\big(\widehat{L}^+\big) > n/10\big] \\ &\leq (\log n)^{-3}/2. \end{aligned} \tag{93}$$

We prove similarly that $Q_\omega\big[\tau\big(\widehat{L}^-\big) > n/10\big] \leq (\log n)^{-3}/2$ for all $n \geq n_3$ and $\omega \in E_3^{(n)} \cap E_5^{(n)}$, using (9) instead of (8). This, together with (93), proves Lemma 4.3. □

We now prove that the invariant measure outside $\big]\widehat{L}^-, \widehat{L}^+\big[$ is small for $n \geq n_3$.

Lemma 4.4. *We have,*

$$\forall n \geq n_3,\ \forall \omega \in E_3^{(n)} \cap E_5^{(n)}, \qquad \widehat{\nu}\big(\big[M^-, \widehat{L}^-\big]\big) + \widehat{\nu}\big(\big[\widehat{L}^+, M^+\big]\big) \leq (\log n)^{-4}. \tag{94}$$

Proof: Let $n \geq n_3$ and $\omega \in E_3^{(n)} \cap E_5^{(n)}$. As explained in Remark 4.2, due to $E_3^{(n)}$, $x_i(V, h_n - C_1 \log_2 n) = x_i(V, \log n) = x_i$ for every $i \in \{-1, 0, 1, 2\}$. So when $b_{\log n} \leq 0$, there is no left $(h_n - C_1 \log_2 n)$-extremum in $]x_0(V, h_n - C_1 \log_2 n), x_1(V, h_n - C_1 \log_2 n)[=]x_0, x_1[=]b_{\log n}, M^+[$. Similarly, when $b_{\log n} > 0$, there is no left $(h_n - C_1 \log_2 n)$-extremum in $]x_1, x_2[=]b_{\log n}, M^+[$.

We first prove that

$$\min_{[\widehat{L}^+, M^+]} V \geq V(b_{\log n}) + C_1 \log_2 n. \tag{95}$$

Assume that $\min_{[\widehat{L}^+, M^+]} V < V(b_{\log n}) + C_1 \log_2 n$, and let $u \in \big[\widehat{L}^+, M^+\big]$ be such that $V(u) = \min_{[\widehat{L}^+, M^+]} V$, and $y := \min\{\ell \in [b_{\log n}, u], V(\ell) = \max_{[b_{\log n}, u]} V\}$, so $y \geq \widehat{L}^+$. Notice that

$V(y) \geq V\big(\widehat{L}^+\big) \geq V(b_{\log n}) + h_n$ and $V(y) \geq V(b_{\log n}) + h_n \geq V(u) - C_1 \log_2 n + h_n$, so y would be a left $(h_n - C_1 \log_2 n)$-maximum for V. Since $b_{\log n} < y < u \leq M^+$, this contradicts the remark before (95). So, (95) is true. We prove similarly that

$$\min_{[M^-, \widehat{L}^-]} V \geq V(b_{\log n}) + C_1 \log_2 n. \tag{96}$$

We have by (95) and since $\omega \in E_5^{(n)}$ and $\widehat{\mu}_n(2\mathbb{Z}) = \widehat{\mu}_n(2\mathbb{Z}+1) = \sum_{i=M^-}^{M^+-1} e^{-V(i)} \geq e^{-V(b_{\log n})}$,

$$\begin{aligned} \widehat{\nu}\big(\big[\widehat{L}^+, M^+\big]\big) &\leq \big[M^+ - \widehat{L}^+ + 1\big] \max_{x \in [\widehat{L}^+, M^+]} \big(e^{-V(x)} + e^{-V(x-1)}\big) e^{V(b_{\log n})} \\ &\leq 3(\log n)^3 \big(1 + \varepsilon_0^{-1}\big)(\log n)^{-C_1} \leq (\log n)^{-4}/2 \end{aligned} \tag{97}$$

since $n \geq n_3$ and $C_1 > 20$, and where we used $-V(x-1) \leq -V(x) + \log(\varepsilon_0^{-1})$, $x \in \mathbb{Z}$ by (16). We prove similarly that $\widehat{\nu}\big(\big[M^-, \widehat{L}^-\big]\big) \leq (\log n)^{-4}/2$ for all $n \geq n_3$ and $\omega \in E_3^{(n)} \cap E_5^{(n)}$ thanks to (96). This, together with (97) proves (94). □

We can now prove that, with large enough probability, the coupling (i.e. $\widehat{S} = S$) occurs quickly, and lasts at least until time n.

Lemma 4.5. *We have,*

$$\forall n \geq n_3,\ \forall \omega \in E_3^{(n)} \cap E_5^{(n)}, \qquad Q_\omega\big[\tau_{\widehat{S}=S} > n/10\big] \leq 2(\log n)^{-3}, \tag{98}$$

and

$$\forall n \geq n_3,\ \forall \omega \in E_3^{(n)}, \qquad Q_\omega\big[\tau_{\text{exit}} \leq n\big] \leq (\log n)^{-3}. \tag{99}$$

Proof: Let $n \geq n_3$, and $\omega \in E_3^{(n)} \cap E_5^{(n)}$. We have by Lemma 4.3,

$$\begin{aligned} & Q_\omega\big[\tau_{\widehat{S}=S} > n/10\big] \\ \leq\ & Q_\omega\big[\tau\big(\widehat{L}^-\big) \vee \tau\big(\widehat{L}^+\big) < \tau_{\widehat{S}=S}\big] + Q_\omega\big[\tau\big(\widehat{L}^-\big) \vee \tau\big(\widehat{L}^+\big) > n/10\big] \\ \leq\ & Q_\omega\big[\tau\big(\widehat{L}^-\big) < \tau_{\widehat{S}=S},\ \widehat{S}_0 < \widehat{b}(n)\big] + Q_\omega\big[\tau\big(\widehat{L}^+\big) < \tau_{\widehat{S}=S},\ \widehat{S}_0 \geq \widehat{b}(n)\big] + (\log n)^{-3}. \end{aligned}$$

Now, observe that a.s. under Q_ω, $S_0 = \widehat{b}(n)$ by (90) and has the same parity as n by (86), and $\widehat{S}_0$ also has the same parity as n by (90) and (89). Hence the process $\big(\widehat{S}_k - S_k\big)_{k\in\mathbb{N}}$ starts at $\big(\widehat{S}_0 - \widehat{b}(n)\big) \in (2\mathbb{Z})$, and it only makes jumps belonging to $\{-2, 0, 2\}$, so up to time $\tau_{\widehat{S}=S} - 1$ it is < 0 (resp. > 0) on $\big\{\widehat{S}_0 < \widehat{b}(n)\big\}$ (resp. on $\big\{\widehat{S}_0 > \widehat{b}(n)\big\}$), and in particular at time $\tau\big(\widehat{L}^-\big)$ on $\big\{\tau\big(\widehat{L}^-\big) < \tau_{\widehat{S}=S},\ \widehat{S}_0 < \widehat{b}(n)\big\}$ (resp. at time $\tau\big(\widehat{L}^+\big)$ on $\big\{\tau\big(\widehat{L}^+\big) < \tau_{\widehat{S}=S},\ \widehat{S}_0 > \widehat{b}(n)\big\} = \big\{\tau\big(\widehat{L}^+\big) < \tau_{\widehat{S}=S},\ \widehat{S}_0 \geq \widehat{b}(n)\big\}$; for the last equality, notice that $\tau_{\widehat{S}=S} = 0$ on $\big\{\widehat{S}_0 = \widehat{b}(n)\big\}$). So,

$$\begin{aligned} & Q_\omega\big[\tau_{\widehat{S}=S} > n/10\big] \\ \leq\ & Q_\omega\big[\tau\big(\widehat{L}^-\big) < \tau_{\widehat{S}=S},\ \widehat{S}_{\tau(\widehat{L}^-)} < \widehat{L}^-\big] + Q_\omega\big[\tau\big(\widehat{L}^+\big) < \tau_{\widehat{S}=S},\ \widehat{S}_{\tau(\widehat{L}^+)} > \widehat{L}^+\big] + (\log n)^{-3} \\ \leq\ & Q_\omega\big[\tau\big(\widehat{L}^-\big) < \tau_{\widehat{S}=S},\ \widehat{S}_{2\lfloor\tau(\widehat{L}^-)/2\rfloor} \leq \widehat{L}^-\big] + Q_\omega\big[\tau\big(\widehat{L}^+\big) < \tau_{\widehat{S}=S},\ \widehat{S}_{2\lfloor\tau(\widehat{L}^+)/2\rfloor} \geq \widehat{L}^+\big] \\ & + (\log n)^{-3} \\ \leq\ & \widehat{\nu}\big(\big[M^-, \widehat{L}^-\big]\big) + \widehat{\nu}\big(\big[\widehat{L}^+, M^+\big]\big) + (\log n)^{-3}. \end{aligned} \tag{100}$$

Indeed, the last inequality is a consequence of the fact that $Q_\omega\big(\widehat{S}_{2k} = x\big) = P_{\widehat{\omega}}^{\widehat{\nu}}\big(\widehat{S}_{2k} = x\big) = \widehat{\nu}(x)$ for all $x \in \mathbb{Z}$ and all (deterministic) $k \in \mathbb{N}$ (see (90) and the explanations after (89)), and from the independence of $\widehat{S}$ with S (and its hitting times $\tau(.)$) up to time $\tau_{\widehat{S}=S}$. Hence, (100) together with Lemma 4.4 prove (98).

Finally, (90), followed by (10) and (11) give for every $n \geq n_3$ and $\omega \in E_3^{(n)}$,

$$\begin{aligned} Q_\omega\big[\tau_{\text{exit}} \leq n\big] &\leq Q_\omega[\tau(M^-) \wedge \tau(M^+) \leq n] = P_\omega^{\widehat{b}(n)}[\tau(M^-) \wedge \tau(M^+) \leq n] \\ &\leq P_\omega^{\widehat{b}(n)}[\tau(M^-) \leq n] + P_\omega^{\widehat{b}(n)}[\tau(M^+) \leq n] \\ &\leq 2(n+1)\varepsilon_0^{-2} \exp[-(\log n + C_2 \log_2 n)] \leq 4\varepsilon_0^{-2}(\log n)^{-C_2} \leq (\log n)^{-3}, \end{aligned} \tag{101}$$

since $\min_{[M^-, b_{\log n}]} V = \min_{[b_{\log n}, M^+]} V = V(b_{\log n})$, $V(M^\pm) - V(b_{\log n}) \geq \log n + C_2 \log_2 n$ on $E_3^{(n)}$, $|b_{\log n} - \widehat{b}(n)| \leq 1$, $|V(u) - V(u-1)| \leq \log(\varepsilon_0^{-1})$ for $u \in \mathbb{Z}$ by (16), $\log n > 2\varepsilon_0^{-1}$ since $n \geq n_3 \geq 3$, and $C_2 > 9$. This proves (99). □

Also, the following lemma will be useful to prove Lemma 4.7 (see (110)).

Lemma 4.6. *We have,*

$$\forall n \geq n_3,\ \forall z \in \mathbb{Z},\ \forall \omega \in E_3^{(n)} \cap E_4^{(n)}(z) \cap E_5^{(n)}, \qquad \widehat{\nu}_n(z) P_\omega\big[\tau(\widehat{b}(n)) \geq n/10\big] \leq (\log n)^{-3}. \tag{102}$$

Proof: Let $n \geq n_3$, $z \in \mathbb{Z}$ and $\omega \in E_3^{(n)} \cap E_4^{(n)}(z) \cap E_5^{(n)}$. We treat separately the three different cases defining $E_4^{(n)}(z)$.

First case: if in addition $\omega \in \{V(z) - V(b_{\log n}) \geq 5 \log_2 n\}$, we have by ellipticity,

$$\widehat{\nu}_n(z) \leq \big(e^{-V(z)} + e^{-V(z-1)}\big) e^{V(b_{\log n})} \leq (1 + \varepsilon_0^{-1}) e^{-[V(z) - V(b_{\log n})]} \leq 2\varepsilon_0^{-1} (\log n)^{-5} \leq (\log n)^{-3}$$

since $n \geq n_3$, which proves (102) in this case.

Second case: if $\omega \in E_-^{(n)} \cap \big\{ \max_{[b_{\log n}, 0]} V < V(x_1) - 9 \log_2 n \big\}$, we have $b_{\log n} = x_0 \leq 0$ and either $\widehat{b}(n) = 1$, or $-(\log n)^3 - 1 \leq \widehat{b}(n) \leq 0 < x_1 < x_2$ since $\omega \in E_5^{(n)}$ and $\big|\widehat{b}(n) - b_{\log n}\big| \leq 1$. We start with this second sub-case $\widehat{b}(n) \leq 0$. We have by (7),

$$\begin{aligned} P_\omega\big[\tau(x_2) < \tau\big(\widehat{b}(n)\big)\big] &\leq \big(\big|\widehat{b}(n)\big| + 1\big) \exp\big[\max_{[\widehat{b}(n), 0]} V - V(x_1)\big] \\ &\leq 2(\log n)^3 \exp\big[-9 \log_2 n + \log \varepsilon_0^{-1}\big] \leq (\log n)^{-4} \end{aligned} \tag{103}$$

by ellipticity since $\big|\widehat{b}(n) - b_{\log n}\big| \leq 1$ and because $n \geq n_3$ so $\log n \geq 2\varepsilon_0^{-1} \geq 2$.

Also, by Lemma 4.1 applied with $\xi_1 = 2C_1 > 19$, $\xi_2 = 1/10$, $\alpha = 3$, $a = \widehat{b}(n) \leq 0 < b = x_1 < c = x_2$, $h = \log n > \widehat{h}_2(2C_1, 1/10)$ because $n \geq n_3$, and $x = 0$, since its hypothesis **(i)** is satisfied because $b_{\log n} \leq 0$ and so x_1 is a left $(\log n)$-maximum, and there is no left $(h - \xi_1 \log h) = (h_n - C_1 \log_2 n)$-extremum in $]x_1, x_2[$ nor in $]x_0, x_1[$ by $E_3^{(n)}$ (as explained after (94) since x_0, x_1 and x_2 are consecutive left $(h_n - C_1 \log_2 n)$-extrema) and so hypotheses **(ii)** and **(iii)** of this lemma are satisfied (e.g. if **(ii)** was not satisfied, there would be a left $(h_n - C_1 \log_2 n)$-maximum in $]x_1, x_2[$), and hypothesis **(iv)** is satisfied with $\alpha = 3$ thanks to $E_5^{(n)}$ and $\delta_1 < 1$, so

$$\forall \omega \in E_3^{(n)} \cap E_5^{(n)}, \quad (b_{\log n} \leq 0 \text{ and } \widehat{b}(n) \leq 0) \Rightarrow P_\omega\big[\tau\big(\widehat{b}(n)\big) \wedge \tau(x_2) \geq n/10\big] \leq (\log n)^{-4}. \tag{104}$$

This and (103) lead to $P_\omega\big[\tau\big(\widehat{b}(n)\big) \geq n/10\big] \leq 2(\log n)^{-4} \leq (\log n)^{-3}$ for every ω of this second subcase since $n \geq n_3$.

We now turn to the other subcase, that is, we assume that $\widehat{b}(n) = 1$. Then, $b_{\log n} = x_0 = 0$ since $\omega \in E_-^{(n)}$ and $\big|\widehat{b}(n) - b_{\log n}\big| \leq 1$. In this subcase we have, using (7), Markov inequality and (8) in the second inequality,

$$P_\omega\big[\tau\big(\widehat{b}(n)\big) \geq n/10\big] = P_\omega\big[\tau(1) \geq n/10\big]$$

$$\begin{aligned}
&\leq P_\omega\big[\tau(x_{-1}) < \tau(1)\big] + P_\omega\big[\tau(x_{-1}) \wedge \tau(1) \geq n/10\big] \\
&\leq \exp[V(x_0) - V(x_{-1})] + 10n^{-1}\varepsilon_0^{-1}(1-x_{-1})^2 \exp[V(0) - \min_{[x_{-1},0]} V] \\
&\leq n^{-1} + 40\varepsilon_0^{-1}n^{-1}(\log n)^6 \leq (\log n)^{-4} \qquad (105)
\end{aligned}$$

for every ω of this subcase since $n \geq n_3$, $H[T_0(V, \log n)] = V(x_{-1}) - V(x_0) \geq \log n$, $|x_{-1}| \leq (\log n)^3$ since $\omega \in E_5^{(n)}$ and $\min_{[x_{-1},0]} V = \min_{[x_{-1},x_0]} V = V(x_0) = V(0) = 0$. So, (102) is proved in this second case (whenever $\widehat{b}(n) = 1$ or not), since $\widehat{\nu}_n(z) \leq 1$, for all $n \geq n_3$, $z \in \mathbb{Z}$ and $\omega \in E_-^{(n)} \cap \big\{ \max_{[b_{\log n},0]} V < V(x_1) - 9\log_2 n \big\} \cap E_3^{(n)} \cap E_5^{(n)}$.

Third case: finally, the proof is similar when $\omega \in E_+^{(n)} \cap \big\{ \max_{[0,b_{\log n}]} V < V(x_0) - 9\log_2 n \big\}$ with x_{-1} instead of x_2 and x_1 exchanged with x_0, which ends the proof of the lemma. □

We now have all the ingredients to approximate the quenched probability $P_\omega(S_n = z)$ by the invariant probability measure $\widehat{\nu}_n(z)$ for $\omega \in E_C^{(n)}(z)$ (defined in (85)), uniformly for $n \geq n_3$ (recall that $P_\omega(S_n = z)$ and $\widehat{\nu}_n(z)$ are equal to 0 if z and n do not have the same parity by (89)).

Lemma 4.7. *We have,*

$$\forall n \geq n_3,\ \forall z \in \mathbb{Z},\ \forall \omega \in E_C^{(n)}(z), \qquad \big|P_\omega(S_n = z) - \widehat{\nu}_n(z)\big| \leq 5(\log n)^{-3}. \qquad (106)$$

Proof: Let $n \geq n_3$, $z \in \mathbb{Z}$ and $\omega \in E_C^{(n)}(z)$. For $u \in \mathbb{Z}$, we define $V_u = V_u^+$ and V_u^- by $V_u(.) := V(u + .) - V(u)$ and $V_u^\pm(.) := V(u \pm .) - V(u)$. Since $\omega \in E_6^{(n)}$, $T_{V^\pm_{b_{\log n}}}(\log n) > \Gamma_n$. Also, $|b_{\log n} - z| \leq \Gamma_n$ because $\omega \in E_7^{(n)}(z)$, so ($M^\pm$ being defined in (87)),

$$z \leq b_{\log n} + \Gamma_n < b_{\log n} + T_{V^+_{b_{\log n}}}(\log n) \leq M^+. \qquad (107)$$

Thus $z < M^+$, and similarly, $z > M^-$, and so $z \in]M^-, M^+[$.

Observe that for $k \in [n/10, n] \cap (2\mathbb{N})$,

$$\begin{aligned}
P_\omega^{\widehat{b}(n)}[S_k = z] = Q_\omega[S_k = z] &\geq Q_\omega\big[S_k = z,\ \tau_{\widehat{S}=S} \leq n/10 \leq k \leq n < \tau_{\text{exit}}\big] \\
&= Q_\omega\big[\widehat{S}_k = z,\ \tau_{\widehat{S}=S} \leq n/10 \leq k \leq n < \tau_{\text{exit}}\big] \\
&\geq Q_\omega\big[\widehat{S}_k = z\big] - Q_\omega\big[\tau_{\widehat{S}=S} > n/10\big] - Q_\omega\big[\tau_{\text{exit}} \leq n\big] \\
&\geq \widehat{\nu}(z) - 3(\log n)^{-3}, \qquad (108)
\end{aligned}$$

where we used (90) in the first equality, $S_k = \widehat{S}_k$ for $k \in \big[\tau_{\widehat{S}=S}, \tau_{\text{exit}}\big[$ in the second one, and $Q_\omega\big[\widehat{S}_k = x\big] = P_{\widehat{\omega}}^{\widehat{\nu}}\big[\widehat{S}_k = x\big] = \widehat{\nu}(x)$ for all $x \in \mathbb{Z}$ since k is even (see (90) and the remark after (89)), and Lemma 4.5 in the last line since $n \geq n_3$ and $\omega \in E_C^{(n)}(z)$.

Similarly, for every $k \in [n/10, n] \cap (2\mathbb{N})$,

$$\begin{aligned}
P_\omega^{\widehat{b}(n)}[S_k = z] \leq& Q_\omega\big[S_k = z,\ \tau_{\widehat{S}=S} \leq n/10,\ \tau_{\text{exit}} > n\big] + Q_\omega\big[\tau_{\widehat{S}=S} > n/10\big] + Q_\omega\big[\tau_{\text{exit}} \leq n\big] \\
\leq& Q_\omega\big[\widehat{S}_k = z\big] + 3(\log n)^{-3} = \widehat{\nu}(z) + 3(\log n)^{-3}. \qquad (109)
\end{aligned}$$

We have, applying the strong Markov property in the second line,

$$\begin{aligned}
P_\omega[S_n = z] &\geq P_\omega\big[S_n = z,\ \tau\big(\widehat{b}(n)\big) < n/10\big] \\
&= E_\omega\big[\mathbf{1}_{\{\tau(\widehat{b}(n)) < n/10\}} P_\omega^{\widehat{b}(n)}[S_k = z]_{|k = n - \tau(\widehat{b}(n))}\big] \\
&\geq E_\omega\big[\mathbf{1}_{\{\tau(\widehat{b}(n)) < n/10\}} \big(\widehat{\nu}(z) - 3(\log n)^{-3}\big)\big]
\end{aligned}$$

$$\begin{aligned} &\geq \widehat{\nu}(z) - \widehat{\nu}(z) P_\omega\big[\tau\big(\widehat{b}(n)\big) \geq n/10\big] - 3(\log n)^{-3} \\ &\geq \widehat{\nu}(z) - 4(\log n)^{-3} \end{aligned} \tag{110}$$

where we used (108) in the second inequality since $\big(n - \tau\big(\widehat{b}(n)\big)\big) \in [9n/10, n] \cap (2\mathbb{N})$ because $\widehat{b}(n)$, and then $\tau\big(\widehat{b}(n)\big)$, has the same parity as n by (86), and Lemma 4.6 in the last inequality, since $n \geq n_3$ and $\omega \in E_C^{(n)}$.

Similarly, using (109) instead of (108), we get

$$\begin{aligned} P_\omega\big[S_n = z,\ \tau\big(\widehat{b}(n)\big) < n/10\big] &\leq E_\omega\big[\mathbf{1}_{\{\tau(\widehat{b}(n)) < n/10\}}\big(\widehat{\nu}(z) + 3(\log n)^{-3}\big)\big] \\ &\leq \widehat{\nu}(z) + 3(\log n)^{-3}. \end{aligned} \tag{111}$$

We now assume that $b_{\log n} \leq 0$, and so $b_{\log n} = x_0$ and $M^+ = x_1$. Also, we have once more

$$P_\omega\big[\tau\big(\widehat{b}(n)\big) \wedge \tau(x_2) \geq n/10\big] \leq (\log n)^{-4}. \tag{112}$$

Indeed this is proved in (104) when $\widehat{b}(n) \neq 1$ since $n \geq n_3$, whereas when $\widehat{b}(n) = 1$, the left hand side of (112) is equal to $P_\omega\big[\tau\big(\widehat{b}(n)\big) \geq n/10\big]$, which is $\leq (\log n)^{-4}$ by (105) since $\widehat{b}(n) = 1 < x_2$ in this case.

Moreover for $0 \leq k \leq n$, using $z < M^+ = x_1 < x_2$ (see (107)), we have by (11) and ellipticity (16), and since $V(x_1) - \min_{[x_1,x_2]} V = H[T_1(V, \log n)] = H[T_1(V, h_n - C_1 \log_2 n)] \geq \log n + C_2 \log_2 n$ because $\omega \in E_3^{(n)}$ (see also Remark 4.2),

$$\begin{aligned} P_\omega^{x_2}(S_k = z) &\leq P_\omega^{x_2}[\tau(x_1) < k] \leq n \exp[-H(T_1[V, \log n)] + \log \varepsilon_0^{-1}] \\ &\leq \varepsilon_0^{-1} (\log n)^{-C_2} \leq (\log n)^{-3} \end{aligned}$$

since $C_2 > 9$ and $\log n > \varepsilon_0^{-1}$ because $n \geq n_3$. Hence by the strong Markov property,

$$\mathsf{P}_\omega\big[S_n = z,\ \tau(x_2) < n/10\big] = \mathsf{E}_\omega\big[\mathbf{1}_{\{\tau(x_2) < n/10\}} P_\omega^{x_2}(S_k = z)_{|k = n - \tau(x_2)}\big] \leq (\log n)^{-3}. \tag{113}$$

Finally, (111), (112) and (113) give

$$\begin{aligned} P_\omega(S_n = z) &\leq P_\omega\big[\tau\big(\widehat{b}(n)\big) \wedge \tau(x_2) \geq n/10\big] + P_\omega\big[S_n = z,\ \tau\big(\widehat{b}(n)\big) < n/10\big] \\ &\qquad + \mathsf{P}_\omega\big[S_n = z,\ \tau(x_2) < n/10\big] \qquad (114) \\ &\leq \widehat{\nu}(z) + 5(\log n)^{-3}. \qquad (115) \end{aligned}$$

We prove similarly this inequality $P_\omega(S_n = z) \leq \widehat{\nu}(z) + 5(\log n)^{-3}$ by symmetry when $b_{\log n} > 0$, exchanging x_0 and x_1 and replacing x_2 by x_{-1} in (112) and (113) since $z > M^- = x_0 > x_{-1}$ in this case, and using (10) instead of (11).

Combining this with (115) and (110) proves (106). □

4.5. **Upper bound of the annealed probability: main contribution.** The aim of this subsection is to give an upper bound of the annealed probability of $\{S_n = z\}$ on the event for which we used the coupling, that is, on $E_C^{(n)}(z)$. More precisely, we prove the following estimate.

Proposition 4.8. *We have, under the hypotheses of Theorem 1.1, as $n \to +\infty$,*

$$\sup_{z \in (2\mathbb{Z}+n)} \left[\mathbb{P}\big(S_n = z, E_C^{(n)}(z)\big) - \frac{2\sigma^2}{(\log n)^2} \varphi_\infty\left(\frac{\sigma^2 z}{(\log n)^2}\right)\right] \leq o\big((\log n)^{-2}\big). \tag{116}$$

The strategy of the proof is to use Lemma 4.7 to dominate $\mathbb{P}\big(S_n = z, E_C^{(n)}(z)\big)$ by some quantity expressed in terms of left $(\log n)$-slopes $T_i(V, \log n)$ for $-1 \le i \le 1$ (see e.g. (121), (122), (124) and (125)), then use our Theorems 2.4 and 2.5 to obtain an expression with $\mathcal{T}^{\uparrow}_{V,\log n}$ and $\mathcal{T}^{\downarrow}_{V,\log n}$, then Lemma 2.6 to make appear the quantity $\mathsf{P}\big(b_{\log n} = z_n^+\big)$ for some $z_n^+ \approx z$, which, in turn, can be approximated by the expression with φ_∞ in (116) thanks to Theorem 1.4.

Proof: We assume that the hypotheses of Theorem 1.1 are satisfied. Let $n \ge n_3$ and $z \in (2\mathbb{Z}+n)$. Using Lemma 4.7 in the last line, $E_C^{(n)}(z)$ being defined in (85), we have

$$\begin{aligned}\mathbb{P}\big(S_n = z, E_C^{(n)}(z)\big) &= \sum_{k=-\Gamma_n}^{\Gamma_n} \mathbb{E}\big[\mathbf{1}_{E_C^{(n)}(z)\cap\{b_{\log n}=z+k\}} P_\omega(S_n = z)\big] \\ &= f_1(n,z) + \sum_{k=-\Gamma_n}^{\Gamma_n} J_0(k,n,z), \end{aligned} \tag{117}$$

where $|f_1(n,z)| \le 5(\log n)^{-3}$ and (writing $E_i^{(n)}(z)$ instead of $E_i^{(n)}$ even for $i \ne 3$ for simplicity),

$$J_0(k,n,z) := \mathbb{E}\big[\mathbf{1}_{E_C^{(n)}(z)\cap\{b_{\log n}=z+k\}} \widehat{\nu}_n(z)\big] = \mathsf{E}\big[\mathbf{1}_{\cap_{i=3}^6 E_i^{(n)}(z)\cap\{b_{\log n}=z+k\}} \widehat{\nu}_n(z)\big]. \tag{118}$$

Notice that, using (107) and in the remark below, we have if $\omega \in E_C^{(n)}(z) \cap \{b_{\log n} = z+k\}$ with $|k| \le \Gamma_n$ ($M^\pm$ being defined in (87)),

$$M^- < z = b_{\log n} - k < M^+. \tag{119}$$

Hence, we have on $E_C^{(n)}(z) \cap \{b_{\log n} = z+k\}$ with $|k| \le \Gamma_n$, using the definitions of $\widehat{\nu}_n$ and $\widehat{\mu}_n$ (see (89) and (88)),

$$\widehat{\nu}_n(z) = \frac{\widehat{\mu}_n(z)}{\widehat{\mu}_n(2\mathbb{Z} + \mathbf{1}_{2\mathbb{N}+1}(n))} = \frac{\widehat{\mu}_n(b_{\log n} - k)}{\sum_{i=M^-}^{M^+-1} e^{-V(i)}} = \frac{e^{-V(b_{\log n}-k)} + e^{-V(b_{\log n}-k-1)}}{\sum_{i=M^-}^{M^+-1} e^{-V(i)}}$$

since z and n have the same parity, and $\widehat{\mu}_n(2\mathbb{Z}) = \widehat{\mu}_n(2\mathbb{Z}+1) = \sum_{i=M^-}^{M^+-1} e^{-V(i)}$, and where we used the definition (88) of $\widehat{\mu}_n$ on $]M^-, M^+[$ and (119) in the last equality.

Now, we define for $j \in \{0,1\}$,

$$J_2^{\pm}(k,n,z,j) := \mathsf{E}\left[\mathbf{1}_{E_\pm^{(n)}\cap\{b_{\log n}=z+k\}\cap E_6^{(n)}} \frac{e^{-V(b_{\log n}-k-j)}}{\sum_{i=M^-}^{M^+-1} e^{-V(i)}}\right]. \tag{120}$$

Notice that for $k \in \mathbb{Z}$ such that $|k| \le \Gamma_n$, if $k \le -z$ then $\{b_{\log n} = z+k\} \subset \{b_{\log n} \le 0\} = E_-^{(n)}$, so $J_0(k,n,z) \le J_2^-(k,n,z,0) + J_2^-(k,n,z,1)$, whereas if $k > -z$, then $\{b_{\log n} = z+k\} \subset \{b_{\log n} > 0\} = E_+^{(n)}$, so $J_0(k,n,z) \le J_2^+(k,n,z,0) + J_2^+(k,n,z,1)$. So we have, thanks to (117),

$$\mathbb{P}\big(S_n = z, E_C^{(n)}(z)\big) \le J_3(n,z,0) + J_3(n,z,1) + 5(\log n)^{-3}, \tag{121}$$

where for $j \in \{0,1\}$,

$$J_3(n,z,j) := \sum_{k=-\Gamma_n}^{\Gamma_n} \Big[J_2^-(k,n,z,j)\mathbf{1}_{\{k+z\le 0\}} + J_2^+(k,n,z,j)\mathbf{1}_{\{k+z>0\}}\Big]. \tag{122}$$

We first consider $k \le -z$, with $|k| \le \Gamma_n$. Hence on $\{b_{\log n} = z+k\}$, we have $b_{\log n} \le 0$, so $\omega \in E_-^{(n)}$, thus $M^- = x_{-1}$, $b_{\log n} = x_0$ and $M^+ = x_1$ (recall that $x_i = x_i(V, \log n)$, $i \in \mathbb{Z}$

in this section). So for $j \in \{0,1\}$, recalling that for $u \in \mathbb{Z}$, $V_u(.) := V(u+.) - V(u)$, and $V_u^-(.) := V(u-.) - V(u)$, we have

$$J_2^-(k,n,z,j) = \mathsf{E}\Big[\mathbf{1}_{E_-^{(n)} \cap \{x_0 = z+k\} \cap E_6^{(n)}} \frac{e^{-V(x_0-k-j)}}{\sum_{i=x_{-1}}^{x_1-1} e^{-V(i)}}\Big]$$
$$= \mathsf{E}\Big[\mathbf{1}_{\left\{V_{x_0}(x_1-x_0) \geq \log n,\ T_{V_{x_0}}(\log n) > \Gamma_n,\ T_{V_{x_0}^-}(\log n) > \Gamma_n\right\}} \frac{e^{-V_{x_0}(-k-j)} \mathbf{1}_{\{x_0 = z+k\}}}{\sum_{i=x_{-1}-x_0}^{x_1-x_0-1} e^{-V_{x_0}(i)}}\Big]. \tag{123}$$

Notice that $(V_{x_0}(i),\ 0 \leq i \leq x_1 - x_0) = \theta(T_0(V, \log n))$ and that $(V_{x_0}^-(i),\ 0 \leq i \leq x_0 - x_{-1}) = (V(x_0 - i) - V(x_0),\ 0 \leq i \leq x_0 - x_{-1}) = (V_{x_{-1}}(x_0 - x_{-1} - i) - V_{x_{-1}}(x_0 - x_{-1}),\ 0 \leq i \leq x_0 - x_{-1}) = \zeta[\theta(T_{-1}(V, \log n))]$, with ζ defined in (24). Also, on the event in (123), $|k| \leq \Gamma_n$ implies that $-k - j \leq \Gamma_n + 1 \leq T_{V_{x_0}}(\log n) \leq x_1 - x_0 = \ell[\theta(T_0(V, \log n))]$, and similarly $k + j \leq x_0 - x_{-1} = \ell[\zeta(\theta(T_{-1}(V, \log n)))]$. Hence, with the following notation for slopes v and t,

$$\varphi_v(t) := \mathbf{1}_{\{t(\ell(t)) \geq \log n,\ T_t(\log n) \wedge T_v(\log n) > \Gamma_n\}} \frac{e^{-t(-k-j)} \mathbf{1}_{\{k+j \leq 0\}} + e^{-v(k+j)} \mathbf{1}_{\{k+j > 0\}}}{\sum_{i=1}^{\ell(v)} e^{-v(i)} + \sum_{i=0}^{\ell(t)-1} e^{-t(i)}}, \tag{124}$$

in which we do not write the dependency on n, k, j to simplify the notations, we have for our fixed n, k and j since $|k| \leq \Gamma_n$,

$$J_2^-(k,n,z,j) = \mathsf{E}\big[\varphi_{\zeta[\theta(T_{-1}(V, \log n))]}[\theta(T_0(V, \log n))] \mathbf{1}_{\{x_0 = z+k\}}\big]. \tag{125}$$

In the rest of this section, all the slopes considered, such as $\mathcal{T}_{V,h}^{\uparrow}$, $\mathcal{T}_{V,h}^{\downarrow}$, $\mathcal{T}_{V_-,h}^{\uparrow *}$, etc, are with $h = \log n$, and we remove this subscript h to simplify the notation. That is, $\mathcal{T}_V^{\uparrow}$ denotes $\mathcal{T}_{V,\log n}^{\uparrow}$, $\mathcal{T}_V^{\downarrow}$ denotes $\mathcal{T}_{V,\log n}^{\downarrow}$, etc. Due to Theorem 2.4 **(i)**, conditionally on $E_-^{(n)}$, $\zeta[\theta(T_{-1}(V, \log n))]$ is independent of $(\theta[T_0(V, \log n)], x_0)$ and has the same law as $\zeta(\mathcal{T}_V^{\downarrow})$ (under P) and so as $\mathcal{T}_{V_-}^{\uparrow *}$ by Proposition 2.12. Hence, we get, since $\varphi_v[\theta(T_0(V, \log n))] \mathbf{1}_{E_+^{(n)}} = 0$ for any v,

$$J_2^-(k,n,z,j) = \mathsf{E}\Big[\mathsf{E}\big(\varphi_v[\theta(T_0(V, \log n))] \mathbf{1}_{\{x_0 = z+k\}}\big)_{|v = \mathcal{T}_{V_-}^{\uparrow *}}\Big].$$

Thus, applying the (renewal) Theorem 2.5 eq. (29) with $h = \log n$, $\varphi = \varphi_v$, $\Delta_0 = \{z+k\}$, $\Delta_1 = \mathbb{Z}$ (notice that $\varphi_v(t) = 0$ if t is a downward slope whereas $\mathbf{1}_{\{t(\ell(t)) \geq \log n\}} = 1$ when t is an upward $(\log n)$-slope), we get, $\mathcal{T}_V^{\uparrow}$ and $\mathcal{T}_{V_-}^{\uparrow *}$ being here independent,

$$\begin{aligned} J_2^-(k,n,z,j) &= \mathsf{E}\left[\frac{\mathsf{E}\big(\sharp\{0 \leq i < \ell(\mathcal{T}_V^{\uparrow}),\ -i = z+k\} \varphi_v(\mathcal{T}_V^{\uparrow})\big)_{|v = \mathcal{T}_{V_-}^{\uparrow *}}}{\mathsf{E}\big[\ell(\mathcal{T}_V^{\uparrow}) + \ell(\mathcal{T}_V^{\downarrow})\big]}\right] \\ &= \mathsf{E}\Bigg(\mathbf{1}_{\left\{T_{\mathcal{T}_V^{\uparrow}}(\log n) \wedge T_{\mathcal{T}_{V_-}^{\uparrow *}}(\log n) > \Gamma_n\right\}} \\ &\qquad \frac{e^{-\mathcal{T}_V^{\uparrow}(-(k+j))} \mathbf{1}_{\{k+j \leq 0\}} + e^{-\mathcal{T}_{V_-}^{\uparrow *}(k+j)} \mathbf{1}_{\{k+j > 0\}}}{\sum_{i=1}^{\ell(\mathcal{T}_{V_-}^{\uparrow *})} e^{-\mathcal{T}_{V_-}^{\uparrow *}(i)} + \sum_{i=0}^{\ell(\mathcal{T}_V^{\uparrow})-1} e^{-\mathcal{T}_V^{\uparrow}(i)}} \frac{\mathbf{1}_{\{-z-k < \ell(\mathcal{T}_V^{\uparrow})\}}}{\mathsf{E}\big[\ell(\mathcal{T}_V^{\uparrow}) + \ell(\mathcal{T}_V^{\downarrow})\big]}\Bigg), \end{aligned} \tag{126}$$

where we used $\sharp\{0 \leq i < \ell(\mathcal{T}_V^{\uparrow}),\ -i = z+k\} = \mathbf{1}_{\{-z-k < \ell(\mathcal{T}_V^{\uparrow})\}}$ when $z + k \leq 0$.

