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# Entropy solutions to a non-conservative and non-strictly hyperbolic diagonal system inspired by dislocation dynamics

Maryam Al Zohbi\*, Stéphane Junca †

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## Abstract

In this work, we study the existence of solutions to a  $2 \times 2$  non-conservative and non-strictly hyperbolic system in one space dimension related to the dynamics of dislocation densities in crystallography, propagating in two opposite directions. For such systems, existence results are mainly established in the sense of viscosity solutions for Hamilton-Jacobi equations. We study this problem for large initial data using the notions of the theory of conservation laws by constructing entropy solutions through the means of an adapted Godunov scheme, where the associated Riemann problem enjoys new features, more elementary waves than usual, and loss of uniqueness in many cases. The existence is obtained in spaces of functions with bounded fractional total variation  $BV^s$ , for all  $0 < s \leq 1$ , such that  $BV^1 = BV$ .

**AMS Classification:** 35L65, 35L67, 35Q35, 26A45, 76N10.

**Key words:** Dislocations, conservation laws, hyperbolic systems, viscosity solutions, entropy solutions, Riemann problem, Fractional  $BV$  spaces.

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# 1 Introduction

We are interested in a coupled system of two equations that is inspired from the dynamics of dislocations. A dislocation is a linear crystallographic defect or irregularity within a crystal structure that contains an abrupt change in the arrangement of its atoms. This concept was introduced by Polanyi, Taylor and Orowan in 1934 as the principle explanation of plastic deformation in materials at the microscopic scale. Under an exterior strain, a dislocation moves according to its Burgers vector which characterizes the intensity and the direction of the defect displacement. We refer to the work of Hirth *et al.* [38], Hull, Bacon [40], and Nabarro [53] for a complete introduction into the theory of dislocations.

Our study is actually motivated by a uni-dimensional model initially proposed in 2 dimensions by Groma, Balogh [34, 35] in order to describe the dynamics of edge dislocations, where the Burgers vector and the line carrying the dislocations are perpendicular. This 2D model is written in a specific geometry, where the dislocations are considered as points in the plane  $(x_1, x_2)$ , propagating to the left and to the right, following two Burgers vectors  $\pm b = \pm(1, 0)$  [17]. In the 1D sub-model, we assume that the dislocation densities depend only on the variable  $x = x_1 + x_2$ , which transforms the 2D model into a 1D one. We refer the reader to El Hajj, Forcadel [28] for more details about the modeling. More precisely, this 1D model is given by

$$\begin{cases} \partial_t \rho^+(t, x) = - \left( (\rho^+ - \rho^-)(t, x) + \alpha \int_0^1 (\rho^+ - \rho^-)(t, y) dy + a(t) \right) \left| \partial_x \rho^+(t, x) \right| & \text{in } (0, +\infty) \times \mathbb{R}, \\ \partial_t \rho^-(t, x) = \left( (\rho^+ - \rho^-)(t, x) + \alpha \int_0^1 (\rho^+ - \rho^-)(t, y) dy + a(t) \right) \left| \partial_x \rho^-(t, x) \right| & \text{in } (0, +\infty) \times \mathbb{R}. \end{cases} \quad (1.1)$$

The functions  $\rho^+$  and  $\rho^-$  are, respectively, the representations of the left-propagating and right-propagating dislocations. Their spatial derivatives  $\partial_x \rho^+$ ,  $\partial_x \rho^-$  represent the dislocation densities of +, - type respectively. The constant  $\alpha$  depends on the elastic coefficients and the material size, while the function  $a(t)$  represents the exterior strain field.

In this work, we study a simplified model of (1.1) where the parameter  $\alpha = 0$  and without

the absolute values. Thus, we introduce the following nonlinear system

$$\begin{cases} \partial_t u + (u - v)\partial_x u = 0, & \text{in } (0, +\infty) \times \mathbb{R}, \\ \partial_t v + (v - u)\partial_x v = 0, & \text{in } (0, +\infty) \times \mathbb{R}. \end{cases} \quad (1.2)$$

The functions  $u = u(t, x)$  and  $v = v(t, x)$  are real valued. We equip this system with the initial data

$$U(0, x) = U_0(x) = (u(0, x), v(0, x)) = (u_0(x), v_0(x)). \quad (1.3)$$

Also, we assume the following regularity for the initial data,

$$U_0 \in BV^s(\mathbb{R}, \mathbb{R}^2), \quad s \in (0, 1], \quad (1.4)$$

where  $BV^s$  is known as the fractional  $BV$  space [13]. The  $BV^s$  framework,  $0 < s \leq 1$ , extends the classical  $BV = BV^1$  one and shares many properties with it such as trace properties as regulated functions, compact embeddings into  $L^1_{loc}$ , and the fractional derivative "s" roughly corresponds to Sobolev derivatives. Nonetheless,  $BV^s$  is not a Sobolev space, but it is much close to the Sobolev space  $W^{s, 1/s}$  [13]. In fact, we have the following inclusions for a bounded interval  $I$  [13],

$$W^{s, +\infty}(I) \subset BV^s(I) \subset W^{s-0, \frac{1}{s}}(I) = \bigcap_{\varepsilon > 0} W^{s-\varepsilon, \frac{1}{s}}(I).$$

Roughly speaking,  $BV^s$  spaces,  $s \in ]0, 1]$ , are a continuum of spaces between  $BV$  and  $L^\infty$  spaces. However, the existence in  $L^\infty$  for system (1.2) remains an open question.

The  $BV^s$  space has been mainly studied in [16, 52]. The  $BV^s$  framework gives optimal results for the regularity of many entropy solutions [32] and mainly for the scalar case [18, 19, 51]. The applications of  $BV^s$  for systems of conservation laws has just started. There are recent results for weakly coupled systems [44], and systems with a linearly degenerate field [10, 14, 37]. Now, the model related to the dynamics of two types of dislocations is a new system where the  $BV^s$  framework is efficient. The formal definition and some of the properties of these spaces are recalled in the appendix.

An interesting feature of (1.2) is that it is a non-conservative and non-strictly hyperbolic system, and yet, we will adapt the tools from the theory of conservation laws in order to establish the existence of  $BV^s$  entropy solutions.

From a different point of view, we remark that system (1.2) can be seen as a system of transport equations with discontinuous velocities, but it is quite a difficult subject [9].

Indeed, first studies for this system have already been obtained by using vanishing viscosity solutions and the theory of Hamilton-Jacobi equations. More precisely, many results have been proven on system (1.1). For instance, in El Hajj [27], the global existence and uniqueness of a solution in the class of non-decreasing  $W^{1,2}_{loc}([0, +\infty) \times \mathbb{R})$  functions has been obtained, based on an energy estimate. However, without any monotony assumptions, El Hajj and Forcadel proved in [28] the global existence and uniqueness of a viscosity solution, for Lipschitz continuous initial data. The authors have also proposed a convergent numerical scheme and established a Crandall-Lions type error estimate. Also

in the framework of viscosity solutions, the global existence of a discontinuous solution was proven in El Hajj *et al.* [29] for  $BV$  initial data. Further more, in El Hajj, Oussaily [30], the global existence of a continuous viscosity solution has been presented, based on an entropy estimate and under a control on the gradient of the solution.

The work presented in this paper is also motivated by a recent result by Al Zohbi *et al.* [3], where the global existence of a discontinuous viscosity solution in a certain weak sense has been established to a diagonal hyperbolic system that is not necessarily strictly hyperbolic of the form

$$\begin{cases} \partial_t u^i(t, x) + \lambda^i(t, x, u(t, x)) \partial_x u^i(t, x) = 0, & \text{in } (0, +\infty) \times \mathbb{R}, \\ u^i(0, x) = u_0^i(x) & \text{in } \mathbb{R}, \end{cases} \quad i = 1, \dots, d, \quad (1.5)$$

for any  $BV$  initial data, assuming the velocities  $\lambda^i$  were only bounded functions. In fact, they were able to show that the upper and lower relaxed semi-limits of Barles and Perthame [6, 7] are a couple of discontinuous viscosity sub- and super-solutions respectively, in the sense introduced for Hamilton-Jacobi equations by Ishii, Koike [41, 42], that is recalled below in Definition 2.2 for system (1.2), and that these relaxed semi-limits are equal almost everywhere in space and uniformly in time. Moreover, in the case of non-decreasing initial data, they were able to prove an absolute equality between the relaxed semi-limits, leading to the existence of a non-decreasing viscosity solution. The key ingredient in [3] was a uniform  $BV$  estimate established for a certain smooth approximate solution that was also inherited to the solution of (1.5), and it was given by

$$TV u^i(t, \cdot)(\mathbb{R}) \leq TV u_0^i(\mathbb{R}), \quad i = 1, \dots, d. \quad (1.6)$$

In fact, this result was generalized to an eikonal system (system (1.5) with an absolute value on the space derivatives) in [2]. For an interested reader, we refer to [1, 4] for more information about viscosity solutions and applications of the methods used in [3].

In the present paper, we wish first to recover some of the properties of the solution obtained in [3]. With that in mind, we will impose condition (1.6) on the constructed solutions of the Riemann problem later on. In other words, we would like to have

$$\begin{aligned} TV u(t, \cdot)(\mathbb{R}) &\leq TV u_0(\mathbb{R}), \\ TV v(t, \cdot)(\mathbb{R}) &\leq TV v_0(\mathbb{R}), \end{aligned} \quad (1.7)$$

for a solution  $U = (u, v)$  of (1.2). What is interesting is that we were able to generalize this decay property in  $BV^s$ . The decay of the fractional total variation is well known for entropy solutions of scalar conservation laws [13], and it is usually proven using some numerical schemes like Wave Front Tracking [43] and the Godunov scheme [21].

We remark here that system (1.2) is diagonal thus hyperbolic with the two eigenvalues

$$\lambda_u(u, v) = u - v = -\lambda_v(u, v). \quad (1.8)$$

The strict hyperbolicity is lost on the diagonal

$$\Delta = \{u = v\}. \quad (1.9)$$

Also, the Riemann invariants are simply  $u$  and  $v$  since the Riemann invariants, when they exist, are used to diagonalize a hyperbolic system for smooth solutions [55].

We would also like to emphasize here that if we replace  $u$  by  $v$  in (1.2), the system remains unchanged. In other words, the system enjoys a symmetry property with respect to the diagonal  $\Delta$  in the  $(u, v)$  state plane. This property will be very useful in solving the Riemann problem later on.

Before presenting formally the main existence result of this work, we would like first to discuss in the following section various possible notions of solutions to system (1.2).

## 2 Various definitions of solutions

Due to the non-conservative form of (1.2), many attempts can be made in order to give sense or meaning to these equations, and hence define a proper solution. This is a well known difficult problem which can lead to many different concepts of solutions [26, 46, 47, 48]. We introduce, in this section, four concepts of solutions for the nonlinear system (1.2).

We begin with the standard vanishing viscosity solution, or *vs* for short, which is the limit of a sequence of smooth solutions to a viscous regularization of (1.2), which is (2.1), as the viscosity term tends to zero. This type of solution is natural to system (1.2) [8], as it not only provides us with existence results, but also with uniqueness in some cases. Then, we move on into the theory of Hamilton-Jacobi equations, since (1.2) falls also into that category, by defining the meaning of a viscosity solution, or *HJs*, in the case of system (1.2). The first existence result of system (1.2) was done with these two notions of solutions, *vs* and *HJs*. After that, moving on into the theory of conservation laws, we introduce the notion of an entropy solution, or *es*, which would be the main type of solution we construct in this paper. Finally, we define a special type of solution that would be called "Volpert *BV* solution". This definition is made up by trying to establish a rigorous meaning to the non-conservative products of (1.2).

Summarizing this long section of definitions of weak solutions:

1. The first definition, *vs*, is a natural way to get existence of solutions. The main properties of *vs* are also used to build a Riemann Solver later.
2. The second definition, *HJs*, allows us to use the theory of Hamilton-Jacobi equations, and it corresponds to the first result of existence of solutions together with the *vs* [3] in the *BV* framework.
3. The third definition, *es*, is the heart of this paper. It provides existence of *BV* and fractional *BV* solutions using the tools of the theory of hyperbolic conservation laws.
4. The fourth and the last definition is an attempt to write the non-conservative system using Volpert calculus for *BV* solutions.

