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Non-coercive Lyapunov–Krasovskii functionals for exponential stability of time-varying delay systems: a switched system approach

Ihab Haidar *

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Abstract

In this paper we deal with a class of time-varying delay systems. We show that the existence of a non-coercive Lyapunov–Krasovskii functional is necessary and sufficient for local exponential stability. This result extends what is recently given in [3] for globally Lipschitz systems to a class of locally Lipschitz systems. A switched system approach is used to elaborate this result.

Keywords Time-delay systems; switched systems; Lyapunov–Krasovskii functionals; exponential stability.

1 Introduction

Different approaches have been developed in the literature of control theory for the stability analysis of time-delay systems (see, e.g., [2, 10, 13, 20]). We refer, for example, to the Lyapunov–Krasovskii approach, the Lyapunov–Razumikhin approach, and recently the trajectory based approach introduced in [15]. Based on these approaches, a variety of stability criteria have been developed. Time-varying delay systems is a very intriguing class of time-delay systems for which some non-intuitive properties are discussed and analyzed in the literature (see, e.g., [4, 14]).

Time-varying delay systems can be seen as a special class of switched systems. This paradigm has been proposed in [12] for discrete-time delays systems, which consists in transforming the system in question into discrete-time switched system in a higher-dimensional space. Continuous-time delay systems can be similarly interpreted as switched systems in infinite-dimensional Banach space. This approach has been used in [5] and [6] to characterize the stability of continuous-time retarded systems in terms of the existence of *coercive* and *non-coercive* Lyapunov–Krasovskii functionals. By non-coercive Lyapunov functional, we mean a positive definite functional decaying along the trajectories of the system which satisfies

$$0 < V(x) \leq \alpha(\|x\|), \quad \forall\, x \in X\backslash\{0\}, \tag{1}$$

where X is the ambient Banach space and α belongs to the class $\mathcal{K}_\infty$ of continuous increasing bijections from $\mathbb{R}_+$ to $\mathbb{R}_+$. Such a functional V would be coercive if there existed $\alpha_0 \in \mathcal{K}_\infty$ such that $V(x) \geq \alpha_0(\|x\|)$ for every $x \in X$.

Various works have been recently devoted to the characterization of the stability of switched systems in finite and infinite-dimensional systems through coercive and non-coercive Lyapunov functionals (see, e.g., [3, 5, 8, 11, 16, 17, 18]). In [3], a characterization of uniform exponential stability in terms of the existence of non-coercive Lyapunov functionals has been developed for abstract forward complet switched system. This result covers the class of globally Lipschitz time-varying delay systems with piecewise-constant

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delays. As explained above, this is possible after representing a nonlinear time-varying delay system as an abstract forward complet switched system. Here we extend this result for a class of locally Lipschitz systems with measurable time-varying delays.

The paper is organized as follows. Section 2 presents the definitions, the assumptions in use and the problem statement. In Section 3 we give a switching representation of the considered class of delay systems and we recall an interesting result in this context. Preliminary results are given in Section 4. The main result concerning the characterization of local exponential stability through non-coercive Lyapunov-Krasovskii functional is given in Section 5.

1.1 Notation

Throughout the paper, we adopt the following notation: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ the set of non-negative real numbers, and $\overline{\mathbb{R}}$ the extended real line. By $(\mathbb{R}^n, \|\cdot\|)$, we denote the n-dimensional Euclidean space, where n is a positive integer and $|\cdot|$ denotes the Euclidean norm of a real vector. Given $\Delta > 0$, $\mathcal{C} = (\mathcal{C}([-\Delta, 0], \mathbb{R}^n), \|\cdot\|_\infty)$ denotes the Banach space of continuous functions mapping $[-\Delta, 0]$ into $\mathbb{R}^n$, where $\|\cdot\|_\infty$ is the norm of uniform convergence. For a function $x : [-\Delta, b) \to \mathbb{R}^n$, with $0 < b \leq +\infty$, for $t \in [0, b)$, $x_t : [-\Delta, 0] \to \mathbb{R}^n$ denotes the history function defined by $x_t(\theta) = x(t + \theta)$, $-\Delta \leq \theta \leq 0$. For a positive real H and given $\phi \in \mathcal{C}$, $\mathcal{C}_H(\phi)$ denotes the subset $\{\psi \in \mathcal{C} : \|\phi - \psi\|_\infty \leq H\}$. We simply denotes $\mathcal{C}_H(0)$ by $\mathcal{C}_H$.