We now assume that $k > -z$, with $|k| \leq \Gamma_n$. We have $b_{\log n} > 0$ on $\{b_{\log n} = z+k\}$, and so $\omega \in E_+^{(n)}$, thus $b_{\log n} = x_1$, $M^- = x_0$ and $M^+ = x_2$. So by (120), for $j \in \{0,1\}$,

$$J_2^+(k,n,z,j) = \mathsf{E}\Big[\mathbf{1}_{\left\{V_{x_1}(x_0 - x_1) \geq \log n,\ x_1 = z+k,\ T_{V_{x_1}}(\log n) \wedge T_{V_{x_1}^-}(\log n) > \Gamma_n\right\}} \frac{e^{-V_{x_1}(-k-j)}}{\sum_{i=x_0-x_1}^{x_2-x_1-1} e^{-V_{x_1}(i)}}\Big].$$

Notice that $(V_{x_1}(i),\ 0 \le i \le x_2 - x_1) = \theta(T_1(V, \log n))$ and that $(V_{x_1}(-i),\ 0 \le i \le x_1 - x_0) = (V(x_1 - i) - V(x_1),\ 0 \le i \le x_1 - x_0) = \zeta[\theta(T_0(V, \log n))]$. Hence, with

$$\varphi_v^+(t) := \mathbf{1}_{\{t(\ell(t)) \ge \log n,\ T_t(\log n) \wedge T_v(\log n) > \Gamma_n\}} \frac{e^{-v(-k-j)}\mathbf{1}_{\{k+j \le 0\}} + e^{-t(k+j)}\mathbf{1}_{\{k+j>0\}}}{\sum_{i=1}^{\ell(t)} e^{-t(i)} + \sum_{i=0}^{\ell(v)-1} e^{-v(i)}},$$

in which we do not write the dependency on n, k, j to simplify the notations, we have

$$J_2^+(k,n,z,j) \quad = \quad \mathsf{E}\big[\varphi^+_{\theta[T_1(V,\log n)]}[\zeta(\theta(T_0(V,\log n)))]\mathbf{1}_{\{x_1 = z+k\}}\big].$$

Since due to Theorem 2.4 **(ii)**, conditionally on $E_+^{(n)}$, $\theta(T_1(V, \log n))$ has the law $\mathscr{L}\big(\mathcal{T}_V^{\uparrow}\big)$, and is independent of $(\theta(T_0(V, \log n)), x_1)$, we have,

$$J_2^+(k,n,z,j) = \mathsf{E}\big[\mathsf{E}\big(\varphi_v^+[\zeta(\theta(T_0(V,\log n)))]\mathbf{1}_{\{x_1=z+k\}}\big)_{|v=\mathcal{T}_V^{\uparrow}}\big],$$

since $\varphi_v^+[\zeta(\theta(T_0(V, \log n)))]\mathbf{1}_{E_-^{(n)}} = 0$ for any v. Thus, applying the (renewal) Theorem 2.5 with $h = \log n$, $\varphi = \varphi_v^+ \circ \zeta$, $\Delta_0 = \mathbb{Z}$, $\Delta_1 = \{z + k\}$ (we use once more that $\varphi_v^+ \circ \zeta(t) = 0$ when t is a (translated) upward slope, since in this case $\zeta(t)$ is a downward slope), we get

$$J_2^+(k,n,z,j) = \mathsf{E}\left[\frac{\mathsf{E}\big(\sharp\{0 \le i < \ell\big(\mathcal{T}_V^{\downarrow}\big),\ \ell\big(T_V^{\downarrow}\big) - i = z+k\}\varphi_v^+ \circ \zeta\big(\mathcal{T}_V^{\downarrow}\big)\big)_{|v=\mathcal{T}_V^{\uparrow}}}{\mathsf{E}\big[\ell\big(\mathcal{T}_V^{\uparrow}\big) + \ell\big(\mathcal{T}_V^{\downarrow}\big)\big]}\right].$$

Recall that, by Proposition 2.12 **(ii)**, $\zeta(\mathcal{T}_V^{\downarrow}) =_{law} \mathcal{T}_{V_-}^{\uparrow *}$. Hence, $\mathcal{T}_V^{\uparrow}$ and $\mathcal{T}_{V_-}^{\uparrow *}$ being independent, and using $\ell\big(\mathcal{T}_V^{\downarrow}\big) = \ell\big(\zeta\big(\mathcal{T}_V^{\downarrow}\big)\big)$, we get

$$\begin{aligned} J_2^+(k,n,z,j) \quad = \quad & \mathsf{E}\Bigg(\mathbf{1}_{\left\{T_{\mathcal{T}_{V_-}^{\uparrow *}}(\log n) > \Gamma_n\right\}}\mathbf{1}_{\left\{T_{\mathcal{T}_V^{\uparrow}}(\log n) > \Gamma_n\right\}} \\ & \frac{e^{-\mathcal{T}_V^{\uparrow}(-(k+j))}\mathbf{1}_{\{k+j\le 0\}} + e^{-\mathcal{T}_{V_-}^{\uparrow *}(k+j)}\mathbf{1}_{\{k+j>0\}}}{\sum_{i=1}^{\ell(\mathcal{T}_{V_-}^{\uparrow *})} e^{-\mathcal{T}_{V_-}^{\uparrow *}(i)} + \sum_{i=0}^{\ell(\mathcal{T}_V^{\uparrow})-1} e^{-\mathcal{T}_V^{\uparrow}(i)}} \frac{\mathbf{1}_{\{z+k \le \ell(\mathcal{T}_{V_-}^{\uparrow *})\}}}{\mathsf{E}\big[\ell\big(\mathcal{T}_V^{\uparrow}\big) + \ell\big(\mathcal{T}_V^{\downarrow}\big)\big]}\Bigg), \end{aligned} \tag{127}$$

where we used $\sharp\{0 \le i < \ell\big(\mathcal{T}_V^{\downarrow}\big),\ \ell\big(T_V^{\downarrow}\big) - i = z+k\} = \mathbf{1}_{\{z+k \le \ell(\mathcal{T}_V^{\downarrow})\}}$ which becomes $\mathbf{1}_{\{z+k \le \ell(\mathcal{T}_{V_-}^{\uparrow *})\}}$ since $z + k > 0$ and $\mathbf{1}_{\{t(\ell(t)) \ge \log n\}} = 1$ for $t = \zeta\big(\mathcal{T}_V^{\downarrow}\big)$. Notice that the only difference between this formula and (126) is that $\mathbf{1}_{\{-z-k<\ell(\mathcal{T}_V^{\uparrow})\}}$ is replaced by $\mathbf{1}_{\{z+k \le \ell(\mathcal{T}_{V_-}^{\uparrow *})\}}$.

We now define

$$\begin{aligned} z_n^+ \quad &:= \quad \begin{cases} z + \Gamma_n & \text{if } z \le -\Gamma_n, \\ 0 & \text{if } -\Gamma_n < z \le \Gamma_n, \\ z - \Gamma_n & \text{if } z > \Gamma_n, \end{cases} \\ \psi_k\big(\mathcal{T}_V^{\uparrow}, \mathcal{T}_{V_-}^{\uparrow *}, z\big) \quad &:= \quad \begin{cases} \mathbf{1}_{\{-z-k<\ell(\mathcal{T}_V^{\uparrow})\}} & \text{if } z \le -\Gamma_n, \\ \mathbf{1}_{\{0<\ell(\mathcal{T}_V^{\uparrow})\}} & \text{if } -\Gamma_n < z \le \Gamma_n, \\ \mathbf{1}_{\{z+k \le \ell(\mathcal{T}_{V_-}^{\uparrow *})\}} & \text{if } z > \Gamma_n. \end{cases} \end{aligned}$$

Notice that in the case $z \le -\Gamma_n$, we have $z + k \le 0$ for every k in the sum in (122), so, using (126), we have for each $j \in \{0, 1\}$ (the inequality being an equality in this first case $z \le -\Gamma_n$),

$$J_3(n,z,j) \quad \le \quad \mathsf{E}\Bigg(\mathbf{1}_{\left\{T_{\mathcal{T}_V^{\uparrow}}(\log n) > \Gamma_n\right\}}\mathbf{1}_{\left\{T_{\mathcal{T}_{V_-}^{\uparrow *}}(\log n) > \Gamma_n\right\}}$$

$$\left.\frac{\sum_{k=-\Gamma_n}^{\Gamma_n}\big(e^{-\mathcal{T}_V^{\uparrow}(-(k+j))}\mathbf{1}_{\{k+j\le 0\}}+e^{-\mathcal{T}_{V_-}^{\uparrow *}(k+j)}\mathbf{1}_{\{k+j>0\}}\big)\psi_k\big(\mathcal{T}_V^{\uparrow},\mathcal{T}_{V_-}^{\uparrow *},z\big)}{\Big(\sum_{i=0}^{\ell(\mathcal{T}_V^{\uparrow})-1}e^{-\mathcal{T}_V^{\uparrow}(i)}+\sum_{i=1}^{\ell(\mathcal{T}_{V_-}^{\uparrow *})}e^{-\mathcal{T}_{V_-}^{\uparrow *}(i)}\Big)\mathsf{E}\big[\ell\big(\mathcal{T}_V^{\uparrow}\big)+\ell\big(\mathcal{T}_V^{\downarrow}\big)\big]}\right). \quad (128)$$

When $z>\Gamma_n$, we have $z+k>0$ for every k in the sum in (122). So, combining (122) and (127), inequality (128) remains true in this case (and is actually an equality in this second case).

Finally, assume that $-\Gamma_n<z\le\Gamma_n$. In this case, notice that the quantity $\mathbf{1}_{\{-z-k<\ell(\mathcal{T}_V^{\uparrow})\}}$ which appears in (126) for $k+z\le 0$, and the quantity $\mathbf{1}_{\{z+k\le\ell(\mathcal{T}_{V_-}^{\uparrow *})\}}$ which appears in (127) for $k+z>0$ are both dominated by $1=\mathbf{1}_{\{0<\ell(\mathcal{T}_V^{\uparrow})\}}=\psi_k\big(\mathcal{T}_V^{\uparrow},\mathcal{T}_{V_-}^{\uparrow *},z\big)$ P-a.s., so $J_2^-(k,n,z,j)$ and $J_2^+(k,n,z,j)$ are dominated by the same formula. So for $j\in\{0,1\}$, (128) also remains true in this case. So, (128) holds for every $z\in\mathbb{Z}$ and every $j\in\{0,1\}$.

Now, we notice that for every $-\Gamma_n\le k\le\Gamma_n$, we have $\psi_k\big(\mathcal{T}_V^{\uparrow},\mathcal{T}_{V_-}^{\uparrow *},z\big)=\mathbf{1}_{\{-z-k<\ell(\mathcal{T}_V^{\uparrow})\}}\le\mathbf{1}_{\{-z-\Gamma_n<\ell(\mathcal{T}_V^{\uparrow})\}}=\mathbf{1}_{\{-z_n^+<\ell(\mathcal{T}_V^{\uparrow})\}}$ when $z\le-\Gamma_n$, also $\psi_k\big(\mathcal{T}_V^{\uparrow},\mathcal{T}_{V_-}^{\uparrow *},z\big)=\mathbf{1}_{\{-z_n^+<\ell(\mathcal{T}_V^{\uparrow})\}}$ when $-\Gamma_n<z\le\Gamma_n$, whereas $\psi_k\big(\mathcal{T}_V^{\uparrow},\mathcal{T}_{V_-}^{\uparrow *},z\big)=\mathbf{1}_{\{z+k\le\ell(\mathcal{T}_{V_-}^{\uparrow *})\}}\le\mathbf{1}_{\{z-\Gamma_n\le\ell(\mathcal{T}_{V_-}^{\uparrow *})\}}=\mathbf{1}_{\{z_n^+\le\ell(\mathcal{T}_{V_-}^{\uparrow *})\}}$ when $z>\Gamma_n$.

Hence, (128) leads to, for every $j\in\{0,1\}$, $n\ge n_3$ and $z\in(2\mathbb{Z}+n)$, as explained below,

$$J_3(n,z,j)\le\frac{\mathsf{P}\big[-z_n^+<\ell\big(\mathcal{T}_V^{\uparrow}\big)\big]}{\mathsf{E}\big[\ell\big(\mathcal{T}_V^{\uparrow}\big)+\ell\big(\mathcal{T}_V^{\downarrow}\big)\big]}\mathbf{1}_{\{z\le\Gamma_n\}}+\frac{\mathsf{P}\big[z_n^+\le\ell(\mathcal{T}_{V_-}^{\uparrow *})\big]}{\mathsf{E}\big[\ell\big(\mathcal{T}_V^{\uparrow}\big)+\ell\big(\mathcal{T}_V^{\downarrow}\big)\big]}\mathbf{1}_{\{z>\Gamma_n\}}=\mathsf{P}\big(b_{\log n}=z_n^+\big). \quad (129)$$

Indeed, we first used $\Gamma_n+1\le T_{\mathcal{T}_V^{\uparrow}}(\log n)\le\ell(\mathcal{T}_V^{\uparrow})$ and similarly $\Gamma_n+1\le\ell(\mathcal{T}_{V_-}^{\uparrow *})$, so that $\sum_{k=-\Gamma_n}^{\Gamma_n}(\cdots+\ldots)\le\big(\sum_{i=0}^{\ell(\mathcal{T}_V^{\uparrow})-1}\cdots+\sum_{i=1}^{\ell(\mathcal{T}_{V_-}^{\uparrow *})}\ldots\big)$ in (128) to get the (first) inequality. Then, to get the following equality, we used eq. (44) of Lemma 2.6 when $z\le\Gamma_n$, and $\ell\big(\mathcal{T}_{V_-}^{\uparrow *}\big)=_{law}\ell\big(\zeta\big(\mathcal{T}_V^{\downarrow}\big)\big)=\ell\big(\mathcal{T}_V^{\downarrow}\big)$ by Proposition 2.12 **(ii)** and eq. (43) of Lemma 2.6 when $z>\Gamma_n$.

Now, let $\varepsilon>0$. By Theorem 1.4, there exists $n_4\ge n_3$ such that, for every $j\in\{0,1\}$, $n\ge n_4$ and $z\in(2\mathbb{Z}+n)$,

$$J_3(n,z,j)\le\mathsf{P}\big(b_{\log n}=z_n^+\big)\le\frac{\sigma^2}{(\log n)^2}\varphi_\infty\bigg(\frac{\sigma^2 z_n^+}{(\log n)^2}\bigg)+\varepsilon(\log n)^{-2}.$$

Now, recall that φ_∞ is uniformly continuous on $\mathbb{R}$ since φ_∞ is continuous on $\mathbb{R}$ and $\lim_{\pm\infty}\varphi_\infty=0$. Also, $\sup_{z\in\mathbb{Z}}|\sigma^2 z_n^+(\log n)^{-2}-\sigma^2 z(\log n)^{-2}|\to 0$ as $n\to+\infty$ because $\delta_1<2/3$. Thus, there exists $n_5\ge n_4$ such that for all $n\ge n_5$, $\sup_{z\in\mathbb{Z}}|\varphi_\infty(\sigma^2 z_n^+(\log n)^{-2})-\varphi_\infty(\sigma^2 z(\log n)^{-2})|\le\sigma^{-2}\varepsilon$. Hence,

$$\forall n\ge n_5,\ \forall z\in(2\mathbb{Z}+n),\forall j\in\{0,1\},\quad J_3(n,z,j)\le\frac{\sigma^2}{(\log n)^2}\varphi_\infty\bigg(\frac{\sigma^2 z}{(\log n)^2}\bigg)+\frac{2\varepsilon}{(\log n)^2}. \quad (130)$$

Finally, (121) and (130) lead to, for all $n\ge n_5$,

$$\forall z\in(2\mathbb{Z}+n),\quad\mathbb{P}\big(S_n=z,E_C^{(n)}(z)\big)\le\frac{2\sigma^2}{(\log n)^2}\varphi_\infty\bigg(\frac{\sigma^2 z}{(\log n)^2}\bigg)+\frac{4\varepsilon}{(\log n)^2}+\frac{5}{(\log n)^3}.$$

This gives (116), which proves the proposition. □

5. Proving that some environments or trajectories are negligible

The aim of this section is to prove that $\sup_{z\in\mathbb{Z}}\mathbb{P}\big(S_n=z,\big(E_C^{(n)}(z)\big)^c\big)$ is negligible compared to $(\log n)^{-2}$ as $n\to+\infty$ (recall $E_C^{(n)}(z)$ from (85)). To this aim we give upper bounds of the

probabilities of different events, most of them depending both on the environment and on the walk, except the event considered in Lemma 5.1.

5.1. **Contribution of $\big(E_4^{(n)}(z)\big)^c$.** As a warm up, we start with following estimate.

Lemma 5.1. *There exists $c_9 > 0$ such that*

$$\forall n \geq n_5,\, \forall z \in \mathbb{Z}, \qquad \mathsf{P}\big[\big(E_4^{(n)}(z)\big)^c \cap E_3^{(n)} \cap E_6^{(n)} \cap E_7^{(n)}(z)\big] \leq c_9 (\log_2 n)^3 (\log n)^{-3}.$$

Proof: Let $n \geq n_5$ and $z \in \mathbb{Z}$. We introduce

$$E_8^{(n)}(z) := \big(E_4^{(n)}(z)\big)^c \cap E_3^{(n)} \cap E_6^{(n)} \cap E_7^{(n)}(z), \qquad E_{8,\pm}^{(n)}(z) := E_\pm^{(n)} \cap E_8^{(n)}(z).$$

We first assume that $\omega \in E_{8,-}^{(n)}(z)$ (see Figure 6). Hence, $\omega \in E_-^{(n)} \cap \big(E_4^{(n)}(z)\big)^c$, so $b_{\log n} = x_0(V, \log n)$,

$$V(z) - V(b_{\log n}) < 5 \log_2 n \tag{131}$$

and

$$\begin{aligned} \max_{[b_{\log n}, 0]} V &\geq V[x_1(V, \log n)] - 9 \log_2 n = V(b_{\log n}) + H[T_0(V, \log n)] - 9 \log_2 n \\ &\geq V(b_{\log n}) + \log n + (C_2 - 9) \log_2 n > V(b_{\log n}) + \log n, \end{aligned} \tag{132}$$

since $H[T_0(V, \log n)] = H[T_0(V, h_n - C_1 \log_2 n)] \geq \log n + C_2 \log_2 n$ by Remark 4.2 because $\omega \in E_3^{(n)}$, and where we used $C_2 > 9$.

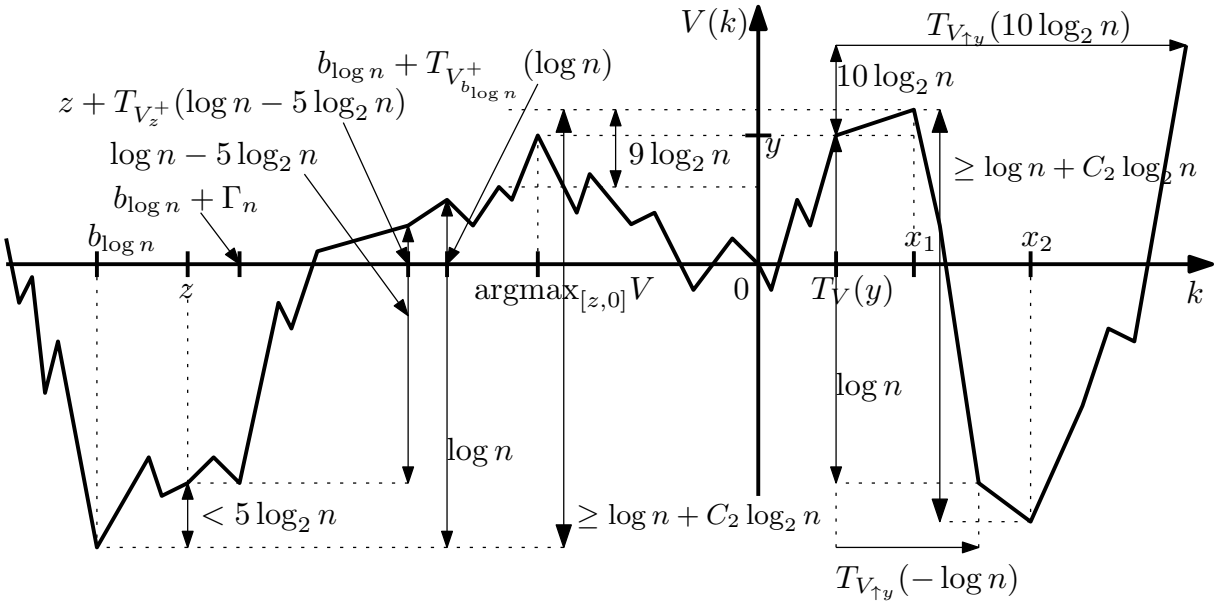

Figure 6. Schema of the potential V on $\omega \in E_{8,-}^{(n)}(z)$, with $x_i = x_i(V, \log n)$ and $y = \max_{[z,0]} V$.

Also, $\omega \in E_6^{(n)} \cap E_7^{(n)}(z)$, so as in (107), using (132) in the last inequality,

$$b_{\log n} - T_{V^-_{b_{\log n}}}(\log n) < z < b_{\log n} + T_{V^+_{b_{\log n}}}(\log n) \leq 0, \tag{133}$$

where for $x \in \mathbb{Z}$, $V_x^\pm(k) = V(x \pm k) - V(x)$, $k \in \mathbb{N}$, as before. This and (132) also lead to

$$\max_{[z,0]} V = \max_{[b_{\log n}, 0]} V \geq V[x_1(V, \log n)] - 9 \log_2 n. \tag{134}$$

We now introduce, for $y \geq 0$, $V_{\uparrow y}(k) := V[k + T_V(y)] - V[T_V(y)]$, $k \in \mathbb{N}$, and

$$\begin{aligned} E^{(n)}_{9,\pm}(z) &:= \big\{T_{V_z^{\pm}}(\log n - 5\log_2 n) < T_{V_z^{\pm}}(-5\log_2 n)\big\}, \\ E^{(n)}_{10}(y) &:= \big\{T_{V_{\uparrow y}}(-\log n) < T_{V_{\uparrow y}}(10\log_2 n)\big\}, \end{aligned}$$

where $\log n - 5\log_2 n > h_n > 0$ since $n \geq n_5 \geq n_3$ and $C_1 > 20$. Due to (131) and (133) and since $b_{\log n}$ is a left $(\log n)$-minimum, we have $\omega \in E^{(n)}_{9,-}(z) \cap E^{(n)}_{9,+}(z)$.

Also, notice that, using (134), $V[x_1(V, \log n)] = \max_{[0, x_2(V,\log n)]} V = \max_{[b_{\log n}, x_2(V,\log n)]} V \geq \max_{[b_{\log n}, 0]} V = \max_{[z,0]} V$ and $H[T_1(V, \log n)] = H[T_1(V, h_n - C_1 \log_2 n)] \geq \log n + C_2 \log_2 n$ with $C_2 > 9$ (by Remark 4.2 since $\omega \in E^{(n)}_3$). So, after hitting $\big[\max_{[z,0]} V, +\infty\big[$, the potential $(V(u),\ u \geq 0)$ cannot take values larger than $V[x_1(V, \log n)] \leq \max_{[z,0]} V + 9\log_2 n$ (see (134)) before going (down) to $x_2(V, \log n)$ with $V[x_2(V, \log n)] = V[x_1(V, \log n)] - H[T_1(V, \log n)] \leq V[x_1(V, \log n)] - \log n - C_2 \log_2 n \leq \max_{[z,0]} V - \log n$ by (134) and since $C_2 > 9$. Hence, $\omega \in E^{(n)}_{10}\big(\max_{[z,0]} V\big)$.

Finally, $z + T_{V_z^+}(\log n - 5\log_2 n) \leq b_{\log n} + T_{V^+_{b_{\log n}}}(\log n) \leq 0$ by (131) and (133), and $E^{(n)}_{9,-}(z) \cap E^{(n)}_{9,+}(z) \cap \{z + T_{V_z^+}(\log n - 5\log_2 n) \leq 0\}$ depend only on $V^- = (V(k),\ k \leq 0)$. Hence, conditioning by V^- to get the third line, using (17) for the forth, the independence of V_z^- and V_z^+ and $C_0 < \log_2 n$ since $n \geq n_5 \geq n_3$ for the fifth, and once more (17) for the sixth, we get for every $n \geq n_5$ and $z \in \mathbb{Z}$,

$$\begin{aligned} & \mathsf{P}\big[E^{(n)}_{8,-}(z)\big] \\ \leq\ & \mathsf{P}\big[E^{(n)}_{9,-}(z) \cap E^{(n)}_{9,+}(z) \cap \big\{z + T_{V_z^+}(\log n - 5\log_2 n) \leq 0\big\} \cap E^{(n)}_{10}\big(\max_{[z,0]} V\big)\big] \\ =\ & \mathsf{E}\big[\mathbf{1}_{E^{(n)}_{9,-}(z) \cap E^{(n)}_{9,+}(z) \cap \{z + T_{V_z^+}(\log n - 5\log_2 n) \leq 0\}} \mathsf{P}\big(E^{(n)}_{10}\big(\max_{[z,0]} V\big) | V^-\big)\big] \\ \leq\ & \mathsf{E}\big[\mathbf{1}_{E^{(n)}_{9,-}(z) \cap E^{(n)}_{9,+}(z)} (10\log_2 n + C_0)(\log n + 10\log_2 n + C_0)^{-1}\big] \\ \leq\ & \mathsf{P}\big[E^{(n)}_{9,-}(z)\big]\mathsf{P}\big[E^{(n)}_{9,+}(z)\big](11\log_2 n)(\log n)^{-1} \\ \leq\ & (6\log_2 n)^2(11\log_2 n)(\log n)^{-3}. \end{aligned} \tag{135}$$

We show similarly that $\mathsf{P}\big[E^{(n)}_{8,+}(z)\big] \leq 396(\log_2 n)^3(\log n)^{-3}$ for every $n \geq n_5$ and $z \in \mathbb{Z}$. This, combined with (135), ends the proof of the lemma. □

5.2. **Case when $b_{\log n}$ is far from z without subvalleys or small valleys.** In this subsection, we prove that the event constituted by environments and trajectories such that $b_{\log n}$ is far from z and $S_n = z$ while $E^{(n)}_3 \cap E^{(n)}_5$ holds is negligible. More precisely, we prove the following proposition.

Proposition 5.2. *There exist $c_{10} > 0$ and $n_6 \geq n_5$ such that, for all $n \geq n_6$,*

$$\forall z \in \mathbb{Z}, \qquad \mathbb{P}\big(S_n = z, |z - b_{\log n}| > \Gamma_n, E^{(n)}_3, E^{(n)}_5\big) \leq c_{10}(\log n)^{-2-\delta_1/2}. \tag{136}$$

Before giving a complete proof, we first introduce the different cases considered.

Organisation of the proof: We consider separately the case $\tau(b_{\log n}) \leq n$ (see Lemma 5.3) and the case $\tau(b_{\log n}) > n$ (see Lemmas 5.4, 5.5 and 5.6) since in this second case, we prove (see (150)) that with large enough probability, $\tau[x_2(V, \log n)] \leq n$ on $E^{(n)}_-$ and similarly $\tau[x_{-1}(V, \log n)] \leq n$ on $E^{(n)}_+$. So in the first case $\tau(b_{\log n}) \leq n$, S goes before time n to the bottom $b_{\log n}$ of the central

valley of height at least $\log n$, whereas in the second case $\tau(b_{\log n}) > n$, S goes before time n to the bottom of a neighbour valley of height at least $\log n$ with large probability. Figure 7 gives the schema of a potential for which S can go before time n, with relatively comparable quenched probability, to each of the bottoms of the two valleys "surrounding" the origin, $x_0(V, \log n)$ and $x_2(V, \log n)$ in this figure.

5.2.1. *Case when $\tau(b_{\log n}) \le n$.* In this subsection, we consider the case $\tau(b_{\log n}) \le n$ of Proposition 5.2, since for this case we can use an inequality coming from the reversibility of S. More precisely, we prove the following lemma.

Lemma 5.3. *There exists $c_{11} > 0$ and $n_6 \ge n_5$ such that for all $n \ge n_6$,*

$$\forall z \in \mathbb{Z}, \qquad \mathbb{P}\big(S_n = z, |b_{\log n} - z| > \Gamma_n, \tau(b_{\log n}) \le n, E_3^{(n)}, E_5^{(n)}\big) \le c_{11}(\log n)^{-2-\delta_1/2}. \tag{137}$$

Proof: In this proof, $\mathcal{T}_V^{\uparrow}$ and $\mathcal{T}_V^{\downarrow}$ denote respectively $\mathcal{T}_{V,\log n}^{\uparrow}$ and $\mathcal{T}_{V,\log n}^{\downarrow}$. By Lemma 2.15 applied with $h = \log n$, there exists $n_6 \ge n_5$ such that for all $n \ge n_6$, $\mathsf{E}\big[\ell\big(\mathcal{T}_V^{\uparrow}\big) + \ell\big(\mathcal{T}_V^{\downarrow}\big)\big] \ge c_7(\log n)^2$. Let $n \ge n_6$ and $z \in \mathbb{Z}$. We separate the proof into different cases, first when $z \notin [M^-, M^+]$, then when $z \in [M^-, M^+] - \big]\widehat{L}^-, \widehat{L}^+\big[$ and finally when $z \in \big]\widehat{L}^-, \widehat{L}^+\big[$, this last case being cut into four subcases, depending on the signs of $b_{\log n}$ and of $z - b_{\log n}$.

First step: we have, conditioning by ω and applying the strong Markov property at stopping time $\tau(\log n)$, recalling $M^{\pm}$ from (87) (with $x_i = x_i(V, \log n)$, see Figure 5),

$$\begin{aligned}
&\mathbb{P}\big(S_n = z, |b_{\log n} - z| > \Gamma_n, \tau(b_{\log n}) \le n, E_3^{(n)}, E_5^{(n)}, z \notin [M^-, M^+]\big)\\
\le\ &\mathbb{E}\big(\mathbf{1}_{E_3^{(n)}} \mathbf{1}_{\{\tau(b_{\log n}) \le n\}} P_\omega^{b_{\log n}}(S_k \notin [M^-, M^+])_{|k = n - \tau(b_{\log n})}\big)\\
\le\ &\mathsf{E}\big[\mathbf{1}_{E_3^{(n)}} P_\omega^{b_{\log n}}[\tau(M^-) \wedge \tau(M^+) \le n]\big] \le (\log n)^{-3},
\end{aligned} \tag{138}$$

where we used (101), which is still valid on $E_3^{(n)}$ for $n \ge n_3$ with $\widehat{b}(n)$ replaced by $b_{\log n}$, recalling that $n_6 \ge n_3$.

Second step: By reversibility (see (13)), we have for all $y \in \mathbb{Z}$, $k \in \mathbb{N}$ and a.s. every environment ω,

$$P_\omega^{b_{\log n}}(S_k = y) = P_\omega^{y}(S_k = b_{\log n}) \frac{\mu_\omega(y)}{\mu_\omega(b_{\log n})} \le \frac{e^{-V(y)} + e^{-V(y-1)}}{e^{-V(b_{\log n})} + e^{-V(b_{\log n} - 1)}} \le c_{12} e^{-[V(y) - V(b_{\log n})]}$$

with $c_{12} := (1 + \varepsilon_0^{-1})$ by ellipticity. Hence, recalling $M^{\pm}$ from (87) and $\widehat{L}^{\pm}$ from (91) and (92), conditioning by ω and applying the strong Markov property at time $\tau(b_{\log n})$,

$$\begin{aligned}
&\mathbb{P}\big(S_n = z, |b_{\log n} - z| > \Gamma_n, \tau(b_{\log n}) \le n, E_3^{(n)}, E_5^{(n)}, z \in [M^-, M^+]\big)\\
&= \mathbb{E}\Big[\mathbf{1}_{\{|b_{\log n} - z| > \Gamma_n\}} \mathbf{1}_{\{\tau(b_{\log n}) \le n\}} \mathbf{1}_{E_3^{(n)} \cap E_5^{(n)}} \mathbf{1}_{\{z \in [M^-, M^+]\}} P_\omega^{b_{\log n}}[S_k = z]_{|k = n - \tau(b_{\log n})}\Big]\\
&\le \mathsf{E}\Big[\mathbf{1}_{\{|b_{\log n} - z| > \Gamma_n\}} \mathbf{1}_{E_3^{(n)} \cap E_5^{(n)}} \mathbf{1}_{\{z \in [M^-, M^+]\}} c_{12} e^{-[V(z) - V(b_{\log n})]}\Big].
\end{aligned} \tag{139}$$

We cut the expectation in (139) into several parts. We first notice that since $n \ge n_6 \ge n_3$,

$$\mathsf{E}\Big[\mathbf{1}_{\{z \in [M^-, \widehat{L}^-] \cup [\widehat{L}^+, M^+]\}} \mathbf{1}_{E_3^{(n)} \cap E_5^{(n)}} c_{12} e^{-[V(z) - V(b_{\log n})]}\Big] \le c_{12}(\log n)^{-C_1} \le (\log n)^{-3} \tag{140}$$

by (95) and (96), and since $C_1 > 20$ and $\log n > 2\varepsilon_0^{-1}$ because $n \ge n_6 \ge n_3$.