## 2.1 Vanishing viscosity solutions

We begin by introducing the classical parabolic regularization of system (1.2)-(1.3), for  $0 < \varepsilon \leq 1$

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon - v^\varepsilon) \partial_x u^\varepsilon = \varepsilon \partial_{xx} u^\varepsilon, \\ \partial_t v^\varepsilon + (v^\varepsilon - u^\varepsilon) \partial_x v^\varepsilon = \varepsilon \partial_{xx} v^\varepsilon, \end{cases} \quad (2.1)$$

equipped with the regularized initial data

$$u_0^\varepsilon(x) = u_0 \star \rho_\varepsilon(x), \quad \text{and} \quad v_0^\varepsilon(x) = v_0 \star \rho_\varepsilon(x), \quad (2.2)$$

where  $\rho_\varepsilon$  is a standard mollifier in  $\mathbb{R}$  defined as

$$\rho_\varepsilon(\cdot) = \frac{1}{\varepsilon} \rho\left(\frac{\cdot}{\varepsilon}\right), \quad \text{such that} \quad \rho \in C_c^\infty(\mathbb{R}), \quad \text{supp}\{\rho\} \subseteq B(0, 1), \quad \rho \geq 0, \quad \text{and} \quad \int_{\mathbb{R}} \rho = 1.$$

The existence to (2.1)-(2.2) is a classical application of the Fixed Point theorem in Banach spaces. The proof in the more general case of (1.5) can be found in [3]. Indeed, we obtain a smooth Lipschitz solution  $U^\varepsilon = (u^\varepsilon, v^\varepsilon)$  satisfying

$$\|u^\varepsilon\|_{L^\infty((0, +\infty) \times \mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}, \quad \|v^\varepsilon\|_{L^\infty((0, +\infty) \times \mathbb{R})} \leq \|v_0\|_{L^\infty(\mathbb{R})}, \quad (2.3)$$

$$\|\partial_x u^\varepsilon\|_{L^\infty((0, +\infty); L^1(\mathbb{R}))} \leq |u_0|_{BV(\mathbb{R})}, \quad \|\partial_x v^\varepsilon\|_{L^\infty((0, +\infty); L^1(\mathbb{R}))} \leq |v_0|_{BV(\mathbb{R})}, \quad (2.4)$$

$$\|\partial_t u^\varepsilon\|_{L^\infty((0, +\infty); W^{-1,1}(\mathbb{R}))} \leq (1+c) |u_0^i|_{BV(\mathbb{R})}, \quad \|\partial_t v^\varepsilon\|_{L^\infty((0, +\infty); W^{-1,1}(\mathbb{R}))} \leq (1+c) |v_0^i|_{BV(\mathbb{R})},$$

where

$$c = \|u - v\|_{L^\infty((0, +\infty) \times \mathbb{R})},$$

that converges, up to the extraction of a subsequence, as  $\varepsilon \rightarrow 0$  to a function  $U = (u, v)$  strongly in  $(C([0, +\infty); L_{loc}^1(\mathbb{R})))^2$  such that

$$u, v \in L^\infty((0, +\infty) \times \mathbb{R}) \cap L^\infty((0, +\infty); BV(\mathbb{R})) \cap C([0, +\infty); L_{loc}^1(\mathbb{R})).$$

We remark here that the vanishing viscosity method preserves the symmetry property of (1.2). We would also like to point out that (2.4) refers to the TVD (total variation diminishing) property of the  $vvs$ , which is essential in solving the Riemann problem later on. Now, we recall the classical definition of a  $vvs$  to (1.2).

### Definition 2.1. (Vanishing viscosity solution ( $vvs$ for short))

A vanishing viscosity solution of (1.2)-(1.3) is a function  $U = (u, v)$  that is the strong limit, up to a subsequence, as  $\varepsilon \rightarrow 0$  of the smooth solution  $U^\varepsilon = (u^\varepsilon, v^\varepsilon)$  to the parabolic regularized system (2.1)-(2.2).

**Uniqueness of some solutions.** In this part, we will show some primary uniqueness results for the vanishing viscosity solutions. We mention ahead that the following three unique solutions correspond to the three types of single wave solutions presented in Subsection 4.1.

1. If  $u_0 = v_0$  then  $u = v$ : the proof derives from first adding the two equations of (2.1) and then subtracting them to obtain the following system, for  $\mathcal{S}^\varepsilon = u^\varepsilon + v^\varepsilon$  and  $\mathcal{D}^\varepsilon = u^\varepsilon - v^\varepsilon$

$$\begin{cases} \partial_t \mathcal{S}^\varepsilon + \mathcal{D}^\varepsilon \partial_x \mathcal{D}^\varepsilon = \varepsilon \partial_{xx} \mathcal{S}^\varepsilon, \\ \partial_t \mathcal{D}^\varepsilon + \mathcal{D}^\varepsilon \partial_x \mathcal{S}^\varepsilon = \varepsilon \partial_{xx} \mathcal{D}^\varepsilon. \end{cases} \quad (2.5)$$

We can easily see that  $\mathcal{D}^\varepsilon = 0$  is a solution of the second equation of (2.5). Then, inserting this in the first equation yields

$$\partial_t \mathcal{S}^\varepsilon = \varepsilon \partial_{xx} \mathcal{S}^\varepsilon.$$

Passing to the limit in the previous equation as  $\varepsilon \rightarrow 0$ , and using the fact that  $(u, v)$  is a *vis* of (2.1) and that  $\mathcal{D}^\varepsilon = 0$ , we can get for  $\mathcal{S} = u + v$  and  $\mathcal{D} = u - v$

$$\begin{cases} \partial_t \mathcal{S} = 0, \\ \partial_t \mathcal{D} = 0, \end{cases} \Rightarrow \begin{cases} \partial_t u = 0, \\ \partial_t v = 0, \end{cases} \Rightarrow \begin{cases} u = u_0, \\ v = v_0, \end{cases} \Rightarrow u = v.$$

2. If  $v_0$  is constant: in this case, we obtain a Burgers type equation. It is well known that for scalar conservation laws the unique entropy solution is a *vis* [31, 55].
3. If  $u_0$  is constant: similarly as in the previous case.

## 2.2 Viscosity solutions for Hamilton-Jacobi equations

We move on into defining the meaning of a viscosity solution to system (1.2) in the sense of Hamilton-Jacobi equations. We refer the reader to [5, 22, 23, 56] for a complete overview on viscosity solutions.

We denote by  $f^*$  and  $f_*$  the respective upper and lower semi-continuous envelopes of a locally bounded function  $f$  defined on an open domain in  $\mathbb{R}^n$  and given by

$$f^*(X) = \limsup_{Y \rightarrow X} f(Y) \quad \text{and} \quad f_*(X) = \liminf_{Y \rightarrow X} f(Y) \quad \text{for } X \in \mathbb{R}^n.$$

For a vector  $U = (u^1, u^2)$  locally bounded on  $[0, T) \times \mathbb{R}$  for all  $T > 0$ , we write  $U^* = ((u^1)^*, (u^2)^*)$  and  $U_* = ((u^1)_*, (u^2)_*)$ .

**Definition 2.2. (Discontinuous viscosity sub-solution, super-solution and solution (HJs for short))**

Assume that  $U_0 = (u_0^1, u_0^2)$  is locally bounded on  $\mathbb{R}$ . Let  $U = (u^1, u^2)$ ,  $V = (v^1, v^2)$  be two locally bounded functions on  $[0, +\infty) \times \mathbb{R}$  such that  $(v^1)_* \leq (u^1)^*$  and  $(v^2)_* \leq (u^2)^*$ . We say that  $U$  and  $V$  are a couple of discontinuous viscosity sub- and super- solutions of (1.2) if they satisfy the following two conditions

- (i) •  $(u^i)^*(0, x) \leq (u_0^i)^*(x)$ , for  $i = 1, 2$  and all  $x \in \mathbb{R}$ .
- $(v^i)_*(0, x) \geq (u_0^i)_*(x)$ , for  $i = 1, 2$  and all  $x \in \mathbb{R}$ .



(ii) • Whenever a test function  $\phi \in C^1((0, +\infty) \times \mathbb{R})$ ,  $i = 1, 2$  and  $(u^i)^* - \phi$  attains a local maximum at  $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}$ , then we have for  $j = 1, 2$  such that  $j \neq i$

$$\min \left\{ \partial_t \phi(t_0, x_0) + ((u^i)^*(t_0, x_0) - r^j) \partial_x \phi(t_0, x_0) : (v^j)_*(t_0, x_0) \leq r^j \leq (u^j)^*(t_0, x_0) \right\} \leq 0. \quad (2.6)$$

• Whenever  $\phi \in C^2((0, +\infty) \times \mathbb{R})$ ,  $i = 1, \dots, d$  and  $(v^i)_* - \phi$  attains a local minimum at  $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}$ , then we have for  $j = 1, 2$  such that  $j \neq i$

$$\max \left\{ \partial_t \phi(t_0, x_0) + ((v^i)_*(t_0, x_0) - r^j) \partial_x \phi(t_0, x_0) : (v^j)_*(t_0, x_0) \leq r^j \leq (u^j)^*(t_0, x_0) \right\} \geq 0. \quad (2.7)$$

Finally, we call a function  $U$  a discontinuous viscosity solution of (1.2) if  $U^*$  and  $U_*$  verify conditions (i) and (ii).

Noting that the minimum and the maximum in (2.6) and (2.7) are attained, since the sets

$$[(v^j)_*(t_0, x_0), (u^j)^*(t_0, x_0)], \quad j = 1, 2,$$

are non-empty and compact.

## 2.3 Entropy solutions

It is well known that a conservation law would sometimes induce another for smooth solutions. In order to see when we might have this for system (1.2), we first rewrite it in the form

$$\partial_t U + A(U) \partial_x U = 0, \quad (2.8)$$

where  $U = (u, v)$ , and  $A(U)$  is the matrix

$$A(U) = \begin{pmatrix} u - v & 0 \\ 0 & v - u \end{pmatrix}.$$

For  $\eta = \eta(U) \in \mathbb{R}$  and  $\psi = \psi(U) \in \mathbb{R}$ , we multiply (2.8) by  $\nabla \eta(U)$  to obtain the following conservative system from the non-conservative one as in [46],

$$\partial_t \eta(U) + \partial_x \psi(U) = 0, \quad (2.9)$$

when  $Q$  satisfies the following condition

$$\nabla \psi(U) = \nabla \eta(U) \cdot A(U). \quad (2.10)$$

The functions  $\eta$  and  $Q$ , when they exist, are called the entropy function and entropy flux respectively. Now, we can see that a solution of (1.2) would be some kind of solution to (2.9). As a means to prove this, we rewrite system (2.1) in the form

$$\partial_t U^\varepsilon + A(U^\varepsilon) \partial_x U^\varepsilon = \varepsilon \partial_{xx} U^\varepsilon, \quad (2.11)$$

then multiplying (2.11) by  $\nabla \eta(U^\varepsilon)$ , we get, by using (2.10)

$$\partial_t \eta(U^\varepsilon) + \partial_x \psi(U^\varepsilon) = \varepsilon \nabla \eta(U^\varepsilon) \partial_{xx} U^\varepsilon. \quad (2.12)$$

If  $\eta$  is convex, then passing to the limit as  $\varepsilon \rightarrow 0$ , and using the fact the  $U$  is bounded, we obtain

$$\partial_t \eta(U) + \partial_x \psi(U) \leq 0, \quad (2.13)$$

in distributional sense. This means that any  $vv$ s of (1.2) satisfies (2.13) in distributional sense, provided that  $\eta$  is convex. If  $\eta$  is concave, we obtain the opposite of (2.13). For more information about entropies we refer the reader to [26, 45, 46, 47, 49, 55].

In the following proposition, we introduce a linear and a strictly convex entropy functions and their associated fluxes for system (1.2).

**Proposition 2.1. (2 Entropies)**

System (1.2) admits the following entropy conditions in distributional sense, for  $\mathcal{S} = u+v$  and  $\mathcal{D} = u - v$

$$\partial_t \mathcal{S} + \partial_x \left( \frac{\mathcal{D}^2}{2} \right) = 0, \quad (2.14)$$

$$\partial_t (2\mathcal{S}^2 + \mathcal{D}^2) + \partial_x (2\mathcal{S}\mathcal{D}^2) \leq 0. \quad (2.15)$$

Of course there is an infinite set of entropies to system (1.2), however, we decided to present only two. The first entropy presented in this proposition is fundamental. It is a linear entropy, so it gives an entropy equality. This equality is the main tool in solving the Riemann problem. The second entropy condition (2.15) is the simplest example of a strictly convex entropy to (1.2), which is a good feature in the classical theory of conservation laws. Indeed, the eigenvalues are genuinely nonlinear. Thus, it is not so surprising to have a strictly convex entropy as in a conservative case [24]. A complete set of entropies can be expressed as polynomial functions in the variables  $\mathcal{S}$  and  $\mathcal{D}$  [54, 55].

**Proof of Proposition 2.1.**

The proof of the first inequality derives from the regularized system. Indeed, adding the two equations of (2.1) gives

$$\partial_t (u^\varepsilon + v^\varepsilon) + \partial_x \left( \frac{(u^\varepsilon - v^\varepsilon)^2}{2} \right) = \varepsilon \partial_{xx} (u^\varepsilon + v^\varepsilon).$$

Multiplying the previous equation by  $\phi \in C_c^\infty((0, T) \times \mathbb{R})$  and integrating over  $(0, T) \times \mathbb{R}$  gives

$$\int_T^0 \int_{\mathbb{R}} (u^\varepsilon + v^\varepsilon) \partial_t \phi + \left( \frac{(u^\varepsilon - v^\varepsilon)^2}{2} \right) \partial_x \phi dx dt = -\varepsilon \int_T^0 \int_{\mathbb{R}} (u^\varepsilon + v^\varepsilon) \partial_{xx} \phi dx dt.$$

Then, by passing to the limit as  $\varepsilon \rightarrow 0$  and using (2.3) and the Dominated Convergence Theorem, we obtain

$$\int_T^0 \int_{\mathbb{R}} \mathcal{S} \partial_t \phi + \left( \frac{\mathcal{D}^2}{2} \right) \partial_x \phi dx dt = 0,$$

which gives (2.14). As for the strictly convex entropy function in (2.15), it comes from trying to solve the equation

$$2(u - v) \partial_{uv} \eta(U) = \partial_u \eta(U) - \partial_v \eta(U), \quad (2.16)$$

which is obtained from Schwarz Identity for an entropy flux  $\psi$  satisfying (2.10).  $\square$

**Definition 2.3. (Entropy solution (es for short))**

We say that a function  $U = (u, v)$  is an entropy solution of (1.2) if  $U \in (L^\infty((0, +\infty) \times \mathbb{R}))^2$ , and for all convex functions  $\eta = \eta(U) \in \mathbb{R}$ , i.e,  $\nabla^2 \eta \geq 0$ , there exists a function  $\psi = \psi(U) \in \mathbb{R}$  satisfying (2.10) such that the couple  $(\eta, \psi)$  satisfies (2.13) in distributional sense.

