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# Alternating submodules for partition algebras, rook algebras, and rook-Brauer algebras

John M. Campbell

## Abstract

Letting  $n \geq 2k$ , the partition algebra  $\mathbb{C}A_{k \geq 2}(n)$  has two one-dimensional subrepresentations that correspond in a natural way to the alternating and trivial characters of the symmetric group  $S_k$ . In 2019, Benkart and Halverson introduced and proved evaluations in the two distinguished bases of  $\mathbb{C}A_k(n)$  for nonzero elements in the one-dimensional regular  $\mathbb{C}A_k(n)$ -submodule that corresponds to the Young symmetrizer  $\sum_{\sigma \in S_k} \sigma$ ; in 2016, Xiao proved an explicit formula for the analogue of the sign representation for the rook monoid algebra. In this article, we lift Xiao's formula to a diagram basis evaluation in the partition algebra  $\mathbb{C}A_k(n)$ . We prove that our diagram basis evaluation for this lifting, which we denote as  $\text{Alt}_k \in \mathbb{C}A_k(n)$ , generates a one-dimensional module under the action of multiplication by arbitrary elements in  $\mathbb{C}A_k(n)$ . Our explicit formula for  $\text{Alt}_k$  gives us a cancellation-free formula for the other one-dimensional regular  $\mathbb{C}A_k(n)$ -module, with regard to Benkart and Halverson's lifting of  $\sum_{\sigma \in S_k} \sigma$ . We then use a sign-reversing involution to evaluate our one-dimensional generators in the orbit basis, and we use our explicit formula for  $\text{Alt}_k$  to lift Young's  $N$ - and  $P$ -functions so as to allow set-partition tableaux as arguments, and we use this lifting to construct Young-type matrix units for  $\mathbb{C}A_2(n)$  and  $\mathbb{C}A_3(n)$ .

## 1 Introduction

We adopt notation, for the most part, concerning partition algebras from [15], writing  $\mathbb{C}A_k(n)$  to denote the partition algebra of order  $k$  with a complex parameter  $n$ . For a partition of the set  $\{1, 2, \dots, k, 1', 2', \dots, k'\}$ , we denote this set-partition as a graph, with vertices labeled with  $\{1, 2, \dots, k\}$  arranged into a top row and with vertices labeled with  $\{1', 2', \dots, k'\}$  arranged into a bottom row. A labeled graph of this form such that its components are precisely the elements in a given set-partition is referred to as a partition diagram corresponding to this set-partition. For such a set-partition  $S$  of  $\{1, 2, \dots, k, 1', 2', \dots, k'\}$ , and for any corresponding partition diagram  $\pi$ , we let  $d_\pi$  or  $d_S$  denote what is referred to as a diagram basis element of  $\mathbb{C}A_k(n)$ . The set of all such expressions of the form  $d_\pi$  forms what is referred to as the diagram basis of  $\mathbb{C}A_k(n)$ , with the multiplicative operation on  $\mathbb{C}A_k(n)$  defined as explained below. In this article, we introduce new results and constructions concerning the representation theory of  $\mathbb{C}A_k(n)$ , building on the recent work of Benkart and Halverson [1] as well as Young's classical construction in the representation theory of symmetric group algebras [9].

For diagram basis elements  $d_\pi$  and  $d_\mu$  in  $\mathbb{C}A_k(n)$ , we take the graph  $\pi$ , with its vertices arranged in the manner we have indicated, and place it on top of  $\mu$ , and then identify the

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bottom vertices of  $\pi$  with the top vertices of  $\mu$  so as to form a new graph, and then we remove the middle row and form new edges if necessary to maintain top and bottom vertices being in the same component. If we let  $\pi \circ \mu$  denote the new partition diagram formed from this procedure, then  $d_\pi d_\mu = n^\ell d_{\pi \circ \mu}$ , where  $\ell$  denotes the number of components removed from the middle row in the formation of  $\pi \circ \mu$ , again letting  $n$  denote a complex parameter. This article is mainly concerned with the following problem: Find an explicit, cancellation-free formula for a nonzero partition algebra element  $x_k \in \mathbb{C}A_k(n)$  such that  $a x_k$  is always a scalar multiple of  $x_k$  for any element  $a \in \mathbb{C}A_k(n)$ , and such that  $x_k$  is a lifting of the alternating subrepresentation of the regular representation for the symmetric group algebra  $\mathbb{C}S_k$ . We succeed in solving this problem, and we apply our construction to lift to partition algebras functions involved in Young's construction of symmetric group algebra matrix units [9, §1].

Given a diagram basis element  $d_\pi$  in  $\mathbb{C}A_k(n)$ , the propagation number of  $\pi$  or of  $d_\pi$  refers to the number of components of  $\pi$  with at least one vertex in the top row and at least one vertex in the bottom row. So, we see that the subalgebra of  $\mathbb{C}A_k(n)$  spanned by diagram basis elements of propagation number  $k$  is isomorphic to the symmetric group algebra  $\mathbb{C}S_k$ . Since the partition algebra  $\mathbb{C}A_k(n)$  is such a rich generalization of the symmetric group algebra  $\mathbb{C}S_k$ , this motivates the pursuit of research based on how the representation theory for symmetric groups can be generalized to partition algebras [1, 2, 3, 14, 15, 20, 21]. We explicitly evaluate generators, which we denote as  $\text{Alt}_k$ , for regular  $\mathbb{C}A_k(n)$ -representations that correspond to the sign representations for  $\mathbb{C}S_k$ . We use bijective arguments to determine cancellation-free formulas for  $\text{Alt}_k$  in terms of the two distinguished bases of  $\mathbb{C}A_k(n)$ .

If we take the subalgebra of  $\mathbb{C}A_k(n)$  given by the linear span of the diagram basis elements in  $\mathbb{C}A_k(n)$  such that each top vertex is adjacent with at most one bottom vertex, and if we then set the complex parameter  $n$  to be equal to 1, then resultant structure is equivalent to what is known as the rook monoid algebra. Our formula for  $\text{Alt}_k$  generalizes a corresponding formula due to Xiao [27] for alternating subrepresentations on rook monoid algebras. Our proof that the linear  $\mathbb{C}$ -span  $\mathcal{L}_{\mathbb{C}}\{\text{Alt}_k\}$  is closed under the action of multiplication by arbitrary elements in  $\mathbb{C}A_k(n)$  generalizes a corresponding result due to Xiao [27] for rook monoid algebras.

For a diagram basis element  $d_\pi$  in a partition algebra, it is often convenient to identify the set-partition or the partition diagram denoted as  $\pi$  with  $d_\pi$ , in terms of our notation. Following [15], we define the relation  $\leq$  so that  $d_\pi \leq d_\mu$  if  $i$  and  $j$  being in the same component of  $\pi$  implies that  $i$  and  $j$  are in the same component of  $\mu$ . We define

$$d = \sum_{d \leq d'} o_{d'}, \quad (1)$$

for a given diagram basis element  $d \in \mathbb{C}A_k(n)$ , so that the set of all expressions of the form  $o_\mu \in \mathbb{C}A_k(n)$  for all possible partition diagrams  $\mu$  forms a basis of  $\mathbb{C}A_k(n)$ , which is referred to as the orbit basis  $\mathbb{C}A_k(n)$ . The diagram basis and the orbit basis are the two canonical bases of  $\mathbb{C}A_k(n)$ . We use a sign-reversing involution to evaluate  $\text{Alt}_k$  in the orbit basis.

For  $k \geq 2$ ,  $\mathbb{C}A_k(n)$  has two one-dimensional subrepresentations. These representations correspond in a natural way to the trivial and alternating subrepresentations of  $\mathbb{C}S_k$ ; see [15]

for a way of formalizing this. For the regular representation of  $\mathbb{C}S_k$ , the trivial subrepresentation is given by the  $\mathbb{C}S_k$ -submodule

$$\mathcal{L}_{\mathbb{C}} \left\{ \sum_{\sigma \in S_k} \sigma \right\}, \quad (2)$$

which is, of course, closed under the action of multiplication by permutations in  $S_k$  and by linear combinations of permutations in  $S_k$ . The alternating subrepresentation of the regular representation of  $\mathbb{C}S_k$  is given by the  $\mathbb{C}S_k$ -submodule

$$\mathcal{L}_{\mathbb{C}} \left\{ \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma \right\}. \quad (3)$$

In a 2019 article from Benkart and Halverson [1], explicit evaluations in the diagram and orbit bases of  $\mathbb{C}A_k(n)$  were given for an element  $y \in \mathbb{C}A_k(n)$  such that

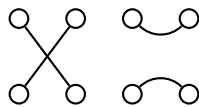
$$\mathcal{L}_{\mathbb{C}} \{y\} \subseteq \mathbb{C}A_k(n) \quad (4)$$

is closed under the action of multiplication by elements in  $\mathbb{C}A_k(n)$  and such that this  $\mathbb{C}A_k(n)$ -module corresponds to (2) in the sense that the character tables are the same when restricted to multiplication by permutations or permuting diagrams. So, the Benkart–Halverson evaluations for the element  $y$  indicated in (4) leads us to consider the following problem: Determine explicit, cancellation-free formulas in the orbit and diagram bases of  $\mathbb{C}A_k(n)$  for an element  $x \in \mathbb{C}A_k(n)$  such that  $\mathcal{L}_{\mathbb{C}} \{x\}$  is closed under the action of multiplication by elements in  $\mathbb{C}A_k(n)$  and such that this regular  $\mathbb{C}A_k(n)$ -module lifts the alternating  $\mathbb{C}S_k$ -module in (3) in the sense that the character tables are the same when restricted to permutations or permuting diagrams. As indicated above, we have successfully solved this problem; it appears that our construction of and evaluations for alternating, regular partition algebra submodules are original. Moreover, apart from our applying our construction/evaluations for such submodules to generalize classic constructions by Alfred Young [9, §1], our interest in  $\text{Alt}_k$  is also motivated by what we refer to as *rook-type algebras*, as we consider below.

## 1.1 Rook-type algebras

A Brauer diagram is a partition diagram such that all blocks have size 2, and the Brauer algebra  $\mathbb{C}B_k(n)$  is spanned by diagrams of this form that form set-partitions of  $\{1, 2, \dots, k, 1', 2', \dots, k'\}$ .

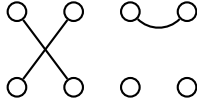
**Example 1.1.** *Using the SageMath [26] convention for denoting diagram basis elements, we have that the partition diagram*



*is a Brauer diagram in  $\mathbb{C}B_4(n) \subseteq \mathbb{C}A_4(n)$ .*

Similarly, the *rook-Brauer algebra*  $\mathbb{C}RB_k(n)$  is the diagram algebra over  $\mathbb{C}$  spanned by order- $k$  diagrams that are either Brauer diagrams or are obtained from Brauer diagrams by removing edges [13].

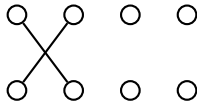
**Example 1.2.** *The partition diagram*



is in  $\mathbb{C}RB_4(n)$ .

The *rook algebra*  $\mathbb{C}R_k(n)$  is a subalgebra of  $\mathbb{C}A_k(n)$  of dimension  $\sum_{i=0}^k \binom{k}{i}^2 i!$  spanned by partial permutations, i.e., diagrams consisting of blocks of size 1 and blocks of size 2 consisting of a vertex in the upper row and a vertex in the lower row [14].

**Example 1.3.** *The partition diagram*



is in  $\mathbb{C}R_4(n)$ .

