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Stability analysis of a Korteweg-de Vries equation with saturated boundary feedback

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Abstract: The stability analysis of the linear and nonlinear Korteweg-de Vries equations in presence of a saturated feedback actuator is studied. The well-posedness is derived by using nonlinear semigroup results, Schauder's and Banach fixed point theorems. The exponential stability is shown thanks to an observability inequality, which is obtained via contradiction and compactness arguments. Finally, an alternative proof of the asymptotic stability of the linear Korteweg-de Vries equation is presented.

Keywords: Partial differential equations; saturation control; boundary control; stability of nonlinear systems

1. INTRODUCTION

The Korteweg-de Vries (KdV) equation $u_t + u_x + u_{xxx} + uu_x = 0$ was introduced by Korteweg and de Vries (1895) to model the propagation of long water waves in a channel. The KdV equation has been widely studied in the last years, in particular its controllability and stabilization properties, see Cerpa (2014); Rosier and Zhang (2009) for a complete introduction of these problems.

From the point of view of stabilization, we can refer to the work of Zhang (1994) in which the boundary exponential stabilization problem in the bounded spatial domain $x \in (0,1)$ was studied. It is well known that the length L of the spatial domain plays an important role in the stabilization and controllability properties of the KdV equation. For example, when $L = 2\pi$ it is possible to find a solution of the linearization around 0 of the KdV equation ($u(t,x) = 1 - \cos(x)$) that has a constant energy. More generally, if $L \in \mathcal{N}$, where $\mathcal{N}$ is the set of critical lengths defined by

$$\mathcal{N} = \left\{ 2\pi\sqrt{(k^2 + kl + l^2)/3} \ ; \ k, l \in \mathbb{N}^* \right\},$$

we can find suitable initial data, such that the solution of the linear KdV equation has a constant energy. In the case of internal stabilization, it is proved in Perla Menzala et al. (2002); Pazoto (2005) that for any critical length we achieve local exponential stability for the nonlinear KdV. In most real-life settings, we have to take into account the saturation in the input control due to some physical or economical constraints. With respect to saturated control in infinite dimensional systems, we can refer to Prieur et al. (2016) where a wave equation with distributed and boundary saturated feedback law was studied, Marx et al. (2017) where the saturated internal stabilization of a single KdV equation was studied and recently Mironchenko et al. (2021) where a saturated feedback control law was derived for a linear reaction-diffusion equation. In our best knowledge, there are no work dealing with the stability of a KdV equation with saturated feedback on the boundary, thus this work gives a sake of completeness on this topic. The plan is to follow the ideas presented in Marx et al. (2017) and Parada et al. (2022) to prove the stability of the KdV equation with saturated actuator. To be more specific, let us define the following saturation map:

$$\mathbf{sat}(s) = \begin{cases} s, & \text{if } |s| \leq M, \\ M\mathbf{sgn}(s), & \text{if } |s| \geq M, \end{cases} \tag{1}$$

where $M > 0$ is the saturation level and **sgn** is the sign function. In this work, we are going to consider the following equations

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & t > 0, \ x \in (0,L), \\ y(t,0) = y(t,L) = 0, & t > 0, \\ y_x(t,L) = \mathbf{sat}(\alpha y_x(t,0)), & t > 0, \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases} \tag{KdVs}$$

and

$$\begin{cases} y_t + y_x + y_{xxx} = 0, & t > 0, \ x \in (0,L), \\ y(t,0) = y(t,L) = 0, & t > 0, \\ y_x(t,L) = \mathbf{sat}(\alpha y_x(t,0)), & t > 0, \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases} \tag{KdVLs}$$

Note that both (KdVs) and (KdVLs) are nonlinear equations due to the action of the saturation. Keep in mind that (KdVLs) is just (KdVs) but without the nonlinear internal term yy_x.

The next theorem is the main result of this work,

Theorem 1. Let $L \notin \mathcal{N}$ and $|\alpha| < 1$, then there exist $C > 0$ and $\mu > 0$ such that:

- For all $y_0 \in L^2(0,L)$, the unique mild solution $y \in C([0,\infty), L^2(0,L)) \cap L^2((0,\infty), H^1(0,L))$ of (KdVLs) satisfies $\|y(t)\|_{L^2(0,L)} \leq Ce^{-\mu t}\|y_0\|_{L^2(0,L)}$.
- For all $y_0 \in L^2(0,L)$, satisfying $\|y_0\|_{L^2(0,L)} \leq \epsilon$ for some $\epsilon > 0$, the unique mild solution $y \in C([0,\infty), L^2(0,L)) \cap L^2((0,\infty), H^1(0,L))$ of (KdVs) satisfies $\|y(t)\|_{L^2(0,L)} \leq Ce^{-\mu t}\|y_0\|_{L^2(0,L)}$.

The article is organized as follows. Section 2 is devoted to prove the well-posedness results associated to (KdVLs) and (KdVs). In Section 3 we study the stability analysis of (KdVLs) and (KdVs) using an observability inequality. In Section 4 we show the asymptotic stability of (KdVLs) by using LaSalle's invariance principle. Finally, we give some conclusions and open problems. From now on, $C > 0$ will be a generic positive constant.

2. WELL-POSEDNESS RESULTS

The goal of this section is to prove the appropriate well-posedness results for (KdVLs) and (KdVs). The proof for (KdVLs) is based on semigroup theory. For (KdVs) we use the result for (KdVLs), then we will add an internal source term in order to take in account the nonlinear term yy_x, and derive some multiplier estimates to obtain the needed regularity.

Consider the operator $A : D(A) \to L^2(0, L)$ defined by

$$Aw := -w' - w''' \tag{2}$$

where

$D(A) = \{w \in H^3(0, L) \cap H_0^1(0, L) : w'(L) = \mathbf{sat}(\alpha w'(0))\}$.

Well-posedness asks to handle the fact that the operator A is nonlinear. We follow the approach presented in Marx et al. (2015); Prieur et al. (2016), i.e., we show that A is a closed, dissipative operator and that for some $\lambda > 0$, $D(A) \subset R(I - \lambda A)$, where R denotes the range of the operator.

2.1 *Well-posedness of* (KdVLs)

In this part, we focus on the study of (KdVLs), in particular we prove the following result.

Proposition 2. Let $y_0 \in L^2(0, L)$ and $|\alpha| < 1$, then there exists a unique mild solution $y \in C([0, \infty), L^2(0, L))$ of (KdVLs). Moreover, if $y_0 \in D(A)$, this solution is classical and $y \in C([0, \infty), D(A)) \cap C^1((0, \infty), L^2(0, L))$.