In fact, we can show that this previous definition is equivalent to the following proposition.

**Proposition 2.2. (Equivalent definition of an es)**

A function  $U = (u, v)$  is called an *es* of (1.2) if it satisfies the conditions of Definition 2.3 and the entropy condition (2.14) in distributional sense.

**Proof of proposition (2.2).**

If we take the two convex entropy functions  $\eta = \mathcal{S}$  and  $\eta = -\mathcal{S}$  with the respective fluxes  $\psi = \mathcal{D}^2/2$  and  $\psi = -\mathcal{D}^2/2$ , we notice that they both satisfy the conditions of Definition 2.3. Then, we can easily obtain (2.14).  $\square$

The presence of a linear entropy is a key tool in this work, since it provides us with a conservative equation for a non-conservative system. Thus, the fundamental Rankine-Hugoniot condition is rigorously written with this conservative equation for the non-conservative system. Despite the fact that this conservative equation is under-determined, it is essential in establishing solutions to the main non-conservative system. In fact, we will first solve the Riemann problem associated to the latter, which in return provide us with special solutions to the former, mainly solutions with constant states in  $u$  or in  $v$ . Then, employing these special solutions in a Godunov scheme, we will be able to build general solutions to system (1.2). We note that in the cases where  $u$  and  $v$  are not constants, we can not make use of equation (2.14) obviously, we would have to go back to the main system, as we will see in the case of a double contact discontinuity introduced in Subsection 4.1.

## 2.4 Volpert BV solutions.

We attempt in this section to make sense of system (1.2). We present directly the following definition, and we explain after how we were able to reach it.

**Definition 2.4. (Volpert BV solution)**

We say that a function  $U = (u, v)$  is a Volpert BV solution to system (1.2) if it is an *es*, and it satisfies the following system for  $\mathcal{D} = u - v$  and  $\mathcal{S} = u + v$

$$\begin{cases} \partial_t \mathcal{S} + \overline{\mathcal{D}} \partial_x \mathcal{D} = 0, & \text{where } \overline{\mathcal{D}} = \frac{\mathcal{D}_l + \mathcal{D}_r}{2}, \\ \partial_t \mathcal{D} + \tilde{\mathcal{D}} \partial_x \mathcal{S} = 0, & \text{where } [\mathcal{S}]^2 \tilde{\mathcal{D}} = \overline{\mathcal{D}} [\mathcal{D}]^2, \end{cases} \quad (2.17)$$

such that  $[\mathcal{S}], [\mathcal{D}]$  represent the jumps in  $\mathcal{S}, \mathcal{D}$  respectively.

We started by adding and subtracting the two equations of the regularized system (2.5). Next, we will see what happens when we try to pass to the limit as  $\varepsilon \rightarrow 0$  in the following.

**Justification of equations (2.17) for a jump solution  $(\mathcal{S}, \mathcal{D})$ .**

We first recall some classical computations in the space of distributions for a jump function with only two values.

A function  $U$  with a jump on the line  $\Gamma : x = st$ , defined by

$$\begin{cases} U(t, x) = U_l, & \text{for } x < st, \\ U(t, x) = U_r, & \text{for } x > st, \end{cases} \quad (2.18)$$

satisfies

$$\begin{cases} \partial_x U = [U]\delta_\Gamma, \\ \partial_t U = -s[U]\delta_\Gamma, \end{cases} \quad (2.19)$$

where  $\delta_\Gamma$ , the Dirac distribution function on the line  $\Gamma$ , is defined by

$$\langle \delta_\Gamma, \phi \rangle = \sqrt{1 + s^2} \int \int \phi(t, st) dt, \quad (2.20)$$

for a test function  $\phi = \phi(t, x)$ . The previous formula is used to obtain the  $2 \times 2$  systems in the framework of Volpert  $BV$  calculus [57] for a solution with a jump.

For a smooth solution, equations (2.17) are satisfied with  $\tilde{\mathcal{D}} = \mathcal{D} = \overline{\mathcal{D}}$ . The problem is to determine carefully  $\tilde{\mathcal{D}}$  and  $\overline{\mathcal{D}}$  for jump solutions.

In fact,  $\overline{\mathcal{D}}$  follows directly from the Volpert  $BV$  calculus for the conservative PDE satisfied by the linear entropy  $\mathcal{S}$  (2.14) when the PDE is written in non-conservative form.

Now, the computation of  $\overline{\mathcal{D}}$  requires the computations in the distributional sense. The first conservative PDE yields

$$s[\mathcal{S}] = \frac{[\mathcal{D}^2]}{2} = [\mathcal{D}]\overline{\mathcal{D}},$$

i.e.

$$s[\mathcal{S}] = \overline{\mathcal{D}}[\mathcal{D}], \quad (2.21)$$

with Rankine-Hugoniot relation or Volpert formula.

For the second equation,  $\partial_t \mathcal{D}$  and  $\partial_x \mathcal{S}$  are computed thanks to the distribution calculus, after simplifications it yields

$$s[\mathcal{D}] = \tilde{\mathcal{D}}[\mathcal{S}]. \quad (2.22)$$

Now, multiplying (2.21) by  $[\mathcal{D}]$  and (2.22) by  $[\mathcal{S}]$  gives

$$s[\mathcal{S}][\mathcal{D}] = \tilde{\mathcal{D}}[\mathcal{S}]^2 = \overline{\mathcal{D}}[\mathcal{D}]^2,$$

which gives the relation between  $\tilde{\mathcal{D}}$  and  $\overline{\mathcal{D}}$

$$\tilde{\mathcal{D}}[\mathcal{S}]^2 = \overline{\mathcal{D}}[\mathcal{D}]^2. \quad (2.23)$$

The essential new result in this formulation is the definition of  $\tilde{\mathcal{D}}$  by (2.23).

Question: What happens when  $[\mathcal{S}] = 0$ ?

A first answer : When  $[\mathcal{S}] = 0$ , the coefficient  $\tilde{\mathcal{D}}$  does not need to be defined in the second equation of (2.17). Thus,  $\tilde{\mathcal{D}}$  can be taken equal to  $\overline{\mathcal{D}}$  for instance, and, as a convention,  $\mathcal{D}$  can be defined as  $\overline{\mathcal{D}}$  on the discontinuity line.

A second answer is presented at the end of the Section 4.

### 3 The main result

The purpose of this work is to provide a general existence result of entropy solutions for the non-conservative and non-strictly hyperbolic system (1.2). The notion of entropy solutions is a cornerstone in the theory of hyperbolic conservation laws and appears to be also efficient for this non-conservative system. Unfortunately, the loss of strict hyperbolicity induces more elementary waves than usual and a loss of the uniqueness for the Riemann problem. Notice also that the uniqueness of vanishing viscosity solutions for System (1.2) is an open problem. Nevertheless, some other properties shared by the vanishing viscosity solutions have been added to build entropy solutions such as the maximum principle and the TVD property. More precisely, Theorem 3.1 states the global existence of entropy  $s$ -TVD solutions to (1.2), where  $s$ -TVD means that the fractional total variation is not increasing. This theorem yields an existence result in all  $BV^s$  for all  $s \in (0, 1]$  which extends the existence result in  $BV$  ( $s = 1$ ) given in [3] for system (1.2).

**Theorem 3.1. (Existence of  $BV^s$  entropy and  $s$ -TVD solutions)**

Assume that  $U_0 \in BV^s(\mathbb{R}, \mathbb{R}^2)$ , and  $s \in (0, 1]$ . Then, system (1.2) admits an entropy  $s$ -TVD solution  $U = (u, v)$  in the sense of Definition 2.3, given by a Godunov scheme, satisfying the fractional  $BV$  in space and Holder in time regularities,

$$U \in L^\infty((0, +\infty); BV^s(\mathbb{R}, \mathbb{R}^2)) \cap Lip^s([0, +\infty), L_{loc}^{1/s}(\mathbb{R}, \mathbb{R}^2)), \quad (3.1)$$

the  $s$ -TVD property,

$$TV^s u(t, \cdot)(\mathbb{R}) \leq TV^s u_0(\mathbb{R}), \quad TV^s v(t, \cdot)(\mathbb{R}) \leq TV^s v_0(\mathbb{R}), \quad \forall t \in [0, +\infty), \quad (3.2)$$

and the maximum principle,

$$\|u\|_{L^\infty((0, +\infty) \times \mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}, \quad \|v\|_{L^\infty((0, +\infty) \times \mathbb{R})} \leq \|v_0\|_{L^\infty(\mathbb{R})}. \quad (3.3)$$

This theorem provides an existence result of solutions to a  $2 \times 2$  hyperbolic system in a non-conservative form.

For non-conservative systems, a general result is performed by Bianchini and Bressan [8] but for strictly hyperbolic systems. Here, the loss of the strict hyperbolicity modifies the structure of the entropy solutions of the Riemann problem and appears to induce loss of uniqueness. There is also the Bressan and Jenssen paper [15] on non-conservative systems where the integral curves of the eigenvector fields are straight lines. Moreover, the Godunov scheme is used in [15]. Our system has the straight lines property since it is a diagonal one. We also use a Godunov scheme to prove the existence. Nonetheless, paper [15] is restricted for strictly hyperbolic systems and cannot be directly applied on system (1.2) studied here.

Now, let us briefly explain shortly the proof strategy for obtaining fractional  $BV$  entropy solutions. To prove this theorem, first we solve the Riemann problem associated to the conservative equation (2.14), as it was explained at the end of Subsection 2.3, where entropy solutions are defined, keeping in mind the TVD property for the constructed solutions. This process would provide us with special solutions to (1.2); solutions with constant states in  $u$  or in  $v$ , and solutions on the diagonal  $\Delta$ . Then, by implementing a Godunov scheme; taking approximate piecewise constant initial data, and solving a series of Riemann problems, we will be able to establish the existence of an approximate solution to (1.2). After that, by using a compactness argument, we can show that the constructed approximate solution converges, up to the extraction of a subsequence, to a function that is an entropy solution to (1.2) in the sense of Definition 2.3. Notice that System (1.2) provides a new system where convergence of the Godunov scheme can be proved.

We dedicate the following section to a major step in our work: solving the Riemann problem associated to system (1.2).

## 4 The Riemann problem (RP)

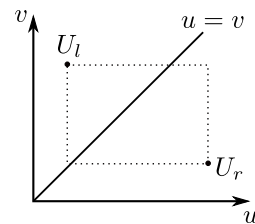
This section is devoted to the detailed presentation of the various solutions to the Riemann problem (RP for short, or RPs for Riemann problems) associated to (2.14). We will prove later on that these solutions are *es* in the sense of Definition 2.3. The RP associated to (2.14) is the initial value problem equipped with the initial data

$$U_0(x) = \begin{cases} U_l = (u_l, v_l) & \text{if } x < 0, \\ U_r = (u_r, v_r) & \text{if } x > 0, \end{cases} \quad (4.1)$$

where  $u_l, v_l, u_r,$  and  $v_r$  are constants, and  $U_l, U_r$  are sufficiently close. We shall solve this problem in the class of functions consisting of constant states, separated by either shock waves, rarefaction waves, or contact discontinuities.

Recall that we would like the Riemann solutions to be TVD (total variation diminishing), in other words, we have to make sure that the TVD condition (1.7) is satisfied for the constructed solutions. Thus, we have the following remark.

**Remark 4.1.** In order to keep condition (1.7) satisfied, the constructed Riemann solutions must be monotonic with respect to each component, and contained in the rectangle of diagonal  $U_l U_r$  with sides parallel to the axes in the  $(u, v)$ -plane, as illustrated in the adjacent figure.



The Riemann solutions are divided into three categories; one wave, two waves, or three wave solutions. For the sake of an organized presentation, we proceed as follows: In subsection 4.1, we introduce first the elementary waves, which are one wave solutions and they form the building block of the RP. They are the unique *es* and *vs*. Then, in Subsection 4.2, we work on one half space, where the solutions are also unique and

mainly composed of two waves. In some cases here however, we may need three waves to establish the Riemann solution. After that, we move on into the more complicated case in Subsection 4.3; crossing the diagonal, where uniqueness is lost in many cases. Finally, we present in Subsection 4.4 a brief study concerning the links between the proposed RP solutions and the various solutions discussed in Section 2.

We note that the elementary waves are not restricted to one side of the diagonal, as we will see in Subsection 4.3. Now, we introduce the following notations.