As in [27],  $\text{Rd}_k$  denotes the set of all rook  $k$ -diagrams, whereas, for each integer  $r$  such that  $0 \leq r \leq k$ ,  $\text{Rd}_k[r]$  is the set of rook  $k$ -diagrams that have exactly  $r$  isolated vertices in each row. For example, the diagram in Example 1.3 is in  $\text{Rd}_4[2]$ .

The evaluations we provide for  $\text{Alt}_k \in \mathbb{C}A_k(n)$ , as defined in (14) below, are such that the complex space  $\mathcal{L}_{\mathbb{C}}\{\text{Alt}_k\}$  is a subspace of  $\mathbb{C}RB_k(n)$  and  $\mathbb{C}R_k(n)$  and is closed under the actions of multiplication by elements in  $\mathbb{C}RB_k(n)$  and  $\mathbb{C}R_k(n)$ . In other words, our evaluations for  $\text{Alt}_k$  also give us formulas for sign representations for rook algebras and rook-Brauer algebras. A construction of the irreducible representations for semisimple algebras of the form  $\mathbb{C}RB_k(n)$  is given in [13], but it is not clear how to obtain the formula in (14) from this construction or from relevantly related literature as in [4, 5, 6, 7, 8, 22]. However, Xiao [27] proved an equivalent version of (14) for the rook monoid algebra, as we describe below. A construction for Specht-type modules for the rook monoid is also given in [11] (cf. [25]), and the irreducibles for the  $q$ -rook partition algebra are given in [12], but these constructions do not provide us with the orbit basis evaluation highlighted in Theorems 3.16 or our cancellation-free diagram basis evaluation for  $\text{Alt}_k$ .

## 1.2 Organization of the article

In Section 2, we briefly review relevant background material. The rest of our article is organized in the following manner:

- Section 3 is mainly devoted to  $\mathbb{C}A_k(n)$ -submodules of the form  $\mathcal{L}_{\mathbb{C}}\{\text{Alt}_k\}$ ;

- In Section 3.1, we prove that  $\mathcal{L}_{\mathbb{C}}\{\mathbf{Alt}_k\}$  is closed under the action of left-multiplication by elements in  $\mathbb{C}A_k(n)$ , following a similar approach relative to [27];
- In Section 3.2, we introduce a sign-reversing involution to prove an explicit, cancellation-free formula for expanding  $\mathbf{Alt}_k$  in the orbit basis;
- Section 4 largely concerns the application of our formula/definition for  $\mathbf{Alt}_k$  in the lifting of Young's  $N$ - and  $P$ -functions so as to be applicable to partition algebras, in the construction of partition algebra matrix units in a non-recursive way (cf. [15]);
- In Section 4.1, we introduce a partition algebra morphism  $\text{Stretch}_{\alpha,k}$  that we employ to define an analogue of Young's  $N$ - and  $P$ -functions, using our definition/formula for  $\mathbf{Alt}_k$ ;
- In Section 4.3, we succeed in applying our liftings of Young's  $N$ - and  $P$ -functions so as to construct full families of Young-type matrix units for the partitions algebras  $\mathbb{C}A_2(n)$  and  $\mathbb{C}A_3(n)$ ; and
- In Section 5, we conclude by considering a conjectural universal formula for Young-type matrix units for partition algebras.

## 2 Background

Let  $\text{GL}_n(\mathbb{C})$  denote the group of invertible  $n \times n$  matrices with complex entries. Let  $S_k$  denote the group of permutations on a set of  $k$  elements. *Schur–Weyl duality* refers to  $\text{GL}_n(\mathbb{C})$  and  $S_k$  generating the centralizer algebras for one another. This was discovered by Issai Schur and introduced in his 1901 thesis [24]. Schur–Weyl duality allows us to use properties of  $S_k$ -representations to determine results on  $\text{GL}_n(\mathbb{C})$ -representations and vice-versa [15].

Let  $W^{\lambda,n}$  denote an irreducible representation for  $\text{GL}_n(\mathbb{C})$ , and let  $S^\lambda$  be an irreducible representation for  $S_k$ . For an  $n$ -dimensional vector space  $V$ , we let  $S_k$  and  $\text{GL}_n(\mathbb{C})$  act on the tensor space  $V^{\otimes k}$ , in the following manner. The symmetric group  $S_k$  acts on  $V^{\otimes k}$  by permuting the tensor positions, and  $\text{GL}_n(\mathbb{C})$  acts diagonally, so that  $m(\mathbf{v}_{i_1} \otimes \mathbf{v}_{i_2} \otimes \cdots \otimes \mathbf{v}_{i_k}) = m\mathbf{v}_{i_1} \otimes m\mathbf{v}_{i_2} \otimes \cdots \otimes m\mathbf{v}_{i_k}$ . These actions give the tensor power  $V^{\otimes k}$  the structure of a bimodule. Schur–Weyl duality relates the structure of these modules through the decomposition of  $V^{\otimes k}$  as a direct sum

$$V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} W^{\lambda,n} \otimes S^\lambda.$$

Schur–Weyl duality may be understood to refer to the  $(\text{GL}_n(\mathbb{C}), S_k)$ -bimodule decomposition given above. From the bimodule structure

$$\text{GL}_n(\mathbb{C}) \curvearrowright (V^{\otimes k}) \curvearrowleft S_k$$

that we have given to  $V^{\otimes k}$ , the action of  $\text{GL}_n(\mathbb{C})$  generates the centralizer algebra  $\text{End}_{S_k}(V^{\otimes k})$ , and, dually, the action of  $S_k$  generates  $\text{End}_{\text{GL}_n(\mathbb{C})}(V^{\otimes k})$ .

If  $G$  is a subgroup of  $\text{GL}_n(\mathbb{C})$ , then  $\mathbb{C}S_k$  is a subalgebra of  $\text{End}_G(V^{\otimes k})$ . For a matrix subgroup  $M$  of  $\text{GL}_n(\mathbb{C})$  containing all the  $n \times n$  permutation matrices, the underlying multiplicative binary operation on  $\text{End}_M(V^{\otimes k})$  may be defined using the graph-theoretic operation

of *diagram multiplication*, as defined above. If  $G \cong S_n$  is the group of all permutation matrices in  $\text{GL}_n(\mathbb{C})$ , and if  $2k \leq n$ , the algebra  $\text{End}_G(V^{\otimes k})$ , in this case, is referred to as a *partition algebra*, and is isomorphic to the algebra  $\mathbb{C}A_k(n)$  defined above. The partition algebra was introduced in the 1990s, in the context of the study of statistical mechanics, by Martin [17, 18, 19, 20] and Jones [16] as a centralizer algebra of the form  $\text{End}_{S_n}(V^{\otimes k})$ .

## 2.1 The Benkart–Halverson subrepresentation

As in [1], we let  $\Psi_{k,n}$  denote the representation from  $\mathbb{C}S_n$  to  $\text{End}(V^{\otimes k})$  given by the diagonal action, letting  $n \geq 2k$ . Again, following [1], we let  $\epsilon_{(n-k,k)}$  denote the primitive central idempotent in  $\mathbb{C}S_n$ . In 2019 [1], an orbit basis expansion was proved for an expression denoted as  $\Xi_{k,n}$ , and it was proved that  $\Xi_{k,n} = \Psi_{k,n}(\epsilon_{(n-k,k)})$ . The expression  $\Psi_{k,n}(\epsilon_{(n-k,k)})$  corresponds to the one-dimensional subrepresentations of  $\mathbb{C}A_k(n)$  indexed by  $(n-k, k)$  [1]. Benkart and Halverson [1] proved that  $\mathcal{L}_{\mathbb{C}}\{\Xi_{k,n}\}$  is closed under multiplication by elements in  $\mathbb{C}A_k(n)$  by showing that this linear span is closed under multiplication by members of a generating set for  $\mathbb{C}A_k(n)$ .

A *rook partition* diagram is a partition diagram  $\pi$  consisting of blocks that are of size 1 or size 2, and such that the number of 2-blocks equals the propagation number of  $\pi$ . In Theorem 6.5 in [1], a formula for evaluating  $\Xi_{k,n} = \Psi_{k,n}(\epsilon_{(n-k,k)})$  in the orbit basis is proved; this formula is equivalent, up to a scalar multiple, to (5) below.

**Definition 2.1.** *Define the element  $\text{Quasi}_k \in \mathbb{C}A_k(n)$  as follows (cf. [1]):*

$$\text{Quasi}_k = \sum_{i=0}^k \sum_d (-1)^i (n - 2k + 1)^{(i)} (k - i)! o_d, \quad (5)$$

where the inner sum is over all rook partition diagrams of propagation number  $i$ , and where our notation for the second factor in the above displayed summand refers to the rising factorial function.

The normalized version of  $\text{Quasi}_2$  is recorded in [21]. As indicated above, for  $k > 1$ , the set of all scalar multiples of  $\text{Quasi}_k$  is closed under the action of multiplication by elements in  $\mathbb{C}A_k(n)$ , as shown in 2019 by Benkart and Halverson [1].

## 2.2 Lifting Young’s construction

As in [9, §1], we define an injective tableau as a partition tableau with  $k$  cells such that the cells of this tableau are labeled with  $\{1, 2, \dots, k\}$ . Among the key “building blocks” used in Young’s construction are expressions of the forms

$$N(T) = \sum_{\beta \in C(T)} \text{sgn}(\beta) \beta \quad (6)$$

and

$$P(T) = \sum_{\alpha \in R(T)} \alpha, \quad (7)$$

for an injective tableau  $T$ , where  $C(T)$  and  $R(T)$  respectively denote the column and row groups of  $T$ ; see [9, §1]. Symmetric group algebra elements of the forms

$$\sum_{\sigma \in S_k} \sigma \quad \text{and} \quad \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma \quad (8)$$

may be referred to as Young's symmetrizers and anti-symmetrizers, respectively. Our interest in partition algebra analogues of expressions as in (6), (7), and (8) is due to our desire to apply such analogues in the study of the semisimple structure of partition algebras. More to the point, since Young's  $N$ - and  $P$ -functions may be thought of as being defined using variants of sums as in  $\sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma$  and  $\sum_{\sigma \in S_k} \sigma$ , respectively, this motivates the lifting of these classical functions, using partition algebra analogues of (8), so as to mimic Young's classical construction of symmetric group algebra matrix units [9, §1]; see [3] for our related work on this subject as applied to 0-propagated subrepresentations of partition algebras.

A standard Young tableau of shape  $\lambda \vdash k$  is a partition tableau of this shape with cells labeled with  $\{1, 2, \dots, k\}$  and with increasing rows and columns. It is common to let  $f^\lambda$  denote the number of such tableaux of shape  $\lambda$ . A basic result in the representation theory of the symmetric group states that the irreducible representations of  $\mathbb{C}S_k$  are indexed by the partitions  $\lambda \vdash k$  and that the dimension and multiplicity of the irreducible representation of  $\mathbb{C}S_k$  corresponding to  $\lambda$  are both equal to  $f^\lambda$ . We intend to make use of a similar result concerning a closely related class of tableaux, in order to construct matrix units for infinite families of semisimple partition algebras.

With regard to the following definition, we define an ordering on subsets  $S, T \subseteq \{1, 2, \dots, k\}$  such that  $S \cap T = \emptyset$  so that  $S < T$  if  $\max(S) < \max(T)$ .

