On the uniform observability of the relative pose estimation problem using bearing measurements and epipolar constraints

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***Abstract*— This paper proposes a comprehensive observability analysis of the relative pose estimation of a monocular camera (moving in three-dimensional space) from bearing measurements and epipolar constraints. It extends our previous work on observer design for the particular case of 3-source points with unknown 3D coordinates. The paper addresses the observability analysis of the more general case of n-source points ($n \geq 3$) using persistence of excitation of the translational motion and bearing references (or equivalently, the position of the origin of the reference frame with respect to the source points). The key contribution of this work is to show that the persistence of excitation is not enough to guarantee uniform observability. In particular, we show that uniform observability also depends on bearing references and the number of observed source points.**

I. Introduction

Robust and efficient pose (i.e., position and orientation) estimation is a key requirement for autonomous navigation of robotic vehicles. The full pose estimation problem has constituted a fruitful source of inspiration for researchers to develop recent advanced theories of invariant/equivariant observers on Lie groups [1], [5], [21], invariant extended Kalman filters (IEKF) [4] and deterministic Riccati observers [9]. The inherent nonlinearity of the state space of a moving rigid body, essentially due to its orientation evolving in the compact Lie group $SO(3)$, which is itself encoded in the larger group $SE(3)$, impedes the design of any continuous observer endowed with *global* asymptotic stability property.

The complexities of pose observer design and the associated analysis are also strongly related to the nature of measurements. When the pose is directly measured, the problem is relatively simple and has been addressed in [3] with solutions guaranteeing almost global stability. The estimation of the relative pose of a moving 'stereo' camera using position measurements of known source points (or landmarks), complemented with proprioceptive sensors measurements, has been carefully investigated in several studies [18], [19], [24], [25]. Observability and stability analysis of all these works concludes that the measurements of at least three non-aligned source points are required to ensure the 'instantaneous observability' of the pose (i.e., the pose can be reconstructed or estimated on a frame-by-frame basis). On the other hand, relative pose estimation of a moving 'monocular' camera by exploiting *bearing* measurements of a set of source points whose coordinates in the reference frame are either known (cf. the classical Perspective-n-Point problem) or unknown (cf. the essential matrix estimation problem) is significantly more challenging than the previously mentioned cases. Several approaches have been opted for dealing with these problems, encompassing algebraic algorithms [6], [22], iterative algorithms based on gradient search [11], nonlinear optimization algorithms [14], [20], [23], EKF algorithms [2], and more recently deterministic Riccati observers [9].

Early studies on the 'static' perspective pose estimation problem have pointed out instantaneous observability conditions associated with the number and disposition of the source points [8]. For instance, classical algorithms resolving the 'static' PnP problem require at least three non-aligned source points, which must not belong to the so-called 'danger cylinder' [10]. The cases of four and more non-aligned source points have also been extensively investigated [7], [16]. More recently, by introducing a novel deterministic Riccati observer framework [9], and by specifying 'uniform observability' conditions guaranteeing the good conditioning of the solutions to the continuous Riccati equation involved in the designed observer (and consequently its robustness and efficiency), the authors have shown that exploiting body motion characteristics through the explicit use of measured velocities allows for enlarging the estimation possibilities. In particular, when the translational velocity is measured in the inertial frame, the observation of a single source point would be enough to estimate the pose, provided that the body motion is sufficiently exciting to guarantee uniform observability. In such a case, all algebraic approaches fail to provide a valid solution.

The present paper investigates the pose estimation problem from bearing measurements of *unknown* source points. Alike the PnP problem, classical approaches (algebraic, iterative, optimization) have been employed to tackle this problem on a frame-by-frame basis without exploiting the rigid body motion. These classical algorithms resolve, from the epipolar constraints of feature correspondences, the so-called 'essential matrix' [15], which can then be decomposed into a rotation matrix and a normalized translation vector [13]. Noticeable algorithms are the eight-point algorithm [12], [15] and Nister's algorithm [22] that calculate the essential matrix from eight and five-point correspondences, respectively. To the best of our knowledge, the five-point algorithm is the algorithm that requires the lowest number of non-aligned source points to ensure the *instantaneous observability* of the

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normalized pose (i.e., normalized position and orientation). In our prior work [17], we showed that uniform observability could be ensured with only three non-aligned source points. Similarly to [9], this remarkable fact is granted by the persistence of excitation condition characterized by the body translational motion and the configuration of the observed source points.

In the present work, a deeper understanding of the uniform (and non-uniform) observability of the Riccati observer proposed in [17], now adapted to an arbitrary number of source points, is investigated. In particular, in the case of three unknown source points considered in [17], we show hereafter that uniform observability is ensured not only by the persistence of excitation condition involving the translational body motion (already pointed out in [17]) but also by the position of the origin of the reference frame with respect to the observed source points.

Other thoughtful analyses on uniform observability singularities associated with the body motion (cf. tridimensional, planar, or unidirectional motions), the number, and the configuration of the source points are also presented. The remainder of the paper is organized as follows. Notations used throughout the paper, recalls on the uniform observability condition, and the Riccati observer framework [9] and its properties are presented in Section II. The estimation problem considered along with the observer presented in [17] are recalled in Section III. A comprehensive observability analysis relying on uniform observability conditions of the system is presented in Section IV. Simulation results that illustrate the relationship between the uniform observability and exponential convergence of the proposed Riccati observers are provided in Section V. Finally, concluding remarks are drawn in Section VI.