Proof. By Rosier (1997) the operator $A_0 : D(A_0) \to L^2(0, L)$, $D(A_0) = \{w \in H^3(0, L) \cap H_0^1(0, L) : w'(L) = 0\}$ such that $A_0 w = Aw$ is closed and by (Khalil and Grizzle, 2002, Page 91) the operator **sat** satisfies

$$\forall (s, \tilde{s}) \in \mathbb{R}^2, \quad |\mathbf{sat}(s) - \mathbf{sat}(\tilde{s})| \le |s - \tilde{s}|, \tag{3}$$

Thus, **sat** is a Lipschitz continuous function and hence the operator A is closed. Now, let us prove that A is dissipative, let u, $v \in D(A)$, then we get

$$\begin{aligned}
&\langle Au - Av, u - v\rangle_{L^2(0,L)} \\
&= \int_0^L (-u' - u''' + v' + v''')(u - v)dx \\
&= -\int_0^L (u - v)'(u - v)dx - \int_0^L (u - v)'''(u - v)dx.
\end{aligned}$$

Now, note that for $f \in H^3(0, L)$

$$\int_0^L f' f dx = \frac{1}{2}\int_0^L (f^2)' dx = \frac{1}{2}[f^2]_0^L, \tag{4}$$

and

$$\begin{aligned}
\int_0^L f''' f dx &= -\int_0^L f'' f' dx + [f'' f]_0^L, \\
&= -\frac{1}{2}\int_0^L (f'^2)' dx + [f'' f]_0^L, \\
&= -\frac{1}{2}[f'^2]_0^L + [f'' f]_0^L.
\end{aligned} \tag{5}$$

Thus, using (4), (5) and that u, $v \in D(A)$ we get

$$\begin{aligned}
&\langle Au - Av, u - v\rangle_{L^2(0,L)} = -\frac{1}{2}[(u - v)'^2]_0^L \\
&= \frac{1}{2}(\mathbf{sat}(\alpha u'(0)) - \mathbf{sat}(\alpha v'(0)))^2 - \frac{1}{2}(u'(0) - v'(0))^2.
\end{aligned}$$

Therefore, denoting

$$C_\alpha = \frac{1 - |\alpha|^2}{2} > 0 \tag{6}$$

which is positive by $|\alpha| < 1$, and using (3)

$$\langle Au - Av, u - v\rangle_{L^2(0,L)} \le -C_\alpha \left|u'(0) - v'(0)\right|^2 \le 0.$$

Thus, we get A dissipative. Now, we are going to prove that, for $\lambda > 0$ small enough $D(A) \subset R(I - \lambda A)$. In other words, for each $u \in D(A)$, there exists $v \in D(A)$ such that $(I - \lambda A)v = u$. Let $u \in D(A)$ and $\tilde{\lambda} = \frac{1}{\lambda}$ we are looking for solutions of

$$\begin{aligned}
&\tilde{\lambda} v + v' + v''' = \tilde{\lambda} u, \\
&v(0) = v(L) = 0, \ v'(L) = \mathbf{sat}(\alpha v'(0)).
\end{aligned} \tag{7}$$

Consider the map $\mathcal{T} : H^2(0, L) \to L^2(0, L)$ defined by $\mathcal{T}(y) = z$, where z is the solution of

$$\begin{aligned}
&\tilde{\lambda} z + z' + z''' = \tilde{\lambda} u, \\
&z(0) = z(L) = 0, \ z'(L) = \mathbf{sat}(\alpha y'(0)).
\end{aligned} \tag{8}$$

Note that the operator $\mathcal{T}$ is well-defined, that is, for all $y \in H^2(0, L)$ there exists a unique solution to (8). Indeed, let $\psi(x) = \frac{\mathbf{sat}(\alpha y'(0))L}{2\pi} \sin\left(\frac{2\pi x}{L}\right)$. Then $\varphi = z - \psi$ satisfies

$$\begin{aligned}
&\tilde{\lambda}\varphi + \varphi' + \varphi''' = f = \tilde{\lambda} u - \tilde{\lambda}\psi - \psi' - \psi''', \\
&\varphi(0) = \varphi(L) = \varphi'(L) = 0.
\end{aligned}$$

that can be written as $(A_0 - \tilde{\lambda} I)\varphi = -f$. But as $\tilde{\lambda} > 0$ then $\tilde{\lambda} \notin \sigma(A_0)$ (see for instance Rosier (1997)). Thus $(A_0 - \tilde{\lambda} I)$ is invertible, and we conclude that φ is well-defined and hence z. Now, we show that $\mathcal{T}$ has a fixed point. Let $C > 0$ to be defined after and consider the set

$$K_C = \{u \in H_0^1(0, L) \ : \ \|u\|_{H_0^1(0,L)} \le C\}. \tag{9}$$

Note that by the Rellich-Kondrachov theorem (Brezis, 2010, Thm 9.19, Page 285) we know that the inclusion $H_0^1(0, L) \hookrightarrow L^2(0, L)$ is compact and then K_C is a bounded set in $H_0^1(0, L)$ and relatively compact in $L^2(0, L)$, moreover as K_C is closed in $L^2(0, L)$ we conclude that K_C is compact in $L^2(0, L)$. Then in order to apply Schauder fixed point theorem (Coron, 2009, Thm B.19), we need to find $C > 0$ such that $\mathcal{T}(H^2(0, L)) \subset K_C$.

Lemma 3. There exists $C > 0$ such that $\mathcal{T}(H^2(0, L)) \subset K_C$.

Proof of Lemma 3. Multiplying the first line of (8) by z and integrating on $(0, L)$ we get,

$$\tilde{\lambda}\int_0^L z^2 dx + \int_0^L z' z dx + \int_0^L z''' z dx = \tilde{\lambda}\int_0^L u z dx. \tag{10}$$

Using (4) and (5) for z and noticing that $z(0) = z(L) = 0$, we obtain:

$$\int_0^L z'zdx = \frac{1}{2}\int_0^L (z^2)'dx = 0,$$

and using $z'(L) = \mathbf{sat}(\alpha y'(0))$

$$\int_0^L z'''zdx = -\frac{1}{2}\int_0^L (z'^2)'dx = \frac{1}{2}z'(0)^2 - \frac{1}{2}\mathbf{sat}(\alpha y'(0))^2,$$

which with (10) yields

$$\tilde{\lambda}\int_0^L z^2dx + \frac{1}{2}z'(0)^2 = \frac{1}{2}\mathbf{sat}(\alpha y'(0))^2 + \tilde{\lambda}\int_0^L uzdx.$$

Then, using Cauchy-Schwarz inequality and the definition of the saturation (1), we get

$$\frac{\tilde{\lambda}}{2}\|z\|^2_{L^2(0,L)} \leq \frac{M^2}{2} + \frac{\tilde{\lambda}}{2}\|u\|^2_{L^2(0,L)}. \tag{11}$$