Third step: Hence, there only remains to treat the case $z \in]\widehat{L}^-, \widehat{L}^+[$, which we divide into 4 subcases, depending on the signs of z and $z - b_{\log n}$. In this step, we write $\mathcal{T}_i := \theta(T_i(V, \log n))$ for $-1 \le i \le 1$ to simplify the notation. First, we have, using $T_{\mathcal{T}_1}(h_n) \le T_{\mathcal{T}_1}(\log n)$ and the

fact that $\{z - b_{\log n} > \Gamma_n,\ b_{\log n} > 0\}$ depends only on $b_{\log n}$ and so is measurable with respect to $\sigma(\mathcal{T}_0, x_0(V, \log n))$ in the first inequality, using the law $\mathcal{T}_V^{\uparrow}$ of $\mathcal{T}_1$ and its independence with $(\mathcal{T}_0, x_0(V, \log n))$ conditionally on $\{b_{\log n} > 0\}$ (i.e. on $T_0(V, \log n)$ being downward) by Theorem 2.4 **(ii)** in the first equality, then the law of $\mathcal{T}_V^{\uparrow}$ ($\mathcal{T}_{V,h}^{\uparrow}$ with $h = \log n$) by Theorem 2.3 with $\Xi_{\log n} = \{T_V(\log n) < T_V(\mathbb{R}_-^*)\}$ as defined in (230) in the second inequality, then Proposition 7.3 in the third one since $\Gamma_n \geq p_4$ because $n \geq n_6 \geq n_3$ and $n_3 \geq \exp(p_5)$, we get

$$\begin{aligned}
&\mathsf{E}\Big[\mathbf{1}_{\{z>b_{\log n}+\Gamma_n\}}\mathbf{1}_{\{b_{\log n}>0\}}\mathbf{1}_{\{z\in]\widehat{L}^-,\widehat{L}^+[\}}\mathbf{1}_{E_3^{(n)}\cap E_5^{(n)}}e^{-[V(z)-V(b_{\log n})]}\Big] \\
\leq\ &\mathsf{E}\Big[\mathbf{1}_{\{z-b_{\log n}>\Gamma_n\}}\mathbf{1}_{\{b_{\log n}>0\}}\mathsf{E}\Big[\mathbf{1}_{\{y_n<T_{\mathcal{T}_1}(\log n)\}}e^{-\mathcal{T}_1(y_n)}\big|\sigma(\mathcal{T}_0, x_0(V,\log n))\Big]_{y_n=z-b_{\log n}}\Big] \\
=\ &\mathsf{E}\Big[\mathbf{1}_{\{z-b_{\log n}>\Gamma_n\}}\mathbf{1}_{\{b_{\log n}>0\}}\mathsf{E}\Big[\mathbf{1}_{\{y_n<T_{\mathcal{T}_V^{\uparrow}}(\log n)\}}e^{-\mathcal{T}_V^{\uparrow}(y_n)}\Big]_{y_n=z-b_{\log n}}\Big] \\
\leq\ &\mathsf{E}\Big[\mathbf{1}_{\{z-b_{\log n}>\Gamma_n\}}\mathsf{E}\Big[\mathbf{1}_{\{y_n<T_V(\log n)\}}e^{-V(y_n)}|\Xi_{\log n}\Big]_{y_n=z-b_{\log n}}\Big] \\
\leq\ &\mathsf{E}\Big[\mathbf{1}_{\{z-b_{\log n}>(\log n)^{4/3+\delta_1}\}}c_{13}(z-b_{\log n})^{-3/2}\Big] \leq c_{13}(\log n)^{-2-3\delta_1/2}. \qquad (141)
\end{aligned}$$

Also, using $x_0(V, \log n) = b_{\log n} < z < \widehat{L}^+ = x_0(V, \log n) + T_{\mathcal{T}_0}(h_n)$ with $h_n \leq \log n$ (on the event of the second line below) and $E_5^{(n)}$ in the first inequality, we have

$$\begin{aligned}
&\mathsf{E}\Big[\mathbf{1}_{\{z>b_{\log n}+\Gamma_n\}}\mathbf{1}_{\{b_{\log n}\leq 0\}}\mathbf{1}_{\{z\in]\widehat{L}^-,\widehat{L}^+[\}}\mathbf{1}_{E_3^{(n)}\cap E_5^{(n)}}e^{-[V(z)-V(b_{\log n})]}\Big] \qquad (142) \\
=\ &\sum_{y\leq 0}\mathsf{E}\Big[\mathbf{1}_{\{z>y+\Gamma_n\}}\mathbf{1}_{\{b_{\log n}=y\}}\mathbf{1}_{\{z\in]\widehat{L}^-,\widehat{L}^+[\}}\mathbf{1}_{E_3^{(n)}\cap E_5^{(n)}}e^{-[V(z)-V(y)]}\Big] \\
\leq\ &\sum_{y\leq 0}\mathbf{1}_{\{z-y>\Gamma_n\}}\mathsf{E}\Big[\mathbf{1}_{\{b_{\log n}=y\}}\mathbf{1}_{\{z-y<T_{\mathcal{T}_0}(\log n)\}}\mathbf{1}_{\{\ell(\mathcal{T}_0)\leq 2(\log n)^{2+\delta_1}\}}e^{-\mathcal{T}_0(z-y)}\Big] \\
=\ &\sum_{y\leq 0}\mathbf{1}_{\{z-y>\Gamma_n\}}\frac{\mathsf{E}\Big[\mathbf{1}_{\{-\ell(\mathcal{T}_V^{\uparrow})<y\}}\mathbf{1}_{\{z-y<T_{\mathcal{T}_V^{\uparrow}}(\log n)\}}\mathbf{1}_{\{\ell(\mathcal{T}_V^{\uparrow})\leq 2(\log n)^{2+\delta_1}\}}e^{-\mathcal{T}_V^{\uparrow}(z-y)}\Big]}{\mathsf{E}\big[\ell(\mathcal{T}_V^{\uparrow})+\ell(\mathcal{T}_V^{\downarrow})\big]},
\end{aligned}$$

where we used, in the last equality, eq. (29) of Theorem 2.5 with $\Delta_0 = \{y\}$, $\Delta_1 = \mathbb{Z}$ and $h = \log n$ and $\sharp\big\{0 \leq i < \ell(\mathcal{T}_V^{\uparrow}),\ -i = y\big\} = \mathbf{1}_{\{-\ell(\mathcal{T}_V^{\uparrow})<y\}}$, for which we recall that for $y \leq 0$, $b_{\log n} = y$ means that $x_0(V, \log n) = y$ and $\mathcal{T}_0(\ell(\mathcal{T}_0)) > 0$, i.e. $\mathcal{T}_0$ is an upward slope.

Then, using the definition of n_6 and $y > -\ell(\mathcal{T}_V^{\uparrow}) \geq -2(\log n)^{2+\delta_1}$ and the law of slopes provided by Theorem 2.3 **(i)** in the first inequality, and Proposition 7.3 in the second inequality since $\Gamma_n \geq p_4$ and $\log n \geq p_5$ because $n \geq n_6 \geq n_3$, we get, with $c_{14} := 3c_7^{-1}c_{13}$,

$$\begin{aligned}
(142)\ &\leq\ \frac{c_7^{-1}}{(\log n)^2}\sum_{y=-\lceil 2(\log n)^{2+\delta_1}\rceil}^{0}\mathbf{1}_{\{z-y>\Gamma_n\}}\mathsf{E}\big[\mathbf{1}_{\{z-y<T_V(\log n)\}}e^{-V(z-y)}|\Xi_{\log n}\big] \\
&\leq\ \frac{c_7^{-1}}{(\log n)^2}\sum_{y=-\lceil 2(\log n)^{2+\delta_1}\rceil}^{0}\mathbf{1}_{\{z-y>(\log n)^{4/3+\delta_1}\}}c_{13}(z-y)^{-3/2} \\
&\leq\ c_7^{-1}(\log n)^{-2}\big[2(\log n)^{2+\delta_1}+1\big]c_{13}\big((\log n)^{4/3+\delta_1}\big)^{-3/2} \leq c_{14}(\log n)^{-2-\delta_1/2}. \qquad (143)
\end{aligned}$$

Notice that $\zeta(\mathcal{T}_{-1}) = \big(V(x_0 - i) - V(x_0),\ 0 \leq i \leq x_0 - x_{-1}\big)$, with $x_j = x_j(V, \log n)$, $j \in \mathbb{Z}$. Moreover, by Theorem 2.4 **(i)**, conditionally on $T_0(V, \log n)$ being upward, i.e. on $\{b_{\log n} \leq 0\}$, $\zeta(\mathcal{T}_{-1})$ is independent of $(\mathcal{T}_0, x_0(V, \log n))$ and has the same law as $\zeta(\mathcal{T}_V^{\downarrow})$, so is equal in law,

by Proposition 2.12, to $\mathcal{T}^{\uparrow *}_{V^-,\log n}$, which law is given by Theorem 2.9 **(i)** applied to V^- (with ζ defined in (24)). Using this in the second inequality, then Proposition 7.3 in the third one, we get since $n \geq n_6$, with $\Xi^{*-}_{\log n} := \{T_{V^-}(\log n) < T^*_{V^-}(]-\infty, 0])\}$, similarly as in (141),

$$\begin{aligned}
&\mathsf{E}\Big[\mathbf{1}_{\{z<b_{\log n}-\Gamma_n\}}\mathbf{1}_{\{b_{\log n}\leq 0\}}\mathbf{1}_{\{z\in]\widehat{L}^-,\widehat{L}^+[\}}\mathbf{1}_{E_3^{(n)}\cap E_5^{(n)}}e^{-[V(z)-V(b_{\log n})]}\Big] \\
\leq\ &\mathsf{E}\Big[\mathbf{1}_{\{z<b_{\log n}-\Gamma_n,\, b_{\log n}\leq 0\}}\mathsf{E}\Big[\mathbf{1}_{\{y_n<T_{\zeta(\mathcal{T}_{-1})}(\log n)\}}e^{-\zeta(\mathcal{T}_{-1})(y_n)}\big|\sigma(\mathcal{T}_0, x_0(V,\log n))\Big]_{y_n=b_{\log n}-z}\Big] \\
\leq\ &\mathsf{E}\Big[\mathbf{1}_{\{b_{\log n}-z>\Gamma_n\}}\mathsf{E}\Big[\mathbf{1}_{\{y_n<T_{V^-}(\log n)\}}e^{-V^-(y_n)}|\Xi^{*-}_{\log n}\Big]_{y_n=b_{\log n}-z}\Big] \\
\leq\ &\mathsf{E}\Big[\mathbf{1}_{\{b_{\log n}-z>\Gamma_n\}}c_{13}(b_{\log n}-z)^{-3/2}\Big] \leq c_{13}(\log n)^{-2-3\delta_1/2}. \qquad (144)
\end{aligned}$$

Also, using $x_1(V,\log n) = b_{\log n} > z > \widehat{L}^- = x_1(V,\log n) - T_{\zeta(\mathcal{T}_0)}(h_n)$ with $h_n \leq \log n$ and $E_5^{(n)}$ in the first inequality, we have

$$\begin{aligned}
&\mathsf{E}\Big[\mathbf{1}_{\{z<b_{\log n}-\Gamma_n\}}\mathbf{1}_{\{b_{\log n}>0\}}\mathbf{1}_{\{z\in]\widehat{L}^-,\widehat{L}^+[\}}\mathbf{1}_{E_3^{(n)}\cap E_5^{(n)}}e^{-[V(z)-V(b_{\log n})]}\Big] \qquad (145) \\
\leq\ &\sum_{y>0}\mathbf{1}_{\{z-y<-\Gamma_n\}}\mathsf{E}\Big[\mathbf{1}_{\{b_{\log n}=y\}}\mathbf{1}_{\{y-z<T_{\zeta(\mathcal{T}_0)}(\log n)\}}\mathbf{1}_{\{\ell(\mathcal{T}_0)\leq 2(\log n)^{2+\delta_1}\}}e^{-\zeta(\mathcal{T}_0)(y-z)}\Big] \\
=\ &\sum_{y>0}\mathbf{1}_{\{z-y<-\Gamma_n\}}\frac{\mathsf{E}\Big[\mathbf{1}_{\{y\leq \ell(\mathcal{T}_V^{\downarrow})\}}\mathbf{1}_{\{y-z<T_{\zeta(\mathcal{T}_V^{\downarrow})}(\log n),\ \ell(\mathcal{T}_V^{\downarrow})\leq 2(\log n)^{2+\delta_1}\}}e^{-\zeta(\mathcal{T}_V^{\downarrow})(y-z)}\Big]}{\mathsf{E}\big[\ell(\mathcal{T}_V^{\uparrow})+\ell(\mathcal{T}_V^{\downarrow})\big]},
\end{aligned}$$

where we used, in the last equality, eq. (29) of Theorem 2.5 with $\Delta_0 = \mathbb{Z}$, $\Delta_1 = \{y\}$ and $h = \log n$ and $\sharp\big\{0 \leq i < \ell(\mathcal{T}_V^{\downarrow}),\, \ell(\mathcal{T}_V^{\downarrow}) - i = y\big\} = \mathbf{1}_{\{y\leq \ell(\mathcal{T}_V^{\downarrow})\}}$ for $y > 0$, for which we recall that for $y > 0$, $b_{\log n} = y$ means that $x_1(V,\log n) = y$ and $\mathcal{T}_0(\ell(\mathcal{T}_0)) < 0$.

Then, using the definition of n_6, $y \leq \ell(\mathcal{T}_V^{\downarrow}) \leq 2(\log n)^{2+\delta_1}$ and Proposition 2.12 in the first inequality, then Theorem 2.9 **(i)** in the equality, and Proposition 7.3 in the second inequality, we get since $\log n \geq p_5$ and $\Gamma_n \geq p_4$ because $n \geq n_6 \geq n_3$, concluding as in (143),

$$\begin{aligned}
(145) &\leq \frac{c_7^{-1}}{(\log n)^2}\sum_{y=1}^{\lfloor 2(\log n)^{2+\delta_1}\rfloor}\mathbf{1}_{\{z-y<-\Gamma_n\}}\mathsf{E}\Big[\mathbf{1}_{\{y-z<T_{\mathcal{T}^{\uparrow *}_{V^-}}(\log n)\}}e^{-\mathcal{T}^{\uparrow *}_{V^-}(y-z)}\Big] \\
&= \frac{c_7^{-1}}{(\log n)^2}\sum_{y=1}^{\lfloor 2(\log n)^{2+\delta_1}\rfloor}\mathbf{1}_{\{y-z>\Gamma_n\}}\mathsf{E}\big[\mathbf{1}_{\{y-z<T_{V^-}(\log n)\}}e^{-V^-(y-z)}|\Xi^{*-}_{\log n}\big] \leq \frac{c_{14}}{(\log n)^{2+\delta_1/2}}. \qquad (146)
\end{aligned}$$

Combining (141), (143), (144) and (146) ensures that, with $c_{15} := 2c_{14} + 2c_{13}$,

$$\mathsf{E}\Big[\mathbf{1}_{\{|b_{\log n}-z|>\Gamma_n\}}\mathbf{1}_{E_3^{(n)}\cap E_5^{(n)}}\mathbf{1}_{\{z\in]\widehat{L}^-,\widehat{L}^+[\}}e^{-[V(z)-V(b_{\log n})]}\Big] \leq c_{15}(\log n)^{-2-\delta_1/2}.$$

This, combined with (140), proves that the right hand side of (139) is $\leq c_{16}(\log n)^{-2-\delta_1/2}$ for all $n \geq n_6$ and $z \in \mathbb{Z}$ with $c_{16} := (c_{15}c_{12} + 1)$. This together with (138) gives (137) since $\delta_1 \in (0, 2/3)$, with $c_{11} := c_{16} + 1$. □

5.2.2. *Case with $b_{\log n}$ far from z, without subvalleys and small valleys when $\tau(b_{\log n}) > n$.* The aim of this subsection is to prove the following lemma.

Lemma 5.4. *There exists a constant $c_{17} > 0$ such that, for all $n \geq n_6$ and all $z \in \mathbb{Z}$,*

$$\mathbb{P}\big(S_n = z, |b_{\log n} - z| > \Gamma_n, \tau(b_{\log n}) > n, E_3^{(n)}, E_5^{(n)}\big) \leq c_{17}(\log_2 n)^3(\log n)^{-3}. \qquad (147)$$

We start with the case $b_{\log n} \le 0$. We first make the following simple remark.

Lemma 5.5. *We have,*

$$\forall n \ge 3,\, \forall z \in \mathbb{Z}, \qquad \mathbb{P}\big(S_n = z,\, b_{\log n} \le 0,\, z < b_{\log n} - \Gamma_n,\, \tau(b_{\log n}) > n\big) = 0. \tag{148}$$

Proof: On $\{b_{\log n} \le 0,\, z < b_{\log n} - \Gamma_n,\, \tau(b_{\log n}) > n\}$, we have $z < b_{\log n} \le 0$, so for S starting from 0 (under P_ω or $\mathbb{P}$), $\tau(z) > \tau(b_{\log n}) > n$ and thus $S_n \ne z$. This leads to (148). □

In order to prove Lemma 5.4, we also have to give an upper bound for the probability of $F_1^{(n)}(z)$, where

$$F_1^{(n)}(z) := \big\{S_n = z,\, b_{\log n} \le 0,\, z > b_{\log n} + \Gamma_n,\, \tau(b_{\log n}) > n\big\} \cap E_3^{(n)} \cap E_5^{(n)}.$$

Loosely speaking, on $E_3^{(n)}$ by Remark 4.2, there are no subvalleys of height larger than $h_n - C_1 \log_2 n$ in the $(\log n)$-central valley $[M^-, M^+]$ and in the two neighbor valleys (of height at least $\log n$) on its left and on its right, and the height of these three valleys is quite larger than $\log n$. In particular, we prove:

Lemma 5.6. *There exists a constant $c_{18} > 0$ such that*

$$\forall n \ge n_6,\, \forall z \in \mathbb{Z}, \qquad \mathbb{P}\big[F_1^{(n)}(z)\big] \le c_{18}(\log_2 n)^3 (\log n)^{-3}. \tag{149}$$

Outline of the proof: See Figure 7 for a schema of the potential. Assume for example that $b_{\log n} \le 0$, so $x_0 = b_{\log n}$, with $x_i := x_i(V, \log n)$, $i \in \mathbb{Z}$, and that $F_1^{(n)}(z)$ holds. Since $\tau(x_0) > n$, we first prove that, by Lemma 4.1, with large probability, $\tau(x_2) \le n$. Second, if z is not in the valley $[x_1, x_3]$, then after first hitting x_2, S has to leave this valley before time n (so that $S_n = z \notin [x_1, x_3]$), which has negligible probability since the height of this valley $[x_1, x_3]$ is quite larger than $\log n$ on $E_3^{(n)}$. Third, if z belongs to the valley $[x_1, x_3]$ with $V(z) \ge V(x_2) + 4\log_2 n$, then the probability that $S_n = z$ is negligible by reversibility, which we can apply to S started at x_2 by strong Markov property. Finally, if z belongs to the valley $[x_1, x_3]$ with $V(z) < V(x_2) + 4\log_2 n$, then $V(z + .) - V(z)$ goes up $\log n$ before going down $4\log_2 n$ on the left and on the right, and conditionally on $(V(k),\ k \ge 0)$, $\max_{[x_0,0]} V - \max_{[0,z]} V = \max_{[x_0,0]} V - V(x_1) \in [-9\log_2 n, 0[$ (otherwise $\tau(x_2) < \tau(x_0)$ would have small probability which would contradict our first step). Since all these three conditions have probability less than $c(\log_2 n)(\log n)^{-1}$ for some $c > 0$ with some independence, this last case is also negligible compared to $(\log n)^{-2}$. We now prove this rigorously.

Proof: Let $n \ge n_6$ and $z \in \mathbb{Z}$. In all the proof, we write x_i for $x_i(V, \log n)$ for every $i \in \mathbb{Z}$.

First step: Applying Lemma 4.1 with $h = \log n$, $\xi_2 = 1$, $a = x_0 < b = x_1 < c = x_2$ (so that **(i)** is satisfied for $\omega \in E_-^{(n)}$), $\xi_1 = 2C_1$ (so **(ii)** and **(iii)** are satisfied since there is no left $(\log n - 2C_1 \log_2 n)$-extremum in $]x_0,\ x_1[$ nor in $]x_1, x_2[$ for $\omega \in E_-^{(n)} \cap E_3^{(n)}$ by Remark 4.2), $\alpha = 3$ (so **(iv)** is satisfied for $\omega \in E_5^{(n)}$ since $0 < \delta_1 < 2/3$) and $x = 0$, we get since $n \ge n_6 \ge n_3$ and so $\log n \ge \widehat{h}_2(2C_1, 1)$,

$$\forall \omega \in E_-^{(n)} \cap E_3^{(n)} \cap E_5^{(n)}, \qquad P_\omega\big[\tau(x_0) \wedge \tau(x_2) \ge n\big] \le (\log n)^{-4}. \tag{150}$$

As a consequence, using $\tau(x_0) = \tau(b_{\log n}) > n$ on $F_1^{(n)}(z)$, we get

$$\mathbb{P}\big[F_1^{(n)}(z) \cap \{\tau(x_2) \ge n\}\big] \le \mathsf{E}\big[\mathbf{1}_{E_-^{(n)} \cap E_3^{(n)} \cap E_5^{(n)}} P_\omega\big(\tau(x_0) \wedge \tau(x_2) \ge n\big)\big] \le (\log n)^{-4}. \tag{151}$$

Second step: There only remains to consider $F_1^{(n)}(z) \cap \{\tau(x_2) < n\}$. This second step focuses on the case $z \notin]x_1, x_3[$. We start with the case $z \le x_1$ $\big($see Figure 7 with $z = z^{(2)}\big)$. In what follows we prove that in this case, the probability that, after hitting x_2, S goes or goes back to $z \in]-\infty, x_1]$ before time n is negligible.

To this aim, using $z \le x_1 < x_2$, then (11) and ellipticity (16) in the second inequality, we have for every $\omega \in E_-^{(n)} \cap E_3^{(n)} \cap \{z \le x_1\}$ and $k \in \{0, \dots, n\}$,

$$\begin{aligned} P_\omega^{x_2}[S_k = z] &\le P_\omega^{x_2}[\tau(x_1) \le \tau(z) \le k] \\ &\le (k+1)\varepsilon_0^{-1} \exp(-H[T_1(V, \log n)]) \le 2\varepsilon_0^{-1} (\log n)^{-C_2} \le (\log n)^{-4} \end{aligned}$$

since $V[x_1] - \min_{[x_1,x_2]} V = H[T_1(V, \log n)] \ge \log n + C_2 \log_2 n$ on $E_3^{(n)}$, $C_2 > 9$ and $n \ge n_6 \ge n_3$. Hence, conditioning by ω then applying the strong Markov property at time $\tau(x_2)$,

$$\begin{aligned} &\mathbb{P}\big[F_1^{(n)}(z) \cap \{\tau(x_2) < n\} \cap \{z \le x_1\}\big] \\ \le\ &\mathbb{E}\Big[\mathbf{1}_{E_-^{(n)} \cap E_3^{(n)} \cap \{z \le x_1\} \cap \{\tau(x_2) < n\}} P_\omega^{x_2}[S_k = z]_{|k = n - \tau(x_2)}\Big] \\ \le\ &(\log n)^{-4}. \end{aligned} \tag{152}$$

Similarly, using (10) instead of (11), we have for large n,

$$\mathbb{P}\big[F_1^{(n)}(z) \cap \{\tau(x_2) < n\} \cap \{z \ge x_3\}\big] \le (\log n)^{-4}. \tag{153}$$

Third step: Now, on $\{x_1 < z < x_3\} \cap \{V(z) \ge V(x_2) + 4\log_2 n\} \cap E_-^{(n)}$ $\big($see Figure 7 with $z = z^{(3)}\big)$, we have by reversibility (see (13)) and ellipticity (16), for $k \in \mathbb{N}$,

$$P_\omega^{x_2}(S_k = z) \le \frac{\mu_\omega(z)}{\mu_\omega(x_2)} \le \varepsilon_0^{-1} \exp[V(x_2) - V(z)] \le \frac{\varepsilon_0^{-1}}{(\log n)^4}.$$

As a consequence, once more conditioning by ω and applying the strong Markov property, proceeding as in (152),

$$\begin{aligned} &\mathbb{P}\big[F_1^{(n)}(z) \cap \{\tau(x_2) < n\} \cap \{x_1 < z < x_3\} \cap \{V(z) \ge V(x_2) + 4\log_2 n\}\big] \\ \le\ &\mathbb{E}\Big[\mathbf{1}_{E_-^{(n)} \cap \{x_1 < z < x_3\} \cap \{V(z) \ge V(x_2) + 4\log_2 n\} \cap \{\tau(x_2) < n\}} P_\omega^{x_2}[S_k = z]_{|k = n - \tau(x_2)}\Big] \\ \le\ &\varepsilon_0^{-1} (\log n)^{-4}. \end{aligned} \tag{154}$$

Forth step: Finally, we study $\big($see Figure 7 with $z = z^{(4)}\big)$,

$$F_2^{(n)}(z) \ := \ F_1^{(n)}(z) \cap \{\tau(x_2) < n\} \cap \{x_1 < z < x_3\} \cap \{V(z) < V(x_2) + 4\log_2 n\}.$$

This set is empty for $z < 0$ because $x_1 > 0$, so we can assume that $z \ge 0$.

We once more define $V_z^{\pm}(k) := V(z \pm k) - V(z)$, $k \in \mathbb{Z}$, and notice that V_z^- and V_z^+ are independent. We also introduce

$$E_{11}^{(n)} := \{\tau(x_2) < \tau(x_0)\}, \qquad E_{12}^{(n)} := \Big\{ \max_{[x_0, 0]} V \le V(x_1) - 9\log_2 n \Big\}.$$

We have by (7), for large n, for all $\omega \in E_-^{(n)} \cap E_5^{(n)} \cap E_{12}^{(n)}$,

$$P_\omega\big(E_{11}^{(n)}\big) = \frac{\sum_{i=x_0}^{-1} e^{V(i)}}{\sum_{i=x_0}^{x_2 - 1} e^{V(i)}} \le \frac{(\log n)^{2+\delta_1} \exp[\max_{[x_0,0]} V]}{\exp[V(x_1)]} \le (\log n)^{-6}.$$

Consequently, since $F_1^{(n)}(z) \cap \{\tau(x_2) < n\} \subset E_{11}^{(n)}$,

$$\mathbb{P}\big[F_2^{(n)}(z) \cap E_{12}^{(n)}\big] \le \mathbb{P}\big[E_-^{(n)} \cap E_5^{(n)} \cap E_{12}^{(n)} \cap E_{11}^{(n)}\big] \le (\log n)^{-6}. \tag{155}$$

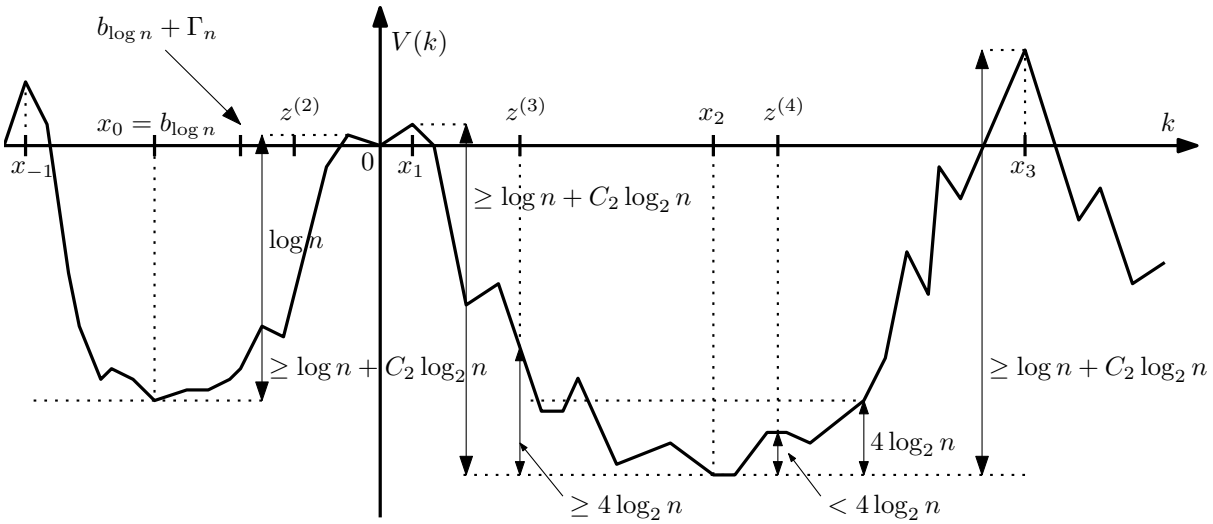

Figure 7. Schema of the potential V with $x_i = x_i(V, \log n)$, and z equal to $z^{(2)}$, $z^{(3)}$ and $z^{(4)}$ respectively for step 2, 3 $\big($on $F_1^{(n)}(z)\big)$ and 4 $\big($on $F_2^{(n)}(z)\big)$.

There remains to study $\mathbb{P}\big[F_2^{(n)}(z) \cap \big(E_{12}^{(n)}\big)^c\big]$. For a process $(v(k),\ k \in \mathbb{N})$ and $y \in \mathbb{N}$, we define $v_y(.) := v(y + .) - v(y)$ and

$$\begin{aligned} E_{13}^{(n)}(v) &:= \{T_v(]-\infty, -\log n]) < T_v(9\log_2 n)\}, \\ E_{14}^{(n)}(z) &:= \big\{T_{V_z^+}(\log n) < T_{V_z^+}(-4\log_2 n)\big\} \cap \big\{T_{V_z^-}(\log n) < T_{V_z^-}(-4\log_2 n)\big\} \\ &\quad \cap \big\{T_{V_z^-}(\log n) \leq z\big\}. \end{aligned}$$

Also for $a \geq 0$, let $V_{1,a}(k) := V^-\big[k + T_{V^-}(a)\big] - V^-\big[T_{V^-}(a)\big]$, $k \in \mathbb{N}$. We claim that

$$F_2^{(n)}(z) \cap \big(E_{12}^{(n)}\big)^c \subset E_{14}^{(n)}(z) \cap E_{13}^{(n)}\big(V_{1,\max(0,\max_{[0,z]} V - 9\log_2 n)}\big). \tag{156}$$

Indeed on $F_2^{(n)}(z) \cap \big(E_{12}^{(n)}\big)^c$, we have $b_{\log n} = x_0 \leq 0$, $z \in]x_1, x_3[$, $\min_{[x_1,x_3]} V = V(x_2) > V(z) - 4\log_2 n$, and $V(x_1) \geq V(x_2) + \log n + C_2 \log_2 n \geq V(z) + \log n$ due to $E_3^{(n)}$ and since $C_2 > 9$, the same being true also for $V(x_3)$ instead of $V(x_1)$. So $V_z^{\pm}$ hits $[\log n, +\infty[$ before $]-\infty, -4\log_2 n]$, so $F_2^{(n)}(z) \cap \big(E_{12}^{(n)}\big)^c$ is included in the first two sets in $E_{14}^{(n)}(z)$.

Also on $F_2^{(n)}(z)$, $x_0 = b_{\log n} \leq 0$, thus $\max_{[x_0,x_2]} V = V(x_1)$, so $\max_{[0,z]} V = V(x_1)$ if $x_1 < z \leq x_2$. Assume now that $x_2 < z < x_3$ and $F_2^{(n)}(z)$ holds. If $\max_{[x_2,z]} V > V(x_1)$, then $\min\{u \in [x_2, z],\ V(u) = \max_{[x_2,z]} V\}$ would be a left $(\log n)$-maximum (because its potential would be greater than $V(x_1) \geq V(x_2) + \log n + C_2 \log_2 n \geq V(z) + \log n$ due to $E_3^{(n)}$ and $C_2 > 9$ as before, and greater than $V(x_2) + \log n$), belonging to $]x_2, x_3[$, which is not possible, so $\max_{[x_2,z]} V \leq V(x_1)$. Hence $\max_{[0,z]} V = V(x_1) \geq V(z) + \log n$ in both cases, so $\max_{[0,z]} V_z^- \geq \log n$, thus $F_2^{(n)}(z) \cap \big(E_{12}^{(n)}\big)^c$ is included in the third set in $E_{14}^{(n)}(z)$.

Finally on $F_2^{(n)}(z) \cap \big(E_{12}^{(n)}\big)^c$, we have $\max_{[x_0,0]} V < V(x_1) = \max_{[x_0,x_2]} V$ by definition of the x_i and of $E_-^{(n)}$, and $\max_{[x_0,0]} V > V(x_1) - 9\log_2 n \geq V(x_0) + \log n$ by definition of $\big(E_{12}^{(n)}\big)^c$ and since $H[T_0(V, \log n)] \geq \log n + C_2 \log_2 n$ with $C_2 > 9$ on $E_3^{(n)}$. Also, we just proved that $\max_{[0,z]} V = V(x_1)$. Hence, starting from 0, V^- first hits $[\max_{[0,z]} V - 9\log_2 n, +\infty[$, then goes down at least $\log n$ before $|x_0|$ and so before going up $9\log_2 n$, so $\omega \in E_{13}^{(n)}\big(V_{1,a}\big)$ with $a = \max_{[0,z]} V - 9\log_2 n$

if $\max_{[0,z]} V - 9\log_2 n \geq 0$. Otherwise, $\max_{[x_0,0]} V < V(x_1) = \max_{[0,z]} V < 9\log_2 n$, with $V(x_0) = V(x_1) - H[T_0(V, \log n)] \leq -\log n + (9 - C_2)\log_2 n < -\log n$ since $C_2 > 9$ due to $E_3^{(n)}$, so $\omega \in E_{13}^{(n)}\big(V^-\big) = E_{13}^{(n)}\big(V_{1,a}\big)$ with $a = 0$. So (156) is proved in every case.

We have in particular, by (17), since $n \geq n_6 \geq n_3$ so $\log_2 n > C_0$,

$$\mathsf{P}[E_{13}^{(n)}(V^-)] = \mathsf{P}\big(T_{V^-}(]-\infty, -\log n]) < T_{V^-}([9\log_2 n, +\infty[)\big) \leq 10(\log_2 n)(\log n)^{-1}. \quad (157)$$

Also, using first the independence between V_z^+ and V_z^-, which have the same law as V and V^- respectively, then applying (17) again, we have since $n \geq n_6 \geq n_3$,

$$\begin{aligned} \mathsf{P}\big(E_{14}^{(n)}(z)\big) &\leq \mathsf{P}\big[T_{V_z^+}(\log n) < T_{V_z^+}(-4\log_2 n)\big]\mathsf{P}\big[T_{V_z^-}(\log n) < T_{V_z^-}(-4\log_2 n)\big] \\ &\leq 25(\log_2 n)^2(\log n)^{-2}. \end{aligned} \quad (158)$$

Hence using (156), then conditioning by $V^+ = (V(k),\ k \geq 0)$, noting that $E_{14}^{(n)}(z)$ and $\max_{[0,z]} V$ depend only on V^+ and for every $a \in \mathbb{R}_+$, $E_{13}^{(n)}\big(V_{1,a}\big)$ only on V^-, which is independent of V^+ and has the same law as $V_{1,a}$, then applying (157) and (158), we get

$$\begin{aligned} \mathbb{P}\big[F_2^{(n)}(z) \cap \big(E_{12}^{(n)}\big)^c\big] &\leq \mathsf{P}\big[E_{14}^{(n)}(z) \cap E_{13}^{(n)}\big(V_{1,\max(0,\max_{[0,z]} V - 9\log_2 n)}\big)\big] \\ &= \mathsf{E}\big[\mathbf{1}_{E_{14}^{(n)}(z)}\mathsf{P}\big[E_{13}^{(n)}\big(V_{1,\max(0,\max_{[0,z]} V - 9\log_2 n)}\big)|V^+\big]\big] \\ &= \mathsf{E}\big[\mathbf{1}_{E_{14}^{(n)}(z)}\mathsf{P}\big[E_{13}^{(n)}\big(V_{1,a}\big)\big]_{|a=\max(0,\max_{[0,z]} V - 9\log_2 n)}\big] \\ &\leq 10(\log_2 n)(\log n)^{-1}\mathsf{P}(E_{14}^{(n)}(z)) \leq 250(\log_2 n)^3(\log n)^{-3}. \end{aligned}$$

This, together with (155) gives $\mathsf{P}\big(F_2^{(n)}(z)\big) \leq 251(\log_2 n)^3(\log n)^{-3}$ for all $n \geq n_6$ and $z \in \mathbb{Z}$.

Conclusion: Combining this with (151), (152), (153), (154) proves (149). □

Proof of Lemma 5.4: We prove, similarly as in Lemmas 5.5 and 5.6 (replacing in particular x_0, x_1, x_2 and x_3 respectively by x_1, x_0, x_{-1} and x_{-2} respectively in its proof, nearly by symmetry) that for every $n \geq n_6$ and every $z \in \mathbb{Z}$,

$$\mathbb{P}\big(S_n = z, b_{\log n} > 0, z > b_{\log n} + \Gamma_n, \tau(b_{\log n}) > n\big) = 0,$$

$$\mathbb{P}\big(S_n = z, b_{\log n} > 0, z < b_{\log n} - \Gamma_n, \tau(b_{\log n}) > n, E_3^{(n)}, E_5^{(n)}\big) \leq c_{18}(\log_2 n)^3(\log n)^{-3}.$$

Combining this with Lemmas 5.5 and 5.6 proves Lemma 5.4 with $c_{17} := 2c_{18}$. □

Proof of Proposition 5.2: This proposition follows directly from Lemmas 5.3 and 5.4 with $c_{10} := c_{11} + c_{17}$, since $(\log_2 n)^3 \leq (\log n)^{1/2}$ for $n \geq n_6 \geq n_3$ and $\delta_1 \in]0, 2/3[$. □

5.3. **Case with at least one subvalley or small valley.** We now focus on the case where some of the valleys (of height $\geq \log n$) close to the origin can be small (i.e. with height $< \log n + C_2 \log_2 n$), or can contain subvalleys of height less than but close to $\log n$. More precisely, the aim of this subsection is to prove the following estimate.

Proposition 5.7. *There exists $n_9 \geq n_6$ and $c_{19} > 0$ such that*

$$\forall n \geq n_9,\ \forall z \in \mathbb{Z}, \qquad \mathbb{P}\big(S_n = z, (E_3^{(n)})^c\big) \leq c_{19}(\log_2 n)^3(\log n)^{-3}.$$

This case can be divided into many different subcases. For example, there can be, or not, a subvalley of height close to $\log n$ inside the $(\log n)$-central valley, either at the right or at the left of $b_{\log n}$, or there can even be two such subvalleys. There can also exist, close to the $(\log n)$-central valley, one or two valleys with height close to $\log n$, larger or smaller than $\log n$, which can trap the random walk $(S_k)_k$ for some time. Also, the height of the $(\log n)$-central valley

can be close to $\log n$, which can enable S to escape it before time n with not so small quenched probability. Taking into account the indexes of the left $(h_n - C_1 \log_2 n)$-slopes considered, i.e. with height less than $\log n + C_2 \log_2 n$, and their height, larger or smaller than $\log n$, the indexes i of the first left h_n-minimum $b_i(V, h_n)$ (defined in (160)) visited by S before time n, of the second one etc, the fact that z is close or far from these left h_n-extrema, this makes dozens of cases. However we will combine together some of these cases, for example with the help of Lemma 5.9 and of the notation $\mathcal{I}_k$ defined in (164) below.

On $\big(E_3^{(n)}\big)^c$, there exists some $i \in \{-10, \dots 10\}$ such that $H[T_i(V, h_n - C_1 \log_2 n)] < \log n + C_2 \log_2 n$. Also, we prove that with large probability, there are no more than two such i. To this aim, we define

$$E_{15}^{(n)} := \big\{ \sharp\{i \in \mathbb{Z},\ -99 \le i \le 99,\ H[T_i(V, h_n - C_1 \log_2 n)] < \log n + C_2 \log_2 n\} \le 2 \big\}.$$

More precisely, we prove the following estimate.