II. Preliminary material

A. Mathematical notations

The following notation will be used throughout the paper:

- The identity matrix and the null matrix of $\mathbb{R}^{n\times n}$ are respectively denoted I_n and 0_n. $0_{m,n}$ denotes the null matrix of size $m \times n$.
- $|x|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$.
- $x^\top$ is the transpose of the vector x, $A^\top$ is the transpose of the matrix A.
- The closed ball in $\mathbb{R}^n$ of radius r is denoted as $\mathbb{B}_r^n$.
- $S^n := \{x \in \mathbb{R}^{n+1}/|x| = 1\}$ is the n-dimensional sphere of radius equal to one.
- The Special orthogonal group of order n is denoted $SO(n) := \{R \in \mathbb{R}^{n\times n}/det(R) = 1, RR^\top = R^\top R = I_n\}$.
- $x_\times$ is the skew-symmetric matrix associated with x, i.e. $\forall x, y \in \mathbb{R}^3\ x_\times y = x \times y$.
- $\pi_x := I_3 - xx^\top = -{x_\times}^2$ with $x \in S^2$ is the orthogonal projection operator in $\mathbb{R}^3$ onto the two-dimensional vector subspace orthogonal to x.
- With f denoting a vector-valued function depending on x and y, and on the time t, we write $f = O(|x|^{k_1}|y|^{k_2})$ with $k_1 \geq 0$ and $k_2 \geq 0$ if there exists $\gamma < \infty$ s.t. for all $t \geq 0$: $|f(x, y, t)|/(|x|^{k_1}|y|^{k_2}) \leq \gamma$ in the neighbourhood of $(x, y) = (0, 0)$. If f only depends on x and t then we write $f = O(|x|^{k_1}|y|)$ if there exists $\gamma < \infty$ s.t. for all $t \geq 0$: $|f(x, t)|/(|x|^{k_1}) \leq \gamma$ in the neighbourhood of $x = 0$.

B. Uniform observability

Consider the following linear time-varying system

$$\begin{cases} \dot{x} = \bar{A}(t)x + \bar{B}(t)u \\ y = \bar{C}(t)x \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^l$, $y \in \mathbb{R}^m$ and $\bar{A}(t)$, $\bar{B}(t)$, $\bar{C}(t)$ denoting continuous matrix-valued functions with adequate dimensions.

Definition 1 (Uniform Observability) *The pair $(\bar{A}(t), \bar{C}(t))$ is called* uniformly observable *if there exist $\delta, \mu > 0$ such that for all $t \geq 0$:*

$$W(t, \delta) := \frac{1}{\delta}\int_t^{t+\delta} \Phi^\top(s, t)\bar{C}^\top(s)\bar{C}(s)\Phi(s, t)\,\mathrm{d}s \geq \mu I_n \tag{1}$$

with $\Phi(s, t)$ the transition matrix associated with $\bar{A}$, i.e. such that $\frac{d}{ds}\Phi(s, t) = \bar{A}(s)\Phi(s, t)$ with $\Phi(t, t) = I_n$.

Matrix $W(t, \delta)$ is the so-called observability Gramian *associated with the pair $(\bar{A}(t), \bar{C}(t))$.*

C. Riccati observer

The observer studied in this paper is based on the Riccati observer framework developed in [9]. Here is a short recall.

Consider a class of nonlinear systems whose state $x = [x_1^\top, x_2^\top]^\top$ lives in $\mathbb{B}_r^{n_1} \times \mathbb{R}^{n_2}$ and whose output y lives in $\mathbb{R}^m$

$$\begin{cases} \dot{x} = A(t)x + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + O(|x_1|^2) + O(|x_1||u_1|) \\ y = C_1(x, t)x_1 + C_2(x, t)x_2 + O(|x_1|^2) + O(|x_1||x_2|) \end{cases} \tag{2}$$

where $A(t)$ is a continuous matrix-valued function uniformly bounded w.r.t. t of the form

$$A(t) = \begin{bmatrix} A_{11}(t) & 0_{n_1\times n_2} \\ A_{21}(t) & A_{22}(t) \end{bmatrix}$$

and $C := [C_1, C_2] \in \mathbb{R}^{m\times(n_1+n_2)}$ is a continuous matrix-valued function uniformly bounded w.r.t. t and uniformly continuous w.r.t. x.

Then apply the input $u = -Ky$ with $K = [K_1^\top, K_2^\top]^\top := PC^\top Q(t)$ and $P \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ a symmetric positive definite matrix solution the Continuous Riccati Equation (CRE)

$$\dot{P} = AP + PA^\top - PC^\top Q(t)CP + V(t)$$

with $P(0) \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ a symmetric positive definite matrix and Q and V bounded continuous symmetric positive semi-definite matrix-valued functions.

Then the Corollary 3.2 in [9] shows that the equilibrium $x = 0$ is locally uniformly exponentially stable when $Q(t)$ and $V(t)$ are both larger than some positive matrix

and the pair $(A^\star(t), C^\star(t)) := (A(t), C(0,t))$ is uniformly observable.

III. Problem statement

A. Kinematics definitions

Let $\mathcal{A}$ be the moving camera-fixed frame and let $\mathring{\mathcal{A}}$ be the reference configuration of this frame. The relative orientation of $\mathcal{A}$ with respect to $\mathring{\mathcal{A}}$ is denoted $R \in SO(3)$ and the relative position of $\mathcal{A}$ with respect to $\mathring{\mathcal{A}}$, expressed in $\mathcal{A}$ (resp. in $\mathring{\mathcal{A}}$) is denoted $\xi \in \mathbb{R}^3$ (resp. $\mathring{\xi}$). One verifies that $\mathring{\xi} = R\xi$. The vector $\Omega \in \mathbb{R}^3$ denotes the instantaneous angular velocity of $\mathcal{A}$ w.r.t. $\mathring{\mathcal{A}}$ expressed in $\mathcal{A}$ and the vector $v \in \mathbb{R}^3$ represents the linear velocity of the camera origin expressed in $\mathcal{A}$. Both velocities are assumed to be measured using gyrometers and a Doppler sensor respectively.