**Definition 2.2.** *Let  $n$  be a positive integer. A set-valued tableau  $T$  of shape  $\lambda \vdash n$  on  $\{1, 2, \dots, k\}$  is a map from the cells of  $\lambda$  to subsets of  $\{1, 2, \dots, k\}$  such that*

$$\{T(i, j) : (i, j) \in \lambda \text{ and } T(i, j) \neq \emptyset\} \quad (9)$$

*is a set partition of  $\{1, 2, \dots, k\}$  and such that  $T(i, j) < T(i+1, j)$  whenever both  $(i, j), (i+1, j) \in \lambda$  and  $T(i, j) \leq T(i, j+1)$  whenever both  $(i, j), (i, j+1) \in \lambda$ ,*

**Example 2.3.** *Let  $n = 8$  and let  $k = 4$ . Note that the inequality  $2k \leq n$  holds. Now, let  $T$  denote the tableau indicated below:*

3						2	14
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*We see that  $T$  is of shape  $(7, 1) \vdash 8$  on  $\{1, 2, 3, 4 = k\}$ . The labels of this tableau are given by a map from the cells of  $\lambda$  to  $2^{\{1, 2, 3, 4\}}$ , and we see that the set indicated in (9), in this case, is equal to the following set partition:  $\{\{2\}, \{1, 4\}, \{3\}\}$ . Also, we find that the rows of  $T$  are weakly increasing, and that its columns are strictly increasing.*

Since the summations in (8) are over sets of permutations, it is unclear as to what might be considered as suitable analogues of the definitions in (6) and (7) if set-partition tableaux



are used as arguments, as opposed to Young tableaux, referring the interested reader to [3, 14] for recent work on the use of set-partition tableaux in the representation theory of partition algebras. As a way of approaching the problem indicated in the preceding sentence, we begin by considering what might be something of a more natural way of expressing the summations in (6) and (7): In particular, letting  $\lambda'$  denote the transpose of an integer partition  $\lambda$ , if we let

$$C_1, C_2, \dots, C_{\ell(\text{shape}(T)')}$$

denote the columns of  $T$ , then the product identity

$$N(T) = N(C_1)N(C_2) \cdots N(C_{\ell(\text{shape}(T)')}) \quad (10)$$

holds, and if we let

$$R_1, R_2, \dots, R_{\ell(\text{shape}(T))}$$

denote the rows of  $T$ , then we have that

$$P(T) = P(R_1)P(R_2) \cdots P(R_{\ell(\text{shape}(T))}) \quad (11)$$

also holds. It is common to define  $N(T)$  and  $P(T)$  as in (10) and (11); see [23, p. 60] and [10], for example.

Informally, for a set-partition tableau  $T$ , if we take a given column  $\mathcal{C}$  of  $T$ , we can think of  $N(\mathcal{C})$  as being defined by taking the alternating subrepresentation of the partition algebra of an appropriate dimension and “stretching” the terms in its diagram basis expansion according to the set-valued labels of  $\mathcal{C}$ ; similarly for  $P(\mathcal{R})$  for a row  $\mathcal{R}$  of  $T$ . We formalize this idea in Section 4, in which we explore, using our definition for  $\text{Alt}_k$ , the idea of constructing full bases of partition algebra matrix units that are defined in a similar way relative to Young’s fundamental formula

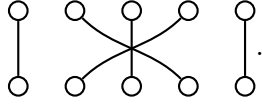
$$e_{i,j}^\lambda = \gamma_i^\lambda \sigma_{i,j}^\lambda (1 - \gamma_{j+1}^\lambda) (1 - \gamma_{j+2}^\lambda) \cdots (1 - \gamma_{f^\lambda}^\lambda), \quad (12)$$

borrowing notation from [9, §1], and where Young’s  $\gamma$ -elements, as in (12), are equal to idempotent scalar multiples of expressions of the form  $N(T)P(T)$  for an injective tableau  $T$  of shape  $\lambda$ ; see [9, §1] for details. In our recent paper [3], we had proved that a direct analogue of (12) holds for non-propagating partition algebra submodules, but the situation becomes much more difficult when dealing analogues of Young’s  $N$ - and  $P$ -functions in full generality, so as to construct matrix units in a meaningfully similar manner relative to (12) for semisimple partition algebras in full generality.

### 3 A lift of Xiao’s anti-symmetrizers

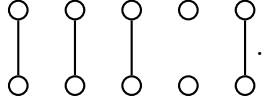
We adopt notation for generating sets for partition algebras from [19, 20]. In this direction, let  $1_{i,j} = \{\{1, 1'\}, \{2, 2'\}, \dots, \{i, j'\}, \{j, i'\}, \dots, \{k, k'\}\}$ .

**Example 3.1.** In  $\mathbb{C}A_5(n)$ , the element  $1_{2,4}$  may be denoted as



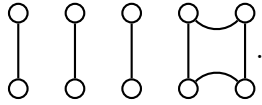
Define  $p_i = \{\{1, 1'\}, \{2, 2'\}, \dots, \{i\}, \{i'\}, \dots, \{k, k'\}\}$ .

**Example 3.2.** In  $\mathbb{C}A_5(n)$ , the element  $p_4$  may be denoted as



Define  $\mathcal{A}^{i,j} = \{\{1, 1'\}, \{2, 2'\}, \dots, \{i, j, i', j'\}, \dots, \{k, k'\}\}$ .

**Example 3.3.** In  $\mathbb{C}A_5(n)$ , the element  $\mathcal{A}^{4,5}$  may be denoted as



One of the most basic results in the theory of partition algebras is given by the property whereby elements of the following forms generate the partition algebra  $\mathbb{C}A_{k \geq 2}(n)$ :  $1_{i,j}$ ,  $p_i$ , and  $\mathcal{A}^{i,j}$  [19, 20]. In order to prove that the  $\mathbb{C}A_k(n)$ -submodules under consideration in this Section are closed under the action of left-multiplication by elements in  $\mathbb{C}A_k(n)$ , we make use of a similar generating set for  $\mathbb{C}A_k(n)$ . In particular, from the generating set consisting of elements of the forms  $1_{i,j}$ ,  $p_i$ , and  $\mathcal{A}^{i,j}$ , it is easily seen that all elements in  $\mathbb{C}A_k(n)$  of the following forms generate  $\mathbb{C}A_k(n)$ : All permutations in  $\mathbb{C}A_k(n)$ ,  $\mathcal{A}_{1,2}$ , and  $p_1$ .

In 2016, Xiao [27] introduced the formula

$$X_2 = \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma + \sum_{\eta \in \text{Rd}_k[1]} \text{sgn}(\eta)\eta \quad (13)$$

as a direct analogue of the anti-symmetrizer for the symmetric group, with the linear span of  $X_2$  giving us a one-dimensional submodule of the rook monoid algebra [27]. As we shall see, (13) naturally corresponds to the partition algebra element

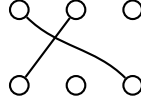
$$n \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma - \sum_{\eta \in \text{Rd}_k[1]} \text{sgn}(\eta)\eta \in \mathbb{C}A_k(n) \quad (14)$$

that is the subject of this Section, where we have to take into account the parameter  $n$  as involved in diagram multiplication in the algebra  $\mathbb{C}A_k(n)$ , and where the sign function in the latter sum in (14) is defined differently for partition algebras.

### 3.1 Alternating regular subrepresentations for partition algebras, rook algebras, and rook-Brauer algebras

We find it convenient to refer to elements in  $\text{Rd}_k[1]$  as *near-permutations*. That is, a near-permutation of order  $k$  may be defined as an order- $k$  rook partition diagram of propagation number  $k - 1$  [27].

**Example 3.4.** *The diagram*



is a near-permutation of order 3.

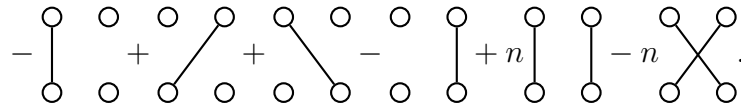
Following Xiao [27], it is easily seen that: Given a near-permutation  $d$  of order  $k \in \mathbb{N}$ , there exist permutations  $\sigma$  and  $\rho$  of order  $k$  such that  $\sigma \cdot \{\{1, 1'\}, \{2, 2'\}, \dots, \{k-1, (k-1)'\}, \{k\}, \{k'\}\} \cdot \rho$  is equal to  $d$ , noting that these permutations are not necessarily unique. Again with regard to [27], it is not difficult to show that: If  $\sigma, \sigma', \rho,$  and  $\rho'$  are permuting diagrams in  $\mathbb{C}A_k(n)$  such that  $\sigma \cdot p_k \cdot \rho = \sigma' \cdot p_k \cdot \rho'$ , then  $\sigma \cdot \rho = \sigma' \cdot \rho'$ . This gives us that the *sign* of a near-permutation, as below, is well-defined [27].

**Definition 3.5.** *Given a near-permutation  $\eta$  in  $\mathbb{C}A_k(n)$ , the sign of  $\eta$ , denoted as  $\text{sgn}(\eta)$ , is defined as  $\text{sgn}(\sigma)\text{sgn}(\rho)$ , where  $\sigma$  and  $\rho$  are any permutations in  $S_k$  such that  $\sigma \cdot p_k \cdot \rho = \eta$  (cf. Definition 3.2 in [27]).*

As indicated above, partition algebra elements of the form indicated in the below definition are a main subject of interest in this article.

**Definition 3.6.** *We define  $\text{Alt} = \text{Alt}_k = n \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma - \sum_{\eta \in \text{Rd}_k[1]} \text{sgn}(\eta)\eta \in \mathbb{C}A_k(n)$  (cf. Proposition 3.6 in [27]).*

**Example 3.7.** *The element  $\text{Alt}_2$  is as below. The idempotent scalar multiple of this same element is recorded in Martin and Woodcock's work [21] on central idempotents for partition algebras:*



Martin and Woodcock's construction from [21] is not directly related to or applicable to our results.

We may factorize  $\text{Alt}_k$  as below:

$$\begin{aligned} \text{Alt}_k &= \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma \right) \left( n \cdot \text{id} - \sum_{i=1}^k p_i \right) \\ &= \left( n \cdot \text{id} - \sum_{i=1}^k p_i \right) \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma \right). \end{aligned} \tag{15}$$

**Lemma 3.8.** *The identity  $\sigma \cdot \text{Alt}_k = \text{sgn}(\sigma) \text{Alt}_k$  holds for all permutations  $\sigma$  in  $\mathbb{C}A_k(n)$  (cf. [27, §3]).*

*Proof.* This follows immediately from the initial factorization formula for  $\text{Alt}_k$  provided above.  $\square$

**Lemma 3.9.** *The product*

$$\begin{array}{c} \text{Diagram 1} \\ \underbrace{\text{Diagram 2} \dots \text{Diagram } k-2}_{k-2} \end{array} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma \quad (16)$$

*vanishes.*

*Proof.* Let  $\sigma$  be a permutation of order  $k$ , denoted as a diagram in  $\mathbb{C}A_k(n)$ . Observe that

$$\begin{array}{c} \text{Diagram 1} \\ \underbrace{\text{Diagram 2} \dots \text{Diagram } k-2}_{k-2} \end{array} \sigma$$

is the diagram obtained from  $\sigma$  by connecting 1, 2,  $\sigma(1)'$ , and  $\sigma(2)'$ . Now, consider the product  $(12)\sigma$ . We see that

$$\begin{array}{c} \text{Diagram 1} \\ \underbrace{\text{Diagram 2} \dots \text{Diagram } k-2}_{k-2} \end{array} \sigma = \begin{array}{c} \text{Diagram 1} \\ \underbrace{\text{Diagram 2} \dots \text{Diagram } k-2}_{k-2} \end{array} (12)\sigma \quad (17)$$

and that  $\sigma$  and  $(12)\sigma$  are of opposite signs. This can be used to construct a sign-reversing involution, in the following manner. Define the  $\phi: S_k \rightarrow S_k$  so that  $\phi(\sigma) = (12)\sigma$ . So, we see that  $\phi$  is an involution and reverses the sign of its argument. So, by rewriting the product in (16) as

$$\sum_{\sigma \in S_k} \text{sgn}(\sigma) \left( \begin{array}{c} \text{Diagram 1} \\ \underbrace{\text{Diagram 2} \dots \text{Diagram } k-2}_{k-2} \end{array} \sigma \right),$$

if we take a given permutation  $\sigma \in S_k$ , we see that the term

$$\text{sgn}(\sigma) \left( \begin{array}{c} \text{Diagram 1} \\ \underbrace{\text{Diagram 2} \dots \text{Diagram } k-2}_{k-2} \end{array} \sigma \right)$$

cancels with

$$\text{sgn}(\phi(\sigma)) \left( \begin{array}{c} \text{Diagram 1} \quad \text{Diagram 2} \quad \dots \quad \text{Diagram } k \\ \underbrace{\hspace{10em}}_{k-2} \end{array} \right),$$

from the equality in (17) together with the fact that  $\sigma$  and  $(12)\sigma$  are of opposite signs. So, from this matching property, together with the fact that  $\phi$  is bijective, we obtain the desired vanishing result.  $\square$

As a consequence of Lemma 3.9, we have that

$$\begin{array}{c} \text{Diagram 1} \quad \text{Diagram 2} \quad \dots \quad \text{Diagram } k \\ \underbrace{\hspace{10em}}_{k-2} \quad \text{Alt}_k \end{array}$$

also vanishes, due to the factorization formula on display in (15).