Lemma 5.8. *There exist $n_7 \ge n_6$ and $c_{20} > 0$ such that,*

$$\forall n \ge n_7, \qquad \mathsf{P}\big[\big(E_{15}^{(n)}\big)^c\big] \le c_{20} (\log_2 n)^3 (\log n)^{-3}. \tag{159}$$

Proof: Due to Lemma 2.16, we have $\mathsf{P}\big[E_{16}^{(n)}(i) \mid b_{\widetilde{h}_n} \le 0\big] = O\big((\log_2 n)(\log n)^{-1}\big)$, $i \in \mathbb{Z}$, where $\widetilde{h}_n = h_n - C_1 \log_2 n$ as before and

$$E_{16}^{(n)}(i) := \big\{ H[T_i(V, h_n - C_1 \log_2 n)] < \log n + C_2 \log_2 n \big\}, \qquad i \in \mathbb{Z}.$$

Hence, using the independence of the translated left $\widetilde{h}_n$-slopes conditionally on $\{b_{\widetilde{h}_n} \le 0\}$ (see Theorem 2.4 **(i)**), we have

$$\begin{aligned} \mathsf{P}\big[\big(E_{15}^{(n)}\big)^c \mid b_{\widetilde{h}_n} \le 0\big] &= \mathsf{P}\big(\cup_{-99 \le i_1 < i_2 < i_3 \le 99} E_{16}^{(n)}(i_1) \cap E_{16}^{(n)}(i_2) \cap E_{16}^{(n)}(i_3) \mid b_{\widetilde{h}_n} \le 0\big) \\ &\le \sum_{-99 \le i_1 < i_2 < i_3 \le 99} \prod_{k=1}^{3} \mathsf{P}\big[E_{16}^{(n)}(i_k) \mid b_{\widetilde{h}_n} \le 0\big] = O\big((\log_2 n)^3 (\log n)^{-3}\big) \end{aligned}$$

as $n \to +\infty$. We prove similarly the same inequality with $b_{\widetilde{h}_n} \le 0$ replaced by $b_{\widetilde{h}_n} > 0$, which proves the lemma. □

We define, for $h > 0$ and $i \in \mathbb{Z}$ (this definition being different from that of [24]),

$$b_i(V, h) := \begin{cases} x_{2i}(V, h) & \text{if } x_0(V, h) \text{ is a left } h\text{-minimum,} \\ x_{2i-1}(V, h) & \text{otherwise.} \end{cases} \tag{160}$$

So, the $b_i(V, h)$, $i \in \mathbb{Z}$, are the left h-minima for V, such that $b_0(V, h) \le 0 < b_1(V, h)$ and $b_i(V, h) < b_{i+1}(V, h)$, $i \in \mathbb{Z}$. We also denote by $M_i(V, h)$ the unique left h-maximum for V between $b_i(V, h)$ and $b_{i+1}(V, h)$. Hence, $M_i(V, h) = x_{j+1}(V, h)$ if $b_i(V, h) = x_j(V, h)$.

We now prove that the probability that z is "close" (in terms of potential) to the bottom $b_j(V, h_n)$ of a valley of height h_n and that $\omega \in \big(E_3^{(n)}\big)^c$ is small. More precisely, we define, for $h > 0$,

$$E_{17}^{(n)}(j, h, z) := \{ M_{j-1}(V, h) \le z \le M_j(V, h),\ V(z) \le V[b_j(V, h)] + 4 \log_2 n \}, \qquad j \in \mathbb{Z}.$$

We now have the following lemma, which is useful to prove Lemma 5.12 (in which we take $h'_n = h_n$) and Lemma 5.13 (in which we take $h'_n = \widetilde{h}_n$) and then Lemma 5.11.

Lemma 5.9. *There exist $c_{21} > 0$ and $n_8 \geq n_7$ such that, whether $h'_n = h_n$ or $h'_n = \widetilde{h}_n := h_n - C_1 \log_2 n$, we have*

$$\forall n \geq n_8,\ \forall z \in \mathbb{Z}, \qquad \mathsf{P}\big[\big(E_3^{(n)}\big)^c \cap \cup_{j=-8}^{8} E_{17}^{(n)}(j, h'_n, z)\big] \leq c_{21}(\log_2 n)^3 (\log n)^{-3}.$$

Loosely speaking, in the case $h'_n = \widetilde{h}_n$, on $E_{17}^{(n)}(j, \widetilde{h}_n, z)$, V_z^+ and V_z^- go up $\widetilde{h}_n - 4\log_2 n$ before going down $-4\log_2 n$, which has probability $O\big(\log_2 n)^2 (\log n)^{-2}\big)$. Also, on $E_3^{(n)}$ one of the left $\widetilde{h}_n$-slopes around the origin has an excess height less than some $C\log_2 n$, which has probability $O\big((\log_2 n)(\log n)^{-1}\big)$, with some independence, which leads to Lemma 5.9 in the case $h'_n = \widetilde{h}_n$, the second case being nearly a consequence of the first one. We now prove this rigorously.

Proof of Lemma 5.9: Let $n \geq n_7$ and $z \in \mathbb{Z}$. We start with the case $h'_n = \widetilde{h}_n$. On the one hand, we notice that for $-13 \leq j \leq 13$, on $\big(E_3^{(n)}\big)^c \cap E_{17}^{(n)}\big(j, \widetilde{h}_n, z\big)$, z belongs to the support $\big[x_k\big(V, \widetilde{h}_n\big), x_{k+1}\big(V, \widetilde{h}_n\big)\big]$ of a left $\widetilde{h}_n$-slope $T'_k := T_k\big(V, \widetilde{h}_n\big)$ with $2j - 2 \leq k \leq 2j$, the value of k depending on $x_0\big(V, \widetilde{h}_n\big)$ being a left $\widetilde{h}_n$-maximum or minimum for V and on $z \leq b_j\big(V, \widetilde{h}_n\big)$ or $z > b_j\big(V, \widetilde{h}_n\big)$, with $T'_k(z) - \inf_{y \in [x_k(V, \widetilde{h}_n), x_{k+1}(V, \widetilde{h}_n)]} T'_k(y) \leq 4\log_2 n$. Hence, using $x_i\big(V, \widetilde{h}_n\big) = x_{i-k}\big(V_z, \widetilde{h}_n\big) + z$, $i \in \mathbb{Z}$ on $\big\{x_k\big(V, \widetilde{h}_n\big) \leq z < x_{k+1}\big(V, \widetilde{h}_n\big)\big\}$ and the definition of $E_3^{(n)}$, we get

$$\begin{aligned}
& \mathsf{P}\big[\big(E_3^{(n)}\big)^c \cap \cup_{j=-13}^{13} E_{17}^{(n)}\big(j, \widetilde{h}_n, z\big)\big] \\
\leq\ & \mathsf{P}\Big(\cup_{k=-28}^{27} \big\{x_k\big(V, \widetilde{h}_n\big) \leq z < x_{k+1}\big(V, \widetilde{h}_n\big)\big\} \cap \Big\{T'_k(z) - \inf_{[0, \ell(T'_k)]} \theta(T'_k) \leq 4\log_2 n\Big\} \\
& \qquad \cap \cup_{i=-10}^{10} \big\{H\big[T_i\big(V, \widetilde{h}_n\big)\big] < \log n + C_2 \log_2 n\big\}\Big) \\
\leq\ & \mathsf{P}\Big(\Big\{ \inf_{[x_0(V_z, \widetilde{h}_n), x_1(V_z, \widetilde{h}_n)]} V_z \geq -4\log_2 n\Big\} \cap \cup_{j=-37}^{38} \big\{H\big[T_j\big(V_z, \widetilde{h}_n\big)\big] < \log n + C_2 \log_2 n\big\}\Big),
\end{aligned}$$

where V_z has the same law as V, so the last probability does not depend on z.

Now, notice that, with $V^{\pm} = (V(\pm y),\ y \in \mathbb{N})$ as before, and $\widetilde{V}_3(k) := V\big[k + T_V\big(\big[\widetilde{h}_n - 4\log_2 n, +\infty\big[\big)\big]$, $k \in \mathbb{N}$, we have

$$E_{18,0}^{(n)} \cap \{b_{\widetilde{h}_n} \leq 0\} \subset E_{19,+}^{(n)} \cap E_{19,-}^{(n)} \cap E_{20}^{(n)}, \tag{161}$$

where for $i \in \mathbb{Z}$ and $h > 0$,

$$\begin{aligned}
E_{18,i}^{(n)} &:= \big\{\inf_{[x_0(V, \widetilde{h}_n), x_1(V, \widetilde{h}_n)]} V \geq -4\log_2 n,\ H\big[T_i\big(V, \widetilde{h}_n\big)\big] < \log n + C_2 \log_2 n\big\}, \\
E_{19,\pm}^{(n)} &:= \{T_{V^{\pm}}\big(\big[\widetilde{h}_n - 4\log_2 n, +\infty\big[\big) < T_{V^{\pm}}(]-\infty, -4\log_2 n[)\}, \\
E_{20}^{(n)} &:= \{T_{\widetilde{V}_3}\big(]-\infty, \log n + C_2\log_2 n - \widetilde{h}_n\big[\big) < T_{\widetilde{V}_3}\big(\big[\log n + C_2 \log_2 n, +\infty\big[\big)\}.
\end{aligned}$$

Using (17) and $n \geq n_7 \geq n_3$, we have $\mathsf{P}\big(E_{19,\pm}^{(n)}\big) \leq 10(\log_2 n)(\log n)^{-1}$ and $\mathsf{P}\big(E_{20}^{(n)} \mid V(k),\ k \leq T_V\big(\big[\widetilde{h}_n - 4\log_2 n, +\infty\big[\big)\big) \leq \big(\log n + C_2\log_2 n - \big(\widetilde{h}_n - 4\log_2 n\big) + C_0\big)\big(\widetilde{h}_n + C_0\big)^{-1} \leq 2(2C_1 + C_2 + 5)(\log_2 n)(\log n)^{-1}$. Hence, using (161), conditioning by $\sigma\big(V(k),\ k \leq T_V\big(\big[\widetilde{h}_n - 4\log_2 n, +\infty\big[\big)$ then using the independence of V^+ and V^-, we have, with $c_{22} := 200(2C_1 + C_2 + 5)$,

$$\mathsf{P}\big(E_{18,0}^{(n)},\ b_{\widetilde{h}_n} \leq 0\big) \leq c_{22}(\log_2 n)^3 (\log n)^{-3}.$$

We get similarly the same result with $b_{\widetilde{h}_n} \leq 0$ replaced by $b_{\widetilde{h}_n} > 0$.

Finally, for $i \neq 0$, using Theorem 2.4 **(i)** since $H\big[\theta\big(T_i\big(V,\widetilde{h}_n\big)\big)\big] = H\big[T_i\big(V,\widetilde{h}_n\big)\big]$,

$$\begin{aligned}
&\mathsf{P}\big(E^{(n)}_{18,i},\, b_{\widetilde{h}_n} \leq 0\big)\\
&= \mathsf{P}\Big(\inf_{[x_0(V,\widetilde{h}_n),x_1(V,\widetilde{h}_n)]} V \geq -4\log_2 n,\, b_{\widetilde{h}_n} \leq 0\Big)\mathsf{P}\big(H\big[T_i\big(V,\widetilde{h}_n\big)\big] < \log n + C_2\log_2 n \mid b_{\widetilde{h}_n} \leq 0\big)\\
&\leq 200c_8(2C_1 + C_2 + C_0)(\log_2 n)^3(\log n)^{-3}
\end{aligned}$$

for large n since the first probability in the second line is $\leq \mathsf{P}\big(E^{(n)}_{19,-}\big)\mathsf{P}\big(E^{(n)}_{19,+}\big)$ and the second one is $\leq c_8\big(\log n + C_2\log_2 n - \widetilde{h}_n\big)\big(\widetilde{h}_n\big)^{-1}$ for large n by Lemma 2.16. We get similarly the same result with $b_{\widetilde{h}_n} \leq 0$ replaced by $b_{\widetilde{h}_n} > 0$, using Theorem 2.4 **(ii)** instead of **(i)**. Thus, there exists some $c_{23} > 0$ and some $n_8 \geq n_7$ such that $\mathsf{P}\big(E^{(n)}_{18,i}\big) \leq c_{23}(\log_2 n)^3(\log n)^{-3}$ for all $n \geq n_8$ and all $-37 \leq i \leq 38$.

Finally, for all $n \geq n_8$ for all $z \in \mathbb{Z}$,

$$\mathsf{P}\big[\big(E^{(n)}_3\big)^c \cap \cup_{j=-13}^{13} E^{(n)}_{17}\big(j,\widetilde{h}_n,z\big)\big] \leq \sum_{i=-37}^{38} \mathsf{P}\big(E^{(n)}_{18,i}\big) \leq 76c_{23}(\log_2 n)^3(\log n)^{-3}, \tag{162}$$

which proves the lemma in the case $h'_n = \widetilde{h}_n$.

We now turn to the case $h'_n = h_n$. Let $z \in \mathbb{Z}$. To this aim, we introduce some notation, which will also be useful in the proof of Lemma 5.10 below. For $j \in \mathbb{Z}$, let

$$\Lambda_j := \sharp\big\{k \in \mathbb{Z},\ x_k\big(V,\widetilde{h}_n\big) \in [x_j(V,h_n), x_{j+1}(V,h_n)[\big\}, \tag{163}$$

which belongs to $(2\mathbb{N}+1)$ since left $\widetilde{h}_n$-maxima and minima alternate and $\widetilde{h}_n < h_n$. If for $j \in \mathbb{Z}$, $\Lambda_j = 2k+1$ with $k > 1$, then $[x_j(V,h_n), x_{j+1}(V,h_n)[= \big[x_\ell\big(V,\widetilde{h}_n\big),\ x_{\ell+2k+1}\big(V,\widetilde{h}_n\big)\big[$ for some $\ell \in \mathbb{Z}$. Also for each $0 \leq i < k$, $H\big[T_{\ell+2i+1}\big(V,\widetilde{h}_n\big)\big] < h_n$, otherwise, if moreover $x_\ell\big(V,\widetilde{h}_n\big)$ is a left h_n-minimum (resp. maximum), then $\widetilde{u} := \min\big\{u \in \big[x_\ell\big(V,\widetilde{h}_n\big), x_{\ell+2i+1}\big(V,\widetilde{h}_n\big)\big],\, V(u) = \max_{[x_\ell(V,\widetilde{h}_n)\leq u\leq x_{\ell+2i+1}(V,\widetilde{h}_n)]} V\big\}$ would be a left h_n-extremum $\big($since $V\big(\widetilde{u}\big) \geq V\big[x_{\ell+2i+2}\big(V,\widetilde{h}_n\big)\big] + h_n \geq V\big[x_\ell\big(V,\widetilde{h}_n\big)\big] + h_n\ \big)$, belonging to $]x_j(V,h_n), x_{j+1}(V,h_n)[$, which is not possible (resp. similar argument with max replaced by min).

Hence on $E^{(n)}_{15}$, $\Lambda_0 \leq 5$, otherwise the support of $T_0(V,h_n)$ would contain the support of at least $(\Lambda_0 - 1)/2 \geq 3$ slopes $T_p\big(V,\widetilde{h}_n\big)$ with height $H\big[T_p\big(V,\widetilde{h}_n\big)\big] < h_n < \log n + C_2\log_2 n$, with at least three of them such that $|p| \leq 5$, which is not possible on $E^{(n)}_{15}$. Also, notice that for $j \geq 1$, $x_j(V,h_n) = x_\ell\big(V,\widetilde{h}_n\big)$ with $1 \leq \ell \leq \Lambda_0 + \cdots + \Lambda_{j-1}$. Thus by induction, $\Lambda_j \leq 5$ for every $0 \leq j \leq 17$, for which we use for $0 < j \leq 17$ the same argument as for Λ_0 with $1 \leq p \leq \Lambda_0 + \cdots + \Lambda_{j-1} + 5$ ($\leq 5(j+1) \leq 90$ by hypothesis of induction). Similarly on $E^{(n)}_{15}$, $\Lambda_j \leq 5$ for every $-17 \leq j \leq 0$, and so for every $-17 \leq j \leq 17$.

Consequently, for the same reasons, on $E^{(n)}_{15}$, if for $-17 \leq j \leq 17$, $\Lambda_j = 3$ (resp. $\Lambda_j = 5$), then the support of $T_j(V,h_n)$ contains the support of at least one (resp. at least two) slope(s) $T_p\big(V,\widetilde{h}_n\big)$ with height $H\big[T_p\big(V,\widetilde{h}_n\big)\big] < h_n < \log n + C_2\log_2 n$ with $|p| < 99$. Thus, $\Lambda_0 + \cdots + \Lambda_j \leq j + 5$ for every $0 \leq j \leq 17$ and $\Lambda_j + \cdots + \Lambda_0 \leq |j| + 5$ for every $-17 \leq j \leq 0$.

Notice that for $-8 \leq j \leq 8$, on $E^{(n)}_{15} \cap E^{(n)}_{17}(j,h_n,z)$, we have $b_j(V,h_n) = x_k(V,h_n)$ with $k \in \{2j-1, 2j\}$ by (160), so $b_j(V,h_n) = x_\ell\big(V,\widetilde{h}_n\big)$ with $1 \leq \ell \leq \Lambda_0 + \cdots + \Lambda_{2j-1} \leq (2j-1) + 5 \leq 20$ if $1 \leq j \leq 8$, and $|\ell| \leq \Lambda_{2j-1} + \cdots + \Lambda_0 \leq |2j-1| + 5 \leq 22$ if $-8 \leq j \leq 0$, using the previous paragraph. So $b_j(V,h_n) = x_\ell\big(V,\widetilde{h}_n\big) = b_{j_0}\big(V,\widetilde{h}_n\big)$ with $\ell \in \{2j_0 - 1, 2j_0\}$, thus $-11 \leq j_0 \leq 10$. Consequently,

there exists $j_1 \in \mathbb{Z}$ such that $z \in \big[M_{j_1-1}\big(V,\widetilde{h}_n\big), M_{j_1}\big(V,\widetilde{h}_n\big)\big] \subset [M_{j-1}(V,h_n), M_j(V,h_n)] \subset \big[x_{\ell-5}\big(V,\widetilde{h}_n\big), x_{\ell+5}\big(V,\widetilde{h}_n\big)\big] \subset \big[x_{-27}\big(V,\widetilde{h}_n\big), x_{25}\big(V,\widetilde{h}_n\big)\big]$ (so that $-13 \le j_1 \le 13$), with $V(z) \le V[b_j(V,h_n)] + 4\log_2 n \le V\big[b_{j_1}\big(V,\widetilde{h}_n\big)\big] + 4\log_2 n$, so the conditions defining $E_{17}^{(n)}\big(j_1,\widetilde{h}_n,z\big)$ are satisfied. Hence,

$$\begin{aligned}&\mathsf{P}\big[\big(E_3^{(n)}\big)^c \cap \cup_{j=-8}^{8} E_{17}^{(n)}(j,h_n,z)\big]\\ \le&\mathsf{P}\big[\big(E_{15}^{(n)}\big)^c\big] + \mathsf{P}\big[\big(E_3^{(n)}\big)^c \cap E_{15}^{(n)} \cap \cup_{j_1=-13}^{13} E_{17}^{(n)}\big(j_1,\widetilde{h}_n,z\big)\big] \le (c_{20}+76c_{23})(\log_2 n)^3(\log n)^{-3}\end{aligned}$$

by Lemma 5.8 and (162) since $n \ge n_8 \ge n_7$, which proves the lemma when $h'_n = h_n$. □

We now introduce some notation. Recall that $\tau[b_i(V,h_n)] < \infty$ $\mathbb{P}$-a.s. for every $i \in \mathbb{Z}$ since $S = (S_k)_k$ is P-almost surely recurrent. We define by induction

$$\begin{aligned}\mathcal{I}_1 &:= \mathbf{1}_{\{\tau[b_1(V,h_n)]<\tau[b_0(V,h_n)]\}},\\ \mathcal{I}_k &:= \sum_{\ell\in\mathbb{Z}-\{\mathcal{I}_j,\,1\le j<k\}} \ell \prod_{i\in\mathbb{Z},\, i\notin\{\mathcal{I}_j,\,1\le j<k\}\cup\{\ell\}} \mathbf{1}_{\{\tau[b_\ell(V,h_n)]<\tau[b_i(V,h_n)]\}}, \qquad k\ge 2. \qquad (164)\end{aligned}$$

In words, $\mathcal{I}_1$ is the index ℓ of the first $b_\ell[V,h_n]$ visited by S, so that $\mathcal{I}_1 = 0$ if $\tau[b_0(V,h_n)] < \tau[b_1(V,h_n)]$ and $\mathcal{I}_1 = 1$ if $\tau[b_1(V,h_n)] < \tau[b_0(V,h_n)]$, which are the only possible cases since $b_0(V,h_n) \le 0 = S_0 < b_1(V,h_n)$ $\mathbb{P}$-a.s. Similarly, $\mathcal{I}_2$ is the index ℓ of the second $b_\ell(V,h_n)$ visited by S, so $\mathcal{I}_2 \ne \mathcal{I}_1$, and more generally $\mathcal{I}_k$ is for $k \in \mathbb{N}^*$ the index ℓ of the k-th $b_\ell(V,h_n)$ visited by S, so that $\mathcal{I}_k \notin \{\mathcal{I}_1,\mathcal{I}_2,\dots,\mathcal{I}_{k-1}\}$. Notice that $\tau[b_{\mathcal{I}_1}(V,h_n)] = \tau[b_0(V,h_n)] \wedge \tau[b_1(V,h_n)]$ is a stopping time under P_ω with the natural filtration of S, and more generally $\tau[b_{\mathcal{I}_k}(V,h_n)]$ is a stopping time for every $k \ge 1$.

Recall that $0 \in [b_0(V,h_n), b_1(V,h_n)[$, that $b_0(V,h_n)$ and $b_1(V,h_n)$ are consecutive left h_n-minima, and $M_0(V,h_n)$ is the only left h_n-maximum between them. So, applying Lemma 4.1 with $h = \log n$, $\xi_2 = 1/10$, $a = b_0(V,h_n) < b = M_0(V,h_n) < c = b_1(V,h_n)$ which satisfy **(i)** due to the previous remark, $\xi_1 = C_1$ so that **(ii)** and **(iii)** are satisfied since there is no left $(h_n = \log n - C_1\log_2 n)$-extremum in $]M_0(V,h_n), b_1(V,h_n)[$ nor in $]b_0(V,h_n), M_0(V,h_n)[$, $\alpha = 3$ (so that **(iv)** is satisfied for $\omega \in E_5^{(n)}$, since $|x_i(V,h_n)| \le |x_i(V,\log n)|$ for every $i \in \mathbb{Z}$ and $\delta_1 < 2/3$) and $x = 0$, we get for $n \ge n_8$ (which implies that $n \ge n_3$ so $\log n \ge \widehat{h}_2(C_1,1/10)$), for almost all $\omega \in E_5^{(n)}$,

$$P_\omega\big[\tau(b_{\mathcal{I}_1}(V,h_n)) \ge n/10\big] = P_\omega\big[\tau(b_0(V,h_n)) \wedge \tau(b_1(V,h_n)) \ge n/10\big] \le (\log n)^{-4}.$$

Consequently, using Lemma 7.1, for $n \ge \max(n_8,p_3) =: n_9$,

$$\mathbb{P}\big[\tau(b_{\mathcal{I}_1}(V,h_n)) \ge n/10\big] \le \mathbb{P}\big[\tau(b_{\mathcal{I}_1}(V,h_n)) \ge n/10, E_5^{(n)}\big] + \mathsf{P}\big[\big(E_5^{(n)}\big)^c\big] \le 2(\log n)^{-3}. \qquad (165)$$

We now prove several lemmas which are useful to prove Proposition 5.7. In what follows, for $i \in \mathbb{Z}$, we write B_i and M_i respectively for $b_i(V,h_n)$ and $M_i(V,h_n)$ (which are defined in and after (160)). We first prove that, with large enough probability, S only visits up to 3 different B_i before time n:

Lemma 5.10. *There exists $c_{24} > 0$ such that,*

$$\forall n \ge n_9, \qquad \mathbb{P}\big[\tau(\mathrm{B}_{\mathcal{I}_4}) \le n\big] \le c_{24}(\log_2 n)^3(\log n)^{-3}.$$

The main idea is that, loosely speaking, on $E_{15}^{(n)}$, S has to cross, before $\tau(\mathrm{B}_{\mathcal{I}_4})$, at least one slope with height at least $\log n + C_2\log_2 n$, which takes more than n units of time with large probability. We now prove this rigorously.

Proof of Lemma 5.10: Let $n \geq n_9$. First, for every $1 \leq k \leq 3$, using $\mathrm{B}_{\mathcal{I}_k} < M_{\mathcal{I}_k} < \mathrm{B}_{\mathcal{I}_k+1} \leq \mathrm{B}_{\mathcal{I}_{k+1}}$ when $\mathcal{I}_k < \mathcal{I}_{k+1}$ in the first inequality, then conditioning by ω then applying the strong Markov property at time $\tau(\mathrm{B}_{\mathcal{I}_k})$ in the following line, and finally (10) and ellipticity in the last line, we have

$$\begin{aligned} p_{1,k,n} &:= \mathbb{P}\big[\tau(\mathrm{B}_{\mathcal{I}_k}, \mathrm{B}_{\mathcal{I}_{k+1}}) \leq n, V(M_{\mathcal{I}_k}) - V(\mathrm{B}_{\mathcal{I}_k}) \geq \log n + C_2 \log_2 n, I_{k+1} > \mathcal{I}_k\big] \\ &\leq \mathbb{P}\big[\tau(\mathrm{B}_{\mathcal{I}_k}, M_{\mathcal{I}_k}) \leq n, V(M_{\mathcal{I}_k}) - V(\mathrm{B}_{\mathcal{I}_k}) \geq \log n + C_2 \log_2 n\big] \\ &= \mathbb{E}\big[\mathbf{1}_{\{V(M_{\mathcal{I}_k}) - V(\mathrm{B}_{\mathcal{I}_k}) \geq \log n + C_2 \log_2 n\}} P_\omega^{\mathrm{B}_{\mathcal{I}_k}}[\tau(M_{\mathcal{I}_k}) \leq n]\big] \\ &\leq 2\varepsilon_0^{-1}(\log n)^{-C_2} \leq (\log n)^{-4} \end{aligned} \tag{166}$$

since $C_2 > 9$ and $n \geq n_9 \geq n_3$. Similarly, using (11) instead of (10) and $\mathrm{B}_{\mathcal{I}_{k+1}} \leq \mathrm{B}_{\mathcal{I}_k - 1} < M_{\mathcal{I}_k - 1} < \mathrm{B}_{\mathcal{I}_k}$ when $I_{k+1} < \mathcal{I}_k$, we have

$$\begin{aligned} p_{2,k,n} &:= \mathbb{P}\big[\tau(\mathrm{B}_{\mathcal{I}_k}, \mathrm{B}_{\mathcal{I}_{k+1}}) \leq n, V(M_{\mathcal{I}_k - 1}) - V(\mathrm{B}_{\mathcal{I}_k}) \geq \log n + C_2 \log_2 n, I_{k+1} < \mathcal{I}_k\big] \\ &\leq (\log n)^{-4} \end{aligned} \tag{167}$$

for every $1 \leq k \leq 3$ since $C_2 > 9$ and $n \geq n_9 \geq n_3$.

We now prove that on $E_{15}^{(n)}$,

$$\sharp\{-6 \leq j \leq 6,\ H[T_j(V, h_n)] < \log n + C_2 \log_2 n\} \leq 2. \tag{168}$$

To this aim, we use (163) and the following paragraphs. We claim that on $E_{15}^{(n)}$, if for some $-6 \leq j \leq 6$, $H[T_j(V, h_n)] < \log n + C_2 \log_2 n$, then the support of $T_j(V, h_n)$ contains at least the support of one $T_k\big(V, \widetilde{h}_n\big)$ with $H\big[T_k\big(V, \widetilde{h}_n\big)\big] < \log n + C_2 \log_2 n$ with $|k| < 99$. Indeed, on $E_{15}^{(n)}$, $x_j(V, h_n) = x_k\big(V, \widetilde{h}_n\big)$ with $|k| \leq \Lambda_0 + \cdots + \Lambda_j \leq 35$ and the support of $T_j(V, h_n)$ contains the support of $T_k\big(V, \widetilde{h}_n\big)$, so $H\big[T_k\big(V, \widetilde{h}_n\big)\big] \leq H[T_j(V, h_n)] < \log n + C_2 \log_2 n$. Since there are at most two slopes $H\big[T_k\big(V, \widetilde{h}_n\big)\big]$, $|k| \leq 35$ with height $< \log n + C_2 \log_2 n$ on $E_{15}^{(n)}$, there are at most two $j \in \{-6, \dots, 6\}$ such that $H[T_j(V, h_n)] < \log n + C_2 \log_2 n$, which proves (168).

Also, $\{\mathcal{I}_j,\ 1 \leq j \leq k\} \subset \{1 - k, \dots, k\}$ for every $k \in \mathbb{N}^*$ by induction, since for $k \geq 2$, $\min\{\mathcal{I}_j,\ 1 \leq j < k\} - 1 \leq \mathcal{I}_k \leq \max\{\mathcal{I}_j,\ 1 \leq j < k\} + 1$, because S only makes ± 1 jumps. So by (160), $\mathrm{B}_{\mathcal{I}_k} = x_{i_k}(V, h_n)$ with $i_k \in \{-5, \dots, 6\}$ when $1 \leq k \leq 3$. Hence, each height $V(M_{\mathcal{I}_k}) - V(\mathrm{B}_{\mathcal{I}_k})$ or $V(M_{\mathcal{I}_k - 1}) - V(\mathrm{B}_{\mathcal{I}_k})$ with $k \in \{1, 2, 3\}$ is equal to some $H[T_j(V, h_n)]$ with $|j| \leq 6$, so at most two of them are less than $\log n + C_2 \log_2 n$ on $E_{15}^{(n)}$ by (168).

Hence, for $n \geq n_9$, using (166) and (167) in the last inequality,

$$\begin{aligned} &\mathbb{P}\big[\tau(\mathrm{B}_{\mathcal{I}_4}) \leq n, E_{15}^{(n)}\big] \\ &\leq \mathbb{P}\big[\cap_{j=1}^3 \{\tau(\mathrm{B}_{\mathcal{I}_j}, \mathrm{B}_{\mathcal{I}_{j+1}}) \leq n\} \cap \cup_{k=1}^3 \big(\{V(M_{\mathcal{I}_k}) - V(\mathrm{B}_{\mathcal{I}_k}) \geq \log n + C_2 \log_2 n, \mathcal{I}_{k+1} > \mathcal{I}_k\} \\ &\qquad\qquad \cup \{V(M_{\mathcal{I}_k - 1}) - V(\mathrm{B}_{\mathcal{I}_k}) \geq \log n + C_2 \log_2 n, \mathcal{I}_{k+1} < \mathcal{I}_k\}\big)\big] \\ &\leq \sum_{k=1}^3 (p_{1,k,n} + p_{2,k,n}) \leq 6(\log n)^{-4}. \end{aligned} \tag{169}$$

This together with Lemma 5.8 proves Lemma 5.10 since $n_9 \geq n_7 \geq n_3$. □

In the following lemma, we study separately the cases in which $z \in [\mathrm{B}_{\mathcal{I}_k - 1}, \mathrm{B}_{\mathcal{I}_k + 1}]$ for $1 \leq k \leq 3$ (in view of Lemma 5.10 since $S_i \in \cup_{k=1}^3 [\mathrm{B}_{\mathcal{I}_k - 1}, \mathrm{B}_{\mathcal{I}_k + 1}]$ for $i \leq \tau(\mathrm{B}_{\mathcal{I}_4})$).

Lemma 5.11. *There exists $c_{25} > 0$ such that, for all $n \geq n_9$, for all $z \in \mathbb{Z}$, for all $1 \leq k \leq 3$,*

$$\mathbb{P}\big(S_n = z, \mathrm{B}_{\mathcal{I}_k - 1} \leq z \leq \mathrm{B}_{\mathcal{I}_k + 1}, \tau(\mathrm{B}_{\mathcal{I}_k}) \leq n, E_5^{(n)}, \big(E_3^{(n)}\big)^c\big) \leq c_{25}(\log_2 n)^3 (\log n)^{-3}.$$

Before proving Lemma 5.11, we introduce some notation. For $i \in \mathbb{Z}$, let (see Figure 8),

$$\begin{aligned} D_i^+ &:= \min\{j \geq M_i,\ V(j) \leq V(\mathrm{B}_i) + 4\log_2 n\}, \\ D_i^- &:= \max\{j \leq M_{i-1},\ V(j) \leq V(\mathrm{B}_i) + 4\log_2 n\}, \end{aligned} \tag{170}$$

so that, by ellipticity, $V(j) \geq V(\mathrm{B}_i) + 4\log_2 n + \log\varepsilon_0$ for each $j \in ([D_i^-, M_{i-1}] \cup [M_i, D_i^+])$.

We cut the proof of Lemma 5.11 into two main parts. First we consider the case $z \in \big[D^-_{\mathcal{I}_k}, D^+_{\mathcal{I}_k}\big]$ in Lemma 5.12, then $z \in \big]D^+_{\mathcal{I}_k}, \mathrm{B}_{\mathcal{I}_k+1}\big]$ in Lemma 5.13, the case $z \in \big[\mathrm{B}_{\mathcal{I}_k-1}, D^-_{\mathcal{I}_k}\big[$ being obtained by symmetry in (198).

Lemma 5.12. *There exists $c_{26} > 0$ such that, for all $n \geq n_9$, for all $z \in \mathbb{Z}$, for all $1 \leq k \leq 3$,*

$$\mathbb{P}\big[S_n = z, D^-_{\mathcal{I}_k} \leq z \leq D^+_{\mathcal{I}_k}, \tau(\mathrm{B}_{\mathcal{I}_k}) \leq n, \big(E_3^{(n)}\big)^c\big] \leq c_{26}(\log_2 n)^3(\log n)^{-3}. \tag{171}$$

Proof: The proof is divided into two cases, one for which we use Lemma 5.9 if $V(z) - V(\mathrm{B}_{\mathcal{I}_k})$ is small enough ($\leq 4\log_2 n$), and one for which we use reversibility if it is larger. More precisely, let $n \geq n_9$ and $z \in \mathbb{Z}$. First, recall that $\{\mathcal{I}_j,\ 1 \leq j \leq k\} \subset \{1-k, \dots, k\}$ for every $k \in \mathbb{N}^*$. So by Lemma 5.9 with $h'_n = h_n$, since $n \geq n_9 \geq n_8$, we have for every $1 \leq k \leq 3$, taking into account all the possible values j of $\mathcal{I}_k$ (see Figure 8 with $z = z^{(5)}$),

$$\begin{aligned} & \mathbb{P}\big[S_n = z, M_{\mathcal{I}_k-1} \leq z \leq M_{\mathcal{I}_k}, V(z) \leq V(\mathrm{B}_{\mathcal{I}_k}) + 4\log_2 n, \big(E_3^{(n)}\big)^c\big] \\ \leq\ & \mathsf{P}\big[\big(E_3^{(n)}\big)^c \cap \cup_{j=-2}^{3}\{M_{j-1} \leq z \leq M_j, V(z) \leq V(\mathrm{B}_j) + 4\log_2 n\}\big] \\ \leq\ & c_{21}(\log_2 n)^3(\log n)^{-3}. \end{aligned} \tag{172}$$

Second, conditioning by ω, then applying the strong Markov property at stopping time $\tau(\mathrm{B}_{\mathcal{I}_k})$ in the first equality, we get (see Figure 8 with $z = z^{(6)}$),

$$\begin{aligned} & \mathbb{P}\big[S_n = z, V(z) \geq V(\mathrm{B}_{\mathcal{I}_k}) + 4\log_2 n + \log\varepsilon_0, \tau(\mathrm{B}_{\mathcal{I}_k}) \leq n\big] \\ =\ & \mathbb{E}\big[\mathbf{1}_{\{V(z) \geq V(\mathrm{B}_{\mathcal{I}_k}) + 4\log_2 n + \log\varepsilon_0\}}\mathbf{1}_{\{\tau(\mathrm{B}_{\mathcal{I}_k}) \leq n\}}P_\omega^{\mathrm{B}_{\mathcal{I}_k}}(S_\ell = z)_{|\ell = n - \tau(\mathrm{B}_{\mathcal{I}_k})}\big] \\ \leq\ & \big(1 + e^{C_0}\big)\varepsilon_0^{-1}(\log n)^{-4}, \end{aligned} \tag{173}$$

since $P_\omega^{\mathrm{B}_{\mathcal{I}_k}}(S_\ell = z) \leq \frac{\mu_\omega(z)}{\mu_\omega(\mathrm{B}_{\mathcal{I}_k})} \leq \big(1 + e^{C_0}\big)\exp(-[V(z) - V(\mathrm{B}_{\mathcal{I}_k})])$ for all $\ell \in \mathbb{N}$ by reversibility and ellipticity (see (13) and (16)).

Finally, notice that if $D^-_{\mathcal{I}_k} \leq z \leq D^+_{\mathcal{I}_k}$, then either $V(z) \geq V(\mathrm{B}_{\mathcal{I}_k}) + 4\log_2 n + \log\varepsilon_0$, either $M_{\mathcal{I}_k-1} \leq z \leq M_{\mathcal{I}_k}$ and $V(z) \leq V(\mathrm{B}_{\mathcal{I}_k}) + 4\log_2 n$ (by the remark after (170) and since $\log\varepsilon_0 \leq 0$). Hence, combining (172) and (173), we get (171), since $n \geq n_9 \geq n_3$. □

We now consider the case $z \in \big]D^+_{\mathcal{I}_k}, \mathrm{B}_{\mathcal{I}_k+1}\big]$ (notice that this interval may be empty). We prove the following lemma.