2 Definitions and problem statement

Let us consider the time-varying delay system

$$\Sigma : \begin{array}{rcll} \dot{x}(t) & = & f(x(t), x(t - \sigma(t))), & a.e.\, t \geq 0, \\ x(\theta) & = & x_0(\theta), & \theta \in [-\Delta, 0], \end{array} \tag{2}$$

where $x(t) \in \mathbb{R}^n$; n is a positive integer; $\sigma \in \mathcal{S}$; $\mathcal{S}$ is the set of Lebesgue measurable time-delay functions $\sigma : \mathbb{R} \to \mathrm{S}$; $\mathrm{S} \subset [0, \Delta]$; Δ positive real (maximum involved time delay); $x_0 \in \mathcal{C}$ is the initial condition; $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz on bounded sets function such that $f(0, 0) = 0$.
For each $s \in \mathrm{S}$, let, with a slight abuse of notation, $f_s : \mathcal{C} \to \mathbb{R}^n$ be the functional which is defined in $\mathcal{C}$ as

$$f_s(\phi) = f(\phi(0), \phi(s)), \quad \phi \in \mathcal{C}.$$

So, system (2) can be equivalently rewritten as the following retarded functional differential equation

$$\begin{array}{rcll} \dot{x}(t) & = & f_{\sigma(t)}(x_t), & a.e.\, t \geq 0, \\ x(\theta) & = & x_0(\theta), & \theta \in [-\Delta, 0]. \end{array} \tag{3}$$

Definition 1. *We say that the system Σ is uniformly globally exponentially stable (UGES for short) if there exist positive reals M and λ such that, for any $x_0 \in \mathcal{C}$ and $\sigma \in \mathcal{S}$, the corresponding solution exists in $\mathbb{R}_+$ and, furthermore, satisfies the inequality*

$$\|x(t, x_0, \sigma)\| \leq M e^{-\lambda t} \|x_0\|_\infty. \tag{4}$$

Definition 2. *We say that the system Σ is uniformly locally exponentially stable (ULES for short) if there exist positive reals M, λ and H such that, for any $x_0 \in \mathcal{C}_H$ and $\sigma \in \mathcal{S}$, the corresponding solution exists in $\mathbb{R}_+$ and, furthermore, satisfies the inequality*

$$\|x(t, x_0, \sigma)\| \leq M e^{-\lambda t} \|x_0\|_\infty. \tag{5}$$

Let us recall the definition of Dini derivative of a functional $V : \mathcal{C} \to \mathbb{R}_+$.

Definition 3. *For a continuous functional $V : \mathcal{C} \to \mathbb{R}_+$ its Dini derivative $\overline{D}V : \mathcal{C} \to \overline{\mathbb{R}}$ is defined, for system Σ, for $\phi \in \mathcal{C}$, as follows,*

$$\overline{D}V(\phi) = \sup_{s\in \mathrm{S}} \limsup_{h\to 0^+} \frac{V(x_h(\phi, s)) - V(\phi)}{h},$$

where $x_h(\phi, s)$ is the solution of Σ starting from ϕ and associated with $\sigma(t) \equiv s$, $s \in \mathrm{S}$.

Let us also recall the following definition about Driver's form derivative [1] of a continuous functional $V : \mathcal{C} \to \mathbb{R}_+$.

Definition 4. *For a continuous functional $V : \mathcal{C} \to \mathbb{R}_+$, its Driver's form derivative, $D^+V : \mathcal{C} \to \overline{\mathbb{R}}$, is defined, for system Σ, for $\phi \in \mathcal{C}$, as follows,*

$$D^+V(\phi) = \sup_{s\in \mathrm{S}} \limsup_{h\to 0^+} \frac{V(\phi_h^{\Sigma,s}) - V(\phi)}{h}, \tag{6}$$

where $\phi_h^{\Sigma,s} \in \mathcal{C}$ is defined, for $h \in [0, \Delta)$ and $\theta \in [-\Delta, 0]$, as follows

$$\phi_h^{\Sigma,s}(\theta) = \begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h) \\ \phi(0) + (\theta + h) f_s(\phi), & \theta \in [-h, 0]. \end{cases} \tag{7}$$