- Notation 4.1.**
1.  $R_u$  (resp.  $R_v$ ) represents a rarefaction wave in  $u$  with  $v$  constant (resp. in  $v$  with  $u$  constant), and  $S_u$  (resp.  $S_v$ ) a shock wave in  $u$  with  $v$  constant (resp. in  $v$  with  $u$  constant).
  2. The  $(-)$ ,  $(+)$ , and  $(0)$  exponents on a certain wave ( $R_u^+$  or  $S_v^-$  for example) refer to the sign of the velocity, i.e., if it is a negative, positive, or a zero speed.
  3. The notation  $S_v R_u$  for instance means that the solution is made up of a shock wave in  $v$  followed by a rarefaction wave in  $u$ .

One can imagine by now that there are many cases to be considered. Nonetheless, the problem possesses a strong tool that would aid in reducing the complexity of the study, which is the symmetry property. To that end, we would like to explain and emphasize on the importance of this property.

For a function  $U = (u, v)$ , we denote by  $Sym(U)$  the symmetric of  $U$  with respect to the diagonal  $\Delta$  such that

$$Sym(U) = (v, u).$$

Roughly speaking, the following property states that, if we consider the solution of the RP associated to a certain case of  $U_l, U_r$ , then the RP solution of  $Sym(U_l), Sym(U_r)$  admits as a path in the  $(u, v)$  plane the symmetric of that associated to  $U_l, U_r$  with respect to the diagonal  $\Delta$ . Formally, this is translated as follows.

**Property 4.1.**

Assume that the solution of the RP between the two states  $U_l = (u_l, v_l)$  and  $U_r = (u_r, v_r)$  associated to (1.2) is of the form  $\xi^1, \xi^1 \xi^2$  or  $\xi^1 \xi^2 \xi^3$ , where  $\xi^i \in \{R_u^\pm, S_u^\pm, R_v^\pm, S_v^\pm\}$  for  $i \in \{1, 2, 3\}$ . Then, the RP solution associated to  $Sym(U_l)$  and  $Sym(U_r)$  is of the form  $\tilde{\xi}^1, \tilde{\xi}^1 \tilde{\xi}^2$ , or  $\tilde{\xi}^1 \tilde{\xi}^2 \tilde{\xi}^3$  respectively, where

$$\tilde{R}_u^\pm = R_v^\pm, \tilde{S}_u^\pm = S_v^\pm, \text{ with } \tilde{\xi}^i = \xi^i, \text{ for } i = 1, 2, 3.$$

We now begin with the first type of Riemann solutions, namely, the elementary waves.

## 4.1 Elementary waves

Consider equation (2.14) equipped with the initial data (4.1). There are three types of elementary waves, which are solutions made up of a single wave, for this problem. First, we set  $v = v_0$ , a constant. Then, equation (2.14) reduces to the scalar conservation law

$$\partial_t u + \partial_x \left( \frac{u^2}{2} - v_0 u \right) = 0, \tag{4.2}$$

with the strictly convex flux  $f_{v_0}(u) = \frac{u^2}{2} - v_0 u$ . The solution of this equation with the initial data

$$u_0(x) = \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0, \end{cases}$$

is classical. Depending on the position of  $u_r$  with respect to  $u_l$ , we either obtain a shock wave or a rarefaction wave. If  $u_l > u_r$ , then the solution is a shock wave given by the formula

$$u(t, x) = \begin{cases} u_l & \text{if } x < st, \\ u_r & \text{if } x > st, \end{cases}$$

where  $s$  is the speed of the shock given by

$$s = \frac{u_l + u_r}{2} - v_0,$$

which is deduced from the Rankine-Hugoniot jump condition

$$s[u] = [f_{v_0}(u)] \Leftrightarrow s(u_l - u_r) = f_{v_0}(u_l) - f_{v_0}(u_r).$$

We require that the following Lax shock condition [25, 45], which is equivalent here to all entropy inequalities, holds for all shocks

$$f'_{v_0}(u_r) < s < f'_{v_0}(u_l).$$

However, if  $u_l < u_r$ , then the solution is a rarefaction wave, i.e., it is a continuous function  $u = u\left(\frac{x}{t}\right)$ , for  $f'_{v_0}(u_l) < \frac{x}{t} < f'_{v_0}(u_r)$ , which is given by solving the equation

$$f'_{v_0}\left(u\left(\frac{x}{t}\right)\right) = \frac{x}{t}.$$

Thus, this solution would be of the following form

$$u(t, x) = \begin{cases} u_l & \text{if } \frac{x}{t} < f'_{v_0}(u_l), \\ \frac{x}{t} + v_0 & \text{if } f'_{v_0}(u_l) < \frac{x}{t} < f'_{v_0}(u_r), \\ u_r & \text{if } \frac{x}{t} > f'_{v_0}(u_r). \end{cases}$$

On the other hand, if we fix  $u = u_0$ , we obtain the equation

$$\partial_t v + \partial_x \left( \frac{v^2}{2} - u_0 v \right) = 0, \tag{4.3}$$

with the initial data

$$v_0(x) = \begin{cases} v_l & \text{if } x < 0, \\ v_r & \text{if } x > 0, \end{cases}$$



and the strictly convex flux  $f_{u_0}(v) = \frac{v^2}{2} - u_0 v$ . As in the case of equation (4.2), depending on the position of  $v_r$  with respect to  $v_l$ , we either obtain a shock wave (if  $v_l > v_r$ ) or a rarefaction wave (if  $v_l < v_r$ ).

Therefore, from the analysis of equations (4.2) and (4.3), we can see that, for a point  $U_l = (u_l, v_l)$  in the state  $(u, v)$ -plane, there are 4 possible waves to consider: a rarefaction wave in  $u$ , a rarefaction wave in  $v$ , a shock wave in  $u$ , or a shock wave in  $v$ . In other words, if  $U_r$  lies on one of the lines emanating from  $U_l$  as shown in Figure 1 (See Notation 4.1 to understand the figure) the solution is one of the 4 previously mentioned waves, which would be called the elementary waves. An example of these solutions is presented in Figure 2.

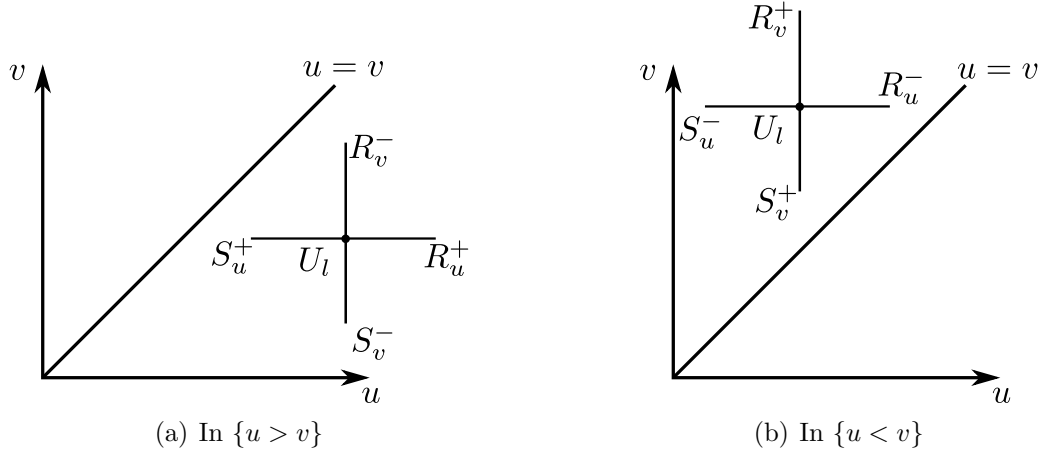


Figure 1: Possible solution paths emanating from  $U_l$ .

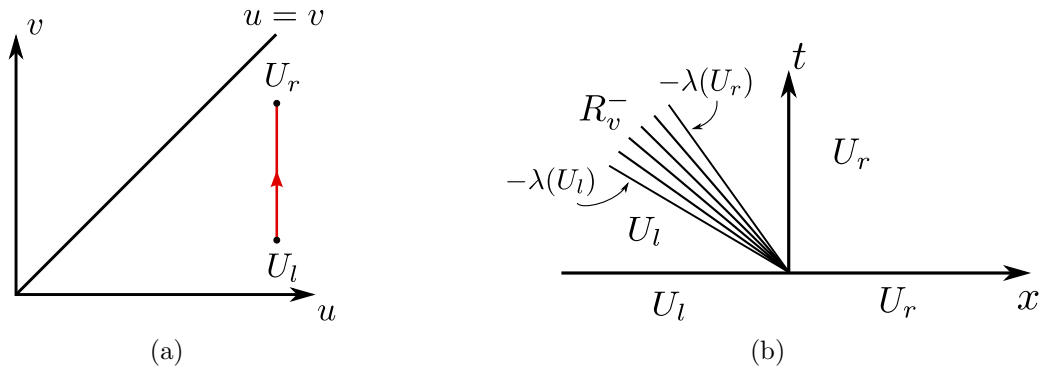


Figure 2: Elementary solution of the form  $R_v^-$ .

The third type of the elementary waves is called a Double Contact Discontinuity, or  $DCD^0$  for short, and it is of zero speed. This type appears if for instance both  $U_l$  and  $U_r$  are situated on the diagonal  $\Delta$ . In this case, we notice that if  $u_l > u_r$ , then there are three

possible Riemann solutions (see the end of Subsection 4.3) of which one is the  $DCD^0$ , unlike the case  $u_l < u_r$  where the solution here is a unique  $DCD^0$ .

**Uniqueness of the elementary waves.** As it was mentioned before, we know that for scalar conservation laws, the solution of the RP is the unique  $es$  of the equation, which is also a  $vvs$  [31, 55]. Thus, the elementary waves are all unique  $es$ .

## 4.2 In one half space

In this part of our study, we restrict ourselves to one side of the diagonal  $\Delta$  in the  $(u, v)$ -plane. We have two cases here, of which uniqueness is guaranteed in both.

Reminder:  $\mathcal{D} = u - v$ .

**Case 1: in  $\{u > v\}$ .** We have

$$\lambda_u(u, v) = \mathcal{D} > 0 \quad \text{and} \quad \lambda_v(u, v) = -\mathcal{D} < 0.$$

If  $U_r$  is not on one of the lines emanating from  $U_l$ , then the solution is given as follows: we connect  $U_l$  to some point  $\bar{U}$  with a negative shock or rarefaction wave, then we connect  $\bar{U}$  to  $U_r$  with a positive shock or rarefaction wave. For instance, if  $u_l < u_r$  and  $v_l > v_r$ , then the solution would be made of  $S_v^-$  connecting  $U_l$  to a  $\bar{U} = (u_l, v_r)$ , then  $R_u^+$  connecting  $\bar{U}$  to  $U_r$ , as it is depicted in Figure 3.

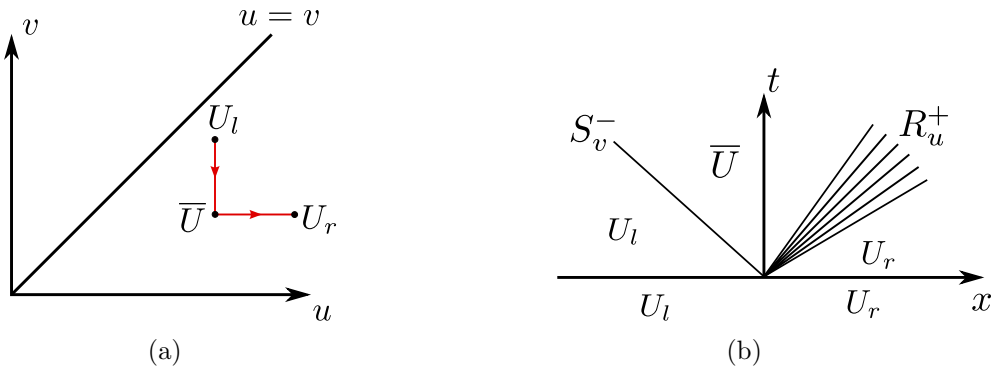


Figure 3: Solution of the form  $S_v^- R_u^+$ .

**Case 2: in  $\{u < v\}$ .** In this case, we have  $\mathcal{D} < 0$ . By property 4.1, we can have all of the possible solutions to the RP. For example, the solution in the case  $u_l = u_r$  and  $v_l < v_r$  is  $R_v^+$ , which is the analogue of the case discussed in Figure 2, where the solution was  $R_v^-$ . Also, the analogous solution of the one presented in Figure 3, which was  $S_v^- R_u^+$ , is  $S_u^- R_v^+$ .

**Critical cases.** There are two critical case here, which are also symmetric, when we have no choice but to walk on the diagonal. One of them is presented in Figure 4. In this case, the solution is  $R_v^- DCD^0 R_u^+$ . For the symmetric case, the solution is  $R_u^- DCD^0 R_v^+$ .