The dynamics of the camera pose (R, ξ) are given by

$$\begin{cases} \dot{R} = R\Omega_\times \\ \dot{\xi} = -\Omega_\times \xi + v \end{cases} \tag{3}$$

The problem considered focuses on the pose estimation of a monocular camera observing N fixed unknown source points from bearing measures. Let p_i^p (resp. $\mathring{p}_i^p$) denote the calibrated projective coordinates of the ith source points on the camera plane w.r.t. the frame $\mathcal{A}$ (resp. $\mathring{\mathcal{A}}$). Define P_i (resp. $\mathring{P}_i$) denotes the 3D coordinates of the ith source point w.r.t. the frame $\mathcal{A}$ (resp. $\mathring{\mathcal{A}}$). If z_i (resp. $\mathring{z}_i$) denotes the third component of P_i (resp. $\mathring{P}_i$), one verifies $p_i^p = P_i/z_i$ (resp. $\mathring{p}_i^p = \mathring{P}_i/\mathring{z}_i$).

Rather than using the perspective outputs p_i^p commonly used in computer vision algorithms, we use bearing measurements

$$\begin{cases} p_i := P_i/|P_i| = p_i^p/|p_i^p| \in S^2 \\ \mathring{p}_i := \mathring{P}_i/|\mathring{P}_i| = \mathring{p}_i^p/|\mathring{p}_i^p| \in S^2 \end{cases}$$

corresponding to projection onto a virtual spherical image and differing from perspective outputs only by scaling. Figure 1 shows a 3D representation of $\mathring{P}_i$, P_i, p_i, p_i^p and ξ.

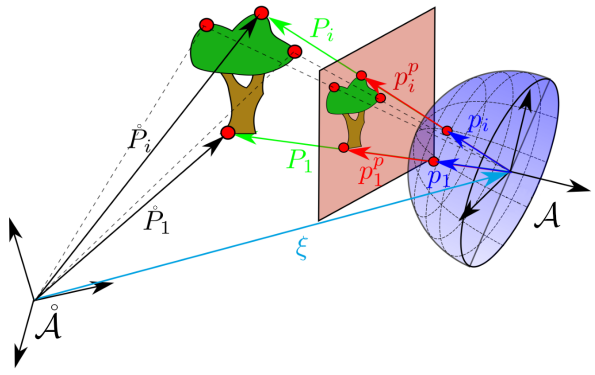

Fig. 1. Intuitive representation of inertial coordinates $\mathring{P}_i$, planar projective coordinates p_i^p and spherical projective coordinates p_i of the ith source point.

Using the relation $P_i = R^\top \mathring{P}_i - \xi$, one deduces for each source points i the following *epipolar* constraint

$$\mathring{p}_i^\top R \xi_\times p_i = 0 \tag{4}$$

In computer vision literature [15], this constraint is usually presented in the form $\mathring{p}_i^\top E p_i = 0$, where $E := R(\xi/|\xi|)_\times$ is the so-called *essential* matrix.

B. Observer design

Let $(\hat{R}, \hat{\xi})$ denote an estimate of the pose (R, ξ). The proposed observer has the following form

$$\begin{cases} \dot{\hat{R}} = \hat{R}\Omega_\times - \hat{R}\sigma_{R\times} \\ \dot{\hat{\xi}} = -\Omega_\times \hat{\xi} + v - \sigma_\xi \end{cases} \tag{5}$$

with initial conditions $\hat{R}(0) \in SO(3)$ and $\hat{\xi}(0) \in \mathbb{R}^3$, and with $\sigma_R, \sigma_\xi \in \mathbb{R}^3$ innovation terms to be designed.

Define the attitude error matrix $\tilde{R} := \hat{R}^\top R$ and the position error vector $\tilde{\xi} := \xi - \hat{\xi}$. From (3) and (5), one deduces the dynamics of the attitude error

$$\dot{\tilde{R}} = -\Omega_\times \tilde{R} + \tilde{R}\Omega_\times + \sigma_{R\times}\tilde{R}$$

From the Rodrigues' formula, the first order approximation of $\tilde{R}$ is given by

$$\tilde{R} = I_3 + \tilde{\lambda}_\times + O(|\tilde{\lambda}|^2)$$

with $\tilde{\lambda} \in \mathbb{B}_2^3$ is equal to twice the vector part of the quaternion associated with the attitude error matrix $\tilde{R}$. One deduces from the two previous relation and from the identity $\forall a, b \in \mathbb{R}^3,\ a_\times b_\times - b_\times a_\times = (a_\times b)_\times$ that

$$\dot{\tilde{\lambda}} = -\Omega_\times \tilde{\lambda} + \sigma_R + O(|\tilde{\lambda}||\sigma_R|) + O(|\tilde{\lambda}|^2) \tag{6}$$

For the position error, one verifies

$$\dot{\tilde{\xi}} = -\Omega_\times \tilde{\xi} + \sigma_\xi \tag{7}$$

Consider now the epipolar constraint (4) and use the fact that $R = \hat{R}\tilde{R}$ and $\xi = \hat{\xi} + \tilde{\xi}$ along with a first order approximation to derive the output equation

$$\begin{aligned} 0 &= \mathring{p}_i^\top \hat{R}\tilde{R}(\hat{\xi} + \tilde{\xi})_\times p_i \\ &= \mathring{p}_i^\top \hat{R}(I_3 + \tilde{\lambda}_\times)(\hat{\xi} + \tilde{\xi})_\times p_i + O(|\tilde{\lambda}|^2) \\ &= \mathring{p}_i^\top \hat{R}\hat{\xi}_\times p_i + \mathring{p}_i^\top \hat{R}\tilde{\lambda}_\times \hat{\xi}_\times p_i + \mathring{p}_i^\top \hat{R}\tilde{\xi}_\times p_i \\ &\qquad + O(|\tilde{\lambda}||\tilde{\xi}|) + O(|\tilde{\lambda}|^2) \end{aligned}$$