Driver's type derivative, by contrast to Dini's one, is an appropriate definition of the derivative of a Lyapunov–Krasovskii functional that does not involve the solution. The relation between Dini's and Driver's type derivatives, in the case of locally Lipschitz Lyapunov-Krasovskii functional V, is clarified by the following: fix $\phi \in \mathcal{C}$ and a right-continuous piecewise-constant function $\sigma \in \mathcal{S}$, and let $x(\cdot)$ be the locally absolutely continuous solution of Σ in a maximal interval $[0, b)$. Inspired from what is already done in [21], in [7] we establish the following equality which holds for every $t \in [0, b)$

$$\limsup_{h\to 0^+} \frac{V((x_t)_h^{\Sigma,\sigma(t)}) - V(x_t)}{h} = \limsup_{h\to 0^+} \frac{V(x_h(x_t, \sigma)) - V(x_t)}{h}, \tag{8}$$

where $(x_t)_h^{\Sigma,\sigma(t)}$ is given using equation (7) and $x_h(x_t, \sigma)$ is the trajectory corresponding to Σ starting from x_t and associated with σ over $[t, t + h)$. By consequence, for every $\phi \in \mathcal{C}$ and $s \in \mathrm{S}$ the following equality holds

$$\limsup_{h\to 0^+} \frac{V((x_t)_h^{\Sigma,s}) - V(x_t)}{h} = \limsup_{h\to 0^+} \frac{V(x_h(x_t, s)) - V(x_t)}{h}, \tag{9}$$

for every $t \in [0, b)$. By applying the supremum over S from the left and the right of equation (9), we obtain that

$$D^+V(x_t) = \overline{D}V(x_t), \quad \forall\, t \in [0, b). \tag{10}$$

Of course this is true if V is locally Lipschitz and σ is piecewise-constant.

Remark 1. *Equality* (8) *holds almost everywhere in* $[0, b)$ *in the case of Lebesgue measurable delay functions.*

3 Switching representation

Let us recall by the following definition (see, e.g. [19]) the notion of strongly continuous nonlinear semigroup.

Definition 5. *Let X be a Banach space and let $T(t) : X \to X$, $t \geq 0$, be a family of nonlinear maps. We say that $(T(t))_{t\geq 0}$ is a strongly continuous nonlinear semigroup if the following properties hold:*

i) For every $x \in X$, it holds that $T(0)x = x$;

ii) For every $t_1, t_2 \geq 0$, it holds that $T(t_1)T(t_2)x = T(t_1 + t_2)x$;

iii) For every $x \in X$, the map $t \mapsto T(t)x$ is continuous.

For each $s \in \mathrm{S}$ and $\phi \in \mathcal{C}$, consider the unique solution $x(\cdot)$ over $[-\Delta, +\infty)$ of Σ with $\sigma(t) = s$ for every $t \geq 0$; the existence and uniqueness of solutions for Σ is discussed in the next section. This defines a family $(T_s(t))_{t\geq 0}$ of nonlinear maps from $\mathcal{C}$ into itself by setting

$$T_s(t)\phi = x_t, \quad \forall\, t \geq 0.$$

According to [10, 22], $(T_s(t))_{t\geq 0}$ is a strongly continuous semigroup of nonlinear operators on $\mathcal{C}$. Let $\sigma \in \mathcal{S}$ be a piecewise constant function which is constantly equal to σ_k over $[t_k, t_{k+1})$, with $0 = t_0 < t_1 < \cdots < t_k < t < t_{k+1}$, for $k \geq 0$. By concatenating the flows $(T_{\sigma_k}(t))_{t\geq 0}$, one can associate with σ the family of nonlinear evolution operators

$$T_\sigma(t) := T_{\sigma_k}(t - t_k)T_{\sigma_{k-1}}(t_k - t_{k-1}) \cdots T_{\sigma_1}(t_1),$$

$t \in [t_k, t_{k+1})$. Then, system Σ can be equivalently represented by the following switched system

$$x_t = T_\sigma(t)\phi, \qquad \phi \in \mathcal{C},\, \sigma \in \mathcal{S}. \tag{11}$$

As a consequence of this switching representation of the nonlinear time-varying delay system Σ, the following results is proved in [3].