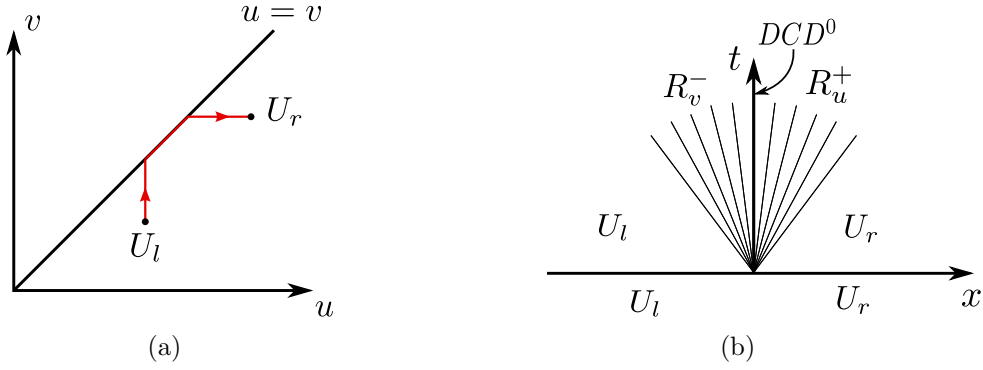


Figure 4: Solution having a Double Contact Discontinuity.

**Remark 4.2.** If we allow for either  $U_l$  or  $U_r$  to be on the diagonal, in other words, if we are in  $\{u \geq v\}$  or  $\{u \leq v\}$ , the previous studies would remain the same.

### 4.3 Crossing the diagonal

We move on into the more interesting cases, namely, when  $U_l$  and  $U_r$  are situated on different sides of the diagonal  $\Delta$ . We will show the detailed construction of the solutions in one case, and then we will list the symmetric solutions briefly for the other case with the aid of Property 4.1.

**For  $U_l \in \{u < v\}$  and  $U_r \in \{u > v\}$ .**

1. If  $v_l = v_r = v$ : the solution is  $R_u^- R_u^+$ . In view of Remark 4.1, we have to stay on the line  $U_l U_r$ . Hence, we can see that the solution is unique here. In fact, the solution is one rarefaction wave in  $u$  that starts with a negative speed and ends with a positive one, taking the value zero on the diagonal evidently. The solution is illustrated in Figure 5.

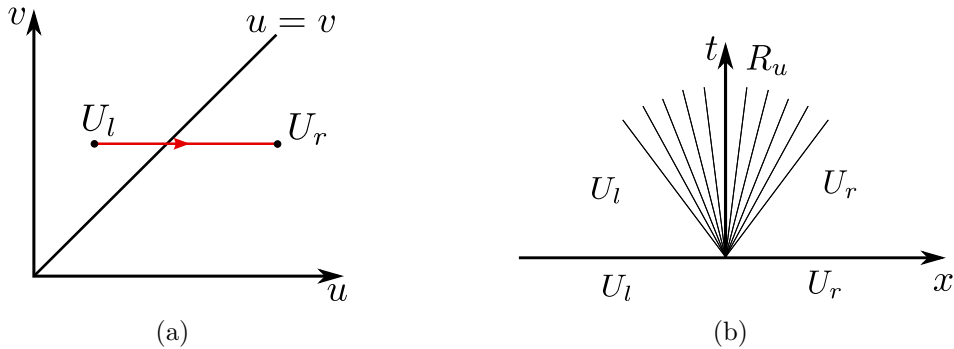


Figure 5: A rarefaction wave in  $u$ .

2. If  $u_l = u_r = u$ : the solution is  $S_v$ , with three possible speeds (positive, negative, or zero), depending on the positions of  $U_l$  and  $U_r$  (See Figure 6). The speed of the

shock is given by

$$s = \frac{v_l + v_r}{2} - u.$$

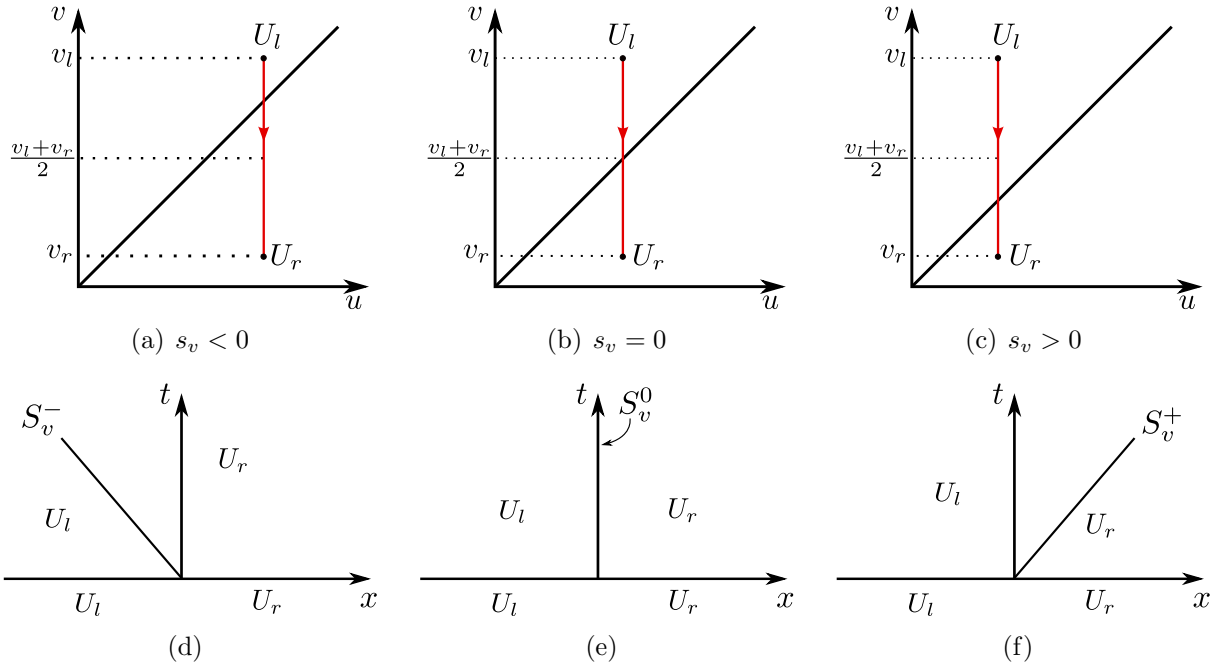


Figure 6: Crossing with  $S_v$  in 3 different speeds.

In all three cases, the solution is unique.

**Remark 4.3.** In Figure 6, we notice that if the major part of the line  $U_l U_r$  lies in the  $\{u < v\}$  half plane, then the speed of shock in  $v$  is positive, and in the opposite case, the speed would be negative. If we cross at the middle, then it is a zero speed.

3. If  $u_l = v_r$  and  $v_l = u_r$ : there are two possible ways to start from  $U_l$ , either a  $S_v^+$  or a  $R_u^-$ . Assume we choose first to walk with a  $S_v^+$  to reach the point  $(u_l, v_r) \in \Delta$ . After that we would have to walk with a  $R_u^+$  that starts with zero speed, which is not possible since the speed of  $S_v^+$  is strictly positive.

If we start now by a  $R_u^-$  to reach the point  $(u_r, v_l) \in \Delta$  with zero speed, after that we would have to continue with a strictly negative shock in  $v$ . This is also impossible.

Hence, we got the idea that the only possible way to have a valid solution would be to cross before reaching the diagonal. Thus, we discovered that the solution in this case would be of the form  $R_u^- S_v R_u^+$ ; a  $R_u^-$  connecting  $U_l$  to a certain point  $\bar{U} = (\bar{u}, v_l)$  before the diagonal, then we cross the diagonal with a  $S_v$  connecting  $\bar{U}$  to a point denoted by  $\tilde{U} = (\bar{u}, v_r)$ , and finally a  $R_u^+$  connecting  $\tilde{U}$  to  $U_r$ . What is lost here is the uniqueness of the solution, as we can see from the following construction.

It is important to pay attention to the succession of the waves' velocities. In other words, we must have

$$\lambda_u(\bar{U}) = \bar{u} - v_l \leq s_v \leq \bar{u} - v_r = \lambda_u(\tilde{U}),$$

where

$$s_v = \frac{v_l + v_r}{2} - \bar{u}.$$

Thus, we obtain the following domain for the point  $\bar{u}$

$$\frac{3v_r + v_l}{4} \leq \bar{u} \leq \frac{3v_l + v_r}{4}.$$

This implies that we can stop  $R_u^-$  at any point  $\bar{u}$  in this interval. The set of possible solutions for this case is illustrated in Figure 7.

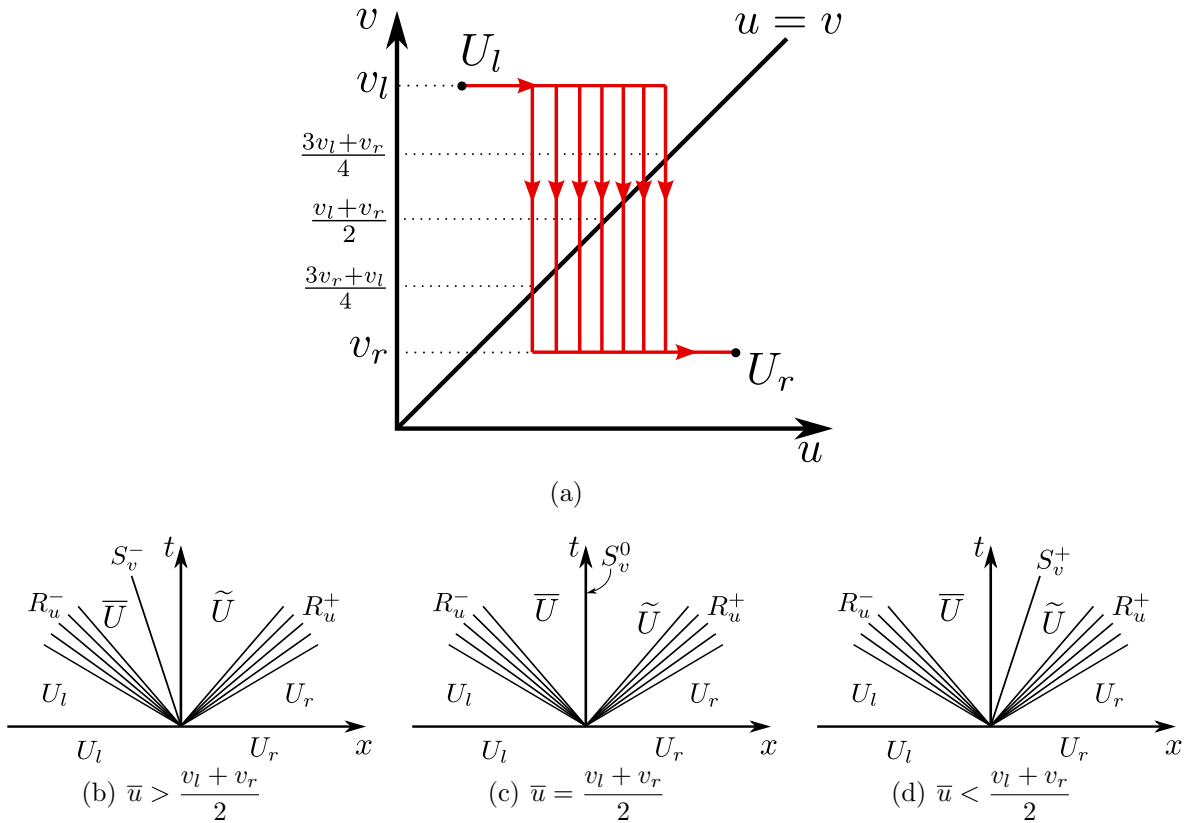


Figure 7: All possible solutions of the form  $R_u^- S_v R_u^+$ .

**Other cases inspired here:** This study remains the same if we move  $U_r$  further away to the right on the line  $\{v = v_r\}$ . A similar study can also be done if we move  $U_r$  upward or downward a little on the line  $\{u = u_r\}$ .

This last case, moving  $U_r$  downward on the line  $\{u = u_r\}$ , admits another kind of solution, which is  $S_v R_u^+$ . In the case where it is a  $S_v^+ R_u^+$ , we should have

$$s_v \leq \lambda_u(\bar{U}) = \lambda_u(u_l, v_r),$$

where  $\bar{U}$  is the intermediary point.

We can also deduce that if  $u_r \leq \frac{3v_r + v_l}{4}$ , then the solution would be unique and of the form  $R_u^- S_v^+$ .

4. If  $u_l > u_r$  and  $v_l > v_r$ : three possible solutions can be found, namely,  $S_v S_u^+$ ,  $S_u^- S_v$ , or  $S_u^- S_v S_u^+$ . For  $S_v S_u^+$ , the speed of  $S_v$  can be negative, positive, or zero, depending on the position of  $U_l$  in the half plane  $\{u > v\}$  (See Figure 8). In the case where it is positive, we should have the condition

$$s_v \leq s_u. \quad (4.4)$$

Similarly for  $S_u^- S_v$ , we should have the opposite of (4.4) when  $S_u$  has a positive speed.