Similarly, multiplying the first line of (8) by xz and integrating on $(0, L)$ we get

$$\tilde{\lambda}\int_0^L xz^2dx + \int_0^L xz'zdx + \int_0^L xz'''zdx = \tilde{\lambda}\int_0^L xuzdx. \tag{12}$$

Performing integration by parts and by the second line of (8) we obtain,

$$\int_0^L xz'zdx = \frac{1}{2}\int_0^L x(z^2)'dx = -\frac{1}{2}\int_0^L z^2dx,$$

and

$$\begin{aligned}\int_0^L xz'''zdx &= -\int_0^L z''zdx - \int_0^L xz''z'dx \\ &= \int_0^L (z')^2dx - \frac{1}{2}\int_0^L x(z'^2)'dx \\ &= \frac{3}{2}\int_0^L (z')^2dx - \frac{L}{2}\mathbf{sat}(\alpha y'(0))^2.\end{aligned}$$

Thus, from (12) we obtain

$$\begin{aligned}&\tilde{\lambda}\int_0^L xz^2dx + \frac{3}{2}\int_0^L (z')^2dx = \frac{1}{2}\int_0^L z^2dx \\ &+\frac{L}{2}\mathbf{sat}(\alpha y'(0))^2 + \tilde{\lambda}\int_0^L xuzdx,\end{aligned}$$

which yields

$$\begin{aligned}&\frac{3}{2}\|z'\|^2_{L^2(0,L)} \leq \left(\frac{L\tilde{\lambda}}{2} + \frac{1}{2}\right)\|z\|^2_{L^2(0,L)} + \frac{LM^2}{2} \\ &+\frac{L\tilde{\lambda}}{2}\|u\|^2_{L^2(0,L)},\end{aligned} \tag{13}$$

and hence using (11) we deduce, $\|z'\|^2_{L^2(0,L)} \leq \tilde{C}$, where $\tilde{C} > 0$ only depends on the level of saturation M, L, $\tilde{\lambda}$ and u which finishes the proof of Lemma 3. □

Then, applying Schauder fixed point theorem to $\mathcal{T}$, we find the solution $v \in H^2(0, L)$ of (7). Moreover, as $u \in D(A)$, then $v \in H^3(0, L)$ and hence by the boundary conditions $v \in D(A)$. We conclude that for all $\lambda > 0$, $D(A) \subset R(A - \lambda I)$. Using (Miyadera, 1992, Thm 4.2, Page 77) A is the generator of a semigroup of contractions $S(t)$, $t > 0$ and by (Miyadera, 1992, Thm 4.2, Page 81) (or (Barbu, 1976, Thm 1.2, Page 102)) for all $T > 0$, $y_0 \in D(A)$, $y = S(t)y_0 \in C([0, T]; D(A)) \cap C^1((0, T); L^2(0, L))$ is a classical solution to (KdVLs). Moreover, by (Miyadera, 1992, Thm 4.10, Page 87) this solution is unique. Also, we can deduce that for all $T > 0$, for all $y_0 \in L^2(0, L)$, there exists a unique mild solution of (KdVLs) $y \in C([0, T]; L^2(0, L))$ and then, the proof of Proposition 2 is finished. □

2.2 Well-posedness of equation (KdVLs) *with a source term*

Now, we deal with the well-posedness of the system (KdVLs) with an extra source term f

$$\begin{cases} y_t + y_x + y_{xxx} = f, & t > 0,\ x \in (0, L), \\ y(t, 0) = y(t, L) = 0, & t > 0, \\ y_x(t, L) = \mathbf{sat}(\alpha y_x(t, 0)), & t > 0, \\ y(0, x) = y_0(x), & x \in (0, L). \end{cases} \tag{14}$$

Formally, this system can be written as $y_t = Ay + f$. The following result can be obtained using (Pavel, 1987, Thm 10.1, Page 129) and (Pavel, 1987, Thm 10.1, Page 132)

Proposition 4. Let $y_0 \in L^2(0, L)$, $|\alpha| < 1$, $f \in L^1((0, \infty), L^2(0, L))$, then there exists a unique mild solution $y \in C([0, \infty), L^2(0, L))$ of (14). Moreover, if $y_0 \in D(A)$ and $f \in W^{1,1}((0, \infty), L^2(0, L))$ this solution is classical and $y \in C([0, \infty), L^2(0, L))$, $y(t) \in D(A)$, for all $t \geq 0$ and $y \in C^1((0, \infty), L^2(0, L))$.

Our idea now is to show that, indeed, the solution of (14) has an extra-regularity that will help us to deal the nonlinear term yy_x. More specifically, let $T > 0$ and define

$$\mathcal{B}_T = C([0, T], L^2(0, L)) \cap L^2((0, T), H^1(0, L)) \tag{15}$$

endowed with the norm

$$\|y\|^2_{\mathcal{B}_T} = \|y\|^2_{C([0,T],L^2(0,L))} + \|y\|^2_{L^2((0,T),H^1(0,L))}.$$

Proposition 5. Let $y_0 \in L^2(0, L)$, $|\alpha| < 1$, $f \in L^1((0, T), L^2(0, L))$ and $T > 0$, consider $y \in C([0, T], L^2(0, L))$, the restriction of the solution of (14) on $[0, T]$, then $y \in \mathcal{B}_T$. Moreover, the following estimate holds,

$$\|y\|_{\mathcal{B}_T} \leq C(1 + \sqrt{T})\left(\|y_0\|_{L^2(0,L)} + \|f\|_{L^1((0,T),L^2(0,L))}\right). \tag{16}$$

Proof. Suppose y is a classical solution of (14), let $T > 0$ and $0 \leq s \leq T$. Several computations are quite similar to those performed in the proof of Lemma 3 and for that reason are omitted here. Multiplying the first line of (14) by y and integrating on $(0, L) \times (0, s)$, we get

$$\begin{aligned}&\int_0^s\int_0^L y_tydxdt + \int_0^s\int_0^L y_xydxdt + \int_0^s\int_0^L y_{xxx}ydxdt \\ &= \int_0^s\int_0^L fydxdt.\end{aligned} \tag{17}$$

Note that,

$$\int_0^s y_tydt = \frac{1}{2}\int_0^s \frac{d}{dt}y^2dt = \frac{1}{2}y^2(s, \cdot) - \frac{1}{2}y_0^2. \tag{18}$$

Hence, using (4) and (5), (17) becomes

$$\begin{aligned}&\frac{1}{2}\int_0^L y(s, x)^2dx + \frac{1}{2}\int_0^s \left(y_x(t, 0)^2 - \mathbf{sat}(\alpha y_x(t, 0))^2\right)dt \\ &= \frac{1}{2}\int_0^L y_0^2dx + \int_0^s\int_0^L fydxdt.\end{aligned} \tag{19}$$

