



**HAL**  
open science

# Non trivial zeros of the Zeta function using the differential equations

W Oukil

► **To cite this version:**

| W Oukil. Non trivial zeros of the Zeta function using the differential equations. 2023. hal-03849860v9

**HAL Id: hal-03849860**

**<https://hal.science/hal-03849860v9>**

Preprint submitted on 20 Aug 2023 (v9), last revised 20 May 2024 (v18)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Non trivial zeros of the Zeta function using the differential equations

W. Oukil

Faculty of Mathematics.

University of Science and Technology Houari Boumediene.

BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

August 20, 2023

## Abstract

In this paper, we investigate a relation between the differential equations and the non trivial zeros of the Zeta function.

**Keywords:** Zeta function, non trivial zeros, Differential equations.

**AMS subject classifications:** 00A05

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . Define the *non trivial zeros* of the function  $\zeta$  in the following sense

**Definition 1.** Consider the function  $\zeta$  defined by the Equation (1). Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function  $\zeta$*  if

$$|\zeta(s)| = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

We prove the following Theorem.

**Theorem 2.** Consider the function  $\zeta$  defined by the Equation (1). The point  $s \in \mathbb{C}$  is a non trivial zero of the function  $\zeta$ , if and only if

$$1 - 4s(1-s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s} = 0.$$

## 2 Main Proposition

In order to simplify the notation, denote

$$B := \{s \in \mathbb{C} / \Re(s) \in (0, 1), \Im(s) \in \mathbb{R}^*\}.$$

For every  $s \in B$ , the Equation (1) is equivalent to,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

The aim is to studies the differential equation of solutions the functions

$$t \mapsto \psi_s(z, t) := t^s \left[ z + \int_1^t u^{-1-s} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 2, is to find  $z_s \in \mathbb{C}$  such that  $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ . Since  $\sup_{t \geq 1} |\psi_s(\frac{1}{1-s}, t)| < +\infty$  if and only if  $|\frac{\zeta(s)}{s}| = 0$ , we obtain  $\frac{1}{1-s} = z_s$ .

For every  $s \in B$  we consider the following differential equation

$$\frac{d}{dt} x = st^{-1}x + t^{-1}\{t\}, \quad (2)$$

$$t \in [1, +\infty)/\mathbb{N}, \quad x(1) = z, \quad z \in \mathbb{C}, \quad x : [1, +\infty) \rightarrow \mathbb{C}.$$

In this paper we derive the functions only on  $[1, +\infty)/\mathbb{N}$ .

**Lemma 3.** *Let be  $s \in B$ . For every  $z \in \mathbb{C}$  there exists a unique continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2). Further,*

$$\psi_s(z, t) = t^s \left[ z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

*Proof.* Let be  $s \in B$  fixed. The function

$$t \mapsto t^s \int_1^t u^{-1-s} \{u\} du, \quad t \geq 1,$$

is  $C^\infty$  on  $[1, +\infty)/\mathbb{N}$  and continuous on  $[1, +\infty)$ . The Equation (2) is a non-homogeneous linear differential equation. For every  $z \in \mathbb{C}$  the unique continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  such that  $\psi_s(z, 1) = z$  is given by

$$\psi_s(z, t) = t^s \left[ z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

□

**Lemma 4.** For every  $s \in B$ , there exists a unique  $z_s \in \mathbb{C}$  such that the solution  $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2) satisfies  $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ . Further,

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du, \text{ and } \psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

*Proof.* Let be  $s \in B$  fixed. There is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm. Let be  $z \in \mathbb{C}$  and suppose that

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty.$$

Since  $\Re(s) \in (0, 1)$ , by the Lemma 3, we have

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty \implies \left| z + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

Then

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Prove that  $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ . Since

$$\left| z_s + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

then

$$z_s + \int_1^t u^{-1-s} \{u\} du = - \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t > 1.$$

By the Lemma 3, we get

$$\psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Then  $\sup_{t \geq 1} |\psi_s(z_s, t)| \leq \frac{1}{\Re(s)}$ . □

**Lemma 5.** Let be  $s \in B$ . For every  $z \in \mathbb{C}$  there exists a unique  $\tilde{z} \in \mathbb{C}$  such that the solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2) satisfies

$$\psi_s(z, t) = \psi_s(z_s, t) + t^s \tilde{z}, \quad \forall t \geq 1.$$

where  $t \mapsto \psi_s(z_s, t)$  is the bounded solution given by the Lemma 4. In addition,

$$\tilde{z} = z + \int_1^{+\infty} u^{-1-s} \{u\} du.$$

*Proof.* Consider

$$\tilde{z} = z + \int_1^{+\infty} u^{-1-s} \{u\} du,$$

By the Lemma 3, we have

$$\psi_s(z, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du + t^s \tilde{z}, \quad \forall t \geq 1.$$

By the Lemma 4, we get

$$\psi_s(z, t) = \psi_s(z_s, t) + t^s \tilde{z}, \quad \forall t \geq 1.$$

□

**Proposition 6.** *Let be  $s \in B$ . Let be  $z \in \mathbb{C}$  and consider the continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2). Then  $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$ , if and only if*

$$z = -\frac{1}{2} \frac{1}{s} + 2(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}.$$

**Notation 7.** Denote

$$p(t) := \int_1^t \left( \{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 1,$$

where we recall that  $\{u\}$  is the fractional part of the real  $u$ .