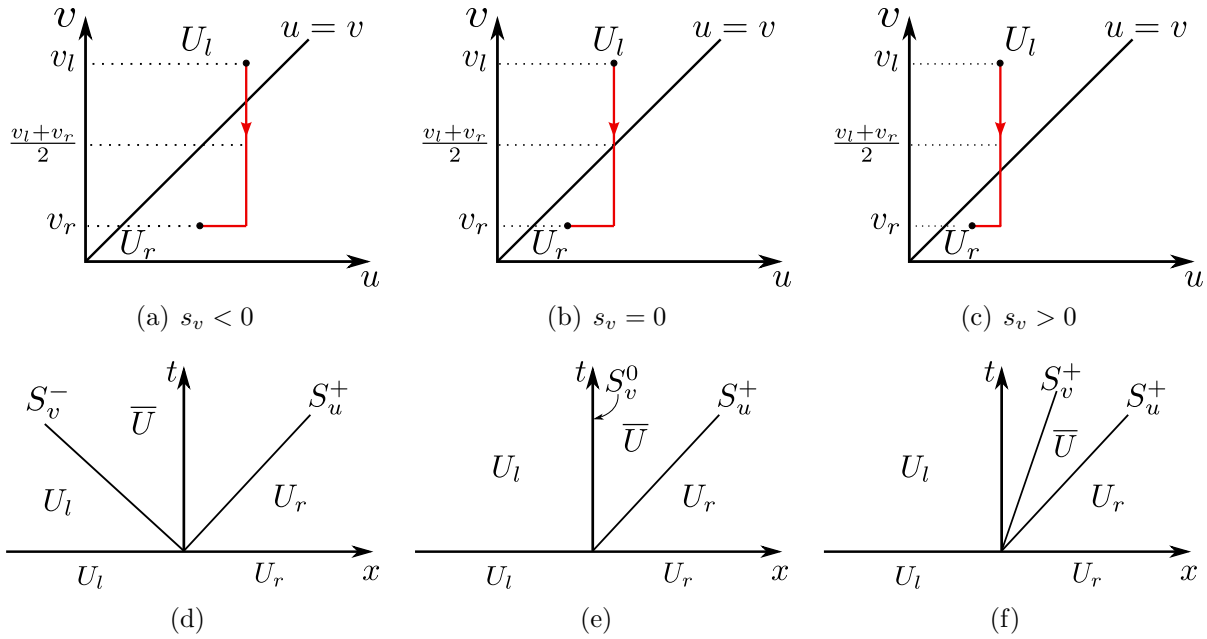


Figure 8: All solutions of the form  $S_v S_u^+$ .

In the case of  $S_u^- S_v S_u^+$ , we walk first with a  $S_u^-$  connecting  $U_l$  to the point  $\bar{U} = (\bar{u}, v_l)$ , then a  $S_v$  connecting  $\bar{U}$  to the point  $\tilde{U} = (\bar{u}, v_r)$ , and finally a  $S_v^+$  connecting  $\tilde{U}$  to  $U_r$ . From the following condition on the speeds

$$s_u^- \leq s_v \leq s_u^+,$$

we deduce that there is a domain of which we can cross with  $S_v$  (See Figure 9), and it is given by

$$\frac{3v_r + v_l - u_r}{3} \leq \bar{u} \leq \frac{3v_l + v_r - u_l}{3}.$$

5. If  $u_l < u_r$  and  $v_l < v_r$ : the solution here is unique and of the form  $R_u^- D C D^0 R_u^+$ .

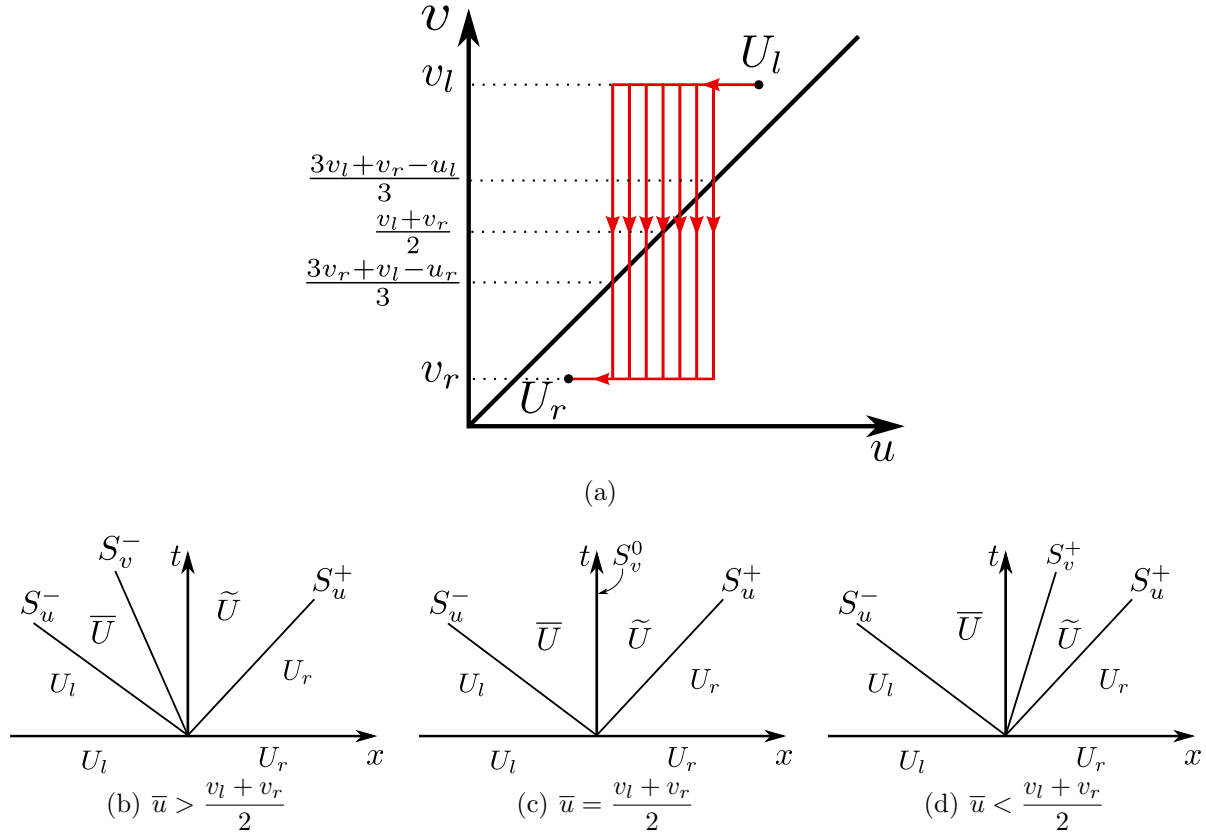


Figure 9: All possible solutions of the form  $S_u^- S_v S_u^+$ .

**For  $U_l \in \{u > v\}$  and  $U_r \in \{u < v\}$ .** We present the symmetric of each case of the previous paragraph, in the same order. The graph of each of the following solutions in the  $(u, v)$ -plane is the symmetric of its analogue from the previous paragraph with respect to the diagonal  $\Delta$ .

1. If  $u_l = u_r = u$ : the unique solution here is  $R_v^- R_v^+$ .
2. If  $v_l = v_r = v$ : the unique solution here is  $S_u$ . Depending on the position of  $U_r$ , the speed is either positive, negative, or zero.
3. If  $u_l = v_r$  and  $v_l = u_r$ : the solution would be  $R_v^- S_u R_v^+$  with several possibilities also for  $S_u$ . In addition, if we move  $U_r$  along  $\{u = u_r\}$  or  $\{v = v_r\}$ , the symmetry of the solutions presented at the end of case (3) from the previous paragraph appear.
4. If  $u_l > u_r$  and  $v_l > u_r$ : there are 3 possible solutions here;  $S_u S_v^+$ ,  $S_v^- S_u$ , or  $S_v^- S_u S_v^+$ .
5. If  $u_l < u_r$  and  $v_l < v_r$ : the solution is unique and of the form  $R_v^- D C D^0 R_v^+$ .

**On the diagonal.** There are two cases here.

1. If  $u_l > u_r$ : there are 3 possible solutions of which one is an EW;  $D C D^0$ ,  $S_u^- S_v^+$ , or  $S_v^- S_u^+$ .

2. If  $u_l < u_r$ : there is a unique EW of the form  $DCD^0$ .

**An entropy solution that is not a TVD one.** All of the constructed solutions in this section are TVD ones, since we maintained condition (1.7) in our study. A counterexample of a solution that is not a TVD one is shown in Figure 10, where  $U_l$  and  $U_r$  are situated on the same line  $\{u = u_r = u_l\}$ . The path shown in the state phase is valid, but we notice that the total variation of the solution is increasing, hence we do not choose such a solution.

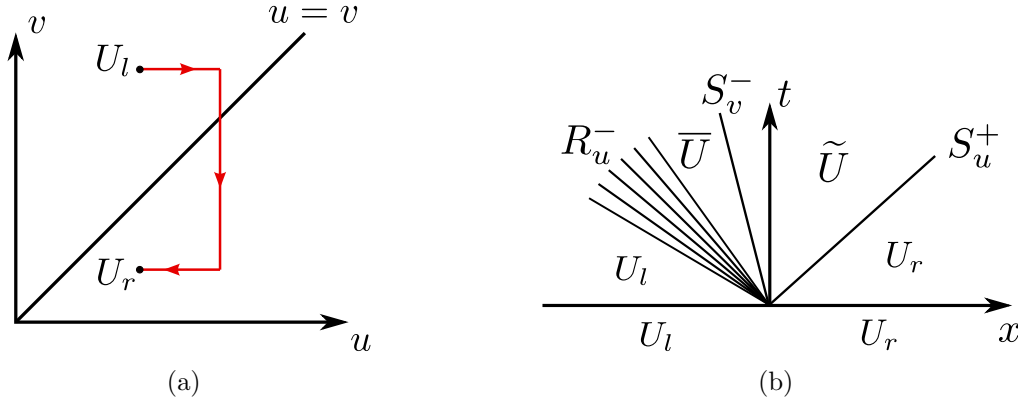


Figure 10: RP solution that is not a TVD one.

In order to sum up this section, we present all of the constructed solutions in Figure 11 for a fixed point  $U_l$  in the  $(u, v)$  state plane. Remark that, using the Property 4.1, we can have all possible solutions for any  $U_l$ .

#### 4.4 Links between RP solutions and other definitions

At the beginning of the paper, various definitions of solutions were proposed for the non-conservative system (1.2). We will study the link between the constructed Riemann solutions and these definitions in this short section.

Recall that the construction of the RP solutions requires the linear entropy  $\mathcal{S}$ , and all of these solutions are *es*. However, as we have discussed in the main result section (Section 3) that this is not enough to select a proper weak solution, we have imposed some properties of the *vvs* on the RP solutions such as the TVD property, and verifying the Maximum Principle. Also, we already know that elementary waves are the unique *vvs* as it was stated at the end of Subsection 4.1. We discuss briefly the link between the RP solutions and the *HJs*. Finally, we justify that the RP solutions are also Volpert *BV* ones.

**Proposed RP solutions are all *es*.** By construction, all the proposed RP solutions are entropy solutions. Classical arguments for strictly hyperbolic systems [25, 39] can be repeated here. More precisely, the elementary waves are vanishing viscosity solutions so are also entropy solutions (see subsection 4.1). The property "to be an entropy solution" is local, so all of the solutions with one, two or three elementary waves are also entropic.



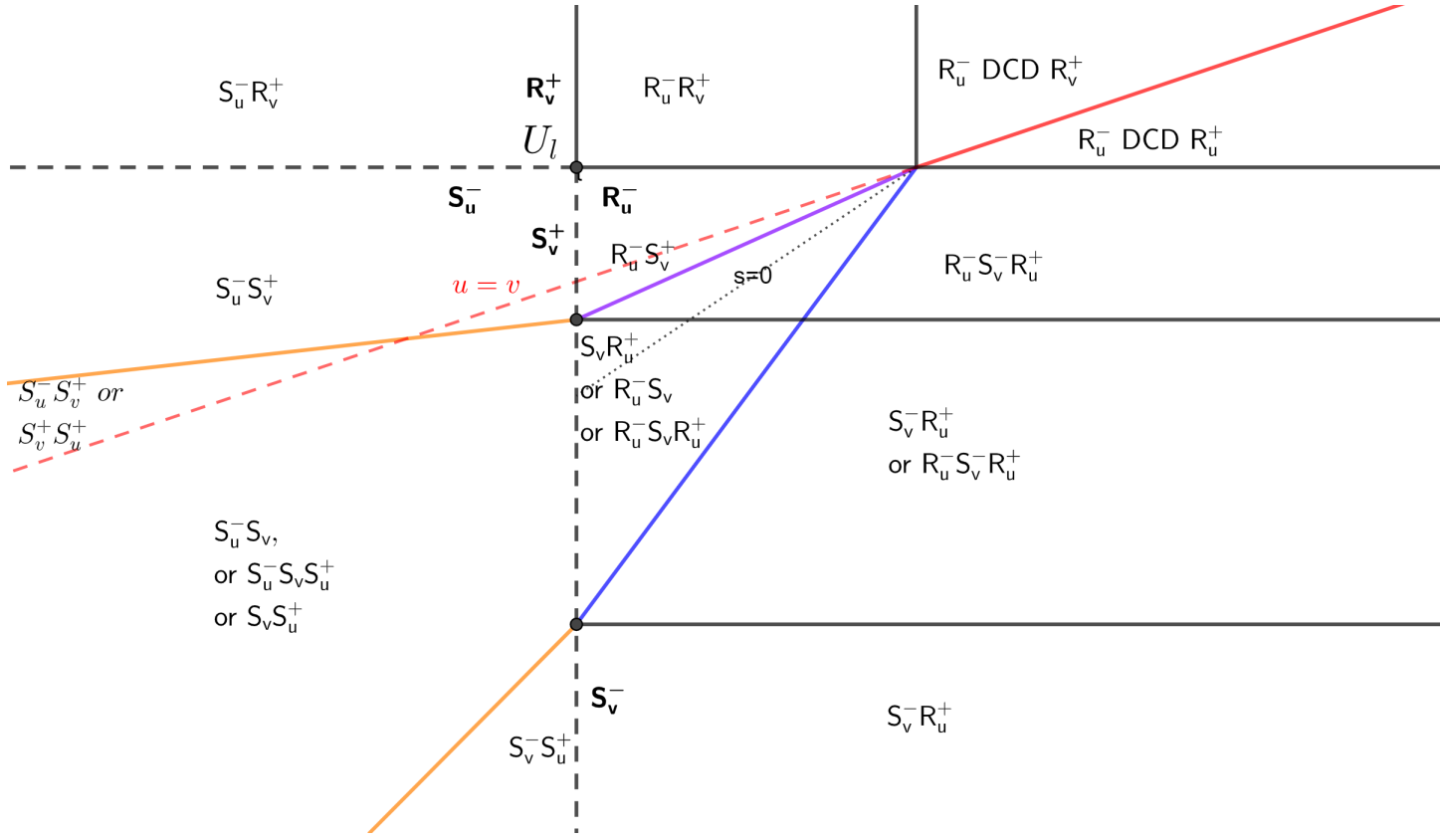


Figure 11: All solutions

**Proposed RP solutions and HJs.** By now we know that, locally, the solutions of the RP are simply elementary waves, which are the unique  $es$  in this case. Moreover, since the definition of viscosity solutions in the sense of HJ equations (Definition 2.2) is given at a point, i.e. it is also local, and in this case the solution is unique, this can imply that the RP solutions are also HJs. However, problems may appear at points of intersection between two waves at the initial time. We leave this problem to another work as it requires a rigorous study.