**Lemma 3.10.** *The product*

$$p_1 \left( n \cdot \text{id} - \sum_{i=1}^k p_i \right) \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma \right) \tag{18}$$

*vanishes (cf. [27, §3]).*

*Proof.* Rewrite the product of the first two factors in (18) as below:

$$\begin{aligned} p_1 \left( n \cdot \text{id} - \sum_{i=1}^k p_i \right) &= n \cdot p_1 - p_1 \sum_{i=1}^k p_i \\ &= n \cdot p_1 - p_1^2 - \sum_{i=2}^k p_1 p_i \\ &= - \sum_{i=2}^k p_1 p_i. \end{aligned}$$

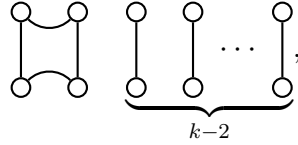
Now, let  $\sigma$  be a permutation in  $\mathbb{C}A_k(n)$ . Letting  $i > 1$ , we know that  $\{1'\}$  is a block in  $p_1 p_i$ . We see that the index  $i$  is such that  $\{i'\}$  is also a singleton block in the bottom row of  $p_1 p_i$ . Now, consider the product  $p_1 p_i \sigma$ . We see that  $p_1 p_i \sigma = p_1 p_i \sigma(1, i)$ . Since  $\sigma$  and  $\sigma(1, i)$  are of opposing signs, this gives us a suitable sign-reversing involution that gives us the desired result.  $\square$

From Lemmas 3.8–3.10, we are led to the following result, giving us a lifting of Xiao’s formula for  $X_2$ .

**Theorem 3.11.** *Let  $d$  be a diagram in  $\mathbb{C}A_k(n)$ . Then, the following identity holds:*

$$d \text{Alt} = \begin{cases} \text{sgn}(d) \text{Alt} & \text{if } d \text{ is a permutation,} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The initial case in the above identity holds by Lemma 3.8. Now, suppose that  $d$  is a non-permuting diagram in  $\mathbb{C}A_k(n)$ . The diagram  $d$  can be written as a product of permutations, copies of



and copies of  $p_1 \in \mathbb{C}A_k(n)$  [19], so the desired result follows from the remaining lemmas, together with our factorization formulas for  $\text{Alt}$ .  $\square$

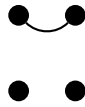
**Corollary 3.12.** *The linear span  $\mathcal{L}_{\mathbb{C}}\{\text{Alt}_k\}$  has the structure of a  $\mathbb{C}R_k$ -submodule under the action of multiplication by elements in  $\mathbb{C}R_k$ , and has the structure of a  $\mathbb{C}RB_k$ -module under the action of multiplication by elements in  $\mathbb{C}RB_k$ .*

*Proof.* This follows in a direct way from Theorem 3.11, since  $\text{Alt}$  is in both  $\mathbb{C}R_k$  and  $\mathbb{C}RB_k$ , and since both  $\mathbb{C}R_k$  and  $\mathbb{C}RB_k$  are contained in  $\mathbb{C}A_k(n)$ .  $\square$

### 3.2 Expansion in the orbit basis

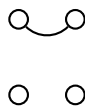
We recall the definition of the orbit basis indicated in (1). We adopt the SageMath [26] convention whereby orbit basis elements are denoted with partition diagrams with black nodes.

**Example 3.13.** *In the partition algebra  $\mathbb{C}A_2(n)$ , the orbit basis element corresponding to the set-partition  $\{\{1, 2\}, \{1'\}, \{2'\}\}$  is denoted as*



and admits the following expansion in the diagram basis of  $\mathbb{C}A_2(n)$ :

On the other hand, the diagram basis element



admits the following expansion in the orbit basis:

$$\begin{array}{c} \circ \quad \circ \\ \circ \quad \circ \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \bullet \quad \bullet \end{array}. \quad (19)$$

We observe that the expansion shown in (19) agrees with the definition of the orbit basis provided in (1).

To convert the expression  $\text{Alt}_k \in \mathbb{C}A_k(n)$  in the orbit basis, our strategy is to use a bijective approach. If we convert each term in the expansion in (14) in the orbit basis, we obtain a variety of expressions that cancel with one another, so it is natural to make use of a sign-reversing involution.

Let  $d$  be a partition diagram in  $S_k \cup \text{Rd}_k[1]$ , being consistent with our notation in (14), and letting the elements in  $S_k$  and  $\text{Rd}_k[1]$  be written as elements in  $\mathbb{C}A_k(n)$ . So, by expanding  $d$  in the orbit basis according to (1), we may identify a term  $o_{d'}$  resulting from this expansion with an expression of the form

$$(d, \{B_1, B_2, \dots, B_m\}), \quad (20)$$

where

$$\{B_1, B_2, \dots, B_m\}$$

is the set-partition of the set of blocks of  $d$  for some  $m \in \mathbb{N}$  such that the partition diagram  $d'$  is given by joining the blocks in  $B_i$ , within the partition diagram  $d$ , for each index  $i$ .

**Example 3.14.** We see that in the expansion of the diagram

$$\begin{array}{c} \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \end{array} \quad (21)$$

in the orbit basis we obtain the term

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \bullet \quad \bullet \quad \bullet \end{array} = o_{\{\{1,2,1',2',3'\},\{3\}\}} \in \mathbb{C}A_3(n). \quad (22)$$

In the set-partition of  $\{1, 2, 3, 1', 2', 3'\}$  illustrated in (21), we join the blocks  $\{1, 1'\}$ ,  $\{2, 2'\}$ , and  $\{3'\}$ , so as to obtain  $\{1, 1', 2, 2', 3'\}$ , and we leave the singleton set  $\{3\}$  as it is. So, in this case, the tuple in (20) is as below:

$$\left( \begin{array}{c} \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \end{array}, \{ \{ \{1, 1'\}, \{2, 2'\}, \{3'\} \}, \{ \{3\} \} \} \right).$$

**Example 3.15.** In the expansion of

$$\begin{array}{c} \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \end{array} \quad (23)$$





Now, suppose that the preceding condition does not hold, and suppose that there exists at least one set  $B_j$  that contains only one 2-set and at least one singleton set. Let  $B_\ell$  denote the lexicographically smallest set of this form, letting  $\{\alpha, \alpha'\}$  denote the unique 2-set in  $B_\ell$  and letting  $s$  denote the lexicographically smallest singleton set in  $B_\ell$ , letting  $s$  be equal to either  $\{\beta\}$  or  $\{\beta'\}$ . In the former case, let  $\phi$  map  $x$  to the pair obtained by changing  $\{\alpha, \alpha'\}$  to  $\{\beta, \alpha'\}$  and changing  $\{\beta\}$  to  $\{\alpha\}$  with respect to both entries in  $x$ , and similarly in the latter case.

Finally, if the above conditions do not hold with respect to a given domain element  $x \in X$ , let  $\phi$  map  $x$  to  $x$ .

We see that if  $\phi(x) \neq x$ , then  $\phi$  switches the sign of  $x \in X$ . So, we see that the only expressions remaining after expanding each term in (14) in the orbit basis are of one of the following forms:  $n \operatorname{sgn}(\sigma) o_\sigma$ , or  $-k \operatorname{sgn}(\sigma) o_\sigma$ , or  $-\operatorname{sgn}(\eta) o_\eta$ .  $\square$

We can also show that

$$\operatorname{Alt}_k = n \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma - \frac{1}{(k-1)!} \sum_{\sigma, \tau \in S_k} \operatorname{sgn}(\sigma \tau) \sigma p_k \tau \quad (24)$$

holds, using something of a similar approach relative to our proof of Theorem 3.16.

## 4 Lifting Young's construction

Again, we let  $2k \leq n$  when considering the representation theory of  $\mathbb{C}A_k(n)$ , in order to ensure that  $\mathbb{C}A_k(n)$  will be semisimple as an algebra. In other words, we want there to be a basis of  $\mathbb{C}A_k(n)$  consisting of expressions of the form  $e_{i,j}^\lambda$ , for indices  $i$  and  $j$  in some set and for expressions  $\lambda$  in some set, such that the matrix unit multiplication rules are satisfied: Explicitly, we want  $e_{i_1, j_1}^\lambda e_{i_2, j_2}^\mu$  to vanish if  $\lambda \neq \mu$  or if  $j_1 \neq i_2$ , and we want the identity

$$e_{i_1, j_1}^\lambda e_{i_2, j_2}^\mu = e_{i_1, j_2}^\lambda$$

to be satisfied if  $\lambda = \mu$  and  $j_1 = i_2$ . In Alfred Young's famous construction of matrix units for symmetric group algebras [9, §1], idempotent elements in  $\mathbb{C}S_k$  were defined using products of the form  $E(T) = N(T)P(T)$ , where  $N(T)$  and  $P(T)$  are as defined in (6) and (7), referring to [9, §1] for details. We refer to these idempotents as Young's  $\gamma$ -elements [9, §1], and these  $\gamma$ -elements are involved in Young's matrix unit formula shown in (12), again referring to [9, §1] for preliminaries. As indicated above, we are interested in constructing an analogue of (12) with set-partition tableaux used in place of Young tableaux, and with the use of our evaluation for  $\operatorname{Alt}_k$ .

In our recent paper [3], we had constructed and proved an analogue of (12) for all non-propagating partition algebra submodules, i.e., for all irreducible  $\mathbb{C}A_k(n)$ -submodules of the regular representation of  $\mathbb{C}A_k(n)$  such that all of the diagram basis elements involved in the expansions of the elements in these  $\mathbb{C}A_k(n)$ -submodules are of propagation number zero. In this case, our analogues of Young's elements of the form  $E(T) = N(T)P(T)$  were such

that the tableaux  $T$  consisted only of one row [3]. In order to determine an analogue of Young's  $\gamma$ -elements using tableaux consisting of more than one row, we consider using the one-dimensional subrepresentations for  $\mathbb{C}A_k(n)$ , including our explicit evaluation for  $\text{Alt}_k$ , as we explore in this Section.