First, note that for all $s \in [0, T]$

$$\int_0^s\int_0^L fydxdt \le \int_0^T \|y(t,\cdot)\|_{L^2(0,L)}\|f(t,\cdot)\|_{L^2(0,L)}dt \le \|y\|_{C([0,T],L^2(0,L))}\|f\|_{L^1((0,T),L^2(0,L))}. \tag{20}$$

Now, using (1), (19) and (20) we get for all $s\in[0,T]$,

$$\frac{1}{2}\int_0^L y(s,x)^2dx + C_\alpha\int_0^s y_x(t,0)^2dt \le \frac{1}{2}\int_0^L y_0^2dx + \|y\|_{C([0,T],L^2(0,L))}\|f\|_{L^1((0,T),L^2(0,L))}. \tag{21}$$

Taking the supremum for $s\in[0,T]$, and with Young's inequality, we get

$$\|y\|^2_{C([0,T],L^2(0,L))} \le C\left(\|y_0\|^2_{L^2(0,L)} + \|f\|^2_{L^1((0,T),L^2(0,L))}\right). \tag{22}$$

Moreover, from (21) taking $s=T$, and using (22) we can derive the following hidden regularity

$$\int_0^T y_x(t,0)^2dt \le C\left(\|y_0\|^2_{L^2(0,L)} + \|f\|^2_{L^1((0,T),L^2(0,L))}\right). \tag{23}$$

Now, we multiply the first line of (14) by xy and integrate on $(0,L)\times(0,T)$ and we get

$$\int_0^T\int_0^L xy_tydxdt + \int_0^T\int_0^L xy_xydxdt + \int_0^T\int_0^L xy_{xxx}ydxdt = \int_0^T\int_0^L xfydxdt. \tag{24}$$

By (18)

$$\int_0^L\int_0^T xy_tydt = \frac{1}{2}\int_0^L xy^2(T,x)dx - \frac{1}{2}\int_0^L xy_0^2dx.$$

Moreover, using the second and third line of (14) we have,

$$\int_0^T\int_0^L xy_xydxdt = \frac{1}{2}\int_0^T\int_0^L x\frac{d}{dx}y^2dxdt = -\frac{1}{2}\int_0^T\int_0^L y^2dxdt,$$

and

$$\begin{aligned}\int_0^T\int_0^L xy_{xxx}ydxdt &= -\int_0^T\int_0^L y_{xx}ydxdt - \int_0^T\int_0^L xy_{xx}y_xdxdt\\ &= \int_0^T\int_0^L y_x^2dxdt - \frac{1}{2}\int_0^T\int_0^L x\frac{d}{dx}y_x^2dxdt\\ &= \frac{3}{2}\int_0^T\int_0^L y_x^2dxdt - \frac{L}{2}\int_0^T \mathbf{sat}(\alpha y_x(t,0)^2dt.\end{aligned}$$

Thus, from (24) we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^L xy(T,x)^2dx + \frac{3}{2}\int_0^T\int_0^L y_x^2dxdt = \frac{1}{2}\int_0^L xy_0^2dx\\ &+\frac{1}{2}\int_0^T\int_0^L y^2dxdt + \frac{L}{2}\int_0^T \mathbf{sat}(\alpha y_x(t,0))^2dt\\ &+\int_0^T\int_0^L xfydxdt.\end{aligned} \tag{25}$$

Note that

$$\int_0^T\int_0^L y^2dxdt \le \int_0^T\left(\sup_{t\in[0,T]}\int_0^L y^2(t,\cdot)dx\right)dt = T\|y\|^2_{C([0,T],L^2(0,L))} \tag{26}$$

Moreover, by (1) and (23)

$$\int_0^T \mathbf{sat}(\alpha y_x(t,0))^2dt \le C\left(\|y_0\|^2_{L^2(0,L)} + \|f\|^2_{L^1((0,T),L^2(0,L))}\right),$$

and by (20) and (22)

$$\int_0^T\int_0^L xfydxdt \le C\left(\|y_0\|^2_{L^2(0,L)} + \|f\|^2_{L^1((0,T),L^2(0,L))}\right).$$

Therefore, joining these estimates, we deduce from (25)

$$\|y_x\|^2_{L^2((0,T),L^2(0,L))} \le C(1+T)\left(\|y_0\|^2_{L^2(0,L)} + \|f\|^2_{L^1((0,T),L^2(0,L))}\right), \tag{27}$$

and thus $y\in L^2((0,T),H^1(0,L))$. Mixing (22) and (27) we deduce (16). The result can be extended by density to mild solutions. □

2.3 Well-posedness of the nonlinear system

Now, we are ready to give the proof of the well-posedness result for (KdVs). For that the next proposition will be crucial, its proof can be founded in Perla Menzala et al. (2002).

Proposition 6. Let $y\in\mathcal{B}_T$. Then $yy_x\in L^1((0,T),L^2(0,L))$ and the map $y\in\mathcal{B}_T\mapsto yy_x\in L^1((0,T),L^2(0,L))$ is continuous. Moreover, there exists $C>0$ such that, for any y, $z\in\mathcal{B}_T$

$$\|yy_x - zz_x\|_{L^1((0,T),L^2(0,L))} \le CT^{1/4}(\|y\|_{\mathcal{B}_T} + \|z\|_{\mathcal{B}_T})\|y-z\|_{\mathcal{B}_T}.$$

The main result of this subsection is the following one.

Proposition 7. Let $y_0\in L^2(0,L)$ and $|\alpha|<1$, then there exists a unique mild solution $y\in C([0,\infty),L^2(0,L))\cap L^2((0,\infty),H^1(0,L))$ of (KdVs).