By setting $y_i = \mathring{p}_i^\top \hat{R}\hat{\xi}_\times p_i$, one gets

$$y_i = \mathring{p}_i^\top \hat{R}(\hat{\xi}_\times p_i)_\times \tilde{\lambda} + \mathring{p}_i^\top \hat{R} p_{i\times} \tilde{\xi} + O(|\tilde{\lambda}||\tilde{\xi}|) + O(|\tilde{\lambda}|^2) \tag{8}$$

From the previous equations (5)-(8), one obtains the compact form of the Riccati observer (2) with

$$\begin{cases} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} \tilde{\lambda} \\ \tilde{\xi} \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} \sigma_R \\ \sigma_\xi \end{bmatrix}, \\ A := \begin{bmatrix} -\Omega_\times & 0_3 \\ 0_3 & -\Omega_\times \end{bmatrix}, \; y := \begin{bmatrix} \mathring{p}_1^\top \hat{R} \hat{\xi}_\times p_1 \\ \vdots \\ \mathring{p}_N^\top \hat{R} \hat{\xi}_\times p_N \end{bmatrix}, \\ C_1 := \begin{bmatrix} \mathring{p}_1^\top \hat{R} (\hat{\xi} \times p_1)_\times \\ \vdots \\ \mathring{p}_N^\top \hat{R} (\hat{\xi} \times p_N)_\times \end{bmatrix}, \; C_2 := \begin{bmatrix} \mathring{p}_1^\top \hat{R} p_{1\times} \\ \vdots \\ \mathring{p}_N^\top \hat{R} p_{N\times} \end{bmatrix} \end{cases}$$

Innovation terms introduced in (5) can then be deduced: $\sigma_R = -K_1 y$ and $\sigma_\xi = -K_2 y$.

IV. Observability analysis

Theorem 3.1 and Corollary 3.2 in [9] emphasize that if both matrices $Q(t)$ and $V(t)$ are larger that some positive matrix, the uniform observability of the pair $(A^\star(t), C^\star(t)) := (A(0,t), C(0,t))$ is a sufficient condition for $x = 0$ to be locally exponentially stable.

One verifies that $A^\star(t) = A(t)$ and

$$C^\star(t) = C(0,t) = \begin{bmatrix} |\mathring{P}_1| \frac{\mathring{\xi}^\top}{|\mathring{P}_1|} \pi_{\mathring{p}_1} R & \frac{\mathring{\xi}^\top}{|\mathring{P}_1|} \mathring{p}_{1\times} R \\ \vdots & \vdots \\ |\mathring{P}_N| \frac{\mathring{\xi}^\top}{|\mathring{P}_N|} \pi_{\mathring{p}_N} R & \frac{\mathring{\xi}^\top}{|\mathring{P}_N|} \mathring{p}_{N\times} R \end{bmatrix} \quad (9)$$

The proof of Theorem 1 in [17] gives an expression of the observability Gramian (1) of the pair $(A^\star(t), C^\star(t))$

$$W(t,\delta) = \begin{bmatrix} R^\top(t) & 0_3 \\ 0_3 & R^\top(t) \end{bmatrix} H(t,\delta) \begin{bmatrix} R(t) & 0_3 \\ 0_3 & R(t) \end{bmatrix} \quad (10)$$

with

$$H(t,\delta) := \frac{1}{\delta} \int_t^{t+\delta} M(s) \, \mathrm{d}s \quad (11)$$

and

$$M(s) := \sum_{i=1}^{N} \begin{bmatrix} |\mathring{P}_i|^2 \pi_{\mathring{p}_i} \frac{\mathring{\xi}\mathring{\xi}^\top}{|\mathring{P}_i|^2} \pi_{\mathring{p}_i} & |\mathring{P}_i| \pi_{\mathring{p}_i} \frac{\mathring{\xi}\mathring{\xi}^\top}{|\mathring{P}_i|^2} \mathring{p}_{i\times} \\ -|\mathring{P}_i| \mathring{p}_{i\times} \frac{\mathring{\xi}\mathring{\xi}^\top}{|\mathring{P}_i|^2} \pi_{\mathring{p}_i} & -\mathring{p}_{i\times} \frac{\mathring{\xi}\mathring{\xi}^\top}{|\mathring{P}_i|^2} \mathring{p}_{i\times} \end{bmatrix}$$

It is straightforward to verify that the matrix $H(t,\delta)$ can be decomposed as follows

$$H(t,\delta) = \sum_{i=1}^{N} Z_i^\top(t,\delta) Z_i(t,\delta) \quad (12)$$

with

$$Z_i(t,\delta) := \Pi_i(t,\delta)^{\frac{1}{2}} \mathring{p}_{i\times} \begin{bmatrix} -|\mathring{P}_i| \mathring{p}_{i\times} & I_3 \end{bmatrix}$$

and

$$\Pi_i(t,\delta) := \frac{1}{\delta} \int_t^{t+\delta} \frac{\mathring{\xi}(s)\mathring{\xi}^\top(s)}{|\mathring{P}_i - \mathring{\xi}(s)|^2} \, \mathrm{d}s \quad (13)$$

The following definitions are useful to characterize the uniform observability analyses depending on the chosen origin of the reference frame $\mathring{\mathcal{A}}$ with respect to the observed source points.

Definition 2 (Danger cylinder) *A danger cylinder is a circular cylinder generated by the circle passing through three source points whose axis is orthogonal to the plane containing the source points. We said that a 3D point $P = (x, y, z)$ is on the danger cylinder if it belongs to the surface of the danger cylinder.*

Definition 3 (Horopter curve) *A horopter curve is the intersection of a circular cylinder and an elliptic cone. A 3D point $P = (x, y, z)$ belongs to a horopter if*

$$\begin{cases} y^2 + z^2 - az = 0 \\ k(y^2 + z^2) - axy = 0 \end{cases}$$

with $a, k \in \mathbb{R}$ the parameters of the horopter curve $\mathcal{H}_{a,k}$.