Young's  $N$ -function is such that

$$N \left( \begin{array}{|c|} \hline k \\ \hline \vdots \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \right) = \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma, \quad (25)$$

and for distinct labels  $s_1, s_2, \dots, s_k$  in  $\mathbb{N}$ , we have that

$$N \left( \begin{array}{|c|} \hline s_k \\ \hline \vdots \\ \hline s_2 \\ \hline s_1 \\ \hline \end{array} \right) = \sum_{\rho \in S_{\{s_1, s_2, \dots, s_k\}}} \text{sgn}(\rho)\rho, \quad (26)$$

letting  $S_T$  denote the group of all bijections on a finite set  $T$ , with  $S_{\{1, 2, \dots, k\}} = S_k$ . So, if we compare (25) with (26), and if we let permutations be denoted as permutation diagrams, we can think of the right-hand side of (26) as being obtained from that of (25) by “replacing”  $i$  with  $s_i$  in a given diagram, for each index  $i$ , and similarly for expressions of the following form:

$$P \left( \begin{array}{|c|c|c|c|} \hline s_1 & s_2 & \cdots & s_k \\ \hline \end{array} \right).$$

Now, let us consider the problem of finding a suitable way of expressing

$$N \left( \begin{array}{|c|c|c|c|} \hline k \\ \hline \vdots \\ \hline 2 \\ \hline 1 \\ \hline \square & \square & \cdots & \square \\ \hline \end{array} \right) = N \left( \begin{array}{|c|} \hline k \\ \hline \vdots \\ \hline 2 \\ \hline 1 \\ \hline \square \\ \hline \end{array} \right), \quad (27)$$

adopting the convention whereby the labels of set-partition tableaux may be denoted without “curly brackets”. The arguments of the  $N$ -function indicated in (27) are meant to illustrate set-partition tableaux, and we adopt the convention whereby empty cells in the first row of a set-partition tableau may be removed, for the sake of convenience. Since we have determined explicit evaluations for partition algebra analogues of the alternating representations of symmetric group algebras, it would be appropriate to let (27) be equal to an element in the alternating  $\mathbb{C}A_k(n)$ -submodule, and then mimic the “substitution” approach suggested

in (26), as outlined below. To evaluate an expression of the form

$$N \begin{pmatrix} S_k \\ \vdots \\ S_2 \\ S_1 \\ \square \end{pmatrix}, \quad (28)$$

where  $\{S_1, S_2, \dots, S_k\}$  is a set-partition of a finite subset of  $\mathbb{N}$ , we want to somehow “replace” each label  $i$  in (27) with  $S_i$ . This has led us to introduce the notion of a “Stretch” operator, as defined in Section 4.1.

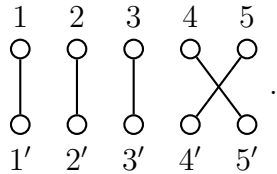
In much the same way that permutations may be denoted in a simplified way using cycle notation without specifying the order of a given symmetric group, it may be convenient to denote partition diagrams with labels that do not form a set of the form  $\{1, 2, \dots, k, 1', 2', \dots, k'\}$ . This leads us to Definition 4.1 below, noting that: For a set  $S$  of natural numbers, we may let  $S'$  denote the set of primed elements in  $S$ .

**Definition 4.1.** *Let  $\pi$  be a set-partition of a set of the form  $S \cup S'$  where  $S$  is a finite set of natural numbers. For fixed  $k \in \mathbb{N}$  such that  $k \geq \max(S)$ , we define the diagram basis element of order  $k$  corresponding to  $\pi$  to be the element in the diagram basis of  $\mathbb{C}A_k(n)$  corresponding to the set-partition obtained by adding blocks of the form  $\{i, i'\}$  to  $\pi$  for natural numbers  $i \notin S$ , with  $i \leq k$ . We denote this as  $\delta_k(\pi)$ .*

**Example 4.2.** *If we let  $\pi$  denote the set-partition of  $\{\{4, 5'\}, \{5, 4'\}\}$ , then the diagram basis element of order 5 corresponding to  $\pi$  is none other than*

$$\delta_5(\pi) = d_{\{\{1,1'\},\{2,2'\},\{3,3'\},\{4,5'\},\{5,4'\}\}} \in \mathbb{C}A_5(n).$$

We may denote this diagram basis element as below:



**Remark 4.3.** *Let  $S_1$  and  $S_2$  be disjoint sets of natural numbers. Let  $\pi_1$  and  $\pi_2$  respectively denote set-partitions of  $S_1 \cup S_1'$  and  $S_2 \cup S_2'$ . Let  $k \geq \max(S_1 \cup S_2)$ . Since  $\pi_1$  and  $\pi_2$  are on disjoint vertex sets, then we must have that  $\delta_k(\pi_1)\delta_k(\pi_2) = \delta_k(\pi_2)\delta_k(\pi_1)$ , in much the same way that disjoint cycles commute.*

For a given tableau  $T$ , we let  $\text{col}_i(T)$  denote the  $i^{\text{th}}$  column of  $T$  for a given index  $i$ , and we write  $\text{row}_j(T)$  in place of the  $j^{\text{th}}$  row of  $T$ , letting  $j$  be a suitable index. We let  $\text{shape}(T)$  denote the shape of a partition tableau  $T$ , i.e., the integer partition  $\lambda$  such that the  $i^{\text{th}}$  entry in this partition is the number of cells in the  $i^{\text{th}}$  row of  $T$ .

## 4.1 Set-compositions and “Stretch” operators

The concept of a *set-composition*, as defined below, is, of course, to be often involved in our work.

**Definition 4.4.** A set-composition of a set  $S$  is a tuple of disjoint nonempty subsets of  $S$  such that the union of the sets in this tuple equals  $S$ .

**Example 4.5.** The ordered 3-tuple  $(\{3\}, \{1, 4\}, \{2, 5\})$  is a set-composition of  $\{1, 2, 3, 4, 5\}$  that is distinct from the set-composition  $(\{3\}, \{2, 5\}, \{1, 4\})$ .

**Remark 4.6.** As before, with regard to our definition of the term set-partition tableau, we adopt the convention whereby finite sets of natural numbers are ordered or arranged according to the relation whereby  $S_i < S_j$  if and only if  $\max(S_i) < \max(S_j)$ , for two such sets  $S_i$  and  $S_j$ .

Given a set-composition  $\alpha$ , we let  $\ell(\alpha)$  denote the length of  $\alpha$ , i.e., the number of entries in  $\alpha$ , and we write  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$ , as in with the usual notation for integer compositions. We henceforward let the entries of set-compositions be finite subsets of  $\mathbb{N}$ . We also note that may write  $\bigcup \alpha$  in place of  $\alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_{\ell(\alpha)}$ .

**Definition 4.7.** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$  be a set-composition of a finite set of natural numbers, writing  $m = \ell(\alpha)$ . Let  $k \geq \max(\bigcup \alpha)$ . Define

$$\text{Stretch}_{\alpha,k}: \mathbb{C}A_m(n) \rightarrow \mathbb{C}A_k(n) \quad (29)$$

as follows. Let  $d_\pi$  be an element of the diagram basis of the domain in (29), where  $\pi$  is a set-partition of  $\{1, 2, \dots, m\} \cup \{1', 2', \dots, m'\}$ , writing  $\pi = \{\pi_1, \pi_2, \dots, \pi_{\ell(\pi)}\}$ . Then

$$\text{Stretch}_{\alpha,k}(d_\pi) = \delta_k \left( \left\{ \bigcup_{\substack{i \in \pi_j \\ i \text{ is unprimed}}} \alpha_i \cup \bigcup_{i' \in \pi_j} \alpha_{i'} : 1 \leq j \leq \ell(\pi) \right\} \right).$$

We extend this definition linearly, so as to obtain a well-defined function on the domain in (29).

Recall that we may identify a given diagram basis element  $d_\pi$  with a graph denoting  $\pi$ . Informally, the “Stretch” operator is such that it replaces each vertex in this diagram with a clique, and in such a way so that the property of being in the same block is preserved.

**Example 4.8.** From the above definition, we may obtain the following:

$$\begin{aligned} & \text{Stretch}_{(\{3,4\}, \{1\}, \{9\}, \{5,7\}), 10} \left( \begin{array}{cc} \text{---} & \text{---} \\ \circ & \circ \\ \text{---} & \text{---} \\ \circ & \circ \\ \text{---} & \text{---} \\ \circ & \circ \end{array} \right) \\ &= \text{Stretch}_{(\{3,4\}, \{1\}, \{9\}, \{5,7\}), 10} \left( d_{\{\{1,2,1',2'\}, \{3,4\}, \{3'\}, \{4'\}\}} \right) \end{aligned}$$

$$= \delta_{10} (\{\{1, 3, 4, 1', 3', 4'\}, \{5, 7, 9\}, \{5', 7'\}, \{9'\}\})$$

$$= \begin{array}{cccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & & \\ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & & \\ & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & & \\ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & & \\ & 1' & 2' & 3' & 4' & 5' & 6' & 7' & 8' & 9' & 10' & & \end{array} \in \mathbb{C}A_{10}(n).$$

**Lemma 4.9.** *Adopting notation from Definition 4.7, for fixed parameters  $\alpha$  and  $k$ , the function*

$$\text{Stretch}_{\alpha,k} : \mathbb{C}A_{\ell(\alpha)}(n) \rightarrow \mathbb{C}A_k(n)$$

*is an algebra homomorphism.*

*Proof.* By definition, we have that the function  $\text{Stretch}_{\alpha,k}$  is a linear map. Now, let  $d_1$  and  $d_2$  be diagrams in the domain of this function. It remains to prove that the equality

$$\text{Stretch}_{\alpha,k}(d_1 d_2) = \text{Stretch}_{\alpha,k}(d_1) \text{Stretch}_{\alpha,k}(d_2) \quad (30)$$

holds.

Suppose that  $v_1$  and  $v_2$  are in the same block in  $\text{Stretch}_{\alpha,k}(d_1 d_2)$ , where the vertices  $v_1$  and  $v_2$  may be primed or unprimed.

Let  $v_1 \in \alpha_{i_1}^{(j_1)}$  and  $v_2 \in \alpha_{i_2}^{(j_2)}$ , where  $j_1, j_2 \in \{0, 1\}$  indicate a given number of “primes”. From the definition of the “Stretch” operator, it follows that  $i_1^{(j_1)}$  and  $i_2^{(j_2)}$  are in the same block in  $d_1 d_2$ . We recall that the concatenation of two partition diagrams  $\pi_1$  and  $\pi_2$  is denoted as  $\pi_1 * \pi_2$ , and we may deduce that there exists a path joining  $i_1^{(j_1)}$  and  $i_2^{(j_2)}$  in  $d_1 * d_2$ , which implies that there exists a path joining  $v_1$  and  $v_2$  in

$$\text{Stretch}_{\alpha,k}(d_1) * \text{Stretch}_{\alpha,k}(d_2), \quad (31)$$

since the “Stretch” operator preserves the property of being in the same block. So, we have shown that if two vertices  $v_1$  and  $v_2$  satisfying the above conditions are in a common block in  $\text{Stretch}_{\alpha,k}(d_1 d_2)$ , then these same two vertices are in a common block in

$$\text{Stretch}_{\alpha,k}(d_1) \text{Stretch}_{\alpha,k}(d_2),$$

and a similar argument may be used to prove the converse.