Lemma 5.13. *There exists $c_{27} > 0$ such that, for all $n \geq n_9$, for all $z \in \mathbb{Z}$, for all $1 \leq k \leq 3$,*

$$\mathbb{P}\big[S_n = z, D^+_{\mathcal{I}_k} < z \leq \mathrm{B}_{\mathcal{I}_k+1}, \tau(\mathrm{B}_{\mathcal{I}_k}) \leq n, E_5^{(n)}, \big(E_3^{(n)}\big)^c\big] \leq c_{27}(\log_2 n)^3(\log n)^{-3}. \tag{174}$$

Before giving the proof, we introduce some notation. Let $n \geq n_9$ and $z \in \mathbb{Z}$. We define for $i \in \mathbb{Z}$ (see Figure 8),

$$m^+(z,i) := \min\Big\{D_i^+ \leq j \leq z, \quad V(j) = \min\nolimits_{[D_i^+, z]} V\Big\}, \tag{175}$$

with by convention, $\min\emptyset = +\infty$, so $m^+(z,i)$ is defined in every case, even if we use it only when $z \geq D_i^+$.

Idea of the proof: (see Figures 8 and 9 for the different cases). First, loosely speaking, if $V(z)$ is quite larger than the minimum of V in $[D^+_{\mathcal{I}_k}, z]$ $\big($see $E^{(n,z)}_{21,k}$ and Figure 8 below$\big)$ and $n \geq \tau(\mathrm{B}_{\mathcal{I}_k})$, then by reversibility the probability that $S_n = z$ is negligible. So we can assume that $V(z)$ is just slightly higher than $\min_{[D^+_{\mathcal{I}_k}, z]} V$. If moreover on the right of z, the potential V goes up $\widetilde{h}_n$ before going down $4\log_2 n$ $\big($see $E^{(n,z)}_{22,k}$ and Figure 8 below$\big)$, we prove that we are in $\cup_{q=-8}^{8} E^{(n)}_{17}\big(q, \widetilde{h}_n, z\big)$, so, applying Lemma 5.9, the probability of this case is also negligible. Thus we can also assume that on the right of z, the potential V does not go up $\widetilde{h}_n$ before going down $4\log_2 n$ $\big($see $E^{(n,z)}_{23,k}$ below and Figure 9$\big)$. In this case, if $\tau(\mathrm{B}_{\mathcal{I}_k}) + \tau[\mathrm{B}_{\mathcal{I}_k}, m^+(z,\mathcal{I}_k)] > n$, then $S_n \neq z$. Also, we can choose some constant c_{28} such that, applying (12), $\tau(\mathrm{B}_{\mathcal{I}_k}) + \tau[\mathrm{B}_{\mathcal{I}_k}, m^+(z,\mathcal{I}_k)] \in \big[n - n(\log n)^{-c_{28}}, n\big]$ has a negligible probability. Finally, if $\tau(\mathrm{B}_{\mathcal{I}_k}) + \tau[\mathrm{B}_{\mathcal{I}_k}, m^+(z,\mathcal{I}_k)] < n - n(\log n)^{-c_{28}}$, then we prove that quite quickly and in particular before time n (if some very probable additional condition is satisfied, see (186) and (196)), S goes to some place $z_n^{\downarrow}$ with $V(z_n^{\downarrow}) \leq V(z) - 4\log_2 n$, and then the probability that $S_n = z$ is negligible, once more by reversibility. We now prove this rigorously.

Proof of Lemma 5.13: Let $n \geq n_9$, $z \in \mathbb{Z}$ and $1 \leq k \leq 3$. The proof is divided into three main cases, corresponding to the following events, the last one being itself divided into four subcases (which are defined around (183) and (186)):

$$\begin{aligned}
E^{(n,z)}_{21,k} &:= \big\{D^+_{\mathcal{I}_k} < z \leq \mathrm{B}_{\mathcal{I}_k+1}\big\} \cap \Big\{V(z) \geq \min\nolimits_{[D^+_{\mathcal{I}_k}, z]} V + 4\log_2 n\Big\},\\
E^{(n,z)}_{22,k} &:= \big\{D^+_{\mathcal{I}_k} < z \leq \mathrm{B}_{\mathcal{I}_k+1}\big\} \cap \Big\{V(z) < \min\nolimits_{[D^+_{\mathcal{I}_k}, z]} V + 4\log_2 n\Big\}\\
&\qquad\qquad \cap \big\{T_{V^+_z}\big(\big[\widetilde{h}_n, +\infty\big[\big) < T_{V^+_z}(]-\infty, -4\log_2 n]\big)\big\},\\
E^{(n,z)}_{23,k} &:= \big\{D^+_{\mathcal{I}_k} < z \leq \mathrm{B}_{\mathcal{I}_k+1}\big\} \cap \Big\{V(z) < \min\nolimits_{[D^+_{\mathcal{I}_k}, z]} V + 4\log_2 n\Big\}\\
&\qquad\qquad \cap \big\{T_{V^+_z}(]-\infty, -4\log_2 n]) < T_{V^+_z}\big(\big[\widetilde{h}_n, +\infty\big[\big)\big\}.
\end{aligned}$$

where $V^+_z(\ell) = V(z+\ell) - V(z)$, $\ell \in \mathbb{N}$ as before and $\widetilde{h}_n := h_n - C_1\log_2 n = \log n - 2C_1\log_2 n$. See figures 8 and 9.

First case: We consider the event $E^{(n,z)}_{21,k}$.

We have, once more conditioning by ω then applying the strong Markov property at stopping time $\tau(\mathrm{B}_{\mathcal{I}_k})$ in the first equality, then using $\mathrm{B}_{\mathcal{I}_k} \leq D^+_{\mathcal{I}_k} \leq m^+(z,\mathcal{I}_k) \leq z$ on $E^{(n,z)}_{21,k}$ in the second equality, then the strong Markov property at time $\tau[m^+(z,\mathcal{I}_k)]$,

$$\begin{aligned}
&\mathbb{P}\big[S_n = z, \tau(\mathrm{B}_{\mathcal{I}_k}) \leq n, E^{(n,z)}_{21,k}\big]\\
=\ &\mathbb{E}\big[\mathbf{1}_{\{\tau(\mathrm{B}_{\mathcal{I}_k})\leq n\}}\mathbf{1}_{E^{(n,z)}_{21,k}} P^{\mathrm{B}_{\mathcal{I}_k}}_{\omega}(S_\ell = z)_{|\ell = n-\tau(\mathrm{B}_{\mathcal{I}_k})}\big]\\
=\ &\mathbb{E}\big[\mathbf{1}_{\{\tau(\mathrm{B}_{\mathcal{I}_k})\leq n\}}\mathbf{1}_{E^{(n,z)}_{21,k}} P^{\mathrm{B}_{\mathcal{I}_k}}_{\omega}(S_\ell = z, \tau[m^+(z,i)] \leq \ell)_{|i=\mathcal{I}_k,\, \ell = n-\tau(\mathrm{B}_{\mathcal{I}_k})}\big]\\
=\ &\mathbb{E}\big[\mathbf{1}_{\{\tau(\mathrm{B}_{\mathcal{I}_k})\leq n\}}\mathbf{1}_{E^{(n,z)}_{21,k}} \mathsf{E}^{\mathrm{B}_{\mathcal{I}_k}}_{\omega}\big(\mathbf{1}_{\{\tau[m^+(z,i)]\leq \ell\}} P^{m^+(z,i)}_{\omega}(S_t = z)_{|t=\ell-\tau[m^+(z,i)]}\big)_{|i=\mathcal{I}_k,\, \ell = n-\tau(\mathrm{B}_{\mathcal{I}_k})}\big]\\
\leq\ &\big(1 + e^{C_0}\big)(\log n)^{-4} \qquad (176)
\end{aligned}$$

since $P^{m^+(z,\mathcal{I}_k)}_{\omega}(S_t = z) \leq \frac{\mu_\omega(z)}{\mu_\omega[m^+(z,\mathcal{I}_k)]} \leq \big(1 + e^{C_0}\big)\exp(-[V(z) - V(m^+(z,\mathcal{I}_k))]) \leq \big(1 + e^{C_0}\big)(\log n)^{-4}$ for all $t \in \mathbb{N}$ on $E^{(n,z)}_{21,k}$ by reversibility and ellipticity (see (13) and (16)).

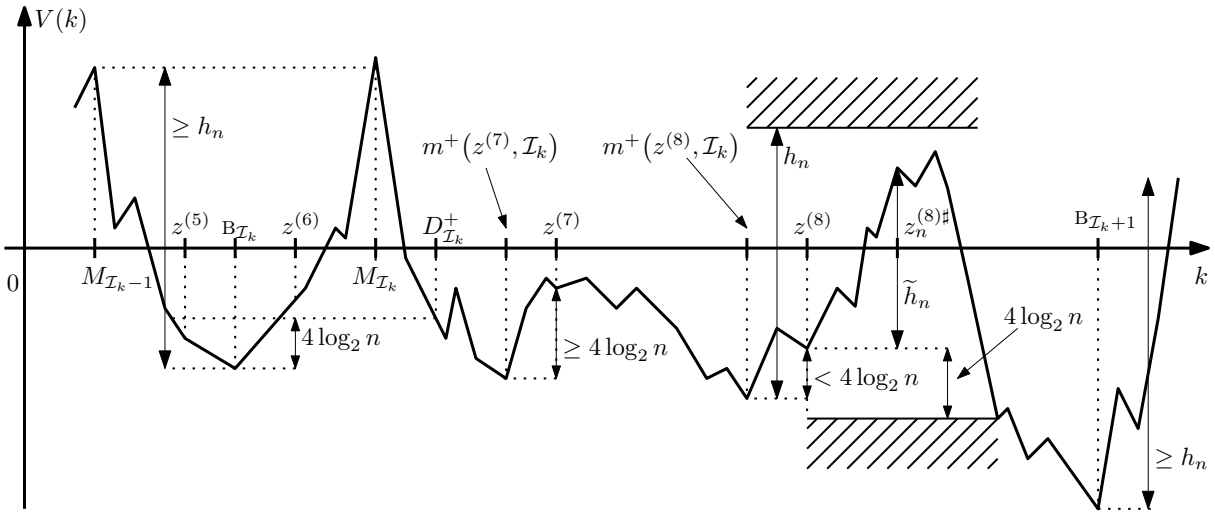

Figure 8. Schema of the potential V, with z represented as $z^{(5)}$ in the first case of the proof of Lemma 5.12, $z^{(6)}$ in the second one, and as $z^{(7)}$ on $E_{21,k}^{(n,z)}$ and $z^{(8)}$ on $E_{22,k}^{(n,z)}$ for the proof of Lemma 5.13.

Second case: We now focus on $E_{22,k}^{(n,z)}$. Notice in particular that $E_{22,k}^{(n,z)}$ includes the case where the potential of z is "close" to the one of $\mathrm{B}_{\mathcal{I}_k+1}$ (with a difference of potential lower than $4\log_2 n$).

We now assume that we are on $E_{22,k}^{(n,z)}$. Hence we have, by definition (170) of $D_{\mathcal{I}_k}^+$,

$$\min_{[M_{\mathcal{I}_k},z]} V = \min_{[D_{\mathcal{I}_k}^+,z]} V > V(z) - 4\log_2 n. \tag{177}$$

Also, $V(M_{\mathcal{I}_k}) = \max_{[M_{\mathcal{I}_k},\mathrm{B}_{\mathcal{I}_k+1}]} V$ and $[M_{\mathcal{I}_k}, z] \subset [M_{\mathcal{I}_k}, \mathrm{B}_{\mathcal{I}_k+1}]$, so

$$\max_{[M_{\mathcal{I}_k},z]} V = V(M_{\mathcal{I}_k}) \geq V(\mathrm{B}_{\mathcal{I}_k}) + h_n \geq V(D_{\mathcal{I}_k}^+) - 4\log_2 n + h_n, \tag{178}$$

since $V(M_{\mathcal{I}_k}) - V(\mathrm{B}_{\mathcal{I}_k}) \geq h_n$ and once more by definition of $D_{\mathcal{I}_k}^+$.

Now, let $z_n^\sharp := z + T_{V_z^+}\big(\big[\widetilde{h}_n, +\infty\big[\big)$. By definition of $D_{\mathcal{I}_k}^+$ and due to the first event defining $E_{22,k}^{(n,z)}$, then due to the last two events defining $E_{22,k}^{(n,z)}$, we have

$$\min_{[M_{\mathcal{I}_k},z_n^\sharp]} V = \min_{[D_{\mathcal{I}_k}^+,z_n^\sharp]} V \geq V(z) - 4\log_2 n. \tag{179}$$

There exists a unique index $p \in \mathbb{Z}$ such that $M_{p-1}\big(V,\widetilde{h}_n\big) \leq z < M_p\big(V,\widetilde{h}_n\big)$. So $M_{p-1}\big(V,\widetilde{h}_n\big)$ is the largest left $\widetilde{h}_n$-maximum less than or equal to z. Since $M_{\mathcal{I}_k}$ is a left h_n and then left $\widetilde{h}_n$-maximum and is $\leq z$, we have $M_{\mathcal{I}_k} \leq M_{p-1}\big(V,\widetilde{h}_n\big)$.

Assume that $z_n^\sharp < b_p\big(V,\widetilde{h}_n\big)$. We define

$$b_n^\sharp := \inf\big\{q \in \mathbb{Z},\ q \geq M_{p-1}\big(V,\widetilde{h}_n\big),\ V(q) = \min_{[M_{p-1}(V,\widetilde{h}_n),\, z_n^\sharp]} V\big\}.$$

We would have $M_{p-1}\big(V,\widetilde{h}_n\big) \leq z < z_n^\sharp < b_p\big(V,\widetilde{h}_n\big)$ and so $V(z_n^\sharp) \geq V(z) + \widetilde{h}_n \geq V(b_n^\sharp) + \widetilde{h}_n$ by definition of $z_n^\sharp$ and $b_n^\sharp$, and

$$V\big[M_{p-1}\big(V,\widetilde{h}_n\big)\big] = \max_{[M_{p-1}(V,\widetilde{h}_n),\, b_p(V,\widetilde{h}_n)]} V \geq V(z_n^\sharp) \geq V(b_n^\sharp) + \widetilde{h}_n.$$

Hence, $b_n^\sharp$ would be a left $\widetilde{h}_n$-minimum of V, strictly between $M_{p-1}\big(V,\widetilde{h}_n\big)$ and $b_p\big(V,\widetilde{h}_n\big)$, which is not possible because $M_{p-1}\big(V,\widetilde{h}_n\big)$ and $b_p\big(V,\widetilde{h}_n\big)$ are consecutive left $\widetilde{h}_n$-extrema (see (160) and the comments below). So, $b_p\big(V,\widetilde{h}_n\big) \le z_n^\sharp$.

Thus, $M_{\mathcal{I}_k} \le M_{p-1}\big(V,\widetilde{h}_n\big) \le b_p\big(V,\widetilde{h}_n\big) \le z_n^\sharp$, Hence, $V\big[b_p\big(V,\widetilde{h}_n\big)\big] \ge \min_{[M_{\mathcal{I}_k},z_n^\sharp]} V \ge V(z) - 4\log_2 n$ by (179) for the last inequality, and thus $V(z) \le V\big[b_p\big(V,\widetilde{h}_n\big)\big] + 4\log_2 n$. Hence, using the definition of p, we are in $E_{17}^{(n)}\big(p,\widetilde{h}_n,z\big)$ (defined after (160)).

Also, $M_{\mathcal{I}_k} < z \le \mathrm{B}_{\mathcal{I}_k+1}$ on $E_{22,k}^{(n,z)}$. So, either $z_n^\sharp < \mathrm{B}_{\mathcal{I}_k+1} < M_{\mathcal{I}_k+1}$, either $z_n^\sharp \ge \mathrm{B}_{\mathcal{I}_k+1}$. In the second case, $V(\mathrm{B}_{\mathcal{I}_k+1}) \ge \min_{[M_{\mathcal{I}_k},z_n^\sharp]} V \ge V(z) - 4\log_2 n$ by (179), so $V(z) + \widetilde{h}_n \le V(\mathrm{B}_{\mathcal{I}_k+1}) + \widetilde{h}_n + 4\log_2 n \le V(\mathrm{B}_{\mathcal{I}_k+1}) + h_n$ since $C_1 > 4$, thus $z_n^\sharp \le \mathrm{B}_{\mathcal{I}_k+1} + T_{V^+_{\mathrm{B}_{\mathcal{I}_k+1}}}([h_n,+\infty[) \le M_{\mathcal{I}_k+1}$ by definition of $z_n^\sharp$ and $M_{\mathcal{I}_k+1}$. Hence in every case, $\mathrm{B}_{\mathcal{I}_k} < M_{\mathcal{I}_k} \le b_p\big(V,\widetilde{h}_n\big) \le z_n^\sharp \le M_{\mathcal{I}_k+1} < \mathrm{B}_{\mathcal{I}_k+2}$, and so $b_p\big(V,\widetilde{h}_n\big) \in]\mathrm{B}_{\mathcal{I}_k}, \mathrm{B}_{\mathcal{I}_k+2}[$.

We now also assume that $\omega \in E_{15}^{(n)}$. We recall that since $1 \le k \le 3$, there exists $i_k \in \{-5,\dots,6\}$ such that $\mathrm{B}_{\mathcal{I}_k} = x_{i_k}(V,h_n)$ (as proved before (169)).

Also $\mathrm{B}_{\mathcal{I}_k}$ is a left h_n-minimum and since $\widetilde{h}_n < h_n$, it is a fortiori a left $\widetilde{h}_n$-minimum, so is equal to a $b_j\big(V,\widetilde{h}_n\big)$, with $-4 \le j \le 5$ since $-2 \le \mathcal{I}_k \le 3$ (see before (169)) and on $E_{15}^{(n)}$, as already proved, all the left $\widetilde{h}_n$-minima $b_\ell\big(V,\widetilde{h}_n\big)$ with $|\ell| \le 8$ are also left h_n-minima except at most two of them because $h_n < \log n + C_2 \log_2 n$, thus the number of left $\widetilde{h}_n$-minima in $\big]0, \mathrm{B}_{\mathcal{I}_k}\big]$ if $\mathrm{B}_{\mathcal{I}_k} > 0$ (resp. $\big[\mathrm{B}_{\mathcal{I}_k}, 0\big]$ if $\mathrm{B}_{\mathcal{I}_k} \le 0$) is at most $|\mathcal{I}_k| + 2$ (resp. $|\mathcal{I}_k| + 3$). Also for this last reason, there are no more than three left $\widetilde{h}_n$-minima in $]\mathrm{B}_{\mathcal{I}_k}, \mathrm{B}_{\mathcal{I}_k+2}[$, interval to which $b_p\big(V,\widetilde{h}_n\big)$ belongs as proved previously on $E_{22,k}^{(n,z)}$, so $|p| \le |j| + 3 \le 8$. Since we already proved that we are on $E_{17}^{(n)}\big(p,\widetilde{h}_n,z\big)$, this gives $E_{22,k}^{(n,z)} \cap E_{15}^{(n)} \subset \cup_{q=-8}^{8} E_{17}^{(n)}\big(q,\widetilde{h}_n,z\big)$.

Finally, by Lemmas 5.8 and 5.9, we have since $n \ge n_9 \ge n_8 \ge n_7$,

$$\begin{aligned}
\mathbb{P}\big[E_{22,k}^{(n,z)} \cap \big(E_3^{(n)}\big)^c\big] &\le \mathsf{P}\big[\big(E_{15}^{(n)}\big)^c\big] + \mathbb{P}\big[E_{22,k}^{(n,z)} \cap E_{15}^{(n)} \cap \big(E_3^{(n)}\big)^c\big] \\
&\le \mathsf{P}\big[\big(E_{15}^{(n)}\big)^c\big] + \mathsf{P}\big[\big(E_3^{(n)}\big)^c \cap \cup_{q=-8}^{8} E_{17}^{(n)}\big(q,\widetilde{h}_n,z\big)\big] \\
&\le (c_{20}+c_{21})(\log_2 n)^3 (\log n)^{-3}. \qquad (180)
\end{aligned}$$

Third case: There remains to consider $E_{23,k}^{(n,z)}$. We recall $m^+(z,i)$ from (175), and the definition of the return time $\tau^*(y) := \inf\{k \ge 1 \ : \ S_k = y\}$ for $y \in \mathbb{Z}$.

Using (12) (with $\mathrm{B}_{\mathcal{I}_k} < M_{\mathcal{I}_k} < m^+(z,\mathcal{I}_k)$) in the first line, the Markov property in the second one, (7) in the third one, we have on $E_{23,k}^{(n,z)}$ for every $\ell \in \mathbb{N}$,

$$\begin{aligned}
P_\omega^{\mathrm{B}_{\mathcal{I}_k}}[\tau(m^+(z,i)) = \ell]_{|i=\mathcal{I}_k} &\le P_\omega^{\mathrm{B}_{\mathcal{I}_k}}[\tau(m^+(z,i)) < \tau^*(\mathrm{B}_i)]_{|i=\mathcal{I}_k} \\
&= \omega_{\mathrm{B}_{\mathcal{I}_k}} P_\omega^{\mathrm{B}_{\mathcal{I}_k}+1}[\tau(m^+(z,i)) < \tau(\mathrm{B}_i)]_{|i=\mathcal{I}_k} \\
&\le \exp[V(\mathrm{B}_{\mathcal{I}_k}) - V(M_{\mathcal{I}_k})] \\
&\le \exp(-h_n) = (\log n)^{C_1}/n \qquad (181)
\end{aligned}$$

since $V(M_{\mathcal{I}_k}) - V(\mathrm{B}_{\mathcal{I}_k}) = H[T_q(V,h_n)] \ge h_n$ with q such that $\mathrm{B}_{\mathcal{I}_k} = b_q(V,h_n)$.

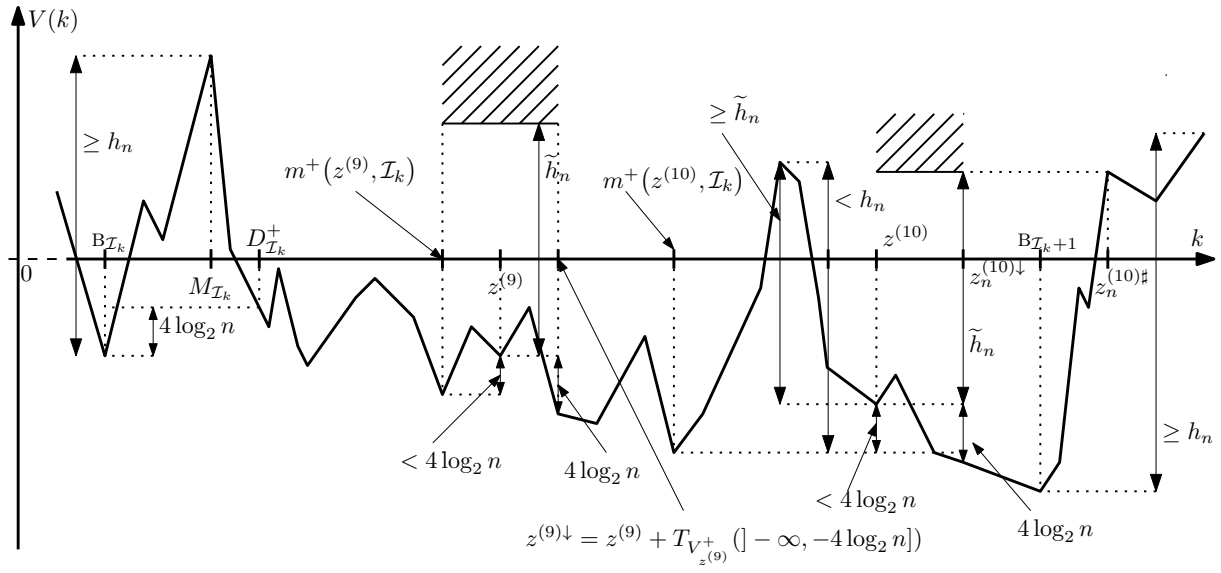

Figure 9. Schema of the potential V, with z equal to $z^{(9)}$ on $E_{23,k}^{(n,z)} \cap E_{27,k}^{(n)}$, and $z^{(10)}$ on $E_{23,k}^{(n,z)} \cap \big(E_{27,k}^{(n)}\big)^c$.

Let $c_{28} := C_1 + 4$. The next step is to prove that

$$\mathbb{P}\big[S_n = z,\, \tau(\mathrm{B}_{\mathcal{I}_k}) \le n,\, E_{23,k}^{(n,z)} \cap E_5^{(n)}\big] \le c_{29}(\log_2 n)^3 (\log n)^{-3} \tag{182}$$

for some constant $c_{29} > 0$. To this aim, we consider the three following events, defined as

$$\begin{aligned} E_{24,k}^{(n,z)} &:= \big\{\tau(\mathrm{B}_{\mathcal{I}_k}) + \tau[\mathrm{B}_{\mathcal{I}_k}, m^+(z,\mathcal{I}_k)] < n - n(\log n)^{-c_{28}}\big\}, \\ E_{25,k}^{(n,z)} &:= \big\{\tau(\mathrm{B}_{\mathcal{I}_k}) + \tau[\mathrm{B}_{\mathcal{I}_k}, m^+(z,\mathcal{I}_k)] \in \big[n - n(\log n)^{-c_{28}}, n\big]\big\}, \\ E_{26,k}^{(n,z)} &:= \big\{\tau(\mathrm{B}_{\mathcal{I}_k}) + \tau[\mathrm{B}_{\mathcal{I}_k}, m^+(z,\mathcal{I}_k)] > n\big\}. \end{aligned} \tag{183}$$

First, we have, conditioning by ω then applying the strong Markov property at stopping time $\tau(\mathrm{B}_{\mathcal{I}_k})$, then summing (181) for all the integers ℓ in $\big[t - n(\log n)^{-c_{28}}, t\big] \cap \mathbb{N}$,

$$\begin{aligned} & \mathbb{P}\big[\tau(\mathrm{B}_{\mathcal{I}_k}) \le n, E_{23,k}^{(n,z)} \cap E_{25,k}^{(n,z)}\big] \\ = \;& \mathbb{E}\Big[\mathbf{1}_{\{\tau(\mathrm{B}_{\mathcal{I}_k}) \le n\} \cap E_{23,k}^{(n,z)}} P_\omega^{\mathrm{B}_{\mathcal{I}_k}}\big(\tau(m^+(z,i)) \in \big[t - n(\log n)^{-c_{28}}, t\big]\big)_{|i=\mathcal{I}_k,\, t=n-\tau(\mathrm{B}_{\mathcal{I}_k})}\Big] \\ \le \;& [n(\log n)^{-c_{28}} + 1](\log n)^{C_1}/n \le 2(\log n)^{-c_{28}+C_1} \le (\log n)^{-3} \end{aligned} \tag{184}$$

since $C_1 - c_{28} = -4$ and $n \ge n_9 \ge n_3$.

Also, on $E_{23,k}^{(n,z)} \cap \{\tau(\mathrm{B}_{\mathcal{I}_k}) \le n\} \cap E_{26,k}^{(n,z)}$, $m^+(z,\mathcal{I}_k) \le z$, and after hitting $\mathrm{B}_{\mathcal{I}_k}$, S does not hit $m^+(z,\mathcal{I}_k) > \mathrm{B}_{\mathcal{I}_k}$ before time n, so $S_n < m^+(z,\mathcal{I}_k) \le z$ thus $S_n \ne z$. Hence,

$$\mathbb{P}\big[S_n = z, \tau(\mathrm{B}_{\mathcal{I}_k}) \le n, E_{23,k}^{(n,z)} \cap E_{26,k}^{(n,z)}\big] \le \mathbb{P}\big[S_n = z, S_n < m^+(z,\mathcal{I}_k) \le z\big] = 0. \tag{185}$$

There only remains to consider $E_{24,k}^{(n,z)}$. To this aim, we introduce

$$E_{27,k}^{(n,z)} := \big\{\max_{[m^+(z,\mathcal{I}_k),\, z]} V \le V(z) + \widetilde{h}_n\big\}. \tag{186}$$

We have, conditioning by ω then applying the strong Markov property at stopping time $\tau(\mathrm{B}_{\mathcal{I}_k})+\tau[\mathrm{B}_{\mathcal{I}_k}, m^+(z,\mathcal{I}_k)]$,

$$\begin{aligned}&\mathbb{P}\big[S_n = z, \tau(\mathrm{B}_{\mathcal{I}_k}) \le n, E_{23,k}^{(n,z)} \cap E_{24,k}^{(n,z)} \cap E_{27,k}^{(n,z)} \cap E_5^{(n)} \cap E_{15}^{(n)}\big]\\ &= \mathbb{E}\Big[\mathbf{1}_{\{\tau(\mathrm{B}_{\mathcal{I}_k})\le n\}\cap E_{23,k}^{(n,z)}\cap E_{24,k}^{(n,z)}\cap E_{27,k}^{(n,z)}\cap E_5^{(n)}\cap E_{15}^{(n)}} P_\omega^{m^+(z,\mathcal{I}_k)}\big(S_t = z\big)_{|t=n-\tau(\mathrm{B}_{\mathcal{I}_k})-\tau[\mathrm{B}_{\mathcal{I}_k},m^+(z,\mathcal{I}_k)]}\Big].\end{aligned} \tag{187}$$

We introduce $z_n^\downarrow := z+T_{V_z^+}(]-\infty,-4\log_2 n])$. Assume that $E_{23,k}^{(n,z)}$ holds and that $\mathrm{B}_{\mathcal{I}_k+1} < z_n^\downarrow$. So we would have $z \le \mathrm{B}_{\mathcal{I}_k+1} < z_n^\downarrow < z_n^\sharp$, and then $V(\mathrm{B}_{\mathcal{I}_k+1}) > V(z) - 4\log_2 n$, so $\max_{[\mathrm{B}_{\mathcal{I}_k+1}, z_n^\sharp]} V = V(z_n^\sharp) \le V(z)+\widetilde{h}_n+C_0 < V(\mathrm{B}_{\mathcal{I}_k+1})+4\log_2 n+\widetilde{h}_n+C_0 < V(\mathrm{B}_{\mathcal{I}_k+1})+h_n$ since $n \ge n_9 \ge n_3$ and $C_1 > 20$, thus $z_n^\sharp < M_{\mathcal{I}_k+1}$. So we would have $z_n^\downarrow \in [\mathrm{B}_{\mathcal{I}_k+1}, M_{\mathcal{I}_k+1}]$ with $V(z_n^\downarrow) \le V(z) - 4\log_2 n < V(\mathrm{B}_{\mathcal{I}_k+1}) = \min_{[\mathrm{B}_{\mathcal{I}_k+1}, M_{\mathcal{I}_k+1}]} V \le V(z_n^\downarrow)$ which is not possible. So, $z_n^\downarrow \le \mathrm{B}_{\mathcal{I}_k+1}$ on $E_{23,k}^{(n,z)}$.

Also on $E_{23,k}^{(n,z)}$, $\min_{[M_{\mathcal{I}_k}, z]} V = V[m^+(z,\mathcal{I}_k)] > V(z) - 4\log_2 n$ as in (177), and $\min_{[z, z_n^\downarrow]} V \ge V(z) - 4\log_2 n + \log(\varepsilon_0)$ by ellipticity. So we have on $E_{23,k}^{(n,z)}$,

$$\min_{[M_{\mathcal{I}_k}, z_n^\downarrow]} V \ge V(z) - 4\log_2 n + \log(\varepsilon_0). \tag{188}$$

Notice that $\max_{[z, z_n^\downarrow]} V < V(z) + \widetilde{h}_n$ on $E_{23,k}^{(n,z)}$. So we have on $E_{23,k}^{(n,z)} \cap E_{27,k}^{(n,z)}$,

$$\max_{[m^+(z,\mathcal{I}_k), z_n^\downarrow]} V \le V(z) + \widetilde{h}_n. \tag{189}$$

Now on $E_{23,k}^{(n,z)} \cap E_{27,k}^{(n,z)} \cap E_5^{(n)} \cap E_{15}^{(n)}$, by Markov inequality and (8), then by (188), (189) and $-(\log n)^3 \le x_{-10}(V, \log n) \le x_{-10}\big(V, \widetilde{h}_n\big) \le M_{-5}\big(V, \widetilde{h}_n\big) \le M_{-3} \le M_{\mathcal{I}_k} < D_{\mathcal{I}_k}^+ \le m^+(z,\mathcal{I}_k) \le z < z_n^\downarrow \le \mathrm{B}_{\mathcal{I}_k+1} \le \mathrm{B}_4 \le b_6\big(V, \widetilde{h}_n\big) \le x_{12}\big(V, \widetilde{h}_n\big) \le x_{12}(V, \log n) \le (\log n)^3$ (because on $E_{15}^{(n)}$ there are similarly as previously, in $[M_{-3}, 0]$, at most two $M_j\big(V, \widetilde{h}_n\big)$ which are not equal to some $M_\ell(V, h_n)$ so $M_{-5}\big(V, \widetilde{h}_n\big) \le M_{-3}$ and similarly $\mathrm{B}_4 \le b_6\big(V, \widetilde{h}_n\big)$), we get

$$\begin{aligned}&P_\omega^{m^+(z,\mathcal{I}_k)}\big[\tau(M_i) \wedge \tau(z_n^\downarrow) \ge 2^{-1} n(\log n)^{-c_{28}}\big]_{|i=\mathcal{I}_k} \\ \le\ & 2n^{-1}(\log n)^{c_{28}} \varepsilon_0^{-1} (z_n^\downarrow - M_{\mathcal{I}_k})^2 \exp\Big(\max_{[m^+(z,\mathcal{I}_k), z_n^\downarrow]} V - \min_{[M_{\mathcal{I}_k}, z_n^\downarrow]} V\Big) \\ \le\ & 8(\log n)^{c_{28}+6} n^{-1} \varepsilon_0^{-2} \exp\big(\widetilde{h}_n + 4\log_2 n\big) = 8(\log n)^{c_{28}-2C_1+10} \varepsilon_0^{-2} \le (\log n)^{-3}\end{aligned} \tag{190}$$

since $c_{28} - 2C_1 + 10 = 14 - C_1 < -6$ and $n \ge n_9 \ge n_3$. Moreover on $E_{23,k}^{(n,z)}$, we get by definition of $E_{23,k}^{(n,z)}$,

$$V(z) < \min_{[D_{\mathcal{I}_k}^+, z]} V + 4\log_2 n \le V\big[D_{\mathcal{I}_k}^+\big] + 4\log_2 n,$$

and as a consequence, using (178) which remains true on $E_{23,k}^{(n,z)}$,

$$V(M_{\mathcal{I}_k}) \ge V(D_{\mathcal{I}_k}^+) - 4\log_2 n + h_n \ge V(z) - 8\log_2 n + h_n. \tag{191}$$

Hence on $E_{23,k}^{(n,z)} \cap E_{27,k}^{(n,z)} \cap E_5^{(n)} \cap E_{15}^{(n)}$, using (7), then (189) and (191),

$$\begin{aligned}&P_\omega^{m^+(z,\mathcal{I}_k)}\big[\tau(M_i) < \tau(z_n^\downarrow)\big]_{|i=\mathcal{I}_k} \\ \le\ & \big(z_n^\downarrow - m^+(z,\mathcal{I}_k)\big) \exp\Big[\max_{[m^+(z,\mathcal{I}_k),\, z_n^\downarrow]} V - V(M_{\mathcal{I}_k})\Big] \\ \le\ & 2(\log n)^3 \exp\big[\widetilde{h}_n + 8\log_2 n - h_n\big] = 2(\log n)^{11-C_1} \le (\log n)^{-3}\end{aligned} \tag{192}$$

since $11 - C_1 < -9$ and $n \geq n_9 \geq n_3$. Consequently on $E^{(n,z)}_{23,k} \cap E^{(n,z)}_{27,k} \cap E^{(n)}_5 \cap E^{(n)}_{15}$,

$$P_\omega^{m^+(z,\mathcal{I}_k)}\big[\tau(z_n^\downarrow) \geq 2^{-1}n(\log n)^{-c_{28}}\big] \leq (190) + (192) \leq 2(\log n)^{-3}. \tag{193}$$

Notice that for every $\ell \in \mathbb{N}$, by reversibility and ellipticity (see (13) and (16)),

$$P_\omega^{z_n^\downarrow}(S_\ell = z) \leq \mu_\omega(z)/\mu_\omega(z_n^\downarrow) \leq \big(1 + e^{C_0}\big)\exp[-V(z) + V(z_n^\downarrow)] \leq \varepsilon_0^{-1}(\log n)^{-4}. \tag{194}$$

On $E^{(n,z)}_{23,k} \cap E^{(n,z)}_{27,k} \cap E^{(n)}_5 \cap E^{(n)}_{15}$, for every $t \geq n(\log n)^{-c_{28}}$, by (193), the strong Markov property and (194), since $n \geq n_9 \geq n_3$,

$$\begin{aligned}
& P_\omega^{m^+(z,\mathcal{I}_k)}\big(S_t = z\big) \\
\leq\ & P_\omega^{m^+(z,\mathcal{I}_k)}\big[\tau(z_n^\downarrow) \geq 2^{-1}n(\log n)^{-c_{28}}\big] + P_\omega^{m^+(z,\mathcal{I}_k)}\big[S_t = z, \tau(z_n^\downarrow) < 2^{-1}n(\log n)^{-c_{28}}\big] \\
\leq\ & 2(\log n)^{-3} + \mathsf{E}_\omega^{m^+(z,\mathcal{I}_k)}\big(\mathbf{1}_{\{\tau(z_n^\downarrow) < 2^{-1}n(\log n)^{-c_{28}}\}} P_\omega^{z_n^\downarrow}(S_\ell = z)_{|\ell = t - \tau(z_n^\downarrow)}\big) \\
\leq\ & 2(\log n)^{-3} + \varepsilon_0^{-1}(\log n)^{-4} \leq 3(\log n)^{-3}.
\end{aligned}$$