Proposition 1 *$(A^\star(t), C^\star(t))$ is not uniformly observable if one the following conditions is fulfilled:*

- *The number of source points is less or equal to two ($N \leq 2$).*
- *All source points are aligned.*
- *There are three non-aligned source points ($N = 3$) and the origin of the reference frame $\mathring{\mathcal{A}}$ belongs to the* danger cylinder *generated by the 3 source points.*
- *There are at least four non-aligned source points ($N \geq 4$) and they are located on a* horopter curve *whose origin corresponds to the origin of the reference frame $\mathring{\mathcal{A}}$.*

Proof: It suffices to show that for all the above cases, the observability Gramian (10) (or equivalently (11)) cannot be positive on all time interval. That is, there exists a non-zero vector $\nu \in \mathbb{R}^6$ such that for all $t, \delta \geq 0$

$$\begin{aligned} \forall i \; & Z_i(t,\delta)\nu = 0 \\ \forall i \; & \Pi_i(t,\delta)^{\frac{1}{2}} \mathring{p}_{i\times} \begin{bmatrix} |\mathring{P}_i| \mathring{p}_{i\times} & -I_3 \end{bmatrix} \nu = 0 \end{aligned} \quad (14)$$

then the pair $(A^\star, C^\star)$ is not uniformly observable.

It remains to show that there exists a non-zero vector $\nu \in \mathbb{R}^6$ such that

$$\forall i \; \mathring{p}_{i\times} \begin{bmatrix} |\mathring{P}_i| \mathring{p}_{i\times} & -I_3 \end{bmatrix} \nu = 0$$

which is equivalent to

$$\forall i \; \pi_{\frac{\mathring{P}_i}{|\mathring{P}_i|}} \begin{bmatrix} \mathring{P}_{i\times} & -I_3 \end{bmatrix} \nu = 0$$

The remaining proof is similar to the one given in [9] for the case of *mobile velocity measurement*. It comes out that if one of the conditions of the proposition is satisfied, then the above system of equations has a non-zero solution leading to the non-uniform observability of the pair $(A^\star, C^\star)$. ■

To provide a comprehensive study of the uniform observability of the pair $(A^\star, C^\star)$ leading to the local uniform observability of the original system, we introduce $\lambda_1^i(t,\delta)$, $\lambda_2^i(t,\delta)$, $\lambda_3^i(t,\delta)$ to denote the eigenvalues of $\Pi_i(t,\delta)$ such that $\lambda_1^i(t,\delta) \leq \lambda_2^i(t,\delta) \leq \lambda_3^i(t,\delta)$.

Definition 4 *The matrix $\Pi_i(t,\delta)$ is called* strongly persistently exciting *if there exist $\delta, \beta > 0$ such that for all $t \geq 0$: $\lambda_1^i(t,\delta) \geq \beta$. It is called* weakly persistently exciting, *if*

$\operatorname{rank}(\Pi_i(t,\delta)) = 2$ *and* $\lambda_2^i(t,\delta) \geq \beta$. *If* $\operatorname{rank}(\Pi_i(t,\delta)) \leq 1$ *then* $\Pi_i(t,\delta)$ *cannot be persistently exciting.*

The definition of the weak persistent of excitation is due to the uniform observability of the pair $(A^\star, C^\star)$ in terms of possible trajectories for which the matrix $\Pi_i(t,\delta)$ even singular does not alter the uniform observability of the system (see Proposition 3).

Proposition 2 *Assume that the camera translational motion is strongly persistently exciting according to Definition 4. That is, there exist* $\delta, \beta > 0$ *such that for all* $t \geq 0$ *and for all* $i = \{1, \ldots, N\}$ $\lambda_1^i(t,\delta) \geq \beta$. *If* $N \geq 3$ *and none of the conditions of Proposition 1 is fulfilled, then the pair* $(A^\star(t), C^\star(t))$ *is uniformly observable.*

Proof: Using the fact that $\Pi_i(t,\delta) \geq \lambda_1^i(t,\delta) I_3 \geq \beta I_3$, direct application of Theorem 1 of [17] with N source points shows that $\nu = 0 \in \mathbb{R}^6$ is the unique solution to:

$$\forall i,\ \mathring{p}_{i\times} \begin{bmatrix} |\mathring{P}_i| \mathring{p}_{i\times} & -I_3 \end{bmatrix} \nu = 0, \tag{15}$$

which in turn implies the uniform observability condition of the pair $(A^\star(t), C^\star(t))$.

According to [9], if none of the conditions of Proposition 1 is fulfilled, $\nu = 0 \in \mathbb{R}^6$ is the unique solution which in turn implies the uniform observability the pair $(A^\star(t), C^\star(t))$. ∎

Remark 1 *Note that the strong persistence of excitation of* $\Pi_i(t,\delta)$ *does not automatically imply that the camera's motion is persistent in each direction of the* 3*-dimensional space. Any planar motion that does not contain the reference frame's origin may fulfill the strong persistence of excitation condition.*

Proposition 3 *Assume that the camera's translational motion is weakly persistently exciting according to Definition 4. That is,* $\operatorname{rank}(\Pi_i(t,\delta)) = 2$ *and there exist* $\delta, \beta > 0$ *such that for all* $t \geq 0$ *and for all* $i = \{1, \ldots, N\}$, $\lambda_2^i(t,\delta) \geq \beta$. *If* $N \geq 3$ *such that there exist* 3 *constant vectors* $\mathring{p}_i$ *not orthogonal to* $\ker(\Pi_i(t,\delta))$ *and none of the conditions of Proposition 1 is fulfilled, then the pair* $(A^\star(t), C^\star(t))$ *is uniformly observable.*

Proof: In view of (10), (11) and (13), the uniform observability condition (1) involving the observability Gramian $W(t,\delta)$ is satisfied if there exist $\delta, \mu > 0$ such that for all $t > 0$

$$H(t,\delta) = \sum_{i=1}^{N} \begin{bmatrix} |\mathring{P}_i|^2 \pi_{\mathring{p}_i} \Pi_i \pi_{\mathring{p}_i} & |\mathring{P}_i| \pi_{\mathring{p}_i} \Pi_i \mathring{p}_{i\times} \\ -|\mathring{P}_i| \mathring{p}_{i\times} \Pi_i \pi_{\mathring{p}_i} & -\mathring{p}_{i\times} \Pi_i \mathring{p}_{i\times} \end{bmatrix} \geq \mu I_6$$

where Π_i denotes $\Pi_i(t,\delta)$.