There is a one-to-one correspondence between blocks that are completely contained in the middle row of  $d_1 * d_2$  and blocks that are completely contained in the middle row of (31), as is easily seen through a direct application of Definition 4.7.  $\square$

## 4.2 Using the one-dimensional partition algebra subrepresentations to lift Young's construction

Recall that we let

$$\text{Alt}_k = n \sum_{\sigma \in S_k} \text{sgn}(\sigma) d_\sigma - \sum_{\eta} \text{sgn}(\eta) d_\eta, \quad (32)$$

and that we showed that the one-dimensional space  $\mathcal{L}_{\mathbb{C}}\{\text{Alt}_k\}$  is a  $\mathbb{C}A_k(n)$ -submodule, under the action of left-multiplication by elements in  $\mathbb{C}A_k(n)$ . Noting that the right-hand side of (32) is given in the diagram basis, we may write

$$\text{Alt}_k = n \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma - \sum_{\eta} \text{sgn}(\eta) \eta,$$

giving us an interesting analogue of (25). The unique idempotent element in the subrepresentation  $\mathcal{L}_{\mathbb{C}}\{\text{Alt}_k\}$  is the matrix unit of the form

$$e_{(\emptyset, \emptyset, 1, 1, 1^2, 1^2, \dots, 1^k, 1^k), (\emptyset, \emptyset, 1, 1, 1^2, 1^2, \dots, 1^k, 1^k)}. \quad (33)$$

Since

$$\begin{aligned} (\text{Alt}_k)^2 &= \left( n \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma - \sum_{\eta} \text{sgn}(\eta) \eta \right) \text{Alt}_k \\ &= n \sum_{\sigma \in S_k} \text{sgn}(\sigma) (\sigma \text{Alt}_k) \\ &= n \sum_{\sigma \in S_k} \text{sgn}(\sigma) (\text{sgn}(\sigma) \text{Alt}_k) \\ &= k! n \text{Alt}_k, \end{aligned}$$

we can see that the matrix unit in (33) is also equal to  $\frac{1}{k!n} \text{Alt}_k$ . While it may be more suitable to define

$$N \begin{pmatrix} k \\ \vdots \\ 2 \\ 1 \\ \vdots \end{pmatrix}$$

to be equal to

$$\sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma - \frac{1}{n} \sum_{\eta} \text{sgn}(\eta) \eta \in \mathcal{L}_{\mathbb{C}}\{\text{Alt}_k\}$$

in consideration as to how this partition algebra element resembles the sum in (25), it is convenient for our purposes to instead let (27) be equal to the idempotent element in the alternating representation  $\mathcal{L}_{\mathbb{C}}\{\text{Alt}_k\}$ , as it is often more convenient to work with partition algebra matrix units, as opposed to non-normalized scalar multiples of such matrix units.

We recall the definition of  $\text{Quasi}_k$  given in Definition 2.1 [1]. It was proved in [1] that  $\mathcal{L}_{\mathbb{C}}\{\text{Quasi}_k\}$  is a  $\mathbb{C}A_k(n)$ -submodule, with:

$$d\text{Quasi}_k = \begin{cases} \text{Quasi}_k & \text{if } d \text{ is a permuting diagram,} \\ 0 & \text{otherwise.} \end{cases} \quad (34)$$

The idempotent element in this submodule is equal to the following matrix unit:

$$e_{(\emptyset, \emptyset, 1, 1, 2, 2, \dots, k), (\emptyset, \emptyset, 1, 1, 2, 2, \dots, k)},$$

and it can be shown that the above matrix unit is equal to

$$\frac{(-1)^k}{k!(n-2k+2)^{\binom{k}{2}}} \text{Quasi}_k,$$

following the construction of the  $\mathbb{C}A_k(n)$ -module  $\mathcal{L}_{\mathbb{C}}\{\text{Quasi}_k\}$  given in [1]. We define

$$P \left( \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & k \\ \hline & & \cdots & \\ \hline \end{array} \right) \quad (35)$$

to be the idempotent element in the quasi-trivial representation  $\mathcal{L}_{\mathbb{C}}\{\text{Quasi}_k\}$ , recalling that the diagram basis expansion for the same expression was introduced and proved in [1]. We may also write

$$P \left( \overline{\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & k \\ \hline \end{array}} \right)$$

in place of (35). We let  $\overline{\text{Quasi}_k}$  denote the scalar multiple of  $\text{Quasi}_k$  such that  $\overline{\text{Quasi}_k}$  is idempotent, and similarly for  $\overline{\text{Alt}_k}$ ; we generalize this kind of notation in Definition 4.14 below.

We are now ready to define  $N(T)$  for a single-column set-partition tableau  $T$ , as well as  $P(U)$  in the case whereby  $U$  is a two-row set-partition tableau without labels in the first row.

**Definition 4.10.** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$  be a set-composition such that  $\bigcup \alpha \subseteq \{1, 2, \dots, k\}$ . Define

$$N_k \left( \begin{array}{|c|} \hline \alpha_r \\ \hline \vdots \\ \hline \alpha_2 \\ \hline \alpha_1 \\ \hline \\ \hline \end{array} \right) = N_k \left( \begin{array}{|c|} \hline \alpha_r \\ \hline \vdots \\ \hline \alpha_2 \\ \hline \alpha_1 \\ \hline \end{array} \right) = \text{Stretch}_{\alpha, k} (\overline{\text{Alt}_r}),$$

and define

$$P_k \left( \begin{array}{|c|c|c|c|} \hline \alpha_1 & \alpha_2 & \cdots & \alpha_r \\ \hline & & \cdots & \\ \hline \end{array} \right) = \text{Stretch}_{\alpha, k} (\overline{\text{Quasi}_r}) \quad (36)$$

correspondingly. For the sake of convenience, we may also write

$$P_k \left( \boxed{\alpha_1} \boxed{\alpha_2} \cdots \boxed{\alpha_r} \right) = \text{Stretch}_{\alpha, k} \left( \overline{\text{Quasi}_r} \right).$$

It is often convenient to simply write  $N$  and  $P$  in place of  $N_k$  and  $P_k$ , respectively.

We proceed to generalize the above definition so as to define  $N(T)$  and  $P(T)$  for an arbitrary set-partition tableau  $T$ .

**Definition 4.11.** For a set-partition tableau  $T$  of content  $\{1, 2, \dots, k\}$ , we define  $N(T)$  as follows:

$$N(T) = N_k(\text{col}_1(T)) N_k(\text{col}_2(T)) \cdots N_k(\text{col}_{(\text{shape}(T))_2}(T)) \\ \delta_k(\{S_1, S'_1\}) \delta_k(\{S_2, S'_2\}) \cdots \delta_k(\{S_v, S'_v\}),$$

where  $S_1, S_2, \dots, S_v$  are any labels in the initial row of  $T$ . Similarly, we let

$$P(T) = P_k(\text{row}_2(T)) P_k(\text{row}_3(T)) \cdots P_k(\text{row}_{\ell(\text{shape}(T))}(T)) \\ \delta_k(\{S_1, S'_1\}) \delta_k(\{S_2, S'_2\}) \cdots \delta_k(\{S_v, S'_v\}).$$

**Example 4.12.** Letting  $k = 5$  and  $n = 10$ , we evaluate the expression

$$P \left( \begin{array}{cccccc} \boxed{23} & & & & & \\ \boxed{1} & \boxed{4} & & & & \\ & & & & & \boxed{5} \end{array} \right) \quad (37)$$

in the diagram basis as follows. From Definition 4.10, and since  $\text{Quasi}_1$  and  $\text{Quasi}_2$  have 2 and 15 elements in the respective diagram basis expansions of these expressions, we see that

$$P_k \left( \boxed{23} \right) \quad \text{and} \quad P_k \left( \boxed{1} \boxed{4} \right)$$

also must have 2 and 15 elements, respectively, in the diagram basis expansions of these elements. The partition algebra element in (37) may be written as

$$P_k \left( \boxed{1} \boxed{4} \right) P_k \left( \boxed{23} \right) \delta_k(\{\{5\}, \{5'\}\}). \quad (38)$$

We have that (37) must equal the following:

$$\frac{1}{(n-2)(n-1)n} \left( \begin{array}{cccccc} \circ & \circ & \circ & \circ & \circ & \circ \\ & \cup & & & \cup & \\ \circ & \circ & \circ & \circ & \circ & \circ \\ & \cup & & & \cup & \\ \circ & \circ & \circ & \circ & \circ & \circ \end{array} \right) \\ + \left( \begin{array}{cccccc} \circ & \circ & \circ & \circ & \circ & \circ \\ & \cup & & & \cup & \\ \circ & \circ & \circ & \circ & \circ & \circ \\ & \cup & & & \cup & \\ \circ & \circ & \circ & \circ & \circ & \circ \end{array} \right) \\ - \left( \begin{array}{cccccc} \circ & \circ & \circ & \circ & \circ & \circ \\ & \cup & & & \cup & \\ \circ & \circ & \circ & \circ & \circ & \circ \\ & \cup & & & \cup & \\ \circ & \circ & \circ & \circ & \circ & \circ \end{array} \right)$$





$$\begin{aligned}
& + \frac{1}{n-2} \left( \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ | & | & | & | & | & | & | \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{array} + \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ | & | & | & | & | & | & | \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{array} \right) \\
& + \left( \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ | & | & | & | & | & | & | \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{array} + \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ | & | & | & | & | & | & | \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{array} \right) \\
& + \frac{1}{2} \left( \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ | & | & | & | & | & | & | \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{array} + \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ | & | & | & | & | & | & | \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{array} \right) \\
& - \frac{n}{n-2} \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ | & | & | & | & | & | & | \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{array} .
\end{aligned}$$

**Definition 4.13.** For a set-partition tableau  $T$ , we write  $E_T = E(T) = N(T)P(T)$ .

In our mimicking the formulation of Young's construction given in [9], we want to determine some suitable analogue of Young's  $\gamma$ -expressions. So, we are interested in whether or not partition algebra elements as in Definition 4.13 are idempotent up to a nonzero scalar multiple. This leads us toward the following definition, which we had also used in [3], recalling our above definitions for  $\overline{\text{Quasi}}_k$  and  $\overline{\text{Alt}}_k$ .

**Definition 4.14.** Let  $x \neq 0$  be in a partition algebra. If there is a nonzero scalar  $\alpha$  whereby  $\alpha x$  is idempotent, then  $\bar{x} = \alpha x$  [3].

It turns out that  $E(T)$  is not, in general, idempotent-up-to-a-nonzero-scalar. The first counterexamples are given by the following tableaux, letting  $k = 3$  and  $n = 6$ :

$$\begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & \\ \hline & & \\ \hline \end{array} . \tag{39}$$

However, the property whereby  $E(U)$  is idempotent-up-to-a-nonzero-scalar for an injective tableau  $U$ , which may be proved using the von Neumann Sandwich Lemma, is very important in Young's construction [9, §1]. In Section 4.3 below, we consider a way of dealing with the problem that  $E(T)$  is not, in general, idempotent-up-to-a-nonzero-scalar for a set-valued tableau  $T$ .

### 4.3 Young-type matrix units for $\mathbb{C}A_2(n)$ and $\mathbb{C}A_3(n)$

If we want to mimic the approach that Young had applied in the construction of symmetric group algebra matrix units, with reference to [9, §1], then the question arises as to how we should define a suitable analogue of Young's First Letter Order that may be applied to

set-partition tableaux. It turns out that if we follow the steps introduced below for forming matrix units for  $\mathbb{C}A_2(n)$  and  $\mathbb{C}A_3(n)$ , then the ordering on set-valued tableaux that is used does not matter. So, to begin with, let us adopt some fixed linear ordering on set-partition tableaux.