Finally, this and (187) (on which $t \geq n(\log n)^{-c_{28}}$ thanks to $E^{(n,z)}_{24,k}$) give

$$\mathbb{P}\big[S_n = z, \tau(b_{\mathcal{I}_k}) \leq n, E^{(n,z)}_{23,k} \cap E^{(n,z)}_{24,k} \cap E^{(n,z)}_{27,k} \cap E^{(n)}_5 \cap E^{(n)}_{15}\big] \leq 3(\log n)^{-3}. \tag{195}$$

There only remains to estimate $\mathsf{P}\big[\big(E^{(n,z)}_{27,k}\big)^c \cap E^{(n,z)}_{23,k}\big]$. We define (see Figure 9 with $z = z^{(10)}$),

$$\begin{aligned}
V^-_{2,n} &:= V^-_z[. + T_{V^-_z}([\widetilde{h}_n, +\infty[)] - V^-_z[T_{V^-_z}([\widetilde{h}_n, +\infty[)], \\
V^-_{3,n} &:= V^-_{2,n}[. + T_{V^-_{2,n}}(]-\infty, -\widetilde{h}_n])] - V^-_{2,n}[T_{V^-_{2,n}}(]-\infty, -\widetilde{h}_n])], \\
E^{(n)}_{28} &:= \big\{T_{V^-_z}([\widetilde{h}_n, +\infty[) < T_{V^-_z}(]-\infty, -4\log_2 n[)\big\}, \\
E^{(n)}_{29} &:= \big\{T_{V^-_{2,n}}(]-\infty, -\widetilde{h}_n]) < T_{V^-_{2,n}}([C_1 \log_2 n, +\infty[)\big\}, \\
E^{(n)}_{30} &:= \big\{T_{V^-_{3,n}}([\widetilde{h}_n, +\infty[) < T_{V^-_{3,n}}(]-\infty, -4\log_2 n - C_0[)\big\}.
\end{aligned}$$

Notice that $E^{(n,z)}_{23,k} \cap \big(E^{(n,z)}_{27,k}\big)^c$ is included in $E^{(n)}_{28}$ because $\max_{[0,z-m^+(z,\mathcal{I}_k)]} V^-_z > \widetilde{h}_n$ by (186) and $\min_{[0,z-m^+(z,\mathcal{I}_k)]} V^-_z = V^-_z(z - m^+(z,\mathcal{I}_k)) > -4\log_2 n$ by definitions of $m^+(z,\mathcal{I}_k)$ (see (175)) and of $E^{(n)}_{23,k}$. It is also included in $E^{(n)}_{29}$, otherwise there would be a left h_n-maximum of V in $]m^+(z,\mathcal{I}_k), z[$ and so in $]M_{\mathcal{I}_k}, \mathrm{B}_{\mathcal{I}_k+1}[$ which is not possible. Finally, $E^{(n,z)}_{23,k} \cap \big(E^{(n,z)}_{27,k}\big)^c$ is also included in $E^{(n)}_{30}$ because $\min_{[M_{\mathcal{I}_k}, z]} V > V(z) - 4\log_2 n$ as in (177) and $V(M_{\mathcal{I}_k}) \geq V(z) - 8\log_2 n + h_n \geq V(z) + \widetilde{h}_n + C_0$ by (191) and since $C_1 > 20$ and $n \geq n_9 \geq n_3$. Using the independence of $E^{(n)}_{28}$, $E^{(n)}_{29}$ and $E^{(n)}_{30}$, provided by the strong Markov property, then applying (17), we get

$$\mathsf{P}\big[\big(E^{(n,z)}_{27,k}\big)^c \cap E^{(n,z)}_{23,k}\big] \leq \mathsf{P}\big[E^{(n)}_{28}\big]\mathsf{P}\big[E^{(n)}_{29}\big]\mathsf{P}\big[E^{(n)}_{30}\big] \leq c_{30}(\log_2 n)^3(\log n)^{-3}, \tag{196}$$

with $c_{30} := 10 \times 2(C_1 + 1) \times 12$ since $n \geq n_9 \geq n_3$. This, combined with (195) and Lemma 5.8, gives, where LHS means left hand side and since $n \geq n_9 \geq n_7$,

$$\begin{aligned}
& \mathbb{P}\big[S_n = z, \tau(b_{\mathcal{I}_k}) \leq n, E^{(n,z)}_{23,k} \cap E^{(n,z)}_{24,k} \cap E^{(n)}_5\big] \\
\leq\ & \text{LHS of (195)} + \mathsf{P}\big[\big(E^{(n)}_{15}\big)^c\big] + \mathsf{P}\big[\big(E^{(n,z)}_{27,k}\big)^c \cap E^{(n,z)}_{23,k}\big] \\
\leq\ & (3 + c_{20} + c_{30})(\log_2 n)^3(\log n)^{-3}.
\end{aligned} \tag{197}$$

Combining this, (184) and (185) proves (182) with $c_{29} := c_{20} + c_{30} + 4$ since $n \geq n_3$. Finally, (176), (180) and (182) prove (174) with $c_{27} := \varepsilon_0^{-1} + c_{20} + c_{21} + c_{29}$ for every $n \geq n_9$, $z \in \mathbb{Z}$ and $1 \leq k \leq 3$, which ends the proof of Lemma 5.13. □

Proof of Lemma 5.11: We prove similarly as in the proof of Lemma 5.13 that for all $n \geq n_9$, $z \in \mathbb{Z}$ and $1 \leq k \leq 3$,

$$\mathbb{P}\big[S_n = z, \mathrm{B}_{\mathcal{I}_k - 1} \leq z < D^-_{\mathcal{I}_k}, \tau(\mathrm{B}_{\mathcal{I}_k}) \leq n, E_5^{(n)}, (E_3^{(n)})^c\big] \leq c_{27}(\log_2 n)^3 (\log n)^{-3}. \tag{198}$$

Combining (198), (171) and (174) proves Lemma 5.11 with $c_{25} := c_{26} + 2c_{27}$. □

Proof of Proposition 5.7: Notice that for $n \geq n_9$ and $k \geq 1$, on $\{\tau(\mathrm{B}_{I_k}) \leq n < \tau(\mathrm{B}_{\mathcal{I}_{k+1}})\}$, the random walk S does not reach the B_i with $i \in \mathbb{Z} \setminus \{\mathcal{I}_1, \dots, \mathcal{I}_k\}$ before time n, and so S_n belongs to $\big]\min\{\mathrm{B}_{\mathcal{I}_i - 1}, 1 \leq i \leq k\}, \max\{\mathrm{B}_{\mathcal{I}_i + 1}, 1 \leq i \leq k\}\big[$, which is equal to $\cup_{i=1}^k]\mathrm{B}_{\mathcal{I}_i - 1}, \mathrm{B}_{\mathcal{I}_i + 1}[$. Consequently, using (165) and Lemma 5.10 in the second inequality, for all $n \geq n_9$ and all $z \in \mathbb{Z}$, with $c_{31} := c_{24} + 2$,

$$\begin{aligned}
& \mathbb{P}\big(S_n = z, (E_3^{(n)})^c\big) \\
\leq\ & \mathbb{P}\big[\tau(\mathrm{B}_{\mathcal{I}_1}) > n\big] + \mathbb{P}\big[S_n = z, (E_3^{(n)})^c \cap \cup_{k=1}^3 \{\tau(\mathrm{B}_{I_k}) \leq n < \tau(\mathrm{B}_{\mathcal{I}_{k+1}})\}\big] + \mathbb{P}\big[\tau(\mathrm{B}_{\mathcal{I}_4}) \leq n\big] \\
\leq\ & \sum_{k=1}^3 \mathbb{P}\big(\tau(\mathrm{B}_{I_k}) \leq n < \tau(\mathrm{B}_{\mathcal{I}_{k+1}}), S_n = z \in \cup_{i=1}^k]\mathrm{B}_{\mathcal{I}_i - 1}, \mathrm{B}_{\mathcal{I}_i + 1}[, (E_3^{(n)})^c\big) + c_{31}\frac{(\log_2 n)^3}{(\log n)^3} \\
\leq\ & \sum_{i=1}^3 \sum_{k=i}^3 \mathbb{P}\big(\tau(\mathrm{B}_{I_k}) \leq n < \tau(\mathrm{B}_{\mathcal{I}_{k+1}}), S_n = z \in]\mathrm{B}_{\mathcal{I}_i - 1}, \mathrm{B}_{\mathcal{I}_i + 1}[, (E_3^{(n)})^c\big) + c_{31}\frac{(\log_2 n)^3}{(\log n)^3} \\
=\ & \sum_{i=1}^3 \mathbb{P}\big[\tau(\mathrm{B}_{I_i}) \leq n < \tau(\mathrm{B}_{I_4}), S_n = z \in]\mathrm{B}_{\mathcal{I}_i - 1}, \mathrm{B}_{\mathcal{I}_i + 1}[, (E_3^{(n)})^c\big] + c_{31}\frac{(\log_2 n)^3}{(\log n)^3} \\
\leq\ & c_{19}(\log_2 n)^3 (\log n)^{-3},
\end{aligned}$$

with $c_{19} := 3c_{25} + 3 + c_{31}$ and where we used Lemmas 5.11 and 7.1 in the last line since $n \geq n_9 \geq \max(n_3, p_3)$. This proves Proposition 5.7. □

5.4. **Proof of the upper bound in Theorem 1.1.** Recall $E_C^{(n)}(z)$ from (85). We have, for all $n \geq \max(n_9, p_2)$ and all $z \in \mathbb{Z}$,

$$\begin{aligned}
& \mathbb{P}\big(S_n = z, \big(E_C^{(n)}(z)\big)^c\big) \\
\leq\ & \mathbb{P}\big(S_n = z, \big(E_3^{(n)}\big)^c\big) + \mathsf{P}\big[\big(E_5^{(n)}\big)^c\big] + \mathsf{P}\big[\big(E_6^{(n)}\big)^c \cap E_5^{(n)}\big] \\
& + \mathbb{P}\big[S_n = z, \big(E_7^{(n)}(z)\big)^c, E_3^{(n)}, E_5^{(n)}\big] + \mathsf{P}\big[\big(E_4^{(n)}(z)\big)^c \cap E_3^{(n)} \cap E_6^{(n)} \cap E_7^{(n)}(z)\big] \\
\leq\ & (c_{19} + 2 + c_9)(\log_2 n)^3 (\log n)^{-3} + c_{10}(\log n)^{-2-\delta_1/2}
\end{aligned}$$

by Proposition 5.7, Lemmas 7.1 and 7.2, Proposition 5.2 and Lemma 5.1. This and Proposition 4.8 give, since $\delta_1 \in]0, 2/3[$,

$$\sup_{z \in (2\mathbb{Z}+n)} \left[\mathbb{P}\big(S_n = z\big) - \frac{2\sigma^2}{(\log n)^2}\varphi_\infty\left(\frac{\sigma^2 z}{(\log n)^2}\right)\right] \leq o\big((\log n)^{-2}\big), \tag{199}$$

as $n \to +\infty$, which proves the upper bound in Theorem 1.1. □

6. Proof of the lower bound of Theorem 1.1

Let $\varepsilon > 0$. Since $\lim_{\pm\infty} \varphi_\infty = 0$, we can fix some $A_0 > 0$ such that $\sup_{|x| \geq A_0} |\varphi_\infty(\sigma^2 x)| < \sigma^{-2}\varepsilon$.

In this section, $\mathcal{T}_V^{\uparrow}$ and $\mathcal{T}_V^{\downarrow}$ always denote $\widetilde{h}_n$-slopes, that is, $\mathcal{T}_V^{\uparrow} = \mathcal{T}_{V,\widetilde{h}_n}^{\uparrow}$ and $\mathcal{T}_V^{\downarrow} = \mathcal{T}_{V,\widetilde{h}_n}^{\downarrow}$, where $\widetilde{h}_n = \log n - 2C_1 \log_2 n = h_n - C_1 \log_2 n$ as before. In what follows, we consider independent slopes $Z_{2k}^{\uparrow}$, $-9 \le k \le 9$ and $Z_{2k+1}^{\downarrow}$, $-9 \le k \le 9$, each $Z_{2k}^{\uparrow}$ having the same law as $\mathcal{T}_V^{\uparrow}$ $\big($i.e. $\mathcal{T}_{V,\widetilde{h}_n}^{\uparrow}\big)$, and each $Z_{2k+1}^{\downarrow}$ having the same law as $\mathcal{T}_V^{\downarrow}$ $\big($i.e. $\mathcal{T}_{V,\widetilde{h}_n}^{\downarrow}\big)$.

Recall that ζ is defined in (24). We also introduce $Y_{-1}^{\uparrow} := \zeta\big(Z_{-1}^{\downarrow}\big)$, which is independent of $Z_0^{\uparrow}$, with $Y_{-1}^{\uparrow} =_{law} \zeta\big(\mathcal{T}_V^{\downarrow}\big) =_{law} \mathcal{T}_{V^-,\widetilde{h}_n}^{\uparrow *} =: \mathcal{T}_{V^-}^{\uparrow *}$ by Proposition 2.12, and $\ell\big(Y_{-1}^{\uparrow}\big) = \ell\big(Z_{-1}^{\downarrow}\big)$.

First case: We start with the case $z \le -\Gamma_n$.

Using Lemma 2.6 eq. (44), we have for each $z \in \mathbb{Z}$ such that $z - \Gamma_n \le 0$,

$$J_6(n,z) := \mathsf{P}\big(b_{\widetilde{h}_n} = z - \Gamma_n\big) = \frac{\mathsf{P}\big[-z + \Gamma_n < \ell\big(Z_0^{\uparrow}\big)\big]}{\mathsf{E}\big(\ell\big(Z_0^{\uparrow}\big) + \ell\big(Z_1^{\downarrow}\big)\big)}. \tag{200}$$

Using the uniform continuity of φ_∞ on $\mathbb{R}$ and $\sup_{z\in[-A_0(\log n)^2, A_0(\log n)^2]} \left| \frac{\sigma^2 z}{(\log n)^2} - \frac{\sigma^2\big(z-\Gamma_n\big)}{\big(\widetilde{h}_n\big)^2} \right|$ $= o(1)$ as $n \to +\infty$ since $\delta_1 < 2/3$ and $\widetilde{h}_n \sim_{n\to+\infty} \log n$ in the first inequality, then $\|\varphi_\infty\|_\infty =: \sup_{\mathbb{R}} |\varphi_\infty| < \infty$ and $\widetilde{h}_n \sim_{n\to+\infty} \log n$ in the second one, and finally Theorem 1.4 in the last one, there exists $n_{10} \ge \max(n_9, p_2)$ (with p_2 defined in Lemma 7.2) such that for all $n \ge n_{10}$, for all $z \in [-A_0(\log n)^2, A_0(\log n)^2]$ such that $z - \Gamma_n \le 0$,

$$\begin{aligned}
\frac{\sigma^2}{(\log n)^2}\varphi_\infty\left(\frac{\sigma^2 z}{(\log n)^2}\right) &\le \frac{\sigma^2}{(\log n)^2}\varphi_\infty\left(\frac{\sigma^2\big(z-\Gamma_n\big)}{\big(\widetilde{h}_n\big)^2}\right) + \varepsilon(\log n)^{-2} \\
&\le \frac{\sigma^2}{\big(\widetilde{h}_n\big)^2}\varphi_\infty\left(\frac{\sigma^2\big(z-\Gamma_n\big)}{\big(\widetilde{h}_n\big)^2}\right) + 2\varepsilon(\log n)^{-2} \\
&\le J_6(n,z) + 3\varepsilon(\log n)^{-2}.
\end{aligned} \tag{201}$$

Also for $n \ge n_{10}$, if $|z| > A_0(\log n)^2$, then by definition of A_0,

$$\frac{\sigma^2}{(\log n)^2}\varphi_\infty\left(\frac{\sigma^2 z}{(\log n)^2}\right) \le \frac{\varepsilon}{(\log n)^2} \le J_6(n,z) + 3\varepsilon(\log n)^{-2}. \tag{202}$$

The objective is to approximate progressively this quantity $J_6(n,z)$ by $\mathbb{P}(S_n = z)$, by using Theorems 2.4 **(i)** and 2.5 eq. (29) (see (211) below) and Lemma 4.7. To this aim, we introduce the following events.

$$\begin{aligned}
E_{31}^{(n)} &:= \big\{T_{Z_0^{\uparrow}}\big(\widetilde{h}_n\big) > \Gamma_n\big\} \cap \big\{T_{Y_{-1}^{\uparrow}}\big(\widetilde{h}_n\big) > \Gamma_n\big\}, \\
E_{32}^{(n)} &:= \big\{\forall k \in \big[T_{Z_0^{\uparrow}}\big(\widetilde{h}_n\big), \ell\big(Z_0^{\uparrow}\big)\big],\ Z_0^{\uparrow}(k) \ge 9\log_2 n\big\}, \\
E_{33}^{(n)} &:= \big\{\forall k \in \big[T_{Y_{-1}^{\uparrow}}\big(\widetilde{h}_n\big), \ell\big(Y_{-1}^{\uparrow}\big)\big],\ Y_{-1}^{\uparrow}(k) \ge 9\log_2 n\big\}, \\
E_{34}^{(n)} &:= \cap_{k=-9}^{9}\big\{H\big(Z_{2k}^{\uparrow}\big) \ge \log n + C_2 \log_2 n\big\} \cap \cap_{k=-9}^{9}\big\{H\big(Z_{2k+1}^{\downarrow}\big) \ge \log n + C_2 \log_2 n\big\}, \\
E_{35}^{(n)} &:= \left\{\sum_{k=-9}^{9} \ell\big(Z_{2k}^{\uparrow}\big) + \sum_{k=-9}^{9} \ell\big(Z_{2k+1}^{\downarrow}\big) \le (\log n)^{2+\delta_1}\right\}.
\end{aligned}$$

Recall that $Z_0^{\uparrow} =_{law} \mathcal{T}_V^{\uparrow}$ and $Y_{-1}^{\uparrow} =_{law} \mathcal{T}_{V^-}^{\uparrow *}$ (with $h = \widetilde{h}_n$). So by Lemma 7.2 eq. (228), there exists $n_{11} \ge n_{10}$ such that for all $n \ge n_{11}$,

$$\mathsf{P}\big[\big(E_{31}^{(n)}\big)^c\big] \le \mathsf{P}\big[T_{\mathcal{T}_V^{\uparrow}}\big(\widetilde{h}_n\big) \le \Gamma_n\big] + \mathsf{P}\big[T_{\mathcal{T}_{V^-}^{\uparrow *}}\big(\widetilde{h}_n\big) \le \Gamma_n\big] \le \varepsilon(\log n)^{-2}. \tag{203}$$

Notice that, using Theorem 2.3 **(i)** with its notation and $h=\widetilde{h}_n$, since $Z_0^{\uparrow}\big(T_{Z_0^{\uparrow}}(\widetilde{h}_n)\big)\in\big[\widetilde{h}_n,\widetilde{h}_n+C_0\big[$ by ellipticity (16), there exists $n_{12}\geq n_{11}$ such that for all $n\geq n_{12}$,

$$\begin{aligned}\mathsf{P}\big[(E_{32}^{(n)})^c\big] &= \mathsf{P}\big[\exists k\in\big[T_{Z_0^{\uparrow}}(\widetilde{h}_n),\ell(Z_0^{\uparrow})\big],\ Z_0^{\uparrow}(k)<9\log_2 n\big]\\ &\leq \mathsf{P}\big[T_V\big(9\log_2 n-\widetilde{h}_n\big)<M_{\widetilde{h}_n}^{\sharp},\ \min_{[0,M_{\widetilde{h}_n}^{\sharp}]}V>-\widetilde{h}_n-C_0,\ V\big(M_{\widetilde{h}_n}^{\sharp}\big)\geq 0\big]\\ &\leq \mathsf{P}\big[T_V\big(\widetilde{h}_n-9\log_2 n\big)<T_V(-9\log_2 n-C_0)\big]\leq 22(\log_2 n)(\log n)^{-1},\end{aligned}\tag{204}$$

by the strong Markov property at $T_V\big(9\log_2 n-\widetilde{h}_n\big)$, (16) and (17) since $n\geq n_{12}\geq n_3$. We prove similarly that $\mathsf{P}\big[\big(E_{33}^{(n)}\big)^c\big]\leq 22(\log_2 n)(\log n)^{-1}$ for all $n\geq n_{12}$, using Theorem 2.9 **(i)** instead of Theorem 2.3.

Also, using (61) with $h=\widetilde{h}_n$, there exists $n_{13}\geq n_{12}$ such that, for all $n\geq n_{13}$, and all $-9\leq k\leq 9$, since $n_{13}\geq n_3$,

$$\begin{aligned}&\mathsf{P}\big[H(Z_{2k}^{\uparrow})<\log n+C_2\log_2 n\big]=\mathsf{P}\big[H\big(\mathcal{T}_V^{\uparrow}\big)-\widetilde{h}_n<(2C_1+C_2)\log_2 n\big]\\ \leq\ & 4(2C_1+C_2)(\log_2 n)(\log n)^{-1}.\end{aligned}$$

This remains true for $H\big(Z_{2k}^{\uparrow}\big)$ replaced by $H\big(Z_{2k+1}^{\downarrow}\big)=_{law}H\big(\mathcal{T}_V^{\downarrow}\big)=_{law}H\big(\mathcal{T}_{-V}^{\uparrow}\big)$, $-9\leq k\leq 9$ by Theorem 2.3 **(ii)** (or by the inequality after (61)). Consequently, we get $\mathsf{P}\big[\big(E_{34}^{(n)}\big)^c\big]\leq 152(2C_1+C_2)(\log_2 n)(\log n)^{-1}$ for all $n\geq n_{13}$.

Moreover, we have $\mathsf{P}\big[\big(E_{35}^{(n)}\big)^c\big]\leq 19\mathsf{P}\big[\ell\big(\mathcal{T}_V^{\uparrow}\big)>(\log n)^{2+\delta_1}/50\big]+19\mathsf{P}\big[\ell\big(\mathcal{T}_V^{\downarrow}\big)>(\log n)^{2+\delta_1}/50\big]\leq 38(\log n)^{-8}$ for all $n\geq n_{13}$ by Lemma 7.1 eq. (223) and (224), since $n_{13}\geq n_9\geq p_3$.

Also, using (200) then $\mathsf{P}\big[\big(E_i^{(n)}\big)^c\big]=o(1)$ for $31\leq i\leq 35$, there exists $n_{14}\geq n_{13}$ such that, for all $n\geq n_{14}$ and all $z\leq\Gamma_n$, $\mathsf{E}\big(\ell\big(Z_0^{\uparrow}\big)+\ell\big(Z_1^{\downarrow}\big)\big)\geq c_7(\log n)^2$ by Lemma 2.15 and

$$J_6(n,z)\leq\mathsf{E}\left(\frac{\mathbf{1}_{\{-z+\Gamma_n<\ell(Z_0^{\uparrow})\}}\mathbf{1}_{\cap_{i=31}^{35}E_i^{(n)}}}{\mathsf{E}\big(\ell\big(Z_0^{\uparrow}\big)+\ell\big(Z_1^{\downarrow}\big)\big)}\frac{\Big(\sum_{i=0}^{\ell(Z_0^{\uparrow})-1}e^{-Z_0^{\uparrow}(i)}+\sum_{i=1}^{\ell(Y_{-1}^{\uparrow})}e^{-Y_{-1}^{\uparrow}(i)}\Big)}{\Big(\sum_{i=0}^{\ell(Z_0^{\uparrow})-1}e^{-Z_0^{\uparrow}(i)}+\sum_{i=1}^{\ell(Y_{-1}^{\uparrow})}e^{-Y_{-1}^{\uparrow}(i)}\Big)}\right)+\frac{\varepsilon}{(\log n)^2}.\tag{205}$$

The next step is to deal with the sums in numerator in the previous expectation. Notice that on $E_{32}^{(n)}\cap E_{35}^{(n)}$, we have $\ell\big(Z_0^{\uparrow}\big)\leq(\log n)^3-1$ and so

$$\sum_{i=T_{Z_0^{\uparrow}}(\widetilde{h}_n)}^{\ell(Z_0^{\uparrow})-1}e^{-Z_0^{\uparrow}(i)}\leq\big[\ell(Z_0^{\uparrow})-T_{Z_0^{\uparrow}}(\widetilde{h}_n)\big](\log n)^{-9}\leq(\log n)^{-6}.\tag{206}$$

Similarly, $\sum_{i=T_{Y_{-1}^{\uparrow}}(\widetilde{h}_n)}^{\ell(Y_{-1}^{\uparrow})}e^{-Y_{-1}^{\uparrow}(i)}\leq(\log n)^{-6}$ on $E_{33}^{(n)}\cap E_{35}^{(n)}$ since $\ell\big(Y_{-1}^{\uparrow}\big)=\ell\big(Z_{-1}^{\downarrow}\big)$.

Also using Theorem 2.3 **(i)** since $Z_0^{\uparrow}=_{law}\mathcal{T}_V^{\uparrow}$, then applying Proposition 7.3, for large n, for all $i\geq\Gamma_n$,

$$\mathsf{E}\Big[\exp\big(-Z_0^{\uparrow}(i)\big)\mathbf{1}_{\{i<T_{Z_0^{\uparrow}}(\widetilde{h}_n)\}}\Big]=\mathsf{E}\Big[e^{-V(i)}\mathbf{1}_{\{i<T_V(\widetilde{h}_n)\}}\mid T_V(\widetilde{h}_n)<T_V(\mathbb{R}_-^*)\Big]\leq c_{13}i^{-3/2}.$$

This remains true with $Z_0^{\uparrow}$ and V replaced by $Y_{-1}^{\uparrow}=_{law}\mathcal{T}_{V^-}^{\uparrow *}$ and V_-, and $T_V(\mathbb{R}_-^*)$ by $T_{V^-}^*(\mathbb{R}_-)$ by Theorem 2.9 **(i)** and Proposition 7.3. So there exists $c_{32}>0$ and $n_{15}\geq n_{14}$ such that, for all

$n \geq n_{15}$,

$$\mathsf{E}\left(\sum_{\Gamma_n \leq i < T_{Z_0^\uparrow}(\widetilde{h}_n)} e^{-Z_0^\uparrow(i)}\right) = \sum_{i=\Gamma_n}^{\infty} \mathsf{E}\left[e^{-Z_0^\uparrow(i)} \mathbf{1}_{\left\{i < T_{Z_0^\uparrow}(\widetilde{h}_n)\right\}}\right] \leq \sum_{i=\Gamma_n}^{\infty} \frac{c_{13}}{i^{3/2}} \leq \frac{c_{32}}{(\log n)^{\frac{2}{3}+\frac{\delta_1}{2}}}, \tag{207}$$

since $\Gamma_n = \lfloor (\log n)^{4/3+\delta_1} \rfloor$. This remains true with $Z_0^\uparrow$ replaced by $Y_{-1}^\uparrow$.

Combining (205) with (206), (207), and the corresponding inequalities for V_- and $Y_{-1}^\uparrow$, $\ell\big(Z_0^\uparrow\big) \geq T_{Z_0^\uparrow}\big(\widetilde{h}_n\big) > \Gamma_n$ and $\ell\big(Y_{-1}^\uparrow\big) \geq T_{Y_{-1}^\uparrow}\big(\widetilde{h}_n\big) > \Gamma_n$ on $E_{31}^{(n)}$, $\sum_{i=0}^{\ell(Z_0^\uparrow)-1} e^{-Z_0^\uparrow(i)} \geq 1$ and again $\mathsf{E}\big(\ell\big(Z_0^\uparrow\big) + \ell\big(Z_1^\downarrow\big)\big) \geq c_7(\log n)^2$, there exists $n_{16} \geq n_{15}$ such that, for all $n \geq n_{16}$, for every $j \in \{0,1\}$, for all $z \leq \Gamma_n$ (although J_6 does not depend on j),

$$\begin{aligned} J_6(n,z) &\leq \mathsf{E}\Bigg(\frac{\mathbf{1}_{\{-z+\Gamma_n < \ell(Z_0^\uparrow)\}} \mathbf{1}_{\cap_{i=31}^{35} E_i^{(n)}}}{\mathsf{E}\big(\ell\big(Z_0^\uparrow\big) + \ell\big(Z_1^\downarrow\big)\big)} \\ &\qquad \frac{\Big(\sum_{i=0}^{\Gamma_n-1} e^{-Z_0^\uparrow(i)} + \sum_{i=1}^{\Gamma_n-1} e^{-Y_{-1}^\uparrow(i)}\Big)}{\Big(\sum_{i=0}^{\ell(Z_0^\uparrow)-1} e^{-Z_0^\uparrow(i)} + \sum_{i=1}^{\ell(Y_{-1}^\uparrow)} e^{-Y_{-1}^\uparrow(i)}\Big)}\Bigg) + 2\varepsilon(\log n)^{-2} \\ &\leq J_7(j,n,z) + 2\varepsilon(\log n)^{-2}, \end{aligned} \tag{208}$$

where for $j \in \{0,1\}$,

$$J_7(j,n,z) \tag{209}$$

$$\begin{aligned} &:= \mathsf{E}\Bigg(\frac{\mathbf{1}_{\{-z+\Gamma_n < \ell(Z_0^\uparrow)\}} \mathbf{1}_{\cap_{i=31}^{35} E_i^{(n)}}}{\mathsf{E}\big(\ell\big(Z_0^\uparrow\big) + \ell\big(Z_1^\downarrow\big)\big)} \\ &\quad \frac{\Big(\sum_{k=-\Gamma_n}^{\Gamma_n} \big[e^{-Z_0^\uparrow(-(k+j))} \mathbf{1}_{\{k+j \leq 0\}} + e^{-Y_{-1}^\uparrow(k+j)} \mathbf{1}_{\{k+j>0\}}\big]\Big)}{\Big(\sum_{i=0}^{\ell(Z_0^\uparrow)-1} e^{-Z_0^\uparrow(i)} + \sum_{i=1}^{\ell(Y_{-1}^\uparrow)} e^{-Y_{-1}^\uparrow(i)}\Big)}\Bigg) \\ &\leq \sum_{k=-\Gamma_n}^{\Gamma_n} \mathsf{E}\Bigg(\frac{\mathbf{1}_{\{-z-k<\ell(Z_0^\uparrow)\} \cap \cap_{i=31}^{35} E_i^{(n)}} \Big(e^{-Z_0^\uparrow(-(k+j))} \mathbf{1}_{\{k+j \leq 0\}} + e^{-Y_{-1}^\uparrow(k+j)} \mathbf{1}_{\{k+j>0\}}\Big)}{\mathsf{E}\big(\ell\big(Z_0^\uparrow\big) + \ell\big(Z_1^\downarrow\big)\big)\Big(\sum_{i=0}^{\ell(Z_0^\uparrow)-1} e^{-Z_0^\uparrow(i)} + \sum_{i=1}^{\ell(Y_{-1}^\uparrow)} e^{-Y_{-1}^\uparrow(i)}\Big)}\Bigg). \end{aligned} \tag{210}$$

Now, for $-\Gamma_n \leq k \leq \Gamma_n$, applying Theorems 2.4 **(i)** and 2.5 eq. (29), we have for every nonnegative measurable function φ, since $\{b_{\widetilde{h}_n} = z+k\} = \big\{x_0\big(V,\widetilde{h}_n\big) = z+k\big\} \cap \big\{\theta\big(T_0\big(V,\widetilde{h}_n\big)\big) \in \bigsqcup_{t \in \mathbb{N}^*} \mathbb{R}_+^t\big\} = \big\{x_0\big(V,\widetilde{h}_n\big) = z+k\big\} \cap \big\{V\big(x_0\big(V,\widetilde{h}_n\big)\big) < V\big(x_1\big(V,\widetilde{h}_n\big)\big)\big\}$ and $\sharp\big\{0 \leq i < \ell(\mathcal{T}_V^\uparrow),\ -i = z+k\big\} = \mathbf{1}_{\{-z-k<\ell(\mathcal{T}_V^\uparrow)\}}$ for each $k \in \mathbb{Z}$ such that $z+k \leq 0$,

$$\begin{aligned} &\mathsf{E}\big[\varphi\big(\theta\big(T_i\big(V,\widetilde{h}_n\big)\big), -18 \leq i \leq 19\big) \mathbf{1}_{\{b_{\widetilde{h}_n} = z+k\}}\big] \\ =&\mathsf{E}\Bigg[\varphi\big(Z_{-18}^\uparrow, Z_{-17}^\downarrow, Z_{-16}^\uparrow, \ldots, Z_{-2}^\uparrow, Z_{-1}^\downarrow, Z_0^\uparrow, Z_1^\downarrow, \ldots, Z_{18}^\uparrow, Z_{19}^\downarrow\big) \frac{\mathbf{1}_{\{-z-k<\ell(Z_0^\uparrow)\}}}{\mathsf{E}\big(\ell\big(Z_0^\uparrow\big) + \ell\big(Z_1^\downarrow\big)\big)}\Bigg]. \end{aligned} \tag{211}$$

In the previous equality, $\theta(T_i(V,\widetilde{h}_n))$ becomes $Z_i^\uparrow$ or $Z_i^\downarrow$ depending on the parity of i.

So, since $Y_{-1}^\uparrow = \zeta\big(Z_{-1}^\downarrow\big)$ and $z \leq -\Gamma_n$ in this first case, we get, as explained below,

$$J_7(j,n,z) \leq \sum_{k=-\Gamma_n}^{\Gamma_n} \mathsf{E}\Bigg(\frac{e^{-\theta[T_0(V,\tilde{h}_n)](-(k+j))} \mathbf{1}_{\{k+j \leq 0\}} + e^{-\zeta[\theta(T_{-1}(V,\tilde{h}_n))](k+j)} \mathbf{1}_{\{k+j>0\}}}{\sum_{i=0}^{\ell(T_0(V,\tilde{h}_n))-1} e^{-\theta[T_0(V,\tilde{h}_n)](i)} + \sum_{i=1}^{\ell(T_{-1}(V,\tilde{h}_n))} e^{-\zeta[\theta(T_{-1}(V,\tilde{h}_n))](i)}}$$

$$\left.\mathbf{1}_{\{b_{\widetilde{h}_n}=z+k\}\cap E_3^{(n)}\cap E_5^{(n)}\cap E_6^{(n)}}\right)$$

$$= \mathsf{E}\left(\frac{\sum_{k=-\Gamma_n}^{\Gamma_n} e^{-V(b_{\log n}-k-j)}\mathbf{1}_{\{b_{\log n}=z+k\}\cap E_3^{(n)}\cap E_5^{(n)}\cap E_6^{(n)}}}{\sum_{i=x_{-1}(V,\log n)}^{x_1(V,\log n)-1} e^{-V(i)}}\right), \tag{212}$$

for all $n \geq n_{16}$, $j \in \{0,1\}$ and $z \leq -\Gamma_n$. Indeed, when applying (211) to the quantity after (210), $E_{34}^{(n)}$ corresponds to (i.e. becomes) a set $\widetilde{E}_{34}^{(n)}$ included in $E_3^{(n)}$, on which we have in particular $H\big[T_i\big(V,\widetilde{h}_n\big)\big] \geq \log n + C_2 \log_2 n$ for all $-13 \leq i \leq 13$ and so $x_j\big(V,\widetilde{h}_n\big) = x_j(V,\log n)$ for $j \in \{-12,\dots,12\}$ and so $b_{\widetilde{h}_n} = b_{\log n} = x_0(V,\log n)$ (when $b_{\widetilde{h}_n} = z+k \leq 0$); $Z_0^{\uparrow}$ corresponds to $\theta\big(T_0\big(V,\widetilde{h}_n\big)\big) = \theta\big(T_0\big(V,\log n\big)\big) = \big(V(b_{\log n}+i) - V(b_{\log n}),\ 0 \leq i \leq x_1(V,\log n) - b_{\log n}\big)$ and $Y_{-1}^{\uparrow}$ to $\zeta\big(\theta\big[T_{-1}\big(V,\widetilde{h}_n\big)\big]\big) = (V(b_{\log n}-i) - V(b_{\log n}),\ 0 \leq i \leq b_{\log n} - x_{-1}(V,\log n))$ so $E_{31}^{(n)}$ corresponds to a set included in $E_6^{(n)}$ since $\widetilde{h}_n < \log n$, $E_{35}^{(n)}$ corresponds to a set included in $\big\{\big|x_{-12}\big(V,\widetilde{h}_n\big) - x_{12}\big(V,\widetilde{h}_n\big)\big| \leq (\log n)^{2+\delta_1}\big\}$, and the intersection of this and $\widetilde{E}_{34}^{(n)}$ is itself included in $\big\{\big|x_{-12}\big(V,\log n\big) - x_{12}\big(V,\log n\big)\big| \leq (\log n)^{2+\delta_1}\big\}$, and so in $E_5^{(n)}$, whereas $E_{32}^{(n)}$ and $E_{33}^{(n)}$ are not necessary anymore.