Using the fact that the camera motion is sufficiently persistently exciting along with $\operatorname{rank}(\Pi_i(t,\delta)) = 2\ \forall t \geq 0$, one ensures that it exists $Q \in SO(3)$ such that $\forall t \geq 0$ $\mathring{\xi}(t) := Q^\top \mathring{\xi}(t) = [0, \mathring{\xi}_2(t), \mathring{\xi}_3(t)]^\top$. Recalling (13) and by defining $D := \operatorname{diag}([0,1,1])$ one verifies

$$Q^\top \Pi_i Q - \beta D \geq 0$$

This implies

$$\Pi_i - \beta Q D Q^\top = \Gamma_i^\top \Gamma_i \geq 0$$

with $\Gamma_i := (\Pi_i - \beta Q D Q^\top)^{1/2}$. By denoting $D^\star := Q D Q^\top$ one verifies

$$\begin{aligned} &\begin{bmatrix} |\mathring{P}_i|^2 \pi_{\mathring{p}_i} (\Pi_i - \beta D^\star) \pi_{\mathring{p}_i} & |\mathring{P}_i| \pi_{\mathring{p}_i} (\Pi_i - \beta D^\star) \mathring{p}_{i\times} \\ -|\mathring{P}_i| \mathring{p}_{i\times} (\Pi_i - \beta D^\star) \pi_{\mathring{p}_i} & -\mathring{p}_{i\times} (\Pi_i - \beta D^\star) \mathring{p}_{i\times} \end{bmatrix} \\ &= \begin{bmatrix} |\mathring{P}_i|^2 \pi_{\mathring{p}_i} \Gamma_i^\top \Gamma_i \pi_{\mathring{p}_i} & |\mathring{P}_i| \pi_{\mathring{p}_i} \Gamma_i^\top \Gamma_i \mathring{p}_{i\times} \\ -|\mathring{P}_i| \mathring{p}_{i\times} \Gamma_i^\top \Gamma_i \pi_{\mathring{p}_i} & -\mathring{p}_{i\times} \Gamma_i^\top \Gamma_i \mathring{p}_{i\times} \end{bmatrix} \geq 0 \end{aligned}$$

From there one deduces

$$H(t,\delta) \geq \beta \mathring{N} \tag{16}$$

with

$$\begin{aligned} \mathring{N} &:= \sum_{i=1}^{N} \begin{bmatrix} |\mathring{P}_i|^2 \pi_{\mathring{p}_i} D^\star \pi_{\mathring{p}_i} & |\mathring{P}_i| \pi_{\mathring{p}_i} D^\star \mathring{p}_{i\times} \\ -|\mathring{P}_i| \mathring{p}_{i\times} D^\star \pi_{\mathring{p}_i} & -\mathring{p}_{i\times} D^\star \mathring{p}_{i\times} \end{bmatrix} \\ &= \sum_{i=1}^{N} \begin{bmatrix} |\mathring{P}_i|^2 Q \pi_{\mathring{\bar{p}}_i} D \pi_{\mathring{\bar{p}}_i} Q^\top & |\mathring{P}_i| Q \pi_{\mathring{\bar{p}}_i} D \mathring{\bar{p}}_{i\times} Q^\top \\ -|\mathring{P}_i| Q \mathring{\bar{p}}_{i\times} D \pi_{\mathring{\bar{p}}_i} Q^\top & -Q \mathring{\bar{p}}_{i\times} D \mathring{\bar{p}}_{i\times} Q^\top \end{bmatrix} \\ &= \begin{bmatrix} Q & 0_3 \\ 0_3 & Q \end{bmatrix} \sum_{i=1}^{N} \bar{Z}_i^\top \bar{Z}_i \begin{bmatrix} Q^\top & 0_3 \\ 0_3 & Q^\top \end{bmatrix} \end{aligned}$$

with $\bar{Z}_i := D \begin{bmatrix} |\mathring{P}_i| \pi_{\mathring{\bar{p}}_i} & \mathring{\bar{p}}_{i\times} \end{bmatrix}$, and $\mathring{\bar{p}}_i = Q^\top \mathring{p}_i$

In view of (16) it follows that if $\mathring{N}$ is definite positive, one concludes the uniform observability of the pair $(A^\star, C^\star)$. Thus, it suffices to prove that $\nu^\top \mathring{N} \nu = 0$ with $\nu = [\nu_1^\top, \nu_2^\top]^\top \in \mathbb{R}^6$ implies $\nu = 0$. One has

$$\begin{aligned} 0 = \nu^\top \mathring{N} \nu &= \sum_{i=1}^{N} |\bar{Z}_i \nu|^2 \\ \implies &\forall i\ \bar{Z}_i \nu = 0 \\ \implies &\forall i\ D(|\mathring{P}_i| \pi_{\mathring{\bar{p}}_i} \nu_1 + \mathring{\bar{p}}_{i\times} \nu_2) = 0 \\ \implies &\forall i\ D \mathring{\bar{p}}_{i\times} (|\mathring{P}_i| \mathring{\bar{p}}_{i\times} \nu_1 - \nu_2) = 0 \\ \implies &\forall i\ \exists \alpha_i \in \mathbb{R} |\ \mathring{\bar{p}}_{i\times} (|\mathring{P}_i| \mathring{\bar{p}}_{i\times} \nu_1 - \nu_2) = \alpha_i q \end{aligned}$$