The relation that is defined below is inspired by the insertion algorithm on multiset partitions introduced in [4]. We encourage the interested reader to review the reference [4] to compare the below definition and the results on diagram algebras introduced in [4].

**Definition 4.15.** *Let  $T$  be a set-partition tableau on  $\{1, 2, \dots, k\}$ . The tableau  $T$  must have a label of the form  $\{k\} \cup A$ , where  $A$  may or may not be empty. Let  $T'$  denote the tableau obtained from  $T$  by removing this label. Define*

$$T'' = \begin{cases} T' \text{ with } A \text{ row-inserted starting above the first row} & \text{if } A \neq \emptyset, \\ T' \text{ with an empty cell added by extending the first row} & \text{if } A = \emptyset. \end{cases}$$

We are making use of RSK row insertion to insert labels into set-partition tableaux, along with the label ordering noted in Remark 4.6; we refer the interested reader to classic references as in [23] for exposition on the Robinson–Schensted algorithm.

Now, let  $T_1$  and  $T_2$  be set-partition tableaux on  $\{1, 2, \dots, k\}$ . Letting the dominance ordering on integer partitions be denoted with  $<_{dom}$ , we have that  $T_1 < T_2$  if:

1.  $shape(T_1) <_{dom} shape(T_2)$ ; or
2.  $shape(T_1) = shape(T_2)$  and  $shape(T'_1) <_{dom} shape(T'_2)$ ; or
3.  $shape(T_1) = shape(T_2)$  and  $shape(T'_1) = shape(T'_2)$  and  $shape(T''_1) <_{dom} shape(T''_2)$ .

Otherwise, compare  $T''_1$  and  $T''_2$ , and repeat this process recursively, if necessary.

We leave it as an exercise to show that the ordering on set-partition tableaux given above is well-defined and does indeed give us a total ordering.

**Example 4.16.** *Let  $n = 6$  and  $k = 3$ . There are 10 set-valued tableaux of shape  $(5, 1) \vdash n$  and content  $\{1, 2, k = 3\}$ . Using Definition 4.15, we obtain the linear ordering indicated below on these 10 tableaux.*

$$\begin{array}{ccccccc} \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & & & & 13 \\ \hline \end{array} < \begin{array}{|c|c|c|c|c|} \hline 1 & & & & \\ \hline & & & & 23 \\ \hline \end{array} < \begin{array}{|c|c|c|c|c|} \hline 1 & & & & \\ \hline & & & 2 & 3 \\ \hline \end{array} < \begin{array}{|c|c|c|c|c|} \hline 12 & & & & \\ \hline & & & & 3 \\ \hline \end{array} < \\ \\ \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & & & 1 & 3 \\ \hline \end{array} < \begin{array}{|c|c|c|c|c|} \hline 13 & & & & \\ \hline & & & & 2 \\ \hline \end{array} < \begin{array}{|c|c|c|c|c|} \hline 123 & & & & \\ \hline & & & & \\ \hline \end{array} < \begin{array}{|c|c|c|c|c|} \hline 23 & & & & \\ \hline & & & & 1 \\ \hline \end{array} < \\ \\ \begin{array}{|c|c|c|c|c|} \hline 3 & & & & \\ \hline & & & & 12 \\ \hline \end{array} < \begin{array}{|c|c|c|c|c|} \hline 3 & & & & \\ \hline & & & 1 & 2 \\ \hline \end{array} \end{array}$$

The ordering defined above may be regarded as a natural analogue of Young’s First Letter Ordering, in a variety of ways. In particular, the “last” Young tableau of a given partition shape according to the YFLO relation is always the standard “row-reading” tableau whereby the labels therein are consecutive as “read” from left to right, row by row, and we see that the same kind of phenomenon holds for the last set-partition tableau of a given shape according to Definition 4.15.

**Example 4.17.** According to Definition 4.15, the last tableau of shape  $(6, 2) \vdash 8$  and content  $\{1, 2, 3, 4\}$  is

3	4					1	2

and the last tableau of shape  $(6, 2, 2) \vdash 10$  and content  $\{1, 2, \dots, 5\}$  is as below.

4	5						
2	3						
						1	

Recall that the first counterexamples to  $E(T)$  being idempotent-up-to-a-nonzero scalar are as given in (39). Even when  $\overline{E(T)}$  is well-defined, such expressions do not, in general, form triangular multiplication tables, for  $T$  ranging over all possible standard set-partition tableaux of a given shape, no matter what ordering is to be imposed. However, Young’s  $\gamma$ -expressions do satisfy a triangular multiplication rule, e.g., subject to the YFLO relation, and this forms an important aspect about Young’s construction [9]. So, this leads us to consider how we may construct “triangularized” variants of elements of the form  $\overline{E(T)}$ .

**Example 4.18.** Consider the following evaluations:

$$\overline{E\left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & 12 \\ \hline \end{array}\right)} = \frac{1}{n} \begin{array}{c} \circ \quad \circ \\ \text{---} \\ \circ \quad \circ \end{array},$$

$$\overline{E\left(\begin{array}{|c|c|c|c|} \hline \square & \square & 1 & 2 \\ \hline \end{array}\right)} = \frac{1}{n^2} \begin{array}{c} \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \end{array}.$$

We can see that the above  $\overline{E(T)}$ -expressions do not form a triangular multiplication table.

In our denoting set-partition tableaux ordered according to Definition 4.15, the notation given in the below definition will be useful for our purposes.

**Definition 4.19.** Let  $\lambda$  be an integer partition of order  $n$ , and let  $2k \leq n$ . We let  $g^{\lambda, k}$  denote the number of set-partition tableaux of shape  $\lambda \vdash n$  and of content  $\{1, 2, \dots, k\}$ .

**Example 4.20.** Recalling Example 4.16, we see that  $g^{(5,1),3} = 10$ . Recalling Example 4.18, we see that  $g^{(4),2} = 2$ .

We note that it is important that we specify both  $\lambda$  and  $k$  in the superscript of  $g^{\lambda,k}$ , as in Definition 4.19, since we may have that  $g^{\lambda,\ell_1} \neq g^{\lambda,\ell_2}$  for distinct values  $\ell_1$  and  $\ell_2$ .

**Example 4.21.** *We have that  $g^{(6),2} = 2$  and  $g^{(6),3} = 5$ .*

Much of our research in this article is inspired by and based upon the identity whereby

$$B_{2k} = \sum_{\lambda \vdash n} (g^{\lambda,k})^2$$

for  $2k \leq n$ , which gives us such a direct analogue of the famous *Frobenius–Young identity* whereby

$$k! = \sum_{\lambda \vdash k} (f^\lambda)^2.$$

Since the semisimplicity of algebras of the form  $\mathbb{C}A_k(n)$  is so central to our research, it is convenient for our purposes to always let  $2k \leq n$  if we are letting  $n$  be integral, as this guarantees the semisimplicity of  $\mathbb{C}A_k(n)$ . However, in our research on the representation theory of  $\mathbb{C}A_k(n)$ , we may also let  $n$  be an arbitrary element in  $\mathbb{C}$  apart from from certain integer values less than  $2k$ . So, although we can think of  $n$  as being a complex indeterminate subject to the restriction that  $n$  cannot be equal to certain integer values, in practical situations, it is convenient to let  $n$  be equal to some fixed integer that is sufficiently large, e.g., to illustrate set-partition tableaux. This leads us to the following.

If we assign a fixed value to  $\lambda$  so that  $\lambda$  is of even order, we may write  $g^\lambda$  in place of  $g^{\lambda, \frac{1}{2}|\lambda|}$ , for the sake of convenience. For example, we may write

$$g^{\begin{array}{c} \square \\ \square \square \square \end{array}} = 3$$

and

$$g^{\begin{array}{c} \square \\ \square \square \square \square \square \end{array}} = 10$$

using this notational simplification, according to which we may also let

$$T_1^\lambda < T_2^\lambda < \dots < T_{g^\lambda}^\lambda$$

denote the set-partition tableaux of fixed shape  $\lambda$  on  $\{1, 2, \dots, \frac{1}{2}|\lambda|\}$ , ordered according to Definition 4.15. For example, we may write

$$T_1^{\begin{array}{c} \square \\ \square \square \square \end{array}} < T_2^{\begin{array}{c} \square \\ \square \square \square \end{array}} < T_3^{\begin{array}{c} \square \\ \square \square \square \end{array}}$$

since there are 3 set-partition tableaux of shape  $(3, 1)$  and content  $\{1, 2\}$ .

If we want to consider set-partition tableaux of a given shape  $\lambda$  defined using the parameter  $n$  without *specifying* a value for  $n$ , in our investigating the semisimple structure of  $\mathbb{C}A_k(n)$ , we always let  $\lambda \vdash n$ , and we let it be understood that we are letting  $2k \leq n$ , where  $k$

denotes the order of the partition algebra under consideration, and that the tableaux under consideration are of content  $\{1, 2, \dots, k\}$ . So, for an integer partition  $\lambda$  of a fixed shape denoted with the parameter  $n$ , we may let

$$T_1^{\lambda,k} < T_2^{\lambda,k} < \dots < T_{g^{\lambda,k}}^{\lambda,k}$$

denote the set-partition tableaux of shape  $\lambda$  and content  $\{1, 2, \dots, k\}$  ordered as above, but, for the sake of convenience, if we are fixing a value of  $\lambda$  and writing  $\lambda$  using the parameter  $n$ , we may instead write

$$T_1^\lambda < T_2^\lambda < \dots < T_{g^\lambda}^\lambda \tag{40}$$

under the understanding that we are letting  $2k \leq n$ , and  $T_i^\lambda = T_i^{\lambda,k}$  for each index  $i$  and  $g^\lambda = g^{\lambda,k}$ . For example, we may rewrite

$$T_1^{(n),k} < T_2^{(n),k} < \dots < T_{g^{(n),k}}^{(n),k}$$

as

$$T_1^{(n)} < T_2^{(n)} < \dots < T_{g^{(n)}}^{(n)},$$

as it seems tacit that we may simply let  $n$  be sufficiently large compared to the order  $k$  of the partition algebra under consideration.

Keeping in mind the notational convention indicated in (40), we “tentatively” define analogues of Young’s  $\gamma$ -elements as below, noting that the recursive process indicated in Definition 4.22 is “tentative” in the sense that: In general, proving that an expression of the form

$$E(T_i^\lambda) (1 - \gamma_1^\lambda) (1 - \gamma_2^\lambda) \dots (1 - \gamma_{i-1}^\lambda)$$

is normalizable is difficult, as we shall see.

We employed a version of Definition 4.22 below in [3], but only for flat-shaped tableaux, in the construction of non-propagating matrix units.

**Definition 4.22.** *We let the expressions  $\gamma_1^\lambda, \gamma_2^\lambda, \dots, \gamma_{g^\lambda}^\lambda$  be defined as below, if it is possible to normalize the below expressions according to Definition 4.14:*

$$\gamma_1^\lambda = \overline{E(T_1^\lambda)},$$

$$\gamma_2^\lambda = \overline{E(T_2^\lambda) (1 - \gamma_1^\lambda)},$$

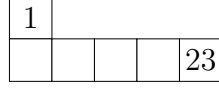
$$\gamma_3^\lambda = \overline{E(T_3^\lambda) (1 - \gamma_1^\lambda) (1 - \gamma_2^\lambda)},$$

*etc.*

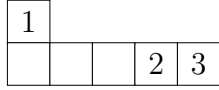
We intend to use the above definition, along with a suitable analogue of Young’s  $\sigma$ -function whereby  $Y_i = \sigma_{ij}^\lambda Y_j$ , for  $\lambda$ -shaped Young tableaux  $Y_i$  and  $Y_j$  in Young’s First Letter Order. This leads us to the following definition, which we had also used in [3] for flat set-partition tableaux.