Proof. Let $y_0\in D(A)$ and R, $\theta>0$ to be chosen later. Consider the closed ball $B_{\mathcal{B}_\theta}(0,R) := \{z\in\mathcal{B}_\theta,\ \|z\|_{\mathcal{B}_\theta}\le R\}$, where $\mathcal{B}_\theta$ is defined by (15). Then, $B_{\mathcal{B}_\theta}(0,R)$ is a complete metric space. Let the map $\Phi:\mathcal{B}_\theta\to\mathcal{B}_\theta$ defined for $z\in\mathcal{B}_\theta$ by $\Phi(z)=y$, where y is the solution of

$$\begin{cases} y_t + y_x + y_{xxx} = -zz_x, & t>0,\ x\in(0,L),\\ y(t,0)=y(t,L)=0, & t>0,\\ y_x(t,L) = \mathbf{sat}(\alpha y_x(t,0)), & t>0,\\ y(0,x)=y_0(x), & x\in(0,L).\end{cases}$$

Note first that y is well-defined, since by Proposition 6, $zz_x\in L^1((0,\theta),L^2(0,L))$. Clearly, $y\in\mathcal{B}_\theta$ is solution of (KdVs) if y is a fixed point of Φ. From Proposition 5 and Proposition 6, we get for all $z\in\mathcal{B}_\theta$

$$\begin{aligned}\|\Phi(z)\|_{\mathcal{B}_\theta} = \|y\|_{\mathcal{B}_\theta} &\le C(1+\theta^{1/2})\left(\|y_0\|_{L^2(0,L)} + \|zz_x\|_{L^1((0,T),L^2(0,L))}\right)\\ &\le C_1(1+\theta^{1/2})\|y_0\|_{L^2(0,L)} + C_2(1+\theta^{1/2})\theta^{1/4}\|z\|^2_{\mathcal{B}_\theta},\end{aligned}$$

for some C_1, $C_2>0$. Then taking $R=3C_1\|y_0\|_{L^2(0,L)}$ and $\theta>0$ small enough such that $C_2(1+\theta^{1/2})\theta^{1/4}R<\frac{1}{3}$ and $(1+\theta^{1/2})<2$, we have that $\|\Phi(z)\|_{\mathcal{B}_\theta}\le R$. Thus, Φ maps the ball $B_{\mathcal{B}_\theta}(0,R)$ into itself. Now, take z_1, $z_2\in B_{\mathcal{B}_\theta}(0,R)$, $y_1=\Phi(z_1)$, $y_2=\Phi(z_2)$, then $w=\Phi(z_1)-\Phi(z_2)$ satisfies

$$\begin{cases} w_t + w_x + w_{xxx} = f, & t>0,\ x\in(0,L),\\ w(t,0)=w(t,L)=0, & t>0,\\ w_x(t,L)=h(t), & t>0,\\ w(0,x)=0, & x\in(0,L). \end{cases} \tag{28}$$

where $f = -z_1 z_{1,x} + z_2 z_{2,x}$ and $h = \mathbf{sat}(\alpha y_{1,x}(t,0)) - \mathbf{sat}(\alpha y_{2,x}(t,0))$. First, note that by (23), $h \in L^2(0,\theta)$ and hence (28) is well-posed (Cerpa, 2014, Prop 3). Moreover,

$$\|w\|_{\mathcal{B}_\theta} \le C\left(\|h\|_{L^2(0,\theta)} + \|f\|_{L^1((0,\theta),L^2(0,L))}\right). \tag{29}$$

We can not use directly the estimates obtained in Section 2.2, because $h \neq \mathbf{sat}(\alpha w_x(t,0))$, but similar estimates can be obtained. If we multiply the first line of (28) by w and integrate on $(0,L)\times(0,\theta)$, we get

$$\int_0^\theta\int_0^L w_t w dxdt + \int_0^\theta\int_0^L w_x w dxdt + \int_0^\theta\int_0^L w_{xxx} w dxdt = \int_0^\theta\int_0^L f w dxdt. \tag{30}$$

Similar computations as in (19) show that

$$\int_0^\theta\int_0^L w_t w dxdt = \frac{1}{2}\left[\int_0^L w^2(t,x)dx\right]_{t=0}^{t=\theta},$$

$$\int_0^\theta\int_0^L w_x w dxdt = 0,$$

and

$$\int_0^\theta\int_0^L w_{xxx} w dxdt = \frac{1}{2}\int_0^\theta w_x^2(t,0)dt - \frac{1}{2}\int_0^\theta h(t)^2 dt.$$

Thus, from (30) we derive

$$\frac{1}{2}\int_0^L w(\theta,x)^2 dx + \frac{1}{2}\int_0^\theta w_x(t,0)^2 dt - \frac{1}{2}\int_0^\theta h(t)^2 dt = \int_0^\theta\int_0^L f w dxdt. \tag{31}$$

Since $h = \mathbf{sat}(\alpha y_{1,x}(t,0)) - \mathbf{sat}(\alpha y_{2,x}(t,0))$, using (3) we derive,

$$\|h\|_{L^2(0,\theta)} = \|\mathbf{sat}(\alpha y_{1,x}(t,0)) - \mathbf{sat}(\alpha y_{2,x}(t,0))\|_{L^2(0,\theta)} \le \|\alpha y_{1,x}(\cdot,0) - \alpha y_{2,x}(\cdot,0)\|_{L^2(0,\theta)} = |\alpha|\|w_x(\cdot,0)\|_{L^2(0,\theta)}. \tag{32}$$

Thus, plugging (32) in (31), we obtain

$$\frac{1}{2}\int_0^L w(\theta,x)^2 dx + C_\alpha\int_0^\theta w_x(t,0)^2 dt \le \int_0^\theta\int_0^L f w dxdt, \tag{33}$$

which implies using Young's inequality

$$\|w\|_{C([0,\theta],L^2(0,L))} \le C\|f\|_{L^1((0,\theta),L^2(0,L))}. \tag{34}$$

Using (34) in (33), we get

$$\int_0^\theta w_x(t,0)^2 dt \le C\|f\|^2_{L^1((0,\theta),L^2(0,L))},$$

and then (32) brings

$$\|h\|_{L^2(0,\theta)} \le C\|f\|_{L^1((0,\theta),L^2(0,L))},$$

which transforms (29) into

$$\|w\|_{\mathcal{B}_\theta} \le C\|f\|_{L^1((0,\theta),L^2(0,L))}. \tag{35}$$

Finally, by (35) and Proposition 6 we conclude:

$$\|\Phi(z_1) - \Phi(z_2)\|_{\mathcal{B}_\theta} = \|w\|_{\mathcal{B}_\theta} \le C_3\theta^{1/4}R\|z_1 - z_2\|_{\mathcal{B}_\theta},$$

for some $C_3 > 0$. Then taking $\theta > 0$, small enough such that $C_3\theta^{1/4}R < 1$, we get that Φ is a contraction, and we conclude by using the Banach Fixed Point theorem (Brezis, 2010, Thm 5.7) and density argument to extend the result for all $y_0 \in L^2(0,L)$. Thus, we have proved the following local in-time well-posedness result:

Proposition 8. Let $y_0 \in L^2(0,L)$, $|\alpha| < 1$ and $T > 0$. Then there exists $0 < T^* \le T$ and a unique mild solution $y \in \mathcal{B}_{T^*}$ of (KdVs) on the time interval $[0,T^*]$. Moreover, there exists $C > 0$ such that $\|y\|_{\mathcal{B}_{T^*}} \le C\|y_0\|_{L^2(0,L)}$.