Theorem 1 ([3])**.** *Let $l > 0$ be such that*

$$\|f_s(\phi_1) - f_s(\phi_2)\| \leq l\|\phi_1 - \phi_2\|,\ \forall\ \phi_1, \phi_2 \in \mathcal{C},\ \forall\ s \in \mathrm{S}. \tag{12}$$

The following statements are equivalent:

i) System Σ is UGES;

ii) there exists a continuous functional $V : \mathcal{C} \to \mathbb{R}_+$ and positive reals p and c such that

$$V(\phi) \leq c\|\phi\|^p, \quad \forall\ \phi \in \mathcal{C},$$

and

$$\overline{D}V(\phi) \leq -\|\phi\|^p, \quad \forall\ \phi \in \mathcal{C}. \tag{13}$$

Theorem 1 argues that if $f_s, s \in \mathrm{S}$, is uniformly globally Lipschitz then the UGES property of system Σ can be characterized by the existence of a non-coercive Lyapunov functional with suitable properties.

The following theorem has not been proven in [3] but it is a straightforward consequence of [3, Theorem 4] applied to the switching representation given by (11) of Σ (see the proof of [3, Theorem 4] for more details).

Theorem 2. *Suppose that inequality* (12) *holds. If there exists $H > 0$ and a continuous functional $V : \mathcal{C}_H \to \mathbb{R}_+$ and positive reals p and c such that*

$$V(\phi) \leq c\|\phi\|^p, \quad \forall\ \phi \in \mathcal{C}_H,$$

and

$$\overline{D}V(\phi) \leq -\|\phi\|^p, \quad \forall\ \phi \in \mathcal{C}_H,$$

then Σ is ULES.

Theorem 2 will be useful to prove the main result of this paper.

Knowing that the global Lipschitzianity assumption is somehow very restrictive, in this paper we will try to relax this property by considering locally Lipschitz systems. An additional regularity property will be considered (see Assumption 1 in Section 4).

4 Preliminary results

The following lemma discusses the existence and uniqueness of the solution of system Σ as well as its continuous dependence on the initial state.

Lemma 1. *For any $\phi \in \mathcal{C}$ and $\sigma \in \mathcal{S}$, there exists, uniquely, a locally absolutely continuous solution $x(t, \phi, \sigma)$ of Σ in a maximal time interval $[0, b)$, with $0 < b \leq +\infty$. If $b < +\infty$, then the solution is unbounded in $[0, b)$. Moreover, for any $\varepsilon > 0$, for any $c \in (0, b)$, there exists $\delta > 0$ such that, for any $\psi \in \mathcal{C}_\delta(\phi)$, the solution $x(t, \psi, \sigma)$ exists in $[0, c]$ and, furthermore, the following inequality holds*

$$\|x(t, \phi, \sigma) - x(t, \psi, \sigma)\| \leq \varepsilon, \quad \forall\, t \in [0, c]. \tag{14}$$

Proof. Let $H > 0$ and take $\phi_1, \phi_2 \in \mathcal{C}_H$. Observe that, for every $s \in \mathrm{S}$, we have

$$\|f_s(\phi_1) - f_s(\phi_2)\| = \|f(\phi_1(0), \phi_1(s)) - f(\phi_2(0), \phi_2(s))\| \leq l(H)\|\phi_1 - \phi_2\|_\infty,$$

where $l(H)$ is the Lipschitz constant associated to f in the ball of center 0 and radius H. Then, $f_s, s \in \mathrm{S}$, is uniformly Lipschitz on bounded subsets of $\mathcal{C}$. In addition, knowing that f is continuous and $\sigma(\cdot)$ is Lebesgue measurable then the map $t \mapsto f_{\sigma(t)}(\phi)$ is Lebesgue measurable. By consequence, system (3) respects the Caratheodory conditions for existence and uniqueness of solutions and the proof is straightforward from the theory of systems described by RFDEs (see, e.g. [10]). □

We introduce the following assumption.

Assumption 1. *The function f is Fréchet differentiable at the origin.*

For each $s \in \mathrm{S}$, let us introduce the operator $L_s : \mathcal{C} \to \mathbb{R}^n$ defined by

$$L_s\phi = L(\phi(0), \phi(s)), \quad \forall\, \phi \in \mathcal{C}, \tag{15}$$

where $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the derivative of f at 0 which is a bounded linear operator.