**Proposed RP solutions are Volpert BV solutions.** For continuous RP solutions it is clear. And for RP solutions with jumps, the definition of Volpert BV solutions is based on Volpert BV calculus for the linear entropy  $\mathcal{S}$  and the associated entropy-flux  $\mathcal{D}$ . Hence, the proposed RP solutions satisfy the non-conservative system (2.17) for Volpert BV solutions.

In the definition of Volpert BV solutions (2.17), the quantity  $\tilde{\mathcal{D}}$  is used and can be different than  $\overline{\mathcal{D}}$ . But the definition of  $\tilde{\mathcal{D}}$  requires that  $[\mathcal{S}] \neq 0$ . Taking advantage of the previous exhaustive study of the Riemann problem, the case  $[\mathcal{S}] = 0$  is again discussed. A first answer to the case  $[\mathcal{S}] = 0$  is mentioned at the end of Subsection 2.4. Now, consider the case  $[\mathcal{S}] = 0$  for the Riemann problem. From (2.23), if  $[\mathcal{S}] = 0$  then  $[\mathcal{D}] = 0$  or  $\overline{\mathcal{D}} = 0$ . For a shock wave, notice that the entropy solution is unique:  $[\mathcal{S}] \neq 0$  and

$[\mathcal{D}] \neq 0$  shocks. The case  $\overline{\mathcal{D}} = 0$  means that  $u_r - v_r = u_l - v_l$ . There is no jump solution like that for the Riemann problem except the  $DCD^0$ . In the  $DCD^0$  case, we have  $u_r = v_r$  and  $u_l = v_l$  so  $[\mathcal{S}] = 0$  and again, as for the first answer in Subsection 2.4, the value of  $\tilde{\mathcal{D}}$  does not matter whether it satisfies the second equation of (2.17) or not.

## 5 Existence of a global solution

The purpose from this section is to establish the existence of a global solution for (1.2)-(1.3). This is achieved via a Godunov scheme that is based on the construction of a sequence of approximate solutions. The key ingredient in this procedure is the previous resolution of the RP.

### 5.1 The Godunov algorithm

The Godunov scheme was mainly used to construct solutions to conservative equations [11, 14, 21]. Here, we adapt this scheme to prove the global existence of an *es* to the non-conservative system (1.2). We will consider a staggered grid in order to verify easily the TVD property. Of course a non-staggered grid can also be used as well. We refer to the book of Leveque [50] for a good introduction into this scheme and its applications.

We consider a mesh discretization

$$\Xi = \{i\Delta x, i \in \mathbb{Z}\}, \quad \Xi_N = \{n\Delta t, n \in \mathbb{N}\},$$

where  $\Delta t, \Delta x$  are positive steps of discretization. The discrete running point is  $(t_n, x_i)$  with  $t_n = n\Delta t$  and  $x_i = i\Delta x$ . For  $C_i$  denoting the  $i$ -th grid cell centered at  $x_{2i}$  and of length  $2\Delta x$  defined as

$$C_i = (x_{2i-1}, x_{2i+1}),$$

we introduce an approximation  $\overline{U}_i^n$  of the average value of the RP solution  $U = (u, v)$  to (1.2) over the  $i$ -th interval  $C_i$  at an instant  $t_n - 0$ , such that, for  $n > 0$ ,

$$\overline{U}_i^n = \frac{1}{2\Delta x} \left( \int_{C_i} u(t_n - 0, x) dx, \int_{C_i} v(t_n - 0, x) dx \right).$$

For  $n = 0$ , the average of the initial data is taken.

The Goudonov algorithm [33] (also known as REA, which is short for reconstruct-evolve-average) is made up of three steps: reconstructing a piecewise constant function  $U^{\Delta x}(t_n, x)$  defined for all  $x$  from the cell averages  $\overline{U}_i^n$ , evolving the hyperbolic system with the initial data  $U^{\Delta x}(t_n, x)$  to obtain the exact solution,  $U^{\Delta x}(t, x) = (u^{\Delta x}(t, x), v^{\Delta x}(t, x))$  for all  $t \in (t_n, t_n + \Delta t)$  and for all  $x \in \mathbb{R}$ , and finally, averaging the exact solution over each grid cell to obtain new cell averages and so on.

This whole process is repeated. The first approximation function is chosen to be a piecewise constant function, since in this way we can evolve the system by solving a series of RPs at each discontinuity point, provided that  $\Delta t$  is small enough to avoid the interactions of waves from two adjacent RPs. Thus, we have to impose the CFL condition

$$\Lambda \frac{\Delta t}{\Delta x} \leq 1, \tag{5.1}$$

where  $\Lambda$  is defined as

$$\Lambda = \sup_{\lambda \in \{\lambda_u, \lambda_v\}} \|\lambda\|_{L^\infty((0, +\infty) \times \mathbb{R})}. \quad (5.2)$$

As usual,  $\Delta t$  is fixed to be of the same order of  $\Delta x$ , here, for instance, the choice is,

$$2\Lambda\Delta t = \Delta x, \quad (5.3)$$

if  $\Lambda > 0$ , which corresponds to a particular CFL condition where the inequality  $\leq 1$  is replaced by the equality  $= 1/2$ .

In many cases, we have seen that the solution of the RP is not unique. This means that there would be several possible solutions to choose from, and since all of our constructed solutions are TVD *es*, there is no mathematical support to use so as to say that some are better than others. Hence, we have to fix a selection criteria in order to choose the solution that would be used in the Godunov algorithm. We do not have a rigorous way for choosing such a solution by now, so we just select arbitrarily.

To implement the Godunov algorithm, we start by approximating the initial data  $U_0$  of (1.2) with  $\bar{U}_i^0$  on each grid cell  $C_i$ . Then, we construct the piecewise constant function  $U^{\Delta x}(t_0, x)$  that is defined for all  $x$  from the cell averages  $\bar{U}_i^0$ . Next, at each point  $x_{2i+1}$ , we solve the RP on the interval  $C_i + \Delta x$ , which would be one of the wave solutions constructed in Section 4, taking into consideration the selection rule we have considered for choosing one RP solution when there is no uniqueness, and the CFL condition (5.1), to obtain  $U^{\Delta x}(t_1 - 0, x)$ , where  $t_1 = \Delta t$ . After that, we average  $U^{\Delta x}(t_1 - 0, x)$  on each interval  $C_i + \Delta x$  to obtain the new cell averages  $\bar{U}_i^1$ . Then, we can reconstruct the solution  $U^{\Delta x}(t_1, x)$  from the cell averages at the instant  $t_1$ , solve the RPs on each interval  $C_i$ , subsequently obtain  $U^{\Delta x}(t_2 - 0, x)$  with  $t_2 = 2\Delta t$ , and so on.

In fact, we consider a *staggered grid*; averaging on  $C_i$  and solving the RP on  $C_i + \Delta x$  for  $t_{2n}$ , and then averaging on  $C_i + \Delta x$  and solving the RP on  $C_i$  for  $t_{2n+1}$ , to follow the evolution of the RP better, since this way we get one RP on each cell and not two halves of two RPs. This grid, which is clarified in Figure 12, is actually more suitable for establishing the  $BV^s$  estimate on the solution of the Godunov scheme (See Lemma 5.2). Notice that the estimates are also valid for the non staggered grid [21].

In the next lemma, the projection step is estimated in  $L^{\frac{1}{s}}$  with the  $BV^s$  semi-norm. It is enough to get the consistency of the Godunov scheme as in [11, 12].

**Lemma 5.1.**

Let  $L > 0$ , and  $f \in BV^s(0, L)$  with  $0 < s \leq 1$ . If we denote by  $\bar{f} = \frac{1}{L} \int_0^L f(x) dx$ , or  $\bar{f} = \frac{f(0+) + f(L-)}{2}$ , then we have

$$\int_0^L |f(x) - \bar{f}|^{\frac{1}{s}} dx \leq L \times TV^s f(0, L).$$

The proof of this lemma can be found in [14].

The following lemma states that the solutions constructed via the Godunov scheme are s-TVD. The proof is done by induction, and it is based on removing the subdivision points

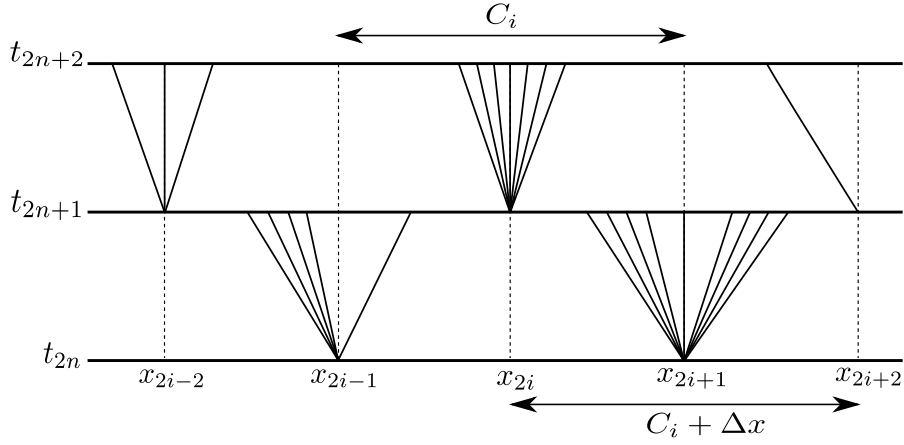


Figure 12: Solving the RP on a staggered grid

that are found on a rarefaction wave, and adding instead some other points in places where the solution is constant. This process does not increase the  $s$ -TV, on the contrary, it almost preserves that of the initial data.

**Lemma 5.2. (Decay of the  $s$ -Total Variation)**

Let  $U_0 = (u_0, v_0)$  be the initial data associated to system (1.2) with compact support in  $\mathbb{R}^2$ . Then, the approximate solution given by the Godunov scheme  $U^{\Delta x} = (u^{\Delta x}, v^{\Delta x})$  satisfies for all time  $t > 0$ ,

$$TV^s u^{\Delta x}(t, \cdot)(\mathbb{R}) \leq TV^s u_0(\mathbb{R}), \quad TV^s v^{\Delta x}(t, \cdot)(\mathbb{R}) \leq TV^s v_0(\mathbb{R}). \quad (5.4)$$

The decay is a direct consequence of the monotonicity of  $u$  and  $v$  for the solution of the Riemann problem. Indeed, the monotonicity insures the constancy of the spatial total variation in the open strip of time  $(t_n, t_{n+1})$ . Then, the projection on constant states reduces the spatial total variation. This is well known for the usual  $BV$  estimate of the Godunov scheme used for a scalar conservation law. For  $BV^s$  estimates, we refer the reader to [14, 21], and for the Wave Front Tracking algorithm to [43].

We now present the proof of the main theorem of this work.

**5.2 Proof of Theorem 3.1.**

The first point is the uniform estimates with respect to  $\Delta x$  of the family of approximate solutions  $U^{\Delta x}$ . The second is the proof of the convergence up to a subsequence to an entropy solution. This second point is detailed.

The proof of  $L^\infty$  estimate (3.3) is easily satisfied for the Riemann problem, see Remark 4.1. Then, taking the mean value at each averaging step in the Godunov scheme does not increase  $L^\infty$  norm. Hence, the  $L^\infty$  norm of the solution never exceeds that of the initial data. That is,  $U^{\Delta x}$  is contained in a bounded ball  $B$  which contains  $U_0$ .