To conclude the proof of the global well-posedness result stated on Proposition 7, we need some a priori estimates, based on a multiplier approach, similar to those presented in Section 2.2. Let $T > 0$ and y a classical solution of (KdVs). Multiplying the first line of (KdVs) by y, integrating on $(0,L)\times(0,s)$ we get

$$\int_0^s\int_0^L y_t y dxdt + \int_0^s\int_0^L y_x y dxdt + \int_0^s\int_0^L y_{xxx} y dxdt + \int_0^s\int_0^L y_x y^2 dxdt = 0. \tag{36}$$

We focus on the term $\int_0^s\int_0^L y_x y^2 dxdt$, the other terms are computing as in (20).

$$\int_0^s\int_0^L y_x y^2 dxdt = \frac{1}{3}\int_0^s\int_0^L \frac{d}{dx}y^3 dxdt = \frac{1}{3}\left[\int_0^s y^3(t,x)dt\right]_{x=0}^{x=L} = 0.$$

Therefore, we derive from (36)

$$\frac{1}{2}\int_0^L y(s,x)^2 dx + \frac{1}{2}\int_0^s \left(y_x(t,0)^2 - \mathbf{sat}(\alpha y_x(t,0))^2\right) = \frac{1}{2}\int_0^L y_0^2 dx. \tag{37}$$

Using the definition of the saturation (1) we derive

$$\frac{1}{2}\int_0^L y(s,x)^2 dx + C_\alpha\int_0^s y_x(t,0)^2 dt \le \frac{1}{2}\int_0^L y_0^2 dx. \tag{38}$$

Then, taking the supremum for $s \in [0,T]$ in (38),

$$\|y\|^2_{C([0,T],L^2(0,L))} \le \|y_0\|^2_{L^2(0,L)}. \tag{39}$$

On the other hand, taking $s = T$ in (38),

$$\int_0^T y_x(t,0)^2 dt \le \frac{1}{2}C_\alpha^{-1}\|y_0\|^2_{L^2(0,L)}. \tag{40}$$

Now, we multiply the first line of (KdVs) by xy and integrate on $(0,L)\times(0,T)$, obtaining

$$\int_0^T\int_0^L x y_t y dxdt + \int_0^T\int_0^L x y_x y dxdt + \int_0^T\int_0^L x y_{xxx} y dxdt + \int_0^T\int_0^L x y^2 y_x dxdt = 0. \tag{41}$$

We focus on the term $\int_0^T\int_0^L x y^2 y_x dxdt$, the other terms are computing as in (25),

$$\int_0^T\int_0^L x y^2 y_x dxdt = \frac{1}{3}\int_0^T\int_0^L x\frac{d}{dx}y^3 dxdt = -\frac{1}{3}\int_0^T y^3 dxdt + \frac{1}{3}\left[\int_0^T x y^3(t,x)dt\right]_{x=0}^{x=L} = -\frac{1}{3}\int_0^T y^3 dxdt.$$

Hence, we infer from (41)

$$\frac{1}{2}\int_0^L xy(T,x)^2dx+\frac{3}{2}\int_0^T\int_0^L y_x^2dxdt=\frac{1}{2}\int_0^L xy_0^2dx$$
$$+\frac{1}{2}\int_0^T\int_0^L y^2dxdt+\frac{L}{2}\int_0^T \mathbf{sat}(\alpha y_x(t,0))^2dt$$
$$+\frac{1}{3}\int_0^T\int_0^L y^3dxdt. \tag{42}$$

For the term involving $\int_0^L y^3(t,x)dx$, by the injection of $H_0^1(0,L)$ into $L^\infty(0,L)$ we know that $\|y\|_{L^\infty(0,L)}\leq C\|y_x\|_{L^2(0,L)}$, for some $C>0$ then

$$\int_0^T\int_0^L y^3(t,x)dxdt\leq\int_0^T\|y\|_{L^\infty(0,L)}\|y\|^2_{L^2(0,L)}dt$$
$$\leq C\int_0^T\|y_x\|_{L^2(0,L)}\|y\|^2_{L^2(0,L)}dt,$$

which implies using (39)

$$\int_0^T\int_0^L y^3(t,x)dxdt\leq C\sqrt{T}\|y_0\|^2_{L^2(0,L)}\|y_x\|_{L^2((0,T),L^2(0,L))}. \tag{43}$$

Thus, from (42) using (26), (39), (40) and (43) we derive

$$\|y_x\|^2_{L^2((0,T),L^2(0,L))}\leq C(1+T)\|y_0\|^2_{L^2(0,L)}$$
$$+C\sqrt{T}\|y_0\|^2_{L^2(0,L)}\|y_x\|_{L^2((0,T),L^2(0,L))}.$$

Using Young's inequality

$$\|y_x\|^2_{L^2((0,T),L^2(0,L))}\leq C(1+T)\|y_0\|^2_{L^2(0,L)}+CT\|y_0\|^4_{L^2(0,L)}. \tag{44}$$

Finally, with (39) and (44) and arguing by density, the proof of Proposition 7 is finished. □

3. EXPONENTIAL STABILITY

In this section, we are going to prove the main result of this work, namely Theorem 1. For all $t>0$, we define the energy of (KdVs) (or (KdVLs)) by

$$E(t)=\frac{1}{2}\int_0^L y(t,x)^2dx. \tag{45}$$

Then, for a classical solution of (KdVs) (or (KdVLs)) performing integration by parts, we get using $|\alpha|<1$

$$\dot{E}(t)=\frac{1}{2}\left(\mathbf{sat}(\alpha y_x(t,0))\right)^2-\frac{1}{2}(y_x(t,0))^2\leq 0. \tag{46}$$

Thus, formally, the energy is a non-increasing function of time. Now, we will prove that indeed, the energy decays exponentially to 0 when $L\notin\mathcal{N}$. Our approach will be based on an observability inequality for the system (KdVLs).

Proposition 9. Let $L\notin\mathcal{N}$, $|\alpha|<1$ and $y_0\in L^2(0,L)$. For any solution y of (KdVLs), the following inequality holds:

$$\int_0^L y_0^2dx\leq C_{obs}\int_0^T y_x^2(t,0)dt, \tag{47}$$

for some $C_{obs}>0$, that does not depend on y_0.