Lemma 2. *Suppose that Assumption 1 holds. Then, for each $s \in \mathrm{S}$, L_s is the Fréchet derivative of f_s at 0. Furthermore, there exists $m > 0$, such that*

$$\|L_s\phi\| \leq m\|\phi\|_\infty, \quad \forall \phi \in \mathcal{C}, \forall s \in \mathrm{S}. \tag{16}$$

Proof. For $s \in \mathrm{S}$ and $\phi \in \mathcal{C}$, we have

$$\begin{aligned}
& \lim_{\phi \to 0, \phi \neq 0} \frac{\|f_s(\phi) - f_s(0) - L_s\phi\|}{\|\phi\|_\infty} \\
= & \lim_{\phi \to 0, \phi \neq 0} \frac{\|f(\phi(0), \phi(s)) - f(0, 0) - L(\phi(0), \phi(s))\|}{\|\phi\|_\infty} \\
\leq & \lim_{\phi \to 0, \phi \neq 0} \frac{\|f(\phi(0), \phi(s)) - f(0, 0) - L(\phi(0), \phi(s))\|}{\max\{\|\phi(0)\|, \|\phi(s)\|\}} \\
= & \lim_{y \in \mathbb{R}^n \times \mathbb{R}^n, y \to 0} \frac{\|f(y) - f(0, 0) - Ly\|}{\|y\|} = 0.
\end{aligned}$$

Furthermore, we have

$$\|L_s\phi\| = \|L(\phi(0), \phi(s))\| \leq m\|\phi\|_\infty, \forall \phi \in \mathcal{C}, \forall s \in \mathrm{S}, \tag{17}$$

for some $m > 0$ independent on ϕ and s. Then for each $s \in \mathrm{S}$, f_s is Fréchet differentiable at 0 and the bounded linear operator L_s is its Fréchet derivative. □

Lemma 3. *Suppose that Assumption 1 holds. Suppose that $\mathcal{S}$ is reduced to piecewise-constant functions. If the linear system*

$$\Sigma_L : \begin{array}{rcll} \dot{\xi}(t) & = & L_{\sigma(t)}\xi_t, & t \geq 0, \\ \xi(\theta) & = & \xi_0(\theta), & \theta \in [-\Delta, 0], \end{array} \tag{18}$$

with $\xi_0 \in \mathcal{C}$, is UGES then system Σ is ULES.

Proof. The proof is a direct consequence of Lemma 2 and [9, Theorem 5.2]. □

Lemma 4. *Suppose that Assumption 1 holds. Suppose that $\mathcal{S}$ is reduced to piecewise-constant signals. If, for system Σ_L, there exist $H, c > 0$ and a continuous functional $V : \mathcal{C}_H \to [0, +\infty)$ such that the following inequalities hold for every $\phi \in \mathcal{C}_H$ then*

(i) $V(\phi) \leq c\|\phi\|_\infty$,

(ii) $D^+V(\phi) \leq -\|\phi\|_\infty$

then system Σ_L is ULES.

Proof. Knowing that $L_s, s \in \mathrm{S}$, is uniformly globally Lipschitz (see inequality (17)) then the proof is a direct consequence of Theorem 2 together with the fact that $D^+V = \overline{D}V$ (see equation (10)). □

5 Main result

The following theorem shows that the existence of a non-coercive Lyapunov–Krasovskii functional is a necessary and sufficient condition for the ULES property of system Σ, provided that Assumption 1 is satisfied.

Theorem 3. *Suppose that Assumption 1 holds. System Σ is ULES if and only if there exist positive reals H, p, c_1, c_2 and locally Lipschitz functional $V : \mathcal{C}_H \to \mathbb{R}_+$, such that the following inequalities hold for any $\phi \in \mathcal{C}_H$*

(i) $V(\phi) \leq c_1\|\phi\|_\infty$,

(ii) $D^+V(\phi) \leq -c_2\|\phi\|_\infty$.