with $q = [1,0,0]^\top \in \ker(D)$. The vector q is orthogonal to the motion plane of $\mathring{\bar{\xi}}(t) := Q^\top \mathring{\xi}(t)$. This implies that, if the ith source point is not orthogonal to $\ker(\Pi_i)$, one ensures that $\mathring{\bar{p}}_i = Q^\top \mathring{p}_i$ is also not orthogonal to q and hence, from the last equation, $\alpha_i = 0$. From there, and according to [9], if none of the conditions of Proposition 1 is fulfilled, one ensures that $\nu^\top \mathring{N} \nu = 0$ with $\nu = [\nu_1^\top, \nu_2^\top]^\top \in \mathbb{R}^6$ implies $\nu = 0$ and concludes on the uniform observability of the pair $(A^\star, C^\star)$. ∎

Remark 2 *Having* $\Pi_i(t,\delta)$ *weakly persistently exciting does not imply that camera motion should be planar. Any linear motion along a straight line that does not contain the origin of frame* $\mathring{\mathcal{A}}$ *may fit the requirements.*

Proposition 4 *If the camera translational motion is not persistently exciting according to Definition 4. Then* $(A^\star(t), C^\star(t))$ *is not uniformly observable.*

Proof: The case where $\text{rank}(\Pi_i(t,\delta)) = 0$ is trivial and omitted here. When $\text{rank}(\Pi_i(t,\delta)) = 1$, one ensures that there exists a constant non-zero vector $u \in S^2$ such that $\Pi_i(t,\delta) = \lambda_3^i(t,\delta) u u^\top$ with $\lambda_3^i(t,\delta) > 0$. Matrix Z_i, that satisfies (12), can be expressed as follow

$$Z_i(t,\delta) = \sqrt{\lambda_3^i(t,\delta)} u^\top \mathring{p}_{i\times} \begin{bmatrix} -|\mathring{P}_i| \mathring{p}_{i\times} & I_3 \end{bmatrix}$$

Now, since any vector $\nu = [0_{1,3}, u^\top]^\top \neq 0$ is a solution to $Z_i(t,\delta)\nu = 0$, one confirms that the pair $(A^\star(t), C^\star(t))$ is not uniformly observable. ■

Remark 3 *The case for which* $\text{rank}(\Pi_i(t,\delta)) = 1$ *corresponds to a static camera's position or moving along a straight line passing through the origin of the frame* $\mathring{\mathcal{A}}$.

V. Simulations

The results demonstrated in this paper are illustrated through simulations performed using Matlab. The angular velocity, which does not affect the observability, is arbitrary set to

$$\Omega(t) = \frac{\pi}{180} \left[5\cos(t),\ 10\cos(2t),\ 45\cos(2t)\right]^\top$$

The orientation obtained by integrating this angular velocity corresponds to a realistic case in which the roll and pitch angles do not exceed $\pm 10°$.

The following parameters have been chosen for the Riccati observer: $Q = 100 I_N$, $V = \text{diag}(0.1 I_3, I_3)$ and $P_0 = 10 I_6$. The initial estimates errors have been set to: $\tilde{\xi}(0) = [4, 5, -5]^\top$ and $\tilde{q}(0) = [0.97, 0.14, -0.06, -0.18]^\top$ (the unit quaternion corresponding to errors in roll, pitch and yaw Euler angles of $-15°$, $10°$ and $20°$ respectively).

The curve of the Lyapunov function candidate $L(x,t) := \frac{1}{2} x^\top P x$, with $x = [\tilde{\lambda}^\top, \tilde{\xi}^\top]^\top$, will illustrate the local exponential stability of the observer, when the observability conditions are fulfilled.

A. 3D motion

The camera is moving on the following trajectory:

$$\mathring{\xi}(t) = \left[8\sin(\pi t/4),\ 12\sin(\pi t/3),\ \sin(\pi t)\right]^\top$$

We consider three source points chosen such that the origin of $\mathring{\mathcal{A}}$ ($\mathring{\xi}(t) = [0,0,0]^\top$) belongs to the danger cylinder generated by the source points: $\mathring{P}_1 = [1,1,-2]^\top$, $\mathring{P}_2 = [0,2,-2]^\top$, $\mathring{P}_3 = [-1,1,-2]^\top$. Figure 2 shows that due to the lack of uniform observability, the Lyapunov function candidate is not exponentially decreasing, and hence, the equilibrium $x = 0$ is not exponentially stable.

Figure 3 represents the Lyapunov function candidate when $\mathring{P}_1 = [1,1,-2.2]^\top$ so that the origin of $\mathring{\mathcal{A}}$ is slightly far from the danger cylinder. It shows that the exponential stability of the observer is ensured.

Now, consider that there are four observed source points: $\mathring{P}_1 = [1,3,-1]^\top$, $\mathring{P}_2 = [1.5,4,-2]^\top$, $\mathring{P}_3 = [6,4,-8]^\top$ and $\mathring{P}_4 = [9,3,-9]^\top$. One can easily verify that these points belong to the horopter curve $\mathcal{H}_{-10,3}$ whose origin is the camera's initial position. Figure 4 illustrates the non-convergence of the observer in this case due to the lack of uniform observability.