Proof. In order to prove the observability inequality, we follow Rosier (1997) and use a contradiction argument. Let us suppose that (47) is false, then there exists $(y_0^n)_{n\in\mathbb{N}}\subset L^2(0,L)$ such that

$$\int_0^L (y_0^n)^2dx=1,\ \text{and}\ \int_0^T (y_x^n(t,0))^2dt\to 0$$

when $n\to\infty$ and where $y^n=S(t)y_0^n$. Now, using (27) with $f=0$ we get that $(y^n)_{n\in\mathbb{N}}$ is bounded in $L^2((0,T),H^1(0,L))$. Then as $y_t^n=-y_x^n-y_{xxx}^n$, we can see that $(y_t^n)_{n\in\mathbb{N}}$ is bounded in $L^2((0,T),H^{-2}(0,L))$, thus by Aubin-Lions lemma $(y^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2((0,T),L^2(0,L))$, therefore we can assume that $y^n\to y$ in $L^2((0,T),L^2(0,L))$. Now, multiplying the first line of (KdVLs) by $(T-t)y$, integrating on $(0,L)\times(0,T)$, we get

$$\int_0^T\int_0^L (T-t)y_tydxdt+\int_0^T\int_0^L (T-t)y_{xxx}ydxdt$$
$$+\int_0^T\int_0^L (T-t)y_xydxdt+\int_0^T\int_0^L (T-t)y^2y_xdxdt=0. \tag{48}$$

Using

$$\int_0^L y_xydx=0,\quad \int_0^L y^2y_xdx=0,$$
$$\int_0^L y_{xxx}ydx=\frac{y_x^2(t,0)}{2}-\frac{\mathbf{sat}(\alpha y_x(t,0))^2}{2},$$

and

$$\int_0^T\int_0^L (T-t)y_tydxdt=\frac{1}{2}\int_0^T\int_0^L (T-t)\frac{d}{dt}y^2dxdt$$
$$=\frac{1}{2}\int_0^T\int_0^L y^2dxdt+\frac{1}{2}\left[\int_0^L (T-t)y^2(t,x)dx\right]_{t=0}^{t=T}$$
$$=\frac{1}{2}\int_0^T\int_0^L y^2dxdt-\frac{1}{2}\int_0^L Ty_0^2dx,$$

we get from (48)

$$\frac{T}{2}\int_0^L y_0^2dx+\frac{1}{2}\int_0^T (T-t)\mathbf{sat}(\alpha y_x(t,0))^2dt$$
$$=\frac{1}{2}\int_0^T\int_0^L y^2dxdt+\frac{1}{2}\int_0^T (T-t)y_x(t,0)^2dt.$$

Therefore,

$$\int_0^L y_0^2dx\leq\frac{1}{T}\|y\|^2_{L^2(0,T,L^2(0,L))}+\int_0^T y_x(t,0)^2dt. \tag{49}$$

Hence, using (49) we derive that $(y_0^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(0,L)$. Let $y_0=\lim_{n\to\infty}y_0^n$, then

$$\int_0^L y_0^2dx=1,\ \text{and}\ \int_0^T y_x(t,0)^2dt=0$$

Thus, as $\mathbf{sat}(0)=0$, we conclude that y solves the following linear equation:

$$\begin{cases} y_t+y_x+y_{xxx}=0, & t>0,\ x\in(0,L),\\ y(t,0)=y(t,L)=0, & t>0,\\ y_x(t,0)=y_x(t,L)=0, & t>0,\end{cases} \tag{50}$$

But by (Rosier, 1997, Lemma 3.4) if $L\notin\mathcal{N}$, there is no function satisfying this system and $\|y_0\|_{L^2(0,L)}=1$, which gives us the contradiction and finishes the proof of Proposition 9. □

Now, we prove our main result.

Proof of Theorem 1. Let $y_0\in D(A)$, using (46) and (3) we get for a classical solution of (KdVLs)

$$\dot{E}(t)\leq -C_\alpha(y_x(t,0))^2,$$

thus, integrating the last expression between 0 and T, we derive

$$E(T) - E(0) \leq -C_\alpha \int_0^T (y_x(t,0))^2 dt.$$

Using the last expression, (46) and (47) we get

$$\begin{aligned} E(T) &\leq E(0) \leq C_{obs} \int_0^T (y_x(t,0))^2 dt \\ &\leq \frac{C_{obs}}{C_\alpha}(E(0) - E(T)), \end{aligned}$$

which implies that

$$E(T) \leq \gamma E(0), \quad \text{with } \gamma = \frac{C_{obs}}{C_\alpha + C_{obs}} < 1. \tag{51}$$

Now as the system is invariant by translation in time, we can repeat this argument on $[(m-1)T, mT]$ for $m = 1, 2, \cdots$ to obtain,

$$E(mT) \leq \gamma E((m-1)T) \leq \cdots \leq \gamma^m E(0).$$

Hence, we have $E(mT) \leq e^{-\mu m T} E(0)$ where $\mu = \frac{1}{T}\ln(\frac{1}{\gamma}) > 0$. Let $t > 0$ then there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and then using again the non-increasing property of the energy we get

$$E(t) \leq E((m-1)T) \leq e^{-\mu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\mu t} E(0).$$

Finally, we conclude the exponential stability of (KdVLs) by a density argument and hence the first point of Theorem 1.

Since (KdVs) is also invariant by translation in time, to prove the second point of Theorem 1, the goal is to prove (51) for a solution of (KdVs). This idea was used in several works, for instance Perla Menzala et al. (2002); Parada et al. (2022). More specifically, we are going to show that $\|y(T)\|_{L^2(0,L)} \leq \tilde{\gamma}\|y_0\|_{L^2(0,L)}$, for some $\tilde{\gamma} \in (0,1)$. Let $\|y_0\|_{L^2(0,L)} \leq \epsilon$ for $\epsilon > 0$ to fix later. Take y the solution of (KdVs), then we can split y as $y = \overline{y} + \tilde{y}$, where

$$\begin{cases} \overline{y}_t + \overline{y}_x + \overline{y}_{xxx} = 0, & t > 0,\ x \in (0,L), \\ \overline{y}(t,0) = \overline{y}(t,L) = 0, & t > 0, \\ \overline{y}_x(t,L) = \mathbf{sat}(\alpha \overline{y}_x(t,0)), & t > 0, \\ \overline{y}(0,x) = y_0(x), & x \in (0,L), \end{cases}$$

and

$$\begin{cases} \tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = -y y_x, & t > 0,\ x \in (0,L), \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, & t > 0, \\ \tilde{y}_x(t,L) = h(t), & t > 0, \\ \tilde{y}(0,x) = 0, & x \in (0,L), \end{cases}$$

where $h(t) = \mathbf{sat}(\alpha y_x(t,0)) - \mathbf{sat}(\alpha \overline{y}_x(t,0))$.