**Definition 4.23.** We set  $T_i^\lambda$  and  $T_j^\lambda$  to be set-partition tableaux on  $\{1, 2, \dots, k\}$  as in (40). We define  $\tau_{i,j}^\lambda$  as the partition diagram in  $\mathbb{C}A_k(n)$  given as follows. Let  $L_1$  and  $L_2$  be labels in  $T_i^\lambda$  and  $T_j^\lambda$ , respectively, such that  $L_1$  and  $L_2$  are not both empty and are in the same position. Then  $L_1 \cup L_2$  is a block in  $\tau_{i,j}^\lambda$  [3].

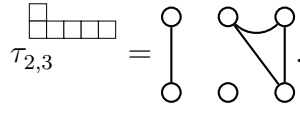
**Example 4.24.** The second entry in the sequence of tableaux in Example 4.16 is



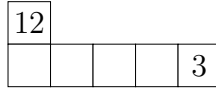
and the next such entry is



Let us compute  $\tau_{2,3}^{(5,1)}$ . We have that  $\emptyset$  and  $\{2\}$  are labels in  $T_2^{(5,1)}$  and  $T_3^{(5,1)}$ , respectively, such that these labels are in the same position and are not both equal to  $\emptyset$ . So, from Definition 4.23, we have that  $\emptyset \cup \{2\}'$  is a block in the  $\tau$ -diagram under consideration. Also, we have that  $\{2, 3\} \cup \{3\}'$  and  $\{1\} \cup \{1\}'$  must be blocks in this diagram, as below:



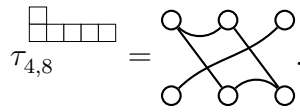
**Example 4.25.** Looking back over Example 4.16, the fourth entry in this sequence is



and the eighth entry in the sequence in Example 4.16 is



We find that:



We are now ready to offer a truly remarkable analogue of Young's fundamental formula for symmetric group algebra matrix units. Letting the expression  $g^\lambda$  be as in (40), and letting  $1 \leq i, j \leq g^\lambda$ , let us write

$$e_{i,j}^\lambda = \tau_{i,j}^\lambda \gamma_j^\lambda (1 - \gamma_{j+1}^\lambda) (1 - \gamma_{j+2}^\lambda) \cdots (1 - \gamma_{g^\lambda}^\lambda). \quad (41)$$

Partition algebra elements of the form indicated in (41) exhibit some very unexpected properties, and computational experiments suggest that these elements often give us a very nice and surprisingly close analogue of Young’s matrix units. We should recall that our construction in Definition 4.22 is “tentative” in the sense that it is unclear as to when it would be possible to normalize the required expressions, according to Definition 4.14, in order to define  $\gamma_i^\lambda$ . How can we show that partition algebra elements given by the product on the right-hand side of (41) are non-vanishing? For the time being, let us consider the elements that (41) gives us in  $\mathbb{C}A_2(n)$  and  $\mathbb{C}A_3(n)$ .

We use the symbol  $\sim$  to denote the equivalence relation given by equality up to a nonzero scalar multiple. Observe that  $0 \sim 0$ , according to this definition. Using a SageMath implementation of (41), we have verified that in the case whereby  $k = 2$ , (41) is always nonzero, and, amazingly, the following matrix unit multiplication formula holds:

$$e_{i_1, i_2}^\lambda e_{i_3, i_4}^\mu \sim \begin{cases} 0 & \text{if } \lambda \neq \mu \\ 0 & \text{if } i_2 \neq i_3 \\ e_{i_1, i_4}^\lambda & \text{if } \lambda = \mu \text{ and } i_2 = i_3 \end{cases} . \quad (42)$$

That is, (41) gives us a complete family of Young-type matrix units for  $\mathbb{C}A_2(n)$  indexed by pairs of set-partition tableaux, as well as explicit matrix unit decompositions of all of the irreducible subrepresentations of  $\mathbb{C}A_2(n)$ . We encourage the reader to consider how strikingly different (41) is compared to the “basic construction” recursion used by Halverson and Ram [15]. In consideration of the many intricate definitions and constructions that we had used to arrive at (41), it is quite remarkable that the elements in (41) are so well-behaved, with regard to the elegant matrix unit multiplication formula in (42).

Now, let us again consider the counterexamples given in (39) to  $E(T)$  being idempotent-up-to-a-nonzero-scalar-multiple. The first counterexamples to the identity in (42) holding are also given by the shapes in (39). For example,

$$\begin{array}{ccc} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} & \not\sim & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} , \\ \\ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} & \not\sim & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} . \end{array}$$

So, we intend to figure out some way of “working around” these kinds of counterexamples. We are inspired to make use of Halverson and Ram’s formula

$$e_{P,Q}^\lambda = \left( \text{id} - \sum_{\substack{\mu \in \hat{A}_\ell \\ |\mu| \leq \ell - 1}} \sum_{P \in \hat{A}_\ell^\mu} e_{P,P}^\mu \right) \psi(s_{P,Q}^\lambda), \quad (43)$$

where  $P$  and  $Q$  denote vacillating tableaux that correspond to standard Young tableaux, i.e., vacillating tableaux that end at level  $\ell$  in the Bratteli diagram for partition algebras



and that are such that the final entry is of order strictly greater than  $\ell - 1$ ; it should also be noted that the matrix units on the right-hand side of (43) are understood to be defined using Halverson and Ram's construction [15]. Informally, the left-hand factor on the right-hand side of the equation (43) comes from the property whereby the sum of idempotent matrix units in a finite-dimensional semisimple algebra must equal the identity, and we want to use this kind of idea to deal with the counterexamples given by the shapes in (39).

Again with  $n = 6$  and  $k = 3$ , define

$$z \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array} = \sum_{\lambda_1 \neq 3} e_{i,i}^\lambda.$$

More specifically, we are taking the sum of all expressions of the form (41), apart from the cases whereby the superscript  $\lambda$  is such that  $\lambda_1$  is not equal to 3. Now, let us consider the variant of the recursion in Definition 4.22 suggested below:

$$\begin{array}{l} \mathbf{y}_1 \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array} = \overline{\left(1 - z \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}\right) E \left( T_1 \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array} \right)}, \\ \mathbf{y}_2 \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array} = \overline{\left(1 - z \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}\right) E \left( T_2 \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array} \right) \left(1 - \mathbf{y}_1 \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}\right)}. \end{array}$$

For an integer partition  $\lambda \vdash n = 6$  other than  $(3, 2, 1)$ , write  $\mathbf{y}_i^\lambda$  to denote  $\gamma_i^\lambda$ . Now, let us consider the following variant of (41), noting the distinction between our notation for  $\mathbf{e}_{i,j}^\lambda$  and  $e_{i,j}^\lambda$ :

$$\mathbf{e}_{i,j}^\lambda = \tau_{i,j}^\lambda \mathbf{y}_j^\lambda (1 - \mathbf{y}_{j+1}^\lambda) (1 - \mathbf{y}_{j+2}^\lambda) \cdots (1 - \mathbf{y}_{g^\lambda}^\lambda). \quad (44)$$

Again, we obtain a full family of matrix units, satisfying the desired multiplicative properties indicated in (42).

It is remarkable that the way in which we have defined  $N(T)$  and  $P(T)$  for set-partition tableaux  $T$  provides us with such a close analogue of Young's fundamental matrix unit formula. How can we generalize the Young-type orthogonal forms that we have determined for  $\mathbb{C}A_2(n)$  and  $\mathbb{C}A_3(n)$ ?

## 5 A conjecturally universal formula for Young-type matrix units for partition algebras

We define the *graded-lexicographic ordering* on integer partitions so that for integer partitions  $\lambda$  and  $\mu$ ,  $\lambda <_{GrLex} \mu$  if  $|\lambda| < |\mu|$  or if  $|\lambda| = |\mu|$  and  $\lambda$  is strictly less than  $\mu$  lexicographically. In particular, we have that

$$() <_{GrLex} (1) <_{GrLex} (1, 1) <_{GrLex} (2) <_{GrLex} (1, 1, 1) <_{GrLex} (2, 1) <_{GrLex} (3) <_{GrLex} \cdots,$$

by definition. We also define the *graded-and-larger-lexicographic* order relation so that for integer partitions  $\lambda$  and  $\mu$ ,  $\lambda <_{GrLar} \mu$  if  $|\lambda| < |\mu|$  or if  $|\lambda| = |\mu|$  and  $\lambda$  is strictly greater than  $\mu$  lexicographically. So, we find that:

$$() <_{GrLar} (1) <_{GrLar} (2) <_{GrLar} (1, 1) <_{GrLar} (3) <_{GrLar} (2, 1) <_{GrLar} (1, 1, 1) <_{GrLar} \cdots .$$

We also recall that we let  $\bar{\lambda}$  denote the integer partition obtained by removing the initial entry of  $\lambda$ .

We have verified computationally that the formula

$$e_{i,j}^\lambda = \tau_{i,j}^\lambda \gamma_j^\lambda (1 - \gamma_{j+1}^\lambda) (1 - \gamma_{j+2}^\lambda) \cdots (1 - \gamma_{g^\lambda}^\lambda). \quad (45)$$

gives us full families of Young-type matrix units for both  $\mathbb{C}A_2(n)$  and  $\mathbb{C}A_3(n)$  according to the universal recursion for  $\gamma$ -elements defined below, as opposed to the “ad hoc” approach that we had previously used.

**Proposition 5.1.** *Letting  $k$  and  $n$  be fixed members of  $\mathbb{N}$ , with  $2k \leq n$ , we let*

$$\gamma_i^{(n)} = \overline{P \left( T_i^{(n)} \right) \left( 1 - \gamma_1^{(n)} \right) \left( 1 - \gamma_2^{(n)} \right) \cdots \left( 1 - \gamma_{i-1}^{(n)} \right)},$$

and for  $\lambda \vdash n$  such that  $\lambda$  is the shape of a non-flat set-partition tableau of content  $\{1, 2, \dots, k\}$ , we redefine  $\gamma_i^\lambda$  so that:

$$\gamma_i^\lambda = \overline{(1 - z^\lambda) P \left( T_i^\lambda \right) \left( 1 - \gamma_1^\lambda \right) \left( 1 - \gamma_2^\lambda \right) \cdots \left( 1 - \gamma_{i-1}^\lambda \right)},$$

where the Halverson–Ram-inspired operator  $(1 - z^\lambda)$  is such that

$$z^\lambda = \sum_{\bar{\mu} <_{GrLar} \bar{\lambda}} e_{i,i}^\mu \quad (46)$$

according to the fundamental formula in (45). According to this construction, for each of the algebras  $\mathbb{C}A_2(n)$  and  $\mathbb{C}A_3(n)$ , each member of the family  $\{e_{i,j}^\lambda\}$  is nonzero and these elements are such that each of the matrix unit multiplication formulas is satisfied up to a nonzero scalar.

One might wonder why the  $<_{GrLar}$  is being used in (46). Interestingly, if we were to replace  $<_{GrLar}$  with  $<_{GrLex}$  in the above proposition, this construction would fail, i.e., the desired matrix unit formulas would not hold up to a nonzero scalar. We leave it as an open problem to prove that our apparently universal formula for partition algebra matrix units, as above, holds in full generality.

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