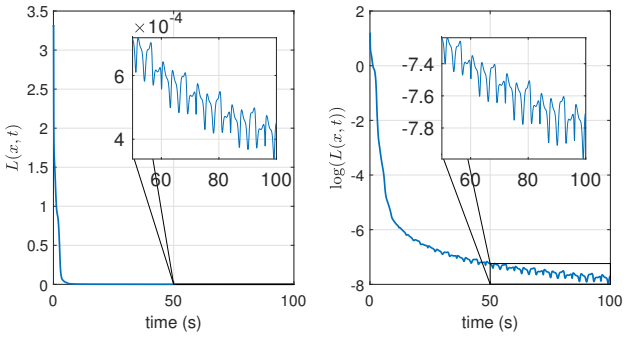

Fig. 2. Lyapunov function candidate $L(x,t) := \frac{1}{2} x^\top P x$ versus time (s) when the initial position of the camera is on the danger cylinder and the camera motion is strongly persistently excited.

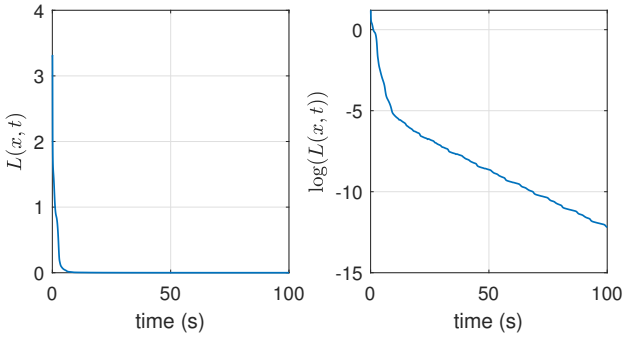

Fig. 3. The Lyapunov function candidate $L(x,t) := \frac{1}{2} x^\top P x$ versus time (s) when the initial position of the camera is not on the danger cylinder and the camera motion is strongly persistently excited.

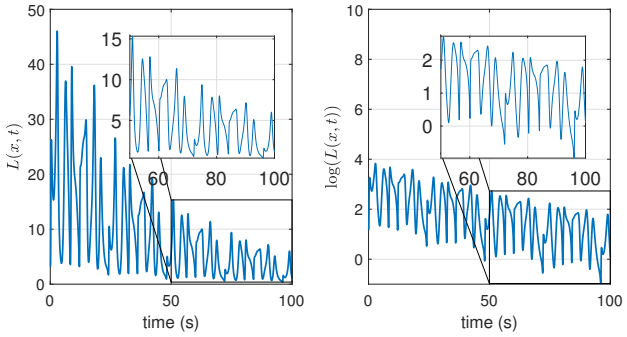

Fig. 4. The Lyapunov function candidate $L(x,t) := \frac{1}{2} x^\top P x$ versus time (s) when the camera motion is strongly persistently excited while the initial position of the camera is located at the origin of a horopter curve crossing the four source points.

B. 2D motion

Consider now the following 2D trajectory of the camera:

$$\mathring{\xi}(t) = \left[8\sin(\pi t/4),\ 12\sin(\pi t/3),\ 0\right]^\top$$

The space $\ker(\Pi_i(t,\delta))$ mentioned in Proposition 3 is here the normal vector to the xy-plane. Figure 5 shows the Lyapunov function candidate when there are 3 source points

$\mathring{P}_1 = [1, 1, -2]^\top$, $\mathring{P}_2 = [0, 3, -2]^\top$, $\mathring{P}_3 = [-1, 1, 0]^\top$ chosen such that the danger cylinder associated with the source points does not cross the initial position of the camera and $\mathring{P}_3$ belongs to the xy-plane. Again, the exponential stability is not ensured due to the lack of uniform observability. When none of the 3 source points belongs to the xy-plane (by setting $\mathring{P}_3 = [-1, 1, -2]^\top$), Figure 6 shows the exponential convergence of the observer error.

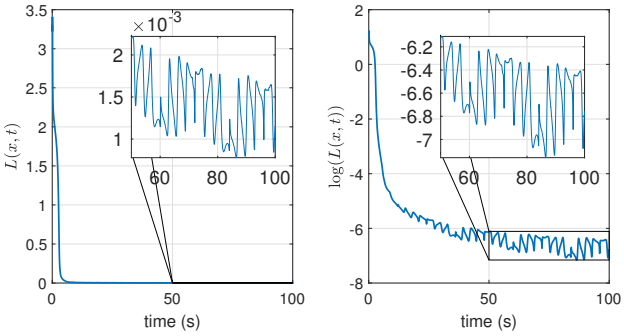

Fig. 5. The Lyapunov function candidate $L(x,t) := \frac{1}{2} x^\top P x$ versus time (s) when the camera motion is weakly persistently excited and only one source point is orthogonal to $\ker(\Pi(t,\delta))$.

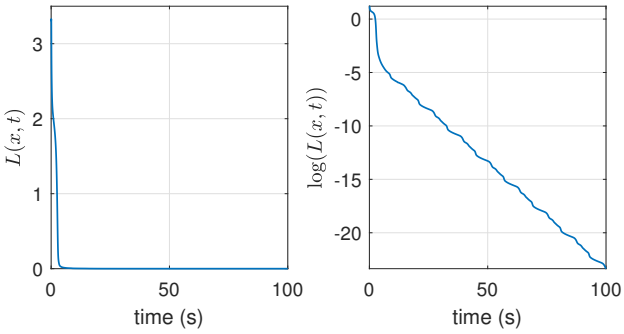

Fig. 6. The Lyapunov function candidate $L(x,t) := \frac{1}{2} x^\top P x$ versus time (s) when the camera motion is weakly persistently excited and no source point is orthogonal to $\ker(\Pi(t,\delta))$.

VI. Conclusions

In this paper, observability analysis of the relative pose observers proposed in [17] has been carried out. Some uniform observability conditions that ensure the exponential stability of the relative pose observers are derived. Situations for which these conditions are not fulfilled are characterized in detail in terms of the position of the reference frame in relation to the location of the source points, their number, and also the translational motion of the camera with respect to the source points. In this work, the location estimation of the source point is not addressed. It will be further discussed in forthcoming studies in relation to the pose-SLAM problem.

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