Notice that $\sum_{k=-\Gamma_n}^{\Gamma_n} e^{-V(b_{\log n}-k-j)} \leq \sum_{i=x_{-1}(V,\log n)}^{x_1(V,\log n)-1} e^{-V(i)} = \sum_{i=M^-}^{M^+-1} e^{-V(i)}$ on $E_6^{(n)} \cap \{b_{\log n} \leq 0\} \cap E_3^{(n)}$ with $M^{\pm}$ defined in (87) since $V(x_{\pm 1}) - V(x_0) \geq \log n + C_0$ for $n \geq n_{16} \geq n_3$. Thus, using Lemma 5.1, there exists $n_{17} \geq n_{16}$ such that, for all $n \geq n_{17}$, $j \in \{0,1\}$ and $z \leq -\Gamma_n$ (writing $E_i^{(n)}(z)$ instead of $E_i^{(n)}$ for $i \neq 3$),

$$J_7(j,n,z) \leq \mathsf{E}\left(\frac{\sum_{k=-\Gamma_n}^{\Gamma_n} e^{-V(z-j)}\mathbf{1}_{\{z=b_{\log n}-k\}\cap\cap_{\ell=3}^6 E_\ell^{(n)}(z)}}{\sum_{i=M^-}^{M^+-1} e^{-V(i)}}\right) + \varepsilon(\log n)^{-2}, \tag{213}$$

where we write $E_\ell^{(n)}(z)$ for $E_\ell^{(n)}$ for $\ell \in \{3,5,6\}$ for convenience. Hence, using (208), then (213), $M^- < z < M^+$ on $E_6^{(n)} \cap \{z = b_{\log n} - k\}$ for $|k| \leq \Gamma_n$ and (88) gives for all $n \geq n_{17}$ and $z \in (2\mathbb{Z}+n)$ such that $z \leq -\Gamma_n$,

$$\begin{aligned} 2J_6(n,z) &\leq J_7(1,n,z) + J_7(0,n,z) + 4\varepsilon(\log n)^{-2} \\ &\leq \mathsf{E}\left(\frac{\sum_{k=-\Gamma_n}^{\Gamma_n} \widehat{\mu}_n(z)\mathbf{1}_{\{z=b_{\log n}-k\}\cap\cap_{\ell=3}^6 E_\ell^{(n)}(z)}}{\sum_{i=M^-}^{M^+-1} e^{-V(i)}}\right) + 6\varepsilon(\log n)^{-2} \\ &= \mathsf{E}\Big(\widehat{\nu}_n(z)\mathbf{1}_{\{|b_{\log n}-z|\leq\Gamma_n\}\cap\cap_{\ell=3}^6 E_\ell^{(n)}(z)}\Big) + 6\varepsilon(\log n)^{-2}, \end{aligned}$$

where we used $\widehat{\mu}_n(2\mathbb{Z}) = \widehat{\mu}_n(2\mathbb{Z}+1) = \sum_{i=M^-}^{M^+-1} e^{-V(i)}$ and the definition (89) of $\widehat{\nu}_n$ since n and z have the same parity.

Applying Lemma 4.7, there exists $n_{18} \geq n_{17}$ such that, for all $n \geq n_{18}$ and all $z \in (2\mathbb{Z}+n)$ such that $z \leq -\Gamma_n$,

$$2J_6(n,z) \leq \mathsf{E}\big(P_\omega[S_n = z] + 5(\log n)^{-3}\big) + 6\varepsilon(\log n)^{-2} \leq \mathbb{P}(S_n = z) + 7\varepsilon(\log n)^{-2}.$$

This, (201) and (202) lead to $\frac{2\sigma^2}{(\log n)^2}\varphi_\infty\big(\frac{\sigma^2 z}{(\log n)^2}\big) \leq \mathbb{P}(S_n = z) + 13\varepsilon(\log n)^{-2}$ for all $n \geq n_{18}$ and all $z \in (2\mathbb{Z}+n)$ such that $z \leq -\Gamma_n$.

Second case: We now consider the case $z > \Gamma_n$. We use the same $Z_{2k}^{\uparrow}$, $Z_{2k+1}^{\downarrow}$, $Y_{-1}^{\uparrow} = \zeta\big(Z_{-1}^{\downarrow}\big)$ and $E_i^{(n)}$ as in the first case.

Using Lemma 2.6 with $x = z + \Gamma_n$, we have when $z + \Gamma_n > 0$,

$$J_6^+(n,z) := \mathsf{P}\big(b_{\widetilde{h}_n} = z + \Gamma_n\big) = \frac{\mathsf{P}\big(z + \Gamma_n \le \ell(Z_{-1}^{\downarrow})\big)}{\mathsf{E}\big(\ell\big(Z_0^{\uparrow}\big) + \ell\big(Z_1^{\downarrow}\big)\big)}. \tag{214}$$

Similarly as in (201), for all $n \ge n_{18}$, for all $z \in]\Gamma_n, A_0(\log n)^2]$,

$$\frac{\sigma^2}{(\log n)^2}\varphi_\infty\left(\frac{\sigma^2 z}{(\log n)^2}\right) \le \frac{\sigma^2}{\big(\widetilde{h}_n\big)^2}\varphi_\infty\left(\frac{\sigma^2\big(z+\Gamma_n\big)}{\big(\widetilde{h}_n\big)^2}\right) + \frac{2\varepsilon}{(\log n)^2} \le J_6^+(n,z) + \frac{3\varepsilon}{(\log n)^2}. \tag{215}$$

Also $\frac{\sigma^2}{(\log n)^2}\varphi_\infty\left(\frac{\sigma^2 z}{(\log n)^2}\right) \le J_6^+(n,z) + \frac{3\varepsilon}{(\log n)^2}$ for all $n \ge n_{18}$ and all $z \ge \max(\Gamma_n,\, A_0(\log n)^2)$ as in (202), and so for all $z > \Gamma_n$.

Similarly as in (205) and (208), using $\mathbf{1}_{\{z+\Gamma_n \le \ell(Z_{-1}^{\downarrow})\}} \le \mathbf{1}_{\{z+k \le \ell(Z_{-1}^{\downarrow})\}}$ instead of $\mathbf{1}_{\{-z+\Gamma_n < \ell(Z_0^{\uparrow})\}} \le \mathbf{1}_{\{-z-k < \ell(Z_0^{\uparrow})\}}$, we get for all $n \ge n_{18}$ and all $z > \Gamma_n$, $J_6^+(n,z) \le J_7^+(j,n,z) + 2\varepsilon(\log n)^{-2}$ for each $j \in \{0,1\}$, where for $j \in \{0,1\}$,

$$\begin{aligned} &J_7^+(j,n,z) \\ &:= \sum_{k=-\Gamma_n}^{\Gamma_n} \mathsf{E}\left(\frac{\mathbf{1}_{\{z+k\le\ell(Z_{-1}^{\downarrow})\}\cap\cap_{i=31}^{35}E_i^{(n)}}\Big(e^{-Z_0^{\uparrow}(-(k+j))}\mathbf{1}_{\{k+j\le 0\}} + e^{-Y_{-1}^{\uparrow}(k+j)}\mathbf{1}_{\{k+j>0\}}\Big)}{\mathsf{E}\big(\ell\big(Z_0^{\uparrow}\big)+\ell\big(Z_1^{\downarrow}\big)\big)\Big(\sum_{i=0}^{\ell(Z_0^{\uparrow})-1} e^{-Z_0^{\uparrow}(i)} + \sum_{i=1}^{\ell(Y_{-1}^{\uparrow})} e^{-Y_{-1}^{\uparrow}(i)}\Big)}\right). \end{aligned} \tag{216}$$

Now, applying Theorems 2.4 **(ii)** and 2.5 eq. (29), we have for every nonnegative measurable function φ, since $\{b_{\widetilde{h}_n} = z+k\} = \big\{x_1\big(V,\widetilde{h}_n\big) = z+k\big\} \cap \big\{\theta\big(T_0\big(V,\widetilde{h}_n\big)\big) \in \bigsqcup_{t\in\mathbb{N}^*}\mathbb{R}_-^t\big\}$ when $z+k>0$,

$$\begin{aligned} &\mathsf{E}\big[\varphi\big(\theta(T_i(V,\widetilde{h}_n)), -17 \le i \le 20\big)\mathbf{1}_{\{b_{\widetilde{h}_n}=z+k\}}\big] \\ =&\mathsf{E}\left[\varphi\big(Z_{-18}^{\uparrow}, Z_{-17}^{\downarrow}, Z_{-16}^{\uparrow}, \dots, Z_{-2}^{\uparrow}, Z_{-1}^{\downarrow}, Z_0^{\uparrow}, \dots, Z_{18}^{\uparrow}, Z_{19}^{\downarrow}\big)\frac{\mathbf{1}_{\{z+k\le\ell(Z_{-1}^{\downarrow})\}}}{\mathsf{E}\big(\ell\big(Z_0^{\uparrow}\big)+\ell\big(Z_1^{\downarrow}\big)\big)}\right]. \end{aligned} \tag{217}$$

In the previous equality, $\theta(T_i(V,\widetilde{h}_n))$ becomes $Z_{i-1}^{\uparrow}$ or $Z_{i-1}^{\downarrow}$ depending on the parity of i.

So, since $Y_{-1}^{\uparrow} = \zeta\big(Z_{-1}^{\downarrow}\big)$ and $z > \Gamma_n$, we get, similarly as in (212), with $b_{\widetilde{h}_n} = b_{\log n} = x_1(V,\log n)$, and using the definition (87) of $M^{\pm}$ on $\{b_{\log n} > 0\}$.

$$\begin{aligned} J_7^+(j,n,z) \;\le\; &\sum_{k=-\Gamma_n}^{\Gamma_n} \mathsf{E}\Bigg(\frac{e^{-\theta[T_1(V,\tilde{h}_n)](-(k+j))}\mathbf{1}_{\{k+j\le 0\}} + e^{-\zeta[\theta(T_0(V,\tilde{h}_n))](k+j)}\mathbf{1}_{\{k+j>0\}}}{\sum_{i=0}^{\ell(T_1(V,\tilde{h}_n))-1} e^{-\theta[T_1(V,\tilde{h}_n)](i)} + \sum_{i=1}^{\ell(T_0(V,\tilde{h}_n))} e^{-\zeta[\theta(T_0(V,\tilde{h}_n))](i)}} \\ &\qquad \mathbf{1}_{\{b_{\widetilde{h}_n}=z+k\}\cap E_3^{(n)}\cap E_5^{(n)}\cap E_6^{(n)}}\Bigg) \\ = \;&\mathsf{E}\left(\frac{\sum_{k=-\Gamma_n}^{\Gamma_n} e^{-V(b_{\log n}-k-j)}\mathbf{1}_{\{b_{\log n}=z+k\}\cap E_3^{(n)}\cap E_5^{(n)}\cap E_6^{(n)}}}{\sum_{i=M^-}^{M^+-1} e^{-V(i)}}\right), \end{aligned} \tag{218}$$

for all $n \ge n_{18}$, $j \in \{0,1\}$ and $z > \Gamma_n$.

We conclude as in the first case that $\frac{2\sigma^2}{(\log n)^2}\varphi_\infty\big(\frac{\sigma^2 z}{(\log n)^2}\big) \le \mathbb{P}(S_n = z) + 13\varepsilon(\log n)^{-2}$ for all $n \ge n_{18}$ and all $z \in (2\mathbb{Z}+n)$ such that $z > \Gamma_n$.

Third case: We finally consider the case $-\Gamma_n < z \le \Gamma_n$.

We use the same notation as in the first case. Notice that (200), (201), (202), (205), (208) and (210) remain valid when $n \geq n_{18}$ and $-\Gamma_n < z \leq \Gamma_n$, with the same definitions of J_6 and J_7. However in this third case, that is, for every $n \geq n_{18}$ and $-\Gamma_n < z \leq \Gamma_n$, for $j \in \{0,1\}$,

$$J_7(j,n,z) \leq J_8(j,n,z) + J_9(j,n,z) + J_{10}(j,n,z), \tag{219}$$

where

$$J_8(j,n,z) := \sum_{k=-\Gamma_n}^{-z} \mathsf{E}\left(\frac{\mathbf{1}_{\{-z-k<\ell(Z_0^\uparrow)\}\cap\cap_{i=31}^{35}E_i^{(n)}} \left(e^{-Z_0^\uparrow(-(k+j))}\mathbf{1}_{\{k+j\leq 0\}} + e^{-Y_{-1}^\uparrow(k+j)}\mathbf{1}_{\{k+j>0\}} \right)}{\mathsf{E}\big(\ell\big(Z_0^\uparrow\big)+\ell\big(Z_1^\downarrow\big)\big)\Big(\sum_{i=0}^{\ell(Z_0^\uparrow)-1} e^{-Z_0^\uparrow(i)} + \sum_{i=1}^{\ell(Y_{-1}^\uparrow)} e^{-Y_{-1}^\uparrow(i)}\Big)} \right),$$

$$J_9(j,n,z) := \sum_{k=-z+1}^{\Gamma_n} \mathsf{E}\left(\frac{\big(1-\mathbf{1}_{\{z+k\leq\ell(Z_{-1}^\downarrow)\}}\big)\mathbf{1}_{\cap_{i=31}^{35}E_i^{(n)}} \left(e^{-Z_0^\uparrow(-(k+j))}\mathbf{1}_{\{k+j\leq 0\}} + e^{-Y_{-1}^\uparrow(k+j)}\mathbf{1}_{\{k+j>0\}} \right)}{\mathsf{E}\big(\ell\big(Z_0^\uparrow\big)+\ell\big(Z_1^\downarrow\big)\big)\Big(\sum_{i=0}^{\ell(Z_0^\uparrow)-1} e^{-Z_0^\uparrow(i)} + \sum_{i=1}^{\ell(Y_{-1}^\uparrow)} e^{-Y_{-1}^\uparrow(i)}\Big)} \right),$$

$$J_{10}(j,n,z) := \sum_{k=-z+1}^{\Gamma_n} \mathsf{E}\left(\frac{\mathbf{1}_{\{z+k\leq\ell(Z_{-1}^\downarrow)\}}\mathbf{1}_{\cap_{i=31}^{35}E_i^{(n)}} \left(e^{-Z_0^\uparrow(-(k+j))}\mathbf{1}_{\{k+j\leq 0\}} + e^{-Y_{-1}^\uparrow(k+j)}\mathbf{1}_{\{k+j>0\}} \right)}{\mathsf{E}\big(\ell\big(Z_0^\uparrow\big)+\ell\big(Z_1^\downarrow\big)\big)\Big(\sum_{i=0}^{\ell(Z_0^\uparrow)-1} e^{-Z_0^\uparrow(i)} + \sum_{i=1}^{\ell(Y_{-1}^\uparrow)} e^{-Y_{-1}^\uparrow(i)}\Big)} \right).$$

We first notice that, since $-\ell(Z_0^\uparrow)+2 \leq -\Gamma_n+1 \leq -z+1 \leq \Gamma_n \leq \ell(Y_{-1}^\uparrow)-1$ on $E_{31}^{(n)}$ for $-\Gamma_n < z \leq \Gamma_n$, and using $\mathbf{1}_{\{z+\Gamma_n\leq\ell(Z_{-1}^\downarrow)\}} \leq \mathbf{1}_{\{z+k\leq\ell(Z_{-1}^\downarrow)\}}$, there exists $n_{19} \geq n_{18}$ such that, for all $n \geq n_{19}$, all $-\Gamma_n < z \leq \Gamma_n$, and all $j \in \{0,1\}$,

$$\begin{aligned} 0 \leq J_9(j,n,z) &\leq \mathsf{E}\left(\frac{1-\mathbf{1}_{\{z+\Gamma_n\leq\ell(Z_{-1}^\downarrow)\}}}{\mathsf{E}\big(\ell\big(Z_0^\uparrow\big)+\ell\big(Z_1^\downarrow\big)\big)} \right) = \mathsf{P}\big(b_{\tilde{h}_n}=0\big) - \mathsf{P}\big(b_{\tilde{h}_n}=z+\Gamma_n\big) \\ &\leq \frac{\sigma^2}{\big(\widetilde{h}_n\big)^2}\left(\varphi_\infty(0) - \varphi_\infty\left(\frac{\sigma^2(z+\Gamma_n)}{\big(\widetilde{h}_n\big)^2} \right) \right) + \varepsilon(\log n)^{-2}/4 \\ &\leq \varepsilon(\log n)^{-2}/2 \end{aligned} \tag{220}$$

by Lemma 2.6, then Theorem 1.4, and finally by continuity of φ_∞ since $\delta_1 < 2/3$ and $|z| \leq \Gamma_n$.

In order to prove an inequality for $J_8(j,n,z)$, we can do the same proof as in the first case from the line following (210) to (213), replacing $\sum_{k=-\Gamma_n}^{\Gamma_n}$ by $\sum_{k=-\Gamma_n}^{-z}$ since $|z| \leq \Gamma_n$ (so $z+k \leq 0$), which gives, for all $n \geq n_{19}$, all $-\Gamma_n < z \leq \Gamma_n$ and all $j \in \{0,1\}$,

$$J_8(j,n,z) \leq \mathsf{E}\left(\frac{\sum_{k=-\Gamma_n}^{-z} e^{-V(z-j)}\mathbf{1}_{\{z=b_{\log n}-k\}\cap\cap_{\ell=3}^{6}E_\ell^{(n)}(z)}}{\sum_{i=M^-}^{M^+-1} e^{-V(i)}} \right) + \varepsilon(\log n)^{-2}/4. \tag{221}$$

In order to prove an inequality for $J_{10}(j,n,z)$, we can do the same proof as in the second case, between the definition (216) of J_7^+ and (218), replacing $\sum_{k=-\Gamma_n}^{\Gamma_n}$ by $\sum_{k=-z+1}^{\Gamma_n}$ since $|z| \leq \Gamma_n$ (so $z+k>0$), then using once more Lemma 5.1 as in (213), we get for all $n \geq n_{19}$, all $-\Gamma_n < z \leq \Gamma_n$ and all $j \in \{0,1\}$,

$$J_{10}(j,n,z) \leq \mathsf{E}\left(\frac{\sum_{k=-z+1}^{\Gamma_n} e^{-V(z-j)}\mathbf{1}_{\{z=b_{\log n}-k\}\cap\cap_{\ell=3}^{6}E_\ell^{(n)}(z)}}{\sum_{i=M^-}^{M^+-1} e^{-V(i)}} \right) + \varepsilon(\log n)^{-2}/4.$$

This, (219), (220) and (221) prove that (213) remains true for all $n \geq n_{19}$, all $-\Gamma_n < z \leq \Gamma_n$ and all $j \in \{0,1\}$.

Since (208), (201) and (202) also remain true, we conclude as in the first case that for all $n \geq n_{19}$ and all $z \in (2\mathbb{Z}+n)$ such that $-\Gamma_n < z \leq \Gamma_n$, we have $\frac{2\sigma^2}{(\log n)^2}\varphi_\infty\big(\frac{\sigma^2 z}{(\log n)^2}\big) \leq \mathbb{P}(S_n = z) + 13\varepsilon(\log n)^{-2}$.

Finally, combining the conclusions of the three cases gives for all $n \geq n_{19}$,

$$\sup_{z\in(2\mathbb{Z}+n)} \left[\frac{2\sigma^2}{(\log n)^2}\varphi_\infty\left(\frac{\sigma^2 z}{(\log n)^2}\right) - \mathbb{P}\big(S_n = z\big)\right] \leq 13\varepsilon(\log n)^{-2},$$

which proves the lower bound in Theorem 1.1. This and (199) prove Theorem 1.1. □

7. Some estimates concerning the environment

7.1. **Probabilities of $\big(E_5^{(n)}\big)^c$ and $(E_6^{(n)})^c$.** The aim of this subsection is to give upper bounds of some probabilities related to the events $E_i^{(n)}$, which are defined between equations (83) and (84).

Lemma 7.1. *There exists $p_3 \geq 2$ such that,*

$$\forall n \geq p_3, \qquad \mathsf{P}\big[(E_5^{(n)})^c\big] \leq (\log n)^{-7}. \tag{222}$$

Also, we have for $n \geq p_3$, with $\widetilde{h}_n = \log n - 2C_1 \log_2 n$ as before,

$$\mathsf{P}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,\widetilde{h}_n}\big) > (\log n)^{2+\delta_1}/50\big] \leq (\log n)^{-8}, \tag{223}$$

$$\mathsf{P}\big[\ell\big(\mathcal{T}^{\uparrow}_{V_-,\widetilde{h}_n}\big) > (\log n)^{2+\delta_1}/50\big] = \mathsf{P}\big[\ell\big(\mathcal{T}^{\downarrow}_{V,\widetilde{h}_n}\big) > (\log n)^{2+\delta_1}/50\big] \leq (\log n)^{-8}. \tag{224}$$

Proof: The idea is to approximate V by a two-sided Brownian motion, in order to transfer to V some results already known for Brownian motions.

To this aim, we recall the definition of h-extrema introduced by Neveu et al. [54] for continuous functions. If w is a continuous function $\mathbb{R} \to \mathbb{R}$, $h > 0$, and $y \in \mathbb{R}$, it is said that w admits an *h-minimum at y* if there exists real numbers u and v such that $u < y < v$, $w(y) = \inf\{w(z),\ z \in [u, v]\}$, $w(u) \geq w(y) + h$ and $w(v) \geq w(y) + h$. It is said that w admits an *h-maximum* at y if $-w$ admits an h-minimum at y. In these two cases we say that w admits an *h-extremum* at y. Notice that contrary to Definition 2.1, all the inequalities are large.

It is known (see [14], Lemma 8) that, when $w = W$ or $w = \sigma W$, almost surely, **(a)** w is continuous on $\mathbb{R}$; **(b)** for every $h > 0$, the set of h-extrema of w can be written $\{x_k(w,h),\ k \in \mathbb{Z}\}$, where $(x_k(w,h))_{k\in\mathbb{Z}}$ is strictly increasing, unbounded from above and below, with $x_0(w,h) \leq 0 < x_1(w,h)$; **(c)** for all $h > 0$ and $k \in \mathbb{Z}$, $x_{k+1}(w,h)$ is an h-maximum if and only if $x_k(w,h)$ is an h-minimum (we use the same notation as for left extrema of V since no confusion is possible).

According to a slightly modified version (see e.g. [24], Lemma 4.3, with $(\log n)^\alpha$ replaced by K and a single potential V instead of two) of the Komlós–Major–Tusnády strong approximation theorem (see Komlós et al. [50]), there exist (strictly) positive constants C_3 and C_4, independent of $K \in \mathbb{N}^*$, such that, possibly in an enlarged probability space, there exists a two-sided standard Brownian motion $(W(t),\ t \in \mathbb{R})$, such that

$$E_{36}(K) := \left\{\sup_{-K\leq t\leq K} \Big|V(\lfloor t\rfloor) - \sigma W(t)\Big| \leq C_3 \log K\right\}$$

satisfies $\mathsf{P}([E_{36}(K)]^c) \leq K^{-C_4}$ for large K.

Let $n \geq n_3$ and $\alpha > 0$, and recall that $0 < \delta_1 < 2/3$. We define $h'_n := \log n + 3C_3(3+8/C_4)\log_2 n$. On $E_{36}\big((\log n)^{3+8/C_4}\big)$, consider, if they exist, two consecutive h'_n-minima for σW, denoted

by $y_i := x_i(\sigma W, h'_n)$ and $y_{i+2} := x_{i+2}(\sigma W, h'_n)$, such that $|y_i| \le \alpha(\log n)^{2+\delta_1}$ and $|y_{i+2}| \le \alpha(\log n)^{2+\delta_1}$. Let $z_{i+1} := \min\big\{k \in [\lfloor y_i \rfloor, \lfloor y_{i+2} \rfloor] \cap \mathbb{Z},\ V(k) = \max_{[\lfloor y_i \rfloor, \lfloor y_{i+2} \rfloor]} V\big\}$. We have, using $\omega \in E_{36}\big((\log n)^{3+8/C_4}\big)$ in the second and forth inequalities, for n large enough so that $(\log n)^{3+8/C_4} > \alpha(\log n)^{2+\delta_1}$,

$$\begin{aligned} V(z_{i+1}) &= \max_{[\lfloor y_i \rfloor, \lfloor y_{i+2} \rfloor]} V \ge V(\lfloor x_{i+1}(\sigma W, h'_n) \rfloor) \ge \sigma W[x_{i+1}(\sigma W, h'_n)] - C_3(3 + 8/C_4)\log_2 n \\ &\ge \sigma W[x_i(\sigma W, h'_n)] + h'_n - C_3(3 + 8/C_4)\log_2 n \\ &\ge V(\lfloor y_i \rfloor) + h'_n - 2C_3(3 + 8/C_4)\log_2 n \ge V(\lfloor y_i \rfloor) + \log n. \end{aligned}$$

We prove similarly that $V(z_{i+1}) \ge V(\lfloor y_{i+2} \rfloor) + \log n$, and so $\lfloor y_i \rfloor < z_{i+1} < \lfloor y_{i+2} \rfloor$. Since $\max_{[\lfloor y_i \rfloor, z_{i+1}[} V < V(z_{i+1})$ and $\max_{]z_{i+1}, \lfloor y_{i+2} \rfloor]} V \le V(z_{i+1})$, z_{i+1} is a left $(\log n)$-maximum for V.

So we have proved that for large n on $E_{36}\big((\log n)^{3+8/C_4}\big)$, between two consecutive h'_n-minima for σW belonging to the interval $\big[-\alpha(\log n)^{2+\delta_1}, \alpha(\log n)^{2+\delta_1}\big]$, there is at least one left $(\log n)$-maximum for V. Notice in particular that for such n, on $E_{36}\big((\log n)^{3+8/C_4}\big)$, if $x_{17}(\sigma W, h'_n) \le \alpha(\log n)^{2+\delta_1}$, then in $[x_1(\sigma W, h'_n), x_{17}(\sigma W, h'_n)]$, there are at least eight consecutive h'_n-minima for σW, and then at least seven left $(\log n)$-maxima for V, and so $x_{13}(V, \log n) \le x_{17}(\sigma W, h'_n) \le \alpha(\log n)^{2+\delta_1}$. Hence for large n,

$$\begin{aligned} & \mathsf{P}\big[x_{13}(V, \log n) > \alpha(\log n)^{2+\delta_1}, E_{36}\big((\log n)^{3+8/C_4}\big)\big] \\ \le\ & \mathsf{P}\big[x_{17}(\sigma W, h'_n) > \alpha(\log n)^{2+\delta_1}\big] \\ \le\ & \sum_{i=0}^{16} \mathsf{P}\left[\ell(T_i(\sigma W, h'_n)) > \frac{\alpha(\log n)^{2+\delta_1}}{17}\right], \end{aligned} \tag{225}$$

where $\ell(T_i(w,h)) := x_{i+1}(w,h) - x_i(w,h)$ for $i \in \mathbb{Z}$, $h > 0$ and any continuous function w, is the *length of the i-th h-slope of w*.

The length of a non central 1-slope of W, that is, $\ell(T_i(W,1))$ for $i \ne 0$, has a density, which is (see [14], eq. (7)) $f_\ell(x) := \pi \sum_{k\in\mathbb{N}} (-1)^k (k+1/2) \exp\big(-\pi^2(k+1/2)^2 x/2\big) \mathbf{1}_{\mathbb{R}_+^*}(x)$. Also, the length of the central 1-slope $\ell(T_0(W,1))$ has a density, which is (see [14], eq. (10)) equal to $f_{\ell(T_0)}(x) := x f_\ell(x)$. Notice that $f_\ell(x) \le (\pi/2)\exp[-\pi^2 x/8]$ for large x. Hence for large x, $f_{\ell(T_0)}(x) \le \exp[-\pi^2 x/10]$ and $f_\ell(x) \le \exp[-\pi^2 x/10]$. Thus, $\mathsf{P}\big[\ell(T_i(W,1)) > u\big) = O(\exp(-\pi^2 u/10))$ as $u \to +\infty$ for any $i \in \mathbb{Z}$, so for large n,

$$\begin{aligned} &\mathsf{P}\big[\ell(T_i(\sigma W, h'_n)) > \alpha(\log n)^{2+\delta_1}/17\big] = \mathsf{P}\big[\ell(T_i(W,1)) > \sigma^2 \alpha(\log n)^{2+\delta_1}/(17(h'_n)^2)\big] \\ &\le \mathsf{P}\big[\ell(T_i(W,1)) > \sigma^2 \alpha(\log n)^{\delta_1}/20\big] = O\big(\exp(-\pi^2\sigma^2\alpha(\log n)^{\delta_1}/200)\big), \end{aligned}$$

as $n \to +\infty$, where we used $\ell(T_i(\sigma W, h'_n)) = \ell(T_i(W, h'_n/\sigma)) =_{law} (h'_n/\sigma)^2 \ell(T_i(W,1))$ by scaling. This, (225) and $\mathsf{P}([E_{36}(K)]^c) \le K^{-C_4}$ for large K lead to

$$\begin{aligned} & \mathsf{P}\big[x_{13}(V, \log n) > \alpha(\log n)^{2+\delta_1}\big] \\ \le\ & O\big(\exp(-\pi^2\sigma^2\alpha(\log n)^{\delta_1}/200)\big) + \mathsf{P}\big[\big(E_{36}\big((\log n)^{3+8/C_4}\big)\big)^c\big] \le (\log n)^{-8} \end{aligned} \tag{226}$$

for large n. We prove similarly that $\mathsf{P}\big[x_{-12}(V, \log n) < -\alpha(\log n)^{2+\delta_1}\big] \le (\log n)^{-8}$. Finally,

$$\mathsf{P}\big[(E_5^{(n)})^c\big] \le \mathsf{P}\big[x_{12}(V, \log n) > (\log n)^{2+\delta_1}\big] + \mathsf{P}\big[x_{-12}(V, \log n) < -(\log n)^{2+\delta_1}\big] \le (\log n)^{-7}$$

for large n, which proves (222).

Since $x_3(V, \widetilde{h}_n) \le x_3(V, \log n) < x_{13}(V, \log n)$, we get

$$\mathsf{P}\big[x_3(V, \widetilde{h}_n) > (\log n)^{2+\delta_1}/50\big] \le \mathsf{P}\big[x_{13}(V, \log n) > (\log n)^{2+\delta_1}/50\big] \le (\log n)^{-8}$$

for large n by (226). Since $x_3\big(V,\widetilde{h}_n\big) > x_3\big(V,\widetilde{h}_n\big) - x_1\big(V,\widetilde{h}_n\big)$, which has the same law as $\ell\big(\mathcal{T}^{\uparrow}_{V,\widetilde{h}_n}\big) + \ell\big(\mathcal{T}^{\downarrow}_{V,\widetilde{h}_n}\big)$, by Theorem 2.4, this gives

$$\mathsf{P}\big[\ell\big(\mathcal{T}^{\uparrow}_{V,\widetilde{h}_n}\big) > (\log n)^{2+\delta_1}/50\big] \le \mathsf{P}\big[x_3(V,\widetilde{h}_n) > (\log n)^{2+\delta_1}/50\big] \le (\log n)^{-8}$$

and similarly $\mathsf{P}\big[\ell\big(\mathcal{T}^{\downarrow}_{V,\widetilde{h}_n}\big) > (\log n)^{2+\delta_1}/50\big] \le (\log n)^{-8}$ for large n. Since $\ell\big(\mathcal{T}^{\uparrow}_{V_-,\widetilde{h}_n}\big) =_{law} \ell\big(\mathcal{T}^{\uparrow}_{-V,\widetilde{h}_n}\big) =_{law} \ell\big(\mathcal{T}^{\downarrow}_{V,\widetilde{h}_n}\big)$, by Theorem 2.3 **(ii)**,

this proves (223) and (224) up to a change of p_3, which ends the proof of the lemma. □

We now turn to the probability of $(E_6^{(n)})^c \cap E_5^{(n)}$.

Lemma 7.2. *Recall that $\delta_1 \in]0, 2/3[$. There exist $c_{33} > 0$ and $p_2 \in \mathbb{N}$ such that*

$$\forall n \ge p_2, \qquad \mathsf{P}\big[(E_6^{(n)})^c \cap E_5^{(n)}\big] \le \exp\big[-c_{33}(\log n)^{2/3-\delta_1}\big] \le (\log n)^{-3}. \tag{227}$$

We now consider left and right $\widetilde{h}_n$-slopes. As $n \to +\infty$,

$$\mathsf{P}\Big[T_{\mathcal{T}^{\uparrow}_{V_\pm,\widetilde{h}_n}}\big(\widetilde{h}_n\big) \le \Gamma_n\Big] = o\big((\log n)^{-2}\big), \qquad \mathsf{P}\Big[T_{\mathcal{T}^{\uparrow *}_{V_\pm,\widetilde{h}_n}}\big(\widetilde{h}_n\big) \le \Gamma_n\Big] = o\big((\log n)^{-2}\big), \tag{228}$$

recalling that T, $T_{\mathcal{T}^{\uparrow}_{V,h}}$, $T_{\mathcal{T}^{\downarrow}_{V,h}}$, $T_{\mathcal{T}^{\uparrow *}_{V,h}}$ and $T_{\mathcal{T}^{\downarrow *}_{V,h}}$ are defined in (14), *Definition 2.2,* (47) *and* (48), *and that $V_\pm(.) = V(\pm.)$.*

Proof: First, for $n \ge n_3$, $b \in \mathbb{Z}$ and $0 < |i| \le \Gamma_n$, we have by Hoeffding's inequality (see [43], Theorem 2),

$$\begin{aligned}\mathsf{P}\big[V(b+i) - V(b) \ge \log n\big] &= \mathsf{P}\big[V(i) \ge \log n\big] \le \exp\big[-2(\log n)^2/\big(|i|\big(2C_0\big)^2\big)\big] \\ &\le \exp\big[-c_{34}(\log n)^2/|i|\big] \le \exp\big[-c_{34}(\log n)^{2/3-\delta_1}\big] \end{aligned} \tag{229}$$

with $c_{34} := 1/(2C_0^2) > 0$, since $V(i)$ is the sum of $|i|$ independent random variables with zero mean, bounded by $\pm C_0$ by ellipticity (see (16)).

Notice that on $\big(E_6^{(n)}\big)^c \cap E_5^{(n)}$, there exists $b = b_{\log n} \in \mathbb{Z}$ and $i \in \mathbb{Z}$ such that $V(b+i) - V(b) \ge \log n$, $|i| \le \Gamma_n$ and $|b| \le (\log n)^{2+\delta_1}$ since $\omega \in E_5^{(n)}$. Thus by (229),

$$\begin{aligned}\mathsf{P}\big[(E_6^{(n)})^c \cap E_5^{(n)}\big] &\le \sum_{|b| \le \lfloor (\log n)^{2+\delta_1} \rfloor} \sum_{|i| \le \lfloor (\log n)^{4/3+\delta_1} \rfloor} \mathsf{P}\big[V(b+i) - V(b) \ge \log n\big] \\ &\le 9(\log n)^5 \exp\big[-c_{34}(\log n)^{2/3-\delta_1}\big],\end{aligned}$$

since $0 < \delta_1 < 2/3$. This proves (227), e.g. with $c_{33} := c_{34}/2$.

Now, notice that, using the law of $\mathcal{T}^{\uparrow}_{V,\widetilde{h}_n}$ provided by Theorem 2.3 **(i)**, then (18) and once more Hoeffding's inequality and $\widetilde{h}_n \sim_{n\to+\infty} \log n$,

$$\begin{aligned}&\mathsf{P}\big[T_{\mathcal{T}^{\uparrow}_{V,\widetilde{h}_n}}\big(\widetilde{h}_n\big) \le \Gamma_n\big] \\ =\ &\mathsf{P}\big[T_V\big(\widetilde{h}_n\big) \le \Gamma_n, T_V\big(\widetilde{h}_n\big) < T_V(\mathbb{R}^*_-)\ \big]/\mathsf{P}\big[T_V\big(\widetilde{h}_n\big) < T_V(\mathbb{R}^*_-)\ \big] \\ \le\ &\frac{2\log n}{c_1}\sum_{i=1}^{\Gamma_n}\mathsf{P}\big[V(i) \ge \widetilde{h}_n\big] \le \frac{2\log n}{c_1}\sum_{i=1}^{\lfloor (\log n)^{4/3+\delta_1}\rfloor}\exp\big[-c_{34}\big(\widetilde{h}_n\big)^2/i\big] \\ \le\ &(2/c_1)(\log n)^3\exp[-c_{34}(\log n)^{2/3-\delta_1}/2]\end{aligned}$$

for large n. This proves (228) for V_+ since $0 < \delta_1 < 2/3$. The proof for V_- is similar. The proof for $T_{\mathcal{T}^{\uparrow *}_{V_\pm,\tilde{h}_n}}$ is the similar, with Theorem 2.9 and c_1^* instead of Theorem 2.3 and c_1. □

7.2. **Laplace transform of V conditioned to stay positive or nonnegative.** The main tools of this subsection are local limit theorems for random walks conditioned to stay positive, by Vatutin and Wachtel ([64], Theorems 4 and 6 and Lemma 12 with $\alpha = 2$ and $\rho = 1/2$).

We define for $h \geq 0$, with T_V and T_V^* defined in (14), and (15),

$$\begin{aligned} \Xi_h &:= \Big\{ \inf_{[1,T_V([h,+\infty[)]} V \geq 0 \Big\} = \{T_V(h) < T_V(\mathbb{R}_-^*)\}, \qquad (230) \\ \Xi_h^* &:= \Big\{ \inf_{[1,T_V([h,+\infty[)]} V > 0 \Big\} = \{T_V(h) < T_V^*(]-\infty,0])\}. \end{aligned}$$

The aim of this subsection is to prove the following uniform upper bound:

Proposition 7.3. *There exist $c_{13} > 0$, $p_4 > 0$ and $p_5 > 0$ such that*

$$\forall x \geq p_4, \forall h \geq p_5, \qquad \mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x<T_V(h)\}} \mid T_V(h) < T_V(\mathbb{R}_-^*)\big] \leq c_{13}x^{-3/2}.$$

This remains true when $T_V(\mathbb{R}_-^*)$ is replaced by $T_V^*(]-\infty,0])$.

Before proving this lemma, we introduce some notation and some technical lemmas. First, let

$$G_x := \{\forall 1 \leq k \leq x,\ V(k) \geq 0\}, \qquad G_x^* := \{\forall 1 \leq k \leq x,\ V(k) > 0\}, \qquad x > 0. \qquad (231)$$

We know (due to the Spitzer and Ròsen theorem, see Vatutin and Wachtel [64] eq. (18), or [10] Theorem 8.9.23, p. 382) that

$$\mathsf{P}[G_x] \sim_{x\to+\infty} c_{35}x^{-1/2}, \qquad \mathsf{P}[G_x^*] \sim_{x\to+\infty} c_{35}^* x^{-1/2}, \qquad (232)$$

where $c_{35} > 0$ and $c_{35}^* > 0$.