Proof. The proof of the necessity part is given in [9, Theorem 5.2]. The proof of the sufficiency part is inspired from [9, Theorem 5.2]. Let $\phi \in \mathcal{C}_H$ and let $\sigma \in \mathcal{S}$ be a piecewise-constant function. Knowing that V is locally Lipschitz, there exist $0 < \delta_0 < H$ and $c_3 = c_3(\delta_0)$ such that, for any $\phi_1, \phi_2 \in \mathcal{C}_{\delta_0}(\phi)$, the following inequality holds

$$|V(\phi_1) - V(\phi_2)| \leq c_3\|\phi_1 - \phi_2\|_\infty. \tag{19}$$

Seeing that, for sufficiently small h, and for any $s \in \mathrm{S}$, we have $\phi_h^{\Sigma_L,s}, \phi_h^{\Sigma,s} \in \mathcal{C}_{\delta_0}(\phi)$, it follows that inequality (19) holds for $\phi_h^{\Sigma,s}$ and $\phi_h^{\Sigma_L,s}$. Now, calculating the Driver's derivative of V along the system Σ_L gives

$$\begin{aligned}
D^+V(\phi) &= \sup_{s\in\mathrm{S}} \limsup_{h\to 0^+} \frac{V(\phi_h^{\Sigma_L,s}) - V(\phi)}{h} \\
&= \sup_{s\in\mathrm{S}} \limsup_{h\to 0^+} \frac{V(\phi_h^{\Sigma_L,s}) + V(\phi_h^{\Sigma,s}) - V(\phi_h^{\Sigma,s}) - V(\phi)}{h} \\
&\leq \sup_{s\in\mathrm{S}} \limsup_{h\to 0^+} \frac{c_3\|\phi_h^{\Sigma,s} - \phi_h^{\Sigma_L,s}\|_\infty}{h} - c_2\|\phi\|_\infty \\
&\leq \sup_{s\in\mathrm{S}} \limsup_{h\to 0^+} \frac{c_3 \sup\limits_{\theta\in[-\Delta,0]} \|\phi_h^{\Sigma,s}(\theta) - \phi_h^{\Sigma_L,s}(\theta)\|}{h} - c_2\|\phi\|_\infty \\
&\leq \sup_{s\in\mathrm{S}} \limsup_{h\to 0^+} \frac{c_3 \sup\limits_{\theta\in[-h,0]} |\theta + h|\|f_s(\phi) - L_s\phi\|}{h} - c_2\|\phi\|_\infty.
\end{aligned}$$

Let $0 < \delta_1 < \delta_0$ be a positive real such that if $\phi \in \mathcal{C}_{\delta_1}$ then the following inequality holds

$$\|f_s(\phi) - L_s\phi\| \leq \frac{c_2}{2c_3}\|\phi\|_\infty. \tag{20}$$

So, therefore, for any $\phi \in \mathcal{C}_{\delta_1}$, we have

$$D^+V(\phi) \leq -\frac{c_2}{2}\|\phi\|_\infty. \tag{21}$$

Thus, the inequalities (i) and (ii) of Lemma 4 are satisfied for system Σ_L, uniformly with respect to piecewise-constant functions $\sigma \in \mathcal{S}$. Then, thanks to Lemma 4, system Σ_L is ULES (hence UGES by linearity) with respect to the class of piecewise-constant functions $\sigma \in \mathcal{S}$. Then, using Lemma 3, system Σ is ULES with respect to the class of piecewise-constant functions $\sigma \in \mathcal{S}$. Now, using the equivalence property provided in [9, Theorem 3.1], that is Σ is ULES with respect to measurable functions σ if and only if it is ULES with respect to piecewise-constant ones, we conclude the proof of the sufficiency part. □

6 Conclusion

In this paper we characterize the local exponential stability of a class of locally Lipschitz uncertain time-varying delay systems through the existence of non-coercive Lyapunov–Krasovskii functionals. This is obtained based on a switched system approach. This result extends what is recently given in [3] for globally Lipschitz delay systems with piecewise-constant delays to locally Lipschitz systems with measurable time-varying delays. This result is given under the assumption that the function defined the dynamics is Fréchet differentiable at the origin. Future works will attempt to relax this regularity condition.

References

[1] Rodney D. Driver. Existence and stability of solutions of a delay-differential system. *Archive for Rational Mechanics and Analysis*, 10(1):401–426, 1962.