**Lemma 8.** *The function  $p$  is a continuous 1-periodic function and satisfies*

$$p(t) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left( \cos(j2\pi t) - 1 \right), \quad \forall t \geq 1.$$

*Proof.* The function  $u \mapsto \{u\}$  is 1-periodic, then there exists a continuous 1-periodic function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_1^t \{u\} du = (t-1) \int_1^2 \{u\} du + p(t), \quad \forall t \geq 1.$$

Since

$$\int_1^2 \{u\} du = \int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_1^t \{u\} du = (t-1)\frac{1}{2} + p(t), \quad \forall t \geq 1.$$

The function  $p$  is a piecewise  $C^\infty$ , continuous on  $\mathbb{R}$  and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on  $\mathbb{R}_+$  to the function  $t \mapsto p(t)$ , where  $(a_k)_k \subset \mathbb{C}$  are the Fourier coefficients of the function  $p$ .

$$\int_0^t \left(\{v\} - \frac{1}{2}\right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left( \int_0^u \left(\{v\} - \frac{1}{2}\right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12} = -\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function  $p$  satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left( \exp(ij2\pi t) - 1 \right), \quad \forall t \geq 0.$$

□

*Proof of the Proposition 6.* Let be  $s \in B$ . Let be  $z \in \mathbb{C}$  and consider the continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2). Define the function  $\tilde{\psi}_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  as

$$\tilde{\psi}_s(z, t) := \int_1^t \psi_s(z, u) du, \quad t \geq 1.$$

The Equation (2) implies

$$\frac{d}{dt} \tilde{\psi}_s(z, t) = t^{-1}(1+s)\tilde{\psi}_s(z, t) + t^{-1} \int_1^t \{u\} du + t^{-1}z, \quad \forall t \geq 1.$$

By the Lemma 8 we have

$$\begin{aligned} \int_1^t \left( \{u\} - \frac{1}{2} \right) du &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left( \cos(j2\pi t) - 1 \right), \\ &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} t^{2k}, \quad \forall t \geq 1. \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} \tilde{\psi}_s(z, t) &= t^{-1} (1+s) \tilde{\psi}_s(z, t) + \frac{1}{2} (1 - (1-2z)t^{-1}) \\ &\quad + 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} t^{2k-1}, \quad \forall t \geq 1. \end{aligned} \quad (3)$$

Integrate, to obtain

$$\begin{aligned} \tilde{\psi}_s(z, t) &= \frac{1}{2} \left( -\frac{1}{s} t + \frac{1-2z}{1+s} \right) \\ &\quad + 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s} t^{2k} \\ &\quad - t^{1+s} \left[ \frac{1}{2} \left( -\frac{1}{s} + \frac{1-2z}{1+s} \right) + 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s} \right]. \end{aligned} \quad (4)$$

By Lemma 5 and the definition of  $\tilde{\psi}_s(z, t)$  we have

$$\tilde{\psi}_s(z, t) = \int_1^t \psi_s(z, u) du = \int_1^t \psi_s(z_s, u) du - \frac{\tilde{z}}{1+s} + t^{1+s} \frac{\tilde{z}}{1+s}, \quad \forall t \geq 1.$$

where  $t \mapsto \psi_s(z_s, t)$  is the bounded solution given by the Lemma 4 and where

$$\tilde{z} = z + \int_1^{+\infty} u^{-1-s} \{u\} du.$$

By identification with the the Equation (4) we get

$$\begin{aligned} \frac{\tilde{z}}{1+s} &= -\frac{1}{2} \left( -\frac{1}{s} + \frac{1-2z}{1+s} \right) \\ &\quad - 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}, \end{aligned}$$

In other words,

$$\int_1^{+\infty} u^{-1-s} \{u\} du = \frac{1}{2} \frac{1}{s} - 2(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}.$$

By the Lemma 4, we have  $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$ , if and only if

$$z = z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du = -\frac{1}{2} \frac{1}{s} + 2(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}.$$

□

### 3 Proof of the Theorem 2

*Proof of the Theorem 2.* Let be  $s \in B$ . By the Equation (1) we have,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Let  $\psi_s(\frac{1}{1-s}, t) : [1, +\infty) \rightarrow \mathbb{C}$  be the unique continuous solution of the differential equation (2). By Lemma 3 we have,

$$|\zeta(s)| = 0 \iff \sup_{t \geq 1} |\psi_s(\frac{1}{1-s}, t)| < +\infty.$$

Thanks to Main Proposition 6 ,  $|\zeta(s)| = 0$  if and only if

$$\frac{1}{1-s} = -\frac{1}{2} \frac{1}{s} + 2(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}.$$

□

### References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).