Now clearly,

$$\|y(T,\cdot)\|_{L^2(0,L)} \leq \|\overline{y}(T,\cdot)\|_{L^2(0,L)} + \|\tilde{y}(T,\cdot)\|_{L^2(0,L)}. \tag{52}$$

Moreover, as $\overline{y}$ satisfies (KdVLs), we get using (51) and $\overline{y}(0,x) = y_0(x)$ that there exists $\gamma < 1$ such that

$$\|\overline{y}(T)\|_{L^2(0,L)} \leq \gamma \|y_0\|_{L^2(0,L)}. \tag{53}$$

Besides,

$$\|\tilde{y}(T,\cdot)\|_{L^2(0,L)} \leq \|\tilde{y}\|_{C([0,T],L^2(0,L))}.$$

To estimate $\|\tilde{y}\|_{C([0,T],L^2(0,L))}$ using (34) we get

$$\|\tilde{y}\|_{C([0,T],L^2(0,L))} \leq C\|y y_x\|_{L^1((0,T),L^2(0,L))},$$

by Proposition 6,

$$\|y y_x\|_{L^1((0,T),L^2(0,L))} \leq C\|y\|^2_{\mathcal{B}_T},$$

using (39)-(44), we derive

$$\|y\|^2_{\mathcal{B}_T} \leq C\left(\|y_0\|^2_{L^2(0,L)} + \|y_0\|^4_{L^2(0,L)}\right).$$

Thus, we get for $\tilde{y}$

$$\|\tilde{y}\|_{C([0,T],L^2(0,L))} \leq C\left(\|y_0\|^2_{L^2(0,L)} + \|y_0\|^4_{L^2(0,L)}\right),$$

and hence

$$\begin{aligned} \|\tilde{y}(T,\cdot)\|_{L^2(0,L)} \leq \big(&C\|y_0\|_{L^2(0,L)} \\ &+ C\|y_0\|^3_{L^2(0,L)}\big)\|y_0\|_{L^2(0,L)}, \end{aligned} \tag{54}$$

Finally, as $\|y_0\|_{L^2(0,L)} \leq \epsilon$ using (53) and (54), we deduce from (52)

$$\begin{aligned} &\|y(T,\cdot)\|_{L^2(0,L)} \leq \|\overline{y}(T,\cdot)\|_{L^2(0,L)} + \|\tilde{y}(T,\cdot)\|_{L^2(0,L)} \\ &\leq \gamma\|y_0\|_{L^2(0,L)} + (C\epsilon + C\epsilon^3)\|y_0\|_{L^2(0,L)} \\ &= (\gamma + C\epsilon + C\epsilon^3)\|y_0\|_{L^2(0,L)}, \end{aligned}$$

we conclude by choosing $\epsilon > 0$ small enough such that $\tilde{\gamma} = \gamma + C\epsilon + C\epsilon^3 < 1$. □

4. ASYMPTOTIC STABILITY VIA LASALLE'S INVARIANCE PRINCIPLE

Another approach to prove the stability of (KdVLs) is to use Lyapunov techniques and LaSalle's invariance principle. This idea was developed in the case of internal saturation in Marx et al. (2015). The Lyapunov function to take in account is the energy defined by (45).

Proposition 10. Let $L \notin \mathcal{N}$ and $|\alpha| < 1$, then for all $y_0 \in L^2(0,L)$, the mild solution y of (KdVLs) satisfies:

$$\|y(t,\cdot)\|_{L^2(0,L)} \to 0, \text{ as } t \to \infty.$$

Proof. First, note that by (46), $\dot{E}(t) \leq 0$ that tells us that for regular initial data the solution of (KdVLs) is stable and moreover $\dot{E}(t) = 0$ if and only if $y_x(t,0) = 0$. In fact, we consider now two cases:

- $|\mathbf{sat}(\alpha y_x(t,0))| = M$. This happens when $|\alpha y_x(t,0)| \geq M$ which implies $|y_x(t,0)| \geq \frac{M}{|\alpha|}$. By (46), $\dot{E}(t) = 0$ if and only if $M = |y_x(t,0)|$. Then $M \geq \frac{M}{|\alpha|}$, which is a contradiction because $|\alpha| < 1$.
- $\mathbf{sat}(\alpha y_x(t,0)) = \alpha y_x(t,0)$. Then $\dot{E}(t) = 0$ if and only if $\alpha^2|y_x(t,0)|^2 = |y_x(t,0)|^2$, which is only possible if $y_x(t,0) = 0$.

To apply LaSalle's invariance principle, we need that the trajectories are compact, in this context, it is possible to show the following result.

Lemma 11. The canonical embedding from $D(A)$, equipped with the graph norm, into $L^2(0,L)$ is compact.

Now as A generates a semigroup of contractions, then from (Brezis, 1973, Thm 3.1 Page 54), we get that for all $t \geq 0$ and $y(0,\cdot) \in D(A)$,

$$\|y(t,\cdot)\|_{L^2(0,L)} \leq \|y(0,\cdot)\|_{L^2(0,L)},$$

$$\|Ay(t,\cdot)\|_{L^2(0,L)} \leq \|Ay(0,\cdot)\|_{L^2(0,L)}.$$

Thus, by Lemma 11, we have that the trajectory $\{v(t) = S(t)v_0, t \geq 0\}$ is pre-compact in $L^2(0,L)$, then by (Slemrod, 1989, Thm, 3,1) the $\omega-$limit set, $\omega[y(0,\cdot)] \subset D(A)$, is not empty and invariant to the semigroup $S(t)$. Take now a regular solution such that $\dot{E}(t) = 0$, that is $y_x(t,0) = 0$ $t \geq 0$. Then y solves (50) and by (Rosier, 1997, Lemma 3.4) if $L \notin \mathcal{N}$ $y(t,x) = 0$ a.e $x \in (0,L)$. Consequently,

$$\|y(t,\cdot)\|_{L^2(0,L)} \to 0, \text{ as } t \to \infty$$

for the regular solution of (KdVLs). It is possible to extend this result to mild solutions by density. □

CONCLUSION

The exponential stability of the nonlinear KdV equation with a saturated feedback on the boundary was addressed. The well-posedness is obtained using nonlinear semigroup theory and fixed point results. The exponential stability was shown via the classical observability approach, and the asymptotic stability of the linear KdV was addressed using LaSalle's invariance principle. An important open question arising in this context is to design a saturated boundary feedback term which gives stability when $L \in \mathcal{N}$. What it is known about the linear KdV equation with boundary conditions on $y(t,0)$, $y(t,L)$ and $y_x(t,L)$ is that it is never exactly controllable from the right side when $L \in \mathcal{N}$ (see Rosier (1997) and Glass and Guerrero (2010)). In particular, concerning the system (KdVs) the feedback term $\alpha y_x(t,0)$, and the strategies employed here are not capable of dealing with critical lengths. A possible solution to this problem could be to take another boundary feedback term acting on the left Dirichlet condition as in Cerpa and Coron (2013), where a backstepping controller was designed. To saturate this control, an idea is to follow Kang and Fridman (2017) and Kang and Fridman (2018).

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