From Lemma 5.2, the family  $U^{\Delta x}$  is bounded in  $(L^\infty((0, +\infty); BV^s(\mathbb{R})))^2$ . Then, as in

[11] for  $s = 1$  and [13, 36, 43] for  $s < 1$ , it follows that

$$\int_{\mathbb{R}} |u^{\Delta x}(t, x) - u^{\Delta x}(s, x)|^{\frac{1}{s}} dx \leq C(|t - s| + \Delta t),$$

where  $C = C(\Lambda, TV^s u_0(\mathbb{R}))$ , and we obtain a similar estimate for  $v^{\Delta x}$ , if the initial data  $u_0$  and  $v_0$  have a compact support. When the initial data are not compactly supported, it suffices to do the estimates on the hyperbolic dependence zones

$$\{(t, x), 0 \leq t \leq T, |x| \leq A + \Lambda(T - t)\},$$

parameterized by positive constants  $A$  and  $T$ . In order to simplify the exposition and write global estimates in  $L^{1/s}(\mathbb{R})$ , we assume that the initial data are compactly supported. Thus, using these estimates, we can obtain a classical compactness argument on  $U^{\Delta x}$ , [55] for  $s = 1$  and [13, 36, 43] for  $s < 1$ . Then, up to the extraction of a subsequence as  $\Delta x \rightarrow 0$ ,  $U^{\Delta x}$  converges to a function  $U = (u, v)$  a.e in  $L^1_{loc}((0, \infty) \times \mathbb{R})$ .

Now, we will show that the limit  $U$  is an entropy solution to (1.2) in the sense of Definition 2.3. In other words, we have to show that for any positive test function  $\phi \in C_c^\infty((0, +\infty) \times \mathbb{R})$ , the following inequality holds for all smooth convex entropy functions  $\eta$  with entropy flux  $\psi$

$$\begin{aligned} 0 \leq & \int_0^{+\infty} \int_{\mathbb{R}} (\eta(U^{\Delta x}(t, x)) \partial_t \phi(t, x) + \psi(U^{\Delta x}(t, x)) \partial_x \phi(t, x)) dx dt \\ & + \int_{\mathbb{R}} \eta(U^{\Delta x}(0, x)) \phi(0, x) dx. \end{aligned} \quad (5.5)$$

Now, the test function  $\phi$  is fixed, thus there exist  $T > 0$  and  $A > 0$  such that

$$\text{supp}(\phi) = (0, T) \times (-A, A).$$

By construction,  $U^{\Delta x}(0, x)$  is an *es* of (1.2) in the strip  $[0, \Delta t) \times \mathbb{R}$ . The problem is that with the first projection, or the new averaging, at the instant  $\Delta t$ , we have an error between  $U^{\Delta x}(\Delta t - 0, x)$  and  $U^{\Delta x}(\Delta t, x) = U^{\Delta x}(\Delta t + 0, x)$ . This means that with each new projection we have an error, but inside the strip  $(t_n, t_{n+1}) \times \mathbb{R}$ , we have an exact solution of (1.2). To fix the notation, an integer  $N$  is chosen such that  $T \leq N\Delta t \leq T + \Delta t$ , so  $n = 0, 1, \dots, N$ . Then, from (5.5), we can get

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} (\eta(U^{\Delta x}(t, x)) \partial_t \phi(t, x) + \psi(U^{\Delta x}(t, x)) \partial_x \phi(t, x)) dx dt \\ & + \int_{\mathbb{R}} \eta(U^{\Delta x}(0, x)) \phi(0, x) dx \geq E = \sum_{n=0}^{N-1} E^n, \end{aligned} \quad (5.6)$$

where

$$E^n = \int_{\mathbb{R}} (\eta(U^{\Delta x}(t_{n+1} - 0, x)) \phi(t_{n+1}, x) - \eta(U^{\Delta x}(t_n + 0, x)) \phi(t_n, x)) dx.$$

The first line in (5.6) is positive on  $(t_n, t_{n+1}) \times \mathbb{R}$ , for every  $n = 0, \dots, N-1$ , since inside each of these strips, the solutions we construct are entropy ones. Hence, we are just left with the integrals on the boundaries at each instant  $t_n$ .

After rearranging the terms, the local error on the time boundary of each cell  $E$  can be

rewritten under the form  $E = \sum_{n=0}^{N-1} \sum_{i \in \mathbb{Z}} E_i^n$ , where

$$\begin{aligned} E_i^n &= \int_{C_i} (\eta(U^{\Delta x}(t_{n+1} - 0, x)) - \eta(U^{\Delta x}(t_{n+1} + 0, x))) \phi(t_{n+1}, x) dx \\ &= \int_{C_i} (\eta(U^{\Delta x}(t_{n+1} - 0, x)) - \eta(\bar{U}_i^{n+1})) \phi(t_{n+1}, x) dx. \end{aligned}$$

By Jensen's inequality for convex functions, we have

$$\eta(\bar{U}_i^{n+1}) \leq \frac{1}{2\Delta x} \int_{C_i} \eta(U^{\Delta x}(t_{n+1} - 0, x)) dx,$$

which gives

$$\int_{C_i} (\eta(U^{\Delta x}(t_{n+1} - 0, x)) - \eta(\bar{U}_i^{n+1})) dx \geq 0.$$

Then, if we write  $\phi(t_{n+1}, x) = \phi(t_{n+1}, x) - \phi(t_{n+1}, x_i) + \phi(t_{n+1}, x_i)$ , we get

$$\begin{aligned} E_i^n &\geq \int_{C_i} (\eta(U^{\Delta x}(t_{n+1} - 0, x)) - \eta(\bar{U}_i^{n+1})) (\phi(t_{n+1}, x) - \phi(t_{n+1}, x_i)) dx \\ &\geq - \int_{C_i} \left| (\eta(U^{\Delta x}(t_{n+1} - 0, x)) - \eta(\bar{U}_i^{n+1})) (\phi(t_{n+1}, x) - \phi(t_{n+1}, x_i)) \right| dx \quad (5.7) \\ &\geq -2\Delta x \|\partial_x \phi\|_{L^\infty((0,T) \times \mathbb{R})} \|\nabla \eta\|_{L^\infty(\mathbb{R}^2)} \int_{C_i} \|U^{\Delta x}(t_{n+1} - 0, x) - \bar{U}_i^{n+1}\| dx, \end{aligned}$$

where  $\|\cdot\|$  is the summation norm defined as  $\|(u, v)\| = |u| + |v|$  for  $u, v \in \mathbb{R}$ , and we have used the Mean Value Theorem in the last line for  $\phi$  and  $\eta$ .

Now, recall that  $[-A, A] = I$  is the uniform compact support in  $x$  of the test function  $\phi$ .

Then, summing over  $i$  in (5.7) gives

$$E^n \geq -C_1 \Delta x \int_I \|U^{\Delta x}(t_{n+1} - 0, x) - \bar{U}^{\Delta x}(t_{n+1} - 0, x)\| dx,$$

where  $C_1 = 2 \|\partial_x \phi\|_{L^\infty((0,T) \times \mathbb{R})} \|\nabla \eta\|_{L^\infty(\mathbb{R}^2)} > 0$ , and  $\bar{U}^{\Delta x}(t_{n+1} - 0, x)$  is defined as the piecewise constant approximation of  $U^{\Delta x}(t_{n+1} - 0, x)$  by taking the averages on each cell.

Now, applying Hölder's inequality on the whole interval  $I$ , we get for  $0 < s \leq 1$

$$E^n \geq -C_2 \Delta x \left[ \left( \int_I |u^{\Delta x} - \bar{u}^{\Delta x}|^{\frac{1}{s}}(t_{n+1} - 0, x) dx \right)^s + \left( \int_I |v^{\Delta x} - \bar{v}^{\Delta x}|^{\frac{1}{s}}(t_{n+1} - 0, x) dx \right)^s \right],$$

where  $C_2 = C_1(2A)^{1-s}$ . Consider one term, the other is similar, with  $p = 1/s$ , and let  $M$  be an integer such that  $(M - 1)\Delta x < A \leq M\Delta x$ . Notice that the interval  $I$  is slightly enlarged by the interval  $[-M\Delta x, M\Delta x]$  such that  $M\Delta x < A + \Delta x$ . We have

$$\int_I |u^{\Delta x} - \bar{u}^{\Delta x}|^p(t_{n+1} - 0, x)dx \leq \sum_i \int_{C_i} |u^{\Delta x} - \bar{u}^{\Delta x}|^p(t_{n+1} - 0, x)dx, \quad \forall C_i \text{ in } [-M\Delta x, M\Delta x].$$

On each cell, Lemma 5.1 is used, where  $|C_i| = 2\Delta x$  is the size of the cell,

$$\begin{aligned} \int_I |u^{\Delta x} - \bar{u}^{\Delta x}|^p(t_{n+1} - 0, x)dx &\leq |C_i| \sum_i TV^s u^{\Delta x}(t_{n+1} - 0, \cdot)(C_i) \\ &\leq 2\Delta x TV^s u^{\Delta x}(t_{n+1} - 0, \cdot)([-M\Delta x, M\Delta x]) \\ &\leq 2\Delta x TV^s u^{\Delta x}(t_{n+1} - 0, \cdot)(\mathbb{R}) \\ &\leq 2\Delta x TV^s u^{\Delta x}(0, \cdot)(\mathbb{R}) \\ &\leq 2\Delta x TV^s u_0(\cdot)(\mathbb{R}). \end{aligned}$$

The last inequalities use the decay of the fractional total variation, Lemma 5.2. Then, for  $0 < \Delta x < 1$ , the inequality for  $E^n$  becomes

$$E^n \geq -C_2 \Delta x (2\Delta x)^s \left[ (TV^s u_0)^s + (TV^s v_0)^s \right].$$

Then, using the CFL condition (5.3), summing over  $n$ , and using the condition  $T \leq N\Delta t \leq T + \Delta t$ , we obtain for  $C_3 = 2\Lambda C_2$

$$E \geq -C_3 (T + \Delta t) (2\Delta x)^s \left[ (TV^s u_0)^s + (TV^s v_0)^s \right].$$

Finally, passing to the limit as  $\Delta x \rightarrow 0$ , we deduce the entropy inequality (5.5).  $\square$

Notice that for  $s = 1$  the proof is simpler than the case  $0 < s < 1$ . We can directly bound the local error  $E_i^n$  by the local total variation. For  $s < 1$ , we need to use the Hölder inequality on the whole interval  $I$  and not directly on the cell.

The proof shows that the error of consistency is of order  $(\Delta x)^s$  which recovers the classical  $BV$  scalar case for  $s = 1$ .

## A $BV^s$ spaces

In this section, we recall few basic properties of functions of bounded  $s$ -variation. Spaces  $BV^s(I)$  with  $0 < s \leq 1$  are in fact a generalization of  $BV(I)$ , the spaces of functions with bounded variation on  $I$ , where  $I$  is a non-empty interval of  $\mathbb{R}$ . These spaces are formally defined as follows.

**Definition A.1. (Definition of  $BV^s(I)$ )**

Let  $I$  be a non-empty interval of  $\mathbb{R}$ . We denote by  $S(I)$  the set of all subdivisions of  $I$ , that is, it is the set of all finite subsets  $\sigma = \{x_0, x_1, \dots, x_n\} \subset I$  with  $x_0 < x_1 < \dots < x_n$ . Let  $\sigma \in S(I)$  and  $u$  be a real function defined on  $I$ . The  $s$ -total variation of  $u$  with respect to  $\sigma$  is

$$TV^s u(\sigma) = \sum_{i=1}^n |u(x_i) - u(x_{i-1})|^{\frac{1}{s}},$$

and the  $s$ -total variation of  $u(\cdot)$  on  $I$  is defined by

$$TV^s u(I) = \sup_{\sigma \in S(I)} TV^s u(\sigma),$$

where the supremum is taken over all subdivisions of  $I$ . We then denote the space

$$BV^s(I) := \{u : I \rightarrow \mathbb{R} : TV^s u(I) < +\infty\}.$$

We then define the  $BV^s$  semi-norm by

$$|u|_{BV^s(I)} = (TV^s u(I))^s.$$

If  $0 < s < t \leq 1$  and  $I$  is not reduced to one point then  $BV^t(I) \subsetneq BV^s(I)$  [13]. The following inclusion is obtained directly from the definition. For an interval  $I$  of  $\mathbb{R}$ , we have

$$BV^s(I) \subset L^\infty(I), \forall s \in (0, 1].$$

Notice that  $\bigcup_{s>0} BV^s$  is strictly smaller than  $L^\infty$  since generalized and bigger  $BV$  spaces denoted by  $BV^\phi$  [20], where  $\phi$  is a function satisfying certain properties, also belong to  $L^\infty$ .

We also recall the following elementary lemma that is quite different from the case  $s = 1$ .

**Lemma A.1.**

If  $0 < s < 1$  and  $(a_i)_{1 \leq i \leq n}$  is a finite sequence of positive real numbers, then

$$\sum_{i=1}^n a_i^{\frac{1}{s}} < \left( \sum_{i=1}^n a_i \right)^{\frac{1}{s}}.$$

This inequality is the converse of the usual triangular inequality. This is the reason why  $BV^s$  estimates require the consideration of all subdivisions of an interval and not only the finest ones, unlike the  $BV$  framework. Nevertheless,  $BV^s$  spaces are well fitted for sharp estimates in the context of scalar conservation laws [13, 19], and here used for a  $2 \times 2$  system.

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