[2] Emilia Fridman. *Introduction to Time-Delay Systems*. Birkhäuser, 2014.

[3] Ihab Haidar, Yacine Chitour, Paolo Mason, and Mario Sigalotti. Lyapunov characterization of uniform exponential stability for nonlinear infinite-dimensional systems. *IEEE Transactions on Automatic Control*, 67(4):1685–1697, 2021.

[4] Ihab Haidar, Paolo Mason, Silviu-Iulian Niculescu, Mario Sigalotti, and Antoine Chaillet. Further remarks on markus-yamabe instability for time-varying delay differential equations. *IFAC-PapersOnLine*, 48(12):33–38, 2015.

[5] Ihab Haidar, Paolo Mason, and Mario Sigalotti. Converse Lyapunov–Krasovskii theorems for uncertain retarded differential equations. *Automatica*, 62:263–273, 2015.

[6] Ihab Haidar, Paolo Mason, and Mario Sigalotti. Stability of interconnected uncertain delay systems: a converse Lyapunov approach. In *Delays and Interconnections: Methodology, Algorithms and Applications*, pages 49–63. Springer, 2019.

[7] Ihab Haidar and Pierdomenico Pepe. ISS characterization of retarded switching systems with relaxed Lyapunov-Krasovskii functionals. Preprint arXiv:2209.06489.

[8] Ihab Haidar and Pierdomenico Pepe. Lyapunov–Krasovskii characterizations of stability notions for switching retarded systems. *IEEE Transactions on Automatic Control*, 66(1):437–443, 2020.

[9] Ihab Haidar and Pierdomenico Pepe. Lyapunov–Krasovskii characterization of the input-to-state stability for switching retarded systems. *SIAM Journal on Control and Optimization*, 59(4):2997–3016, 2021.

[10] Jack K. Hale and Sjoerd M. Verduyn Lunel. *Introduction to functional differential equations*, volume 99. Springer-Verlag, 1993.

[11] Falk M. Hante and Mario Sigalotti. Converse Lyapunov theorems for switched systems in Banach and Hilbert spaces. *SIAM J. Control Optim.*, 49(2):752–770, 2011.

[12] Laurentiu Hetel, Jamal Daafouz, and Claude Iung. Equivalence between the Lyapunov–Krasovskii functional approach for discrete delay systems and the stability conditions for switched systems. *Nonlinear Analysis: Hybrid Systems*, 2(3):697–705, 2008.

[13] V. B. Kolmanovskii and A. D. Myshkis. *Introduction to the Theory and Applications of Functional Differential Equations*, volume 463. Springer, 1999.

[14] James Louisell. New examples of quenching in delay differential equations having time-varying delay. *In Proc. 5th Eur. Control Conf*, 1999.

[15] Frederic Mazenc, Michael Malisoff, and Silviu-Iulian Niculescu. Stability analysis for systems with time-varying delay: trajectory based approach. In *54th IEEE Conference on Decision and Control (CDC)*, pages 1811–1816, 2015.

[16] Andrii Mironchenko and Fabian Wirth. A note on input-to-state stability of linear and bilinear infinite-dimensional systems. In *54th IEEE Conference on Decision and Control (CDC)*, pages 495–500, 2015.

[17] Andrii Mironchenko and Fabian Wirth. Existence of non-coercive Lyapunov functions is equivalent to integral uniform global asymptotic stability. *Mathematics of Control, Signals, and Systems*, 31(2):1–26, 2019.

[18] Andrii Mironchenko and Fabian Wirth. Non-coercive Lyapunov functions for infinite-dimensional systems. *Journal of Differential Equations*, 266(11):7038–7072, 2019.

[19] Isao Miyadera. *Nonlinear Semigroups*, volume 109. American Mathematical Society, Providence, RL, 1991.

[20] Silviu I. Niculescu. *Delay effects on stability: A robust control approach*, volume 269. Berlin: Springer, 2001.

[21] Pierdomenico Pepe. On Liapunov-Krasovskii functionals under carathéodory conditions. *Autom.*, 43:701–706, 2007.

[22] Glenn F. Webb. Autonomous nonlinear functional differential equations and nonlinear semigroups. *Journal of Mathematical Analysis and Applications*, 46(1):1–12, 1974.