The following (uniform) estimates are maybe already known. However we did not find them in the literature, so we provide their proof.

Lemma 7.4. *For large $h > 0$, for every $0 \leq z < h$,*

$$\begin{aligned} \frac{z - \mathsf{E}^z[V(T_V(\mathbb{R}_-^*))]}{h} - \frac{3C_0(z+C_0)}{h^2} &\leq \mathsf{P}^z(\Xi_h) \leq \frac{z - \mathsf{E}^z[V(T_V(\mathbb{R}_-^*))]}{h}, \qquad (233) \\ \frac{z - \mathsf{E}^z[V(T_V^*(\mathbb{R}_-))]}{h} - \frac{3C_0(z+C_0)}{h^2} &\leq \mathsf{P}^z(\Xi_h^*) \leq \frac{z - \mathsf{E}^z[V(T_V^*(\mathbb{R}_-))]}{h}. \qquad (234) \end{aligned}$$

Also, for $z = 0$,

$$h\mathsf{P}[\Xi_h] \to_{h\to+\infty} -\mathsf{E}[V(T_V(\mathbb{R}_-^*))] =: c_1 > 0, \qquad (235)$$
$$h\mathsf{P}[\Xi_h^*] \to_{h\to+\infty} -\mathsf{E}[V(T_V^*(\mathbb{R}_-))] =: c_1^* > 0. \qquad (236)$$

Proof: Let $h > 0$, $U_h := T_V([h,+\infty[) \wedge T_V(\mathbb{R}_-^*)$, and $0 \leq z < h$. Since $(V(k),\ k \geq 0)$ is under P^z a martingale starting at z for its natural filtration due to (3), and $|V(k \wedge U_h)| \leq h + C_0$ a.s. for every $k \in \mathbb{N}$ thanks to ellipticity (16), the optimal stopping theorem gives

$$z = \mathsf{E}^z[V(U_h)] = \mathsf{E}^z[V(T_V([h,+\infty[))\mathbf{1}_{\Xi_h}] + \mathsf{E}^z[V(T_V(\mathbb{R}_-^*))\mathbf{1}_{(\Xi_h)^c}]. \qquad (237)$$

Since $h \leq V(T_V([h,+\infty[)) \leq h + C_0$ a.s. by ellipticity, we have

$$h\mathsf{P}^z[\Xi_h] \leq \mathsf{E}^z[V(T_V([h,+\infty[))\mathbf{1}_{\Xi_h}] \leq (h + C_0)\mathsf{P}^z[\Xi_h]. \qquad (238)$$

Also, $-C_0 \leq V[T_V(\mathbb{R}_-^*)] \leq 0$ a.s. by ellipticity, thus

$$\mathsf{E}^z[V(T_V(\mathbb{R}_-^*))] \leq \mathsf{E}^z[V(T_V(\mathbb{R}_-^*))\mathbf{1}_{(\Xi_h)^c}] = \mathsf{E}^z[V(T_V(\mathbb{R}_-^*))] - \mathsf{E}^z[V(T_V(\mathbb{R}_-^*))\mathbf{1}_{\Xi_h}]$$

$$\leq \quad \mathsf{E}^z[V(T_V(\mathbb{R}^*_-))] + C_0\mathsf{P}^z(\Xi_h). \tag{239}$$

Hence, using first (238) and (237) and then the first inequality in (239),

$$h\mathsf{P}^z[\Xi_h] \leq z - \mathsf{E}^z[V(T_V(\mathbb{R}^*_-))\mathbf{1}_{(\Xi_h)^c}] \leq z - \mathsf{E}^z[V(T_V(\mathbb{R}^*_-))]. \tag{240}$$

Similarly,

$$(h + C_0)\mathsf{P}^z[\Xi_h] \geq z - \mathsf{E}^z[V(T_V(\mathbb{R}^*_-))\mathbf{1}_{(\Xi_h)^c}] \geq z - \mathsf{E}^z[V(T_V(\mathbb{R}^*_-))] - C_0\mathsf{P}^z(\Xi_h),$$

and so for large h for every $0 \leq z < h$, since $z + C_0 \geq z - \mathsf{E}^z[V(T_V(\mathbb{R}^*_-))] \geq z \geq 0$,

$$\begin{aligned}\mathsf{P}^z[\Xi_h] \quad &\geq \quad \frac{z - \mathsf{E}^z[V(T_V(\mathbb{R}^*_-))]}{h + 2C_0} \geq \frac{z - \mathsf{E}^z[V(T_V(\mathbb{R}^*_-))]}{h}\left(1 - \frac{3C_0}{h}\right) \\ &\geq \quad \frac{z - \mathsf{E}^z[V(T_V(\mathbb{R}^*_-))]}{h} - \frac{3C_0(z + C_0)}{h^2}.\end{aligned}$$

This and (240) prove (233). The proof of (234) is similar. We get (235) and (236) as a consequence. □

In order to apply the results of Vatutin et al. ([64], thm. 4 and 6), we introduce some of its notation (see its pages 177 and 179). Let $\chi^+ := V(\tau^+)$, where $\tau^+ := \min\{k \geq 1,\ V(k) > 0\} = T_V(\mathbb{R}^*_+)$, and χ^+_k, $k \geq 1$ be independent copies of χ^+. We can now define the (left-continuous) renewal function

$$H(u) := \mathbf{1}_{\{u>0\}} + \sum_{k=1}^{\infty} \mathsf{P}\big(\chi^+_1 + \cdots + \chi^+_k < u\big), \qquad u \in \mathbb{R}.$$

Also it is well known that $H(x) < \infty$ for every $x \in \mathbb{R}$ (see e.g. [64] Lem. 13).

As in [64] (page 180), we say that the random variable $\log \frac{1-\omega_0}{\omega_0}$ is (ℓ, a)-lattice if its distribution is lattice with span $\ell > 0$ and shift $a \in [0, \ell[$, that is, if ℓ is the maximal real number such that the support of the distribution of $\log \frac{1-\omega_0}{\omega_0}$ is included in the set $(a + \ell\mathbb{Z}) = \{a + k\ell, k \in \mathbb{Z}\}$. We say that the random variable $\log \frac{1-\omega_0}{\omega_0}$ is *non-lattice* if its distribution is not supported in $(a + \ell\mathbb{Z})$ for any $a \in \mathbb{R}$, $\ell > 0$. The two following lemmas are a bit more precise that what is needed in the present paper. They may be of independent interest and will be useful in a work in progress [23].

Lemma 7.5. *Assume that* $\log \frac{1-\omega_0}{\omega_0}$ *is non-lattice. We have for* $p \geq 0$,

$$\mathsf{E}\Big[\big(V(x)\big)^p e^{-V(x)} | G^*_x\Big] \sim_{x\to+\infty} \frac{f_2(p)}{x}, \qquad f_2(p) := \frac{1}{c^*_{35}\sigma\sqrt{2\pi}} \int_0^\infty u^p e^{-u} H(u)\mathrm{d}u \in]0, \infty[. \tag{241}$$

The case $p = 1$ was already proved in Afanasyev et al. ([1], Prop. 2.1) and Hirano ([42] Lemma 5) with different methods.

Proof of Lemma 7.5: We fix $p \geq 0$, and define $\beta_p := \sup_{y\geq 0}(y^p e^{-y/9}) \in]0, \infty[$. We first observe that for large x,

$$\mathsf{E}\big[\big(V(x)\big)^p e^{-V(x)}\mathbf{1}_{\{V(x)\geq 9\log x\}} | G^*_x\big] \leq \beta_p \mathsf{E}\big[e^{-8V(x)/9}\mathbf{1}_{\{V(x)\geq 9\log x\}} | G^*_x\big] \leq \beta_p x^{-8}. \tag{242}$$

Our potential V is a random walk with i.i.d. bounded, non constant and zero mean jumps ρ_x, $x \in \mathbb{Z}$ by (2), (3) (4) and (6), and by hypotheses, its jumps have a non lattice distribution. So we can use the following result ([64], Theorem 4 with $\alpha = 2$, $\beta = 0$ and $c_x \sim_{x\to+\infty} \sigma\sqrt{x}$, as seen in the line after its eq. (3)) and with $g_{2,0}(0) = 1/\sqrt{2\pi}$: for $\Delta > 0$,

$$\sigma\sqrt{x}\mathsf{P}\Big[V(x) \in [y, y + \Delta[\ \Big|\ G^*_x\Big] \sim_{x\to+\infty} \frac{1}{x\mathsf{P}[G^*_x]\sqrt{2\pi}} \int_y^{y+\Delta} H(u)\mathrm{d}u \tag{243}$$

uniformly in $y \in]0, \delta_x\sqrt{x}]$, where $\delta_x \to 0$ as $x \to +\infty$.

We prove that this convergence is in fact uniform in $y \in [0, \delta_x\sqrt{x}]$ as $x \to +\infty$. To this aim, notice that for fixed $x > 0$ and $\Delta > 0$, $\mathsf{P}\big[V(x) \in [y, y+\Delta[\,\big|\, G_x^*\big]$ tends to $\mathsf{P}\big[V(x) \in]0, \Delta] \,\big|\, G_x^*\big] = \mathsf{P}\big[V(x) \in [0, \Delta] \,\big|\, G_x^*\big]$ as $y \to 0$ with $y > 0$, since $\mathsf{P}\big[V(x) = 0 \,\big|\, G_x^*\big] = 0$ by definition (231) of G_x^*. Now, fix some $\varepsilon > 0$. Using the uniformity in $y \in]0, x^{1/4}]$ in (243), there exists $A_\varepsilon > 0$ such that for all $x > A_\varepsilon$, for all $y \in]0, x^{1/4}]$,

$$1 - \varepsilon \le \frac{x\mathsf{P}[G_x^*]\sigma\sqrt{x}\mathsf{P}\big[V(x) \in [y, y+\Delta[\,\big|\, G_x^*\big]}{\frac{1}{\sqrt{2\pi}}\int_y^{y+\Delta} H(u)\mathrm{d}u} \le 1 + \varepsilon. \tag{244}$$

Letting $y \downarrow 0$ in (244) for fixed $x > A_\varepsilon$ and using the convergence before (244), (244) remains true with $[y, y+\Delta[$ and $\int_y^{y+\Delta}$ replaced respectively by $[0, \Delta]$ and $\int_0^\Delta$. Hence,

$$x\mathsf{P}[G_x^*]\sigma\sqrt{x}\mathsf{P}\Big[V(x) \in [0, \Delta] \,\Big|\, G_x^*\Big] \to_{x\to+\infty} \frac{1}{\sqrt{2\pi}}\int_0^\Delta H(u)\mathrm{d}u. \tag{245}$$

Moreover, applying once more (243) with $[y, y+\Delta[$ replaced by $[\Delta - \eta, \Delta + \eta[$ for fixed Δ and $0 < \eta < \Delta$ gives, for large x, $x\mathsf{P}[G_x^*]\sigma\sqrt{x}\mathsf{P}\big[V(x) = \Delta \,\big|\, G_x^*\big] \le (2/\sqrt{2\pi})\int_{\Delta-\eta}^{\Delta+\eta} H(u)\mathrm{d}u \le (4/\sqrt{2\pi})H(2\Delta)\eta$. Since this is true for any $\eta > 0$, we get $x\mathsf{P}[G_x^*]\sigma\sqrt{x}\mathsf{P}\big[V(x) = \Delta \,\big|\, G_x^*\big] \to 0$ as $x \to +\infty$. So, (245) remains true with $[0, \Delta]$ replaced by $[0, \Delta[$. This and (243) prove that the convergence in (243) is in fact uniform in $y \in [0, \delta_x\sqrt{x}]$ as $x \to +\infty$, where $\delta_x \to 0$ as $x \to +\infty$.

So, we have for any $\varepsilon > 0$ and $\Delta > 0$, for large x,

$$\begin{aligned}
\mathsf{E}\Big[\frac{(V(x))^p}{e^{V(x)}}\mathbf{1}_{\{V(x)<9\log x\}} \,\Big|\, G_x^*\Big] &= \sum_{k=0}^{\infty} \mathsf{E}\Big[\frac{(V(x))^p}{e^{V(x)}}\mathbf{1}_{\{V(x)<9\log x\}}\mathbf{1}_{\{V(x)\in[k\Delta,(k+1)\Delta[\}} \mid G_x^*\Big] \\
&\le \sum_{k=0}^{\lfloor 9\Delta^{-1}\log x\rfloor} \frac{((k+1)\Delta)^p}{e^{k\Delta}}\mathsf{P}\big[V(x) \in [k\Delta, (k+1)\Delta[\,\big|\, G_x^*\big] \\
&\le \frac{(1+\varepsilon)}{\sigma x^{3/2}\mathsf{P}[G_x^*]\sqrt{2\pi}} \sum_{k=0}^{\lfloor 9\Delta^{-1}\log x\rfloor} \frac{((k+1)\Delta)^P}{e^{k\Delta}}\int_{k\Delta}^{(k+1)\Delta} H(u)\mathrm{d}u \\
&\le \frac{(1+\varepsilon)^2}{c_{35}^*\sigma\sqrt{2\pi}x} \sum_{k=0}^{\lfloor 9\Delta^{-1}\log x\rfloor} \frac{((k+1)\Delta)^p}{e^{k\Delta}}\Delta H\big[(k+1)\Delta\big],
\end{aligned}$$

where we used (232) and since H is nondecreasing. So,

$$\begin{aligned}
&\limsup_{x\to+\infty}\Big(x\mathsf{E}\big[\big(V(x)\big)^p e^{-V(x)}\mathbf{1}_{\{V(x)<9\log x\}} \mid G_x^*\big]\Big) \\
\le\ & \frac{(1+\varepsilon)^2}{c_{35}^*\sigma\sqrt{2\pi}}e^\Delta \sum_{k=0}^{\infty}\Delta\frac{((k+1)\Delta)^p}{e^{(k+1)\Delta}}H[(k+1)\Delta] \to_{\Delta\to0} \frac{(1+\varepsilon)^2}{c_{35}^*\sigma\sqrt{2\pi}}\int_0^\infty u^p e^{-u}H(u)\mathrm{d}u < \infty,
\end{aligned}$$

since H is a nondecreasing function and $H(x) = O(x^2)$ as $x \to +\infty$ e.g. by ([64] Lem. 13 with $\alpha = 2$ and $\rho = 1/2$ as explained at the end of its p. 181, following from Rogozin [57] and from the Spitzer-Ròzen theorem).

This, combined with (242) gives

$$\limsup_{x\to+\infty}\Big(x\mathsf{E}\big[\big(V(x)\big)^p e^{-V(x)}|G_x^*\big]\Big) \le \frac{1}{c_{35}^*\sigma\sqrt{2\pi}}\int_0^\infty u^p e^{-u}H(u)\mathrm{d}u.$$

Since we get a similar inequality for $\liminf$ and $H \ge 1$ on $]0, \infty[$, this proves (241) and the lemma. □

Lemma 7.6. *Assume that* $\log\frac{1-\omega_0}{\omega_0}$ *is* (h,a)*-lattice for some* $h>0$ *and* $a\in[0,h[$*. We have for* $p\geq 0$*,*

$$\mathsf{E}\big[\big(V(x)\big)^p e^{-V(x)}|G_x^*\big] \sim_{x\to+\infty} \frac{h}{c_{35}^*\sqrt{2\pi}\sigma x}\psi_p[(ax) \bmod h], \tag{246}$$

where $\psi_p(y) := \sum_{k\in\mathbb{N}}(y+kh)^p e^{-(y+kh)}H(y+kh)$*,* $y\in[0,h]$*, is a function bounded on* $[0,h]$ *between two (strictly) positive constants.*

Proof: Let $p\geq 0$, $h>0$ and $a\in[0,h[$, and assume that $\log\frac{1-\omega_0}{\omega_0}$ is (h,a)-lattice. First, notice that for every $y\in[0,h]$, $\psi_p(y)\leq\sum_{k\in\mathbb{N}}(h+kh)^p e^{-kh}H(h+kh) = e^h\psi_p(h)<\infty$ since H is nondecreasing and $H(x)=O(x^2)$ as $x\to+\infty$ as in the previous lemma. Moreover, taking into account only $k=1$, we have $\psi_p(y)\geq h^p e^{-2h}H(h)>0$ for every $y\in[0,h]$, so ψ_p is bounded on $[0,h]$ between two (strictly) positive constants.

Let $\varepsilon>0$. Applying ([64], Theorem 6, extending previous results obtained when $a=0$ by Alili and Doney [2]), again with $\alpha=2$, $\beta=0$, $c_x\sim_{x\to+\infty}\sigma\sqrt{x}$, and $g_{2,0}(0)=1/\sqrt{2\pi}$:

$$\sigma\sqrt{x}\mathsf{P}[V(x)=ax+y\mid G_x^*]\sim_{x\to+\infty}\frac{hH(ax+y)}{\sqrt{2\pi}x\mathsf{P}[G_x^*]} \tag{247}$$

uniformly in $y\in]-ax,-ax+\delta_x\sqrt{x}]\cap(h\mathbb{Z})$, where $\delta_x\to 0$ as $x\to+\infty$. Also, notice that for $y=-ax$ when $x>0$, we have $\mathsf{P}[V(x)=0\mid G_x^*]=0=hH(0)/[\sqrt{2\pi}x\mathsf{P}(G_x^*)\sigma\sqrt{x}]$ by definitions of G_x^* and H. Hence for large x,

$$\begin{aligned}
&\mathsf{E}\big[(V(x))^p e^{-V(x)}\mathbf{1}_{\{V(x)<9\log x\}}\,\big|\,G_x^*\big]\\
=&\sum_{k\in\mathbb{Z},\ ax+kh\geq 0}(ax+kh)^p e^{-(ax+kh)}\mathbf{1}_{\{ax+kh<9\log x\}}\mathsf{P}[V(x)=ax+kh\mid G_x^*]\\
\leq&\sum_{k\in\mathbb{Z},\ 0\leq ax+kh<9\log x}(ax+kh)^p e^{-(ax+kh)}(1+\varepsilon)\frac{hH(ax+kh)}{\sqrt{2\pi}\sigma x^{3/2}\mathsf{P}[G_x^*]}\\
\leq&\frac{(1+\varepsilon)h}{\sqrt{2\pi}\sigma x^{3/2}\mathsf{P}[G_x^*]}\psi_p[(ax) \bmod h]\leq\frac{(1+2\varepsilon)h}{c_{35}^*\sqrt{2\pi}\sigma x}\psi_p[(ax) \bmod h]
\end{aligned}$$

by (247) applied with $\delta_x=9(\log x)/\sqrt{x}$ and (232). This and (242) give for large x,

$$\mathsf{E}\big[\big(V(x)\big)^p e^{-V(x)}|G_x^*\big]\leq\frac{(1+2\varepsilon)h}{c_{35}^*\sqrt{2\pi}\sigma x}\psi_p[(ax) \bmod h]+\beta_p x^{-8}. \tag{248}$$

Similarly as in (248), for large x,

$$\mathsf{E}\big[\big(V(x)\big)^p e^{-V(x)}|G_x^*\big]\geq\frac{(1-2\varepsilon)h}{c_{35}^*\sqrt{2\pi}\sigma x}\Big(\psi_p[(ax) \bmod h]-O(x^{-8})\Big),$$

since $\sum_{k\in\mathbb{Z},\ ax+kh\geq 9\log x}(ax+kh)^p e^{-(ax+kh)}H(ax+kh)=O(x^{-8})$ as $x\to+\infty$ because $H(x)=O(x^2)$ as in the previous lemma. This and (248) prove (246) since $x^{-8}=o\big(\psi_p[(ax) \bmod h]/x\big)$ as $x\to+\infty$ because $\inf_{[0,h]}\psi_p>0$. □

Proof of Proposition 7.3: Let $h>0$ and $x\in\mathbb{N}^*$. We first provide a relation between conditioning by Ξ_h^* and by G_x^*. We have, due to the Markov property,

$$\begin{aligned}
&\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x<T_V(h)\}}\big|\Xi_h^*\big]=\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x<T_V(h)\}}\mathbf{1}_{\{T_V(h)<T_V^*(\mathbb{R}_-)\}}\big]/\mathsf{P}[\Xi_h^*]\\
=&\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{\forall 0<k\leq x,\ 0<V(k)<h\}}\mathbf{1}_{\forall k\in[x,T_V(h)],\ V(k)>0\}}\big]/\mathsf{P}[\Xi_h^*]\\
=&\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{\forall 0<k\leq x,\ 0<V(k)<h\}}\mathsf{P}^{V(x)}\big(T_V(h)<T_V^*(\mathbb{R}_-)\big)\big]/\mathsf{P}[\Xi_h^*].
\end{aligned}\tag{249}$$

Hence for large $h>0$, for every $x\in\mathbb{N}^*$, by (249) and Lemma 7.4 eq. (234),

$$\begin{aligned}\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x<T_V(h)\}}|\Xi_h^*\big] &\le \mathsf{E}\bigg[e^{-V(x)}\mathbf{1}_{\{\forall 0<k\le x,\ 0<V(k)\}}\frac{V(x)-\mathsf{E}^{V(x)}[V(T_V^*(\mathbb{R}_-))]}{h\mathsf{P}[\Xi_h^*]}\bigg]\\ &= \frac{\mathsf{P}[G_x^*]}{h\mathsf{P}[\Xi_h^*]}\mathsf{E}\Big[\big[V(x)-\mathsf{E}^{V(x)}[V(T_V^*(\mathbb{R}_-))]\big]e^{-V(x)}\,\Big|\,G_x^*\Big]. \qquad (250)\end{aligned}$$

Also,

$$\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x=T_V(h)\}}|\Xi_h^*\big]\le\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{V(x)\ge h\}}|\Xi_h^*\big]\le e^{-h}.$$

Let $\varepsilon>0$. By (250), (232) and (236), then by ellipticity (2), there exists $p_6>0$ and $p_7>0$ such that for $x\ge p_6$ and $h\ge p_7$,

$$\begin{aligned}\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x<T_V(h)\}}|\Xi_h^*\big] &\le \mathsf{E}\Big[\big[V(x)-\mathsf{E}^{V(x)}\big[V(T_V^*(\mathbb{R}_-))\big]\big]e^{-V(x)}|G_x^*\Big]\frac{(1+\varepsilon)c_{35}^*}{c_1^*x^{1/2}} \qquad (251)\\ &\le (1+\varepsilon)(c_{35}^*/c_1^*)x^{-1/2}\big[\mathsf{E}\big(V(x)e^{-V(x)}|G_x^*\big)+C_0\mathsf{E}\big(e^{-V(x)}|G_x^*\big)\big].\end{aligned}$$

Thanks to Lemmas 7.5 and 7.6, there exists $p_4>p_6$ such that, for $x\ge p_4$, for each $p\in\{0,1\}$, $\mathsf{E}\big[\big(V(x)\big)^pe^{-V(x)}|G_x^*\big]\le\frac{f_3(p)}{x}$, with $f_3(p):=2f_2(p)$ when $\log\frac{1-\omega_0}{\omega_0}$ is non lattice, and $f_3(p):=2h\sup_{[0,h]}\psi_p/(c_{35}^*\sqrt{2\pi}\sigma)$ if $\log\frac{1-\omega_0}{\omega_0}$ is (h,a)-lattice for some $h>0$ and $a\in[0,h[$. This together with (251) gives for $x\ge p_4$ and $h\ge p_7$,

$$\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x<T_V(h)\}}|\Xi_h^*\big]\ \le\ (1+\varepsilon)(c_{35}^*/c_1^*)\big[f_3(1)+C_0f_3(0)\big]x^{-3/2}. \qquad (252)$$

We now aim to prove a similar inequality, conditioning by Ξ_h instead of Ξ_h^*. There exists $c>0$ such that $\mathsf{P}[V(1)\in[c,2c]]>0$, thanks to (3) and (4). For such a (fixed) c, there exists $p_8\ge p_7$ such that for all $h\ge p_8$, we have $h/10>2c$, $\mathsf{P}(\Xi_{h+2c}^*)/\mathsf{P}(\Xi_h)\le 2c_1^*/c_1$ (by Lemma 7.4) and $\mathsf{P}\big[T_V(h/10)<T_V(]-\infty,-h/10])\big]\ge 1/3$ (e.g. by (17)). So with $\widetilde{V}_1(k):=V(k+1)-V(1)$, $k\ge0$, using the independence of $V(1)$ and $\widetilde{V}_1$, then the independence of $(V(u),\ u\le T_V(h))$ and $\widetilde{V}_2$, defined by $\widetilde{V}_2(k):=V[T_V(h)+k]-V[T_V(h)]$, $k\ge0$, we have for $h\ge p_8$ and for $x\ge p_4$,

$$\begin{aligned}&\mathsf{E}\big[e^{-V(x+1)}\mathbf{1}_{\{x+1<T_V(h+2c)\}}|\Xi_{h+2c}^*\big]=\mathsf{E}\big[e^{-V(x+1)}\mathbf{1}_{\{x+1<T_V(h+2c)\}}\mathbf{1}_{\Xi_{h+2c}^*}\big]/\mathsf{P}(\Xi_{h+2c}^*)\\ \ge\ &\mathsf{E}\big[e^{-V(1)-\widetilde{V}_1(x)}\mathbf{1}_{\{V(1)\in[c,2c]\}}\mathbf{1}_{\{x<T_{\widetilde{V}_1}(h)\}}\mathbf{1}_{\forall y\in[1,T_{\widetilde{V}_1}(h+2c)],\ \widetilde{V}_1(y)\ge0}\big]/\mathsf{P}(\Xi_{h+2c}^*)\\ \ge\ &\frac{e^{-2c}\mathsf{P}[V(1)\in[c,2c]]}{\mathsf{P}(\Xi_{h+2c}^*)}\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x<T_V(h)\}}\mathbf{1}_{\forall y\in[1,T_V(h+2c)],\ V(y)\ge0}\big]\\ \ge\ &\frac{e^{-2c}\mathsf{P}[V(1)\in[c,2c]]}{\mathsf{P}(\Xi_{h+2c}^*)}\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x<T_V(h)\}}\mathbf{1}_{\Xi_h}\mathbf{1}_{T_{\widetilde{V}_2}(h/10)<T_{\widetilde{V}_2}(]-\infty,-h/10])}\big]\\ \ge\ &\frac{\mathsf{P}[V(1)\in[c,2c]]\mathsf{P}(\Xi_h)}{e^{2c}\mathsf{P}(\Xi_{h+2c}^*)}\mathsf{E}\bigg[\frac{\mathbf{1}_{\{x<T_V(h)\}}}{e^{V(x)}}|\Xi_h\bigg]\mathsf{P}\bigg[T_V\Big(\frac{h}{10}\Big)<T_V\Big(\Big]-\infty,-\frac{h}{10}\Big]\Big)\bigg].\end{aligned}$$

So, using the definition of p_8 then (252), we get with $c_{36}:=6e^{2c}c_1^*/(c_1\mathsf{P}[V(1)\in[c,2c]])$, for every $x\ge p_4$ and $h\ge p_8$,

$$\mathsf{E}\big[e^{-V(x)}\mathbf{1}_{\{x<T_V(h)\}}\mid\Xi_h\big]\le c_{36}\mathsf{E}\big[e^{-V(x+1)}\mathbf{1}_{\{x+1<T_V(h+2c)\}}|\Xi_{h+2c}^*\big]\le c_{13}x^{-3/2}$$

for some constant $c_{13}>0$. This and (252) prove Proposition 7.3, up to a change of c_{13}. □

7.3. **Two lemmas about left h-extrema.** For the sake of completeness, we prove the two following lemmas. We recall that $\mathscr{V}$ is defined before (19).

Lemma 7.7. *Let $v\in\mathscr{V}$, and let $h>0$. The left (resp. right) h-minima and left (resp. right) h-maxima for v alternate.*

Proof: Assume that y_1 and y_2 are two left h-minima for v, with $y_1 < y_2$. It is enough to prove that there exists at least a left h-maximum for v between y_1 and y_2. By Definition 2.1, for each $j \in \{1,2\}$, there exists $\alpha_j < y_j < \beta_j$ such that $\min_{[\alpha_j, y_j-1]} v > v(y_j)$, $\min_{[y_j+1,\beta_j]} v \geq v(y_j)$, $v(\alpha_j) \geq v(y_j)+h$ and $v(\beta_j) \geq v(y_j)+h$. We define $x := \min\{u \geq y_1,\ v(u) = \max_{[y_1,\ y_2]} v\}$. The goal is to prove that x is a left h-maximum for v.

Assume that $y_2 \leq \beta_1$. If $\alpha_2 \leq y_1$, then $\alpha_2 \leq y_1 < y_2 \leq \beta_1$, so $v(y_2) \geq \min_{[y_1+1,\beta_1]} v \geq v(y_1)$ and $v(y_1) \geq \min_{[\alpha_2, y_2-1]} v > v(y_2)$, which contradicts $v(y_2) \geq v(y_1)$. So $\alpha_2 > y_1$, thus $y_1 < \alpha_2 < y_2 \leq \beta_1$. We have $v(x) = \max_{[y_1,\ y_2]} v \geq v(\alpha_2) \geq v(y_2) + h$ and $v(x) \geq v(y_2) + h \geq \min_{[y_1+1,\beta_1]} v + h \geq v(y_1) + h$.

Now, assume that $y_2 > \beta_1$ and $\alpha_2 \leq y_1$. Thus, $\alpha_2 \leq y_1 < \beta_1 < y_2$, so $v(y_1) \geq \min_{[\alpha_2, y_2-1]} v > v(y_2)$. We have $v(x) = \max_{[y_1,y_2]} v \geq v(\beta_1) \geq v(y_1) + h \geq v(y_2) + h$.

Finally, assume that $y_2 > \beta_1$ and $\alpha_2 > y_1$. Hence, $y_1 < \beta_1 < y_2$, so $v(x) = \max_{[y_1,y_2]} v \geq v(\beta_1) \geq v(y_1) + h$. Also, $y_1 < \alpha_2 < y_2$, so $v(x) = \max_{[y_1,y_2]} v \geq v(\alpha_2) \geq v(y_2) + h$.

So in every case, we have $v(x) \geq v(y_1) + h$ and $v(x) \geq v(y_2) + h$, with $h > 0$, thus by definition of x, $y_1 < x < y_2$, $\max_{[y_1,x-1]} v < v(x)$ and $\max_{[x+1,y_2]} v \leq v(x)$, so x is a left h-maximum for v such that $y_1 < x < y_2$.

Applying this to $-v$ proves that, if y_1 and y_2 are two left h-maxima for v with $y_1 < y_2$, there exists at least a left h-minimum for v between y_1 and y_2, which concludes the proof of the lemma for left h-extrema. The proof is similar for right ones by symmetry. □

For the following lemma, see definitions (20)–(23), represented in Figure 2.

Lemma 7.8. *Assume that $V \in \mathscr{V}$ (which has probability one if* (2)*,* (3) *and* (4) *are satisfied).* **(i)** *For $i \geq 1$, $m^{(V)}_{2i+1}(h)$ is a left h-minimum for V, and there is no other left h-extremum for V in $\big[\tau^{(V)}_{2i}(h), \tau^{(V)}_{2i+1}(h)\big[$.* **(ii)** *For $i \geq 0$, $m^{(V)}_{2i+2}(h)$ is a left h-maximum for V, and there is no other left h-extremum for V in $\big[\tau^{(V)}_{2i+1}(h), \tau^{(V)}_{2i+2}(h)\big[$.*

Proof: Let $i \geq 1$. First, $m^{(V)}_{2i}(h) < m^{(V)}_{2i+1}(h) < \tau^{(V)}_{2i+1}(h)$ by definition. We also have $V\big(\tau^{(V)}_{2i+1}(h)\big) \geq V\big(m^{(V)}_{2i+1}(h)\big) + h$ by (20) and (21) and $V\big(m^{(V)}_{2i}(h)\big) \geq V\big(\tau^{(V)}_{2i}(h)\big) + h \geq V\big(m^{(V)}_{2i+1}(h)\big) + h$ since $i \geq 1$ by (23), (22) and (21). Also, $\min_{[m^{(V)}_{2i+1}(h)+1, \tau^{(V)}_{2i+1}(h)]} V \geq V\big(m^{(V)}_{2i+1}(h)\big)$ by (21), $\min_{[m^{(V)}_{2i}(h), \tau^{(V)}_{2i}(h)-1]} V > V\big(\tau^{(V)}_{2i}(h)\big) \geq V\big(m^{(V)}_{2i+1}(h)\big)$ by (22), (23) and (21), and $\min_{[\tau^{(V)}_{2i}(h), m^{(V)}_{2i+1}(h)-1]} V > V\big(m^{(V)}_{2i+1}(h)\big)$ by (21). So, $m^{(V)}_{2i+1}(h)$ is a left h-minimum for V.

First case: Assume that there exists a left h-minimum $y \neq m^{(V)}_{2i+1}(h)$ for V in $\big[\tau^{(V)}_{2i}(h), \tau^{(V)}_{2i+1}(h)\big[$, and let $\alpha < y$ and $\beta > y$ be as in Definition 2.1 with $v = V$. Assume first that $y \in \big[\tau^{(V)}_{2i}(h), m^{(V)}_{2i+1}(h)\big[$. If $\beta < m^{(V)}_{2i+1}(h)$, then $V(\beta) \geq V(y) + h$ with $\tau^{(V)}_{2i}(h) \leq y < \beta < \tau^{(V)}_{2i+1}(h)$, which contradicts the definition of $\tau^{(V)}_{2i+1}(h)$. If $\beta \geq m^{(V)}_{2i+1}(h)$, then $y + 1 \leq m^{(V)}_{2i+1}(h) \leq \beta$ so $V\big(m^{(V)}_{2i+1}(h)\big) \geq \min_{[y+1,\beta]} V \geq V(y)$, which contradicts $V(y) \geq \min_{[\tau^{(V)}_{2i}(h), m^{(V)}_{2i+1}(h)-1]} V > V\big(m^{(V)}_{2i+1}(h)\big)$ by (21).

So $y \in \big]m^{(V)}_{2i+1}(h), \tau^{(V)}_{2i+1}(h)\big[$. If $\alpha > m^{(V)}_{2i+1}(h)$, then $V(\alpha) \geq V(y) + h \geq V\big(m^{(V)}_{2i+1}(h)\big) + h$ with $\tau^{(V)}_{2i}(h) \leq m^{(V)}_{2i+1}(h) < \alpha < \tau^{(V)}_{2i+1}(h)$, which contradicts the definition of $\tau^{(V)}_{2i+1}(h)$. If $\alpha \leq m^{(V)}_{2i+1}(h)$, then, since $y > m^{(V)}_{2i+1}(h)$, we have $V\big(m^{(V)}_{2i+1}(h)\big) \geq \min_{[\alpha, y-1]} V > V(y)$ by

definition of α, which contradicts $V(y) \geq \min_{[\tau_{2i}^{(V)}(h),\tau_{2i+1}^{(V)}(h)]} V = V\big(m_{2i+1}^{(V)}(h)\big)$ by (21). So there is no left h-minimum for V in $\big[\tau_{2i}^{(V)}(h), \tau_{2i+1}^{(V)}(h)\big[- \big\{m_{2i+1}^{(V)}(h)\big\}$.

Second case: Now, we assume that there exists a left h-maximum y for V in $\big[\tau_{2i}^{(V)}(h), \tau_{2i+1}^{(V)}(h)\big[$, and let $\alpha < y$ and $\beta > y$ be as in Definition 2.1 for left h-maxima. If $\alpha \geq \tau_{2i}^{(V)}(h)$, then $V(y) \geq V(\alpha) + h$ with $\tau_{2i}^{(V)}(h) \leq \alpha < y < \tau_{2i+1}^{(V)}(h)$, which contradicts the definition of $\tau_{2i+1}^{(V)}(h)$. If $\alpha < \tau_{2i}^{(V)}(h)$, then $V(\alpha) \leq V(y) - h < V\big(\tau_{2i}^{(V)}(h)\big)$ by definition of $\tau_{2i+1}^{(V)}(h)$ since $\tau_{2i}^{(V)}(h) \leq y < \tau_{2i+1}^{(V)}(h)$. So if $m_{2i}^{(V)}(h) \leq \alpha < \tau_{2i}^{(V)}(h)$, then $V(\alpha) < V\big(\tau_{2i}^{(V)}(h)\big)$ contradicts $V\big(\tau_{2i}^{(V)}(h)\big) < \min_{[m_{2i}^{(V)}(h),\tau_{2i}^{(V)}(h)[} V \leq V(\alpha)$, coming from (22) and (23) since $i \geq 1$. Finally if $\alpha < m_{2i}^{(V)}(h)$, then $\alpha < m_{2i}^{(V)}(h) < \tau_{2i}^{(V)}(h) \leq y$ by (22) and (23) since $i \geq 1$, so $V\big(m_{2i}^{(V)}(h)\big) \leq \max_{[\alpha,y-1]} V < V(y) < V\big(\tau_{2i}^{(V)}(h)\big)+h$ by definition of α and (20) since $y \in \big[\tau_{2i}^{(V)}(h), \tau_{2i+1}^{(V)}(h)\big[$, which contradicts $V\big(\tau_{2i}^{(V)}(h)\big) \leq V\big(m_{2i}^{(V)}(h)\big) - h$ coming from (22) and (23) since $i \geq 1$. So there is no left h-maximum for V in $\big[\tau_{2i}^{(V)}(h), \tau_{2i+1}^{(V)}(h)\big[$ for $i \geq 1$.

Thus **(i)** is proved. The proof of **(ii)** is similar. □

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