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| W Oukil. Non trivial zeros of the Zeta function using the differential equations. 2023. hal-03849860v7

HAL Id: hal-03849860

<https://hal.science/hal-03849860v7>

Preprint submitted on 9 Apr 2023 (v7), last revised 20 May 2024 (v18)

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Non trivial zeros of the Zeta function using the differential equations

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April 9, 2023

Abstract

In this paper, we investigate a relation between the differential equations and the non trivial zeros of the Zeta function.

Keywords: Zeta function, Riemann hypothesis, Differential equations.

AMS subject classifications: 00A05

1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{t\}$ is the fractional part of the real t . We prove the following Theorem.

Theorem 1. *Consider the function ζ defined by the Equation (1). For every $\tau \in \mathbb{R}^*$ and $r \in (\frac{1}{2}, 1)$ we have*

$$|\zeta(r + i\tau)| \neq 0.$$

Thanks to the Riemann functional equation we deduce that any *non trivial zero* of the Zeta function has a real part equal to $\frac{1}{2}$, where the *non trivial zeros* are defined in the following sense

Definition 2. Consider the function ζ defined by the Equation (1). Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function ζ* if

$$|\zeta(s)| = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

2 Main Proposition

For every $r \in (0, 1)$ and $\tau \in \mathbb{R}^*$ the Equation (1) implies,

$$\frac{\zeta(r + i\tau)}{r + i\tau} = -\frac{1}{1 - r - i\tau} - \int_1^{+\infty} u^{-i\tau-1-r} \{u\} du.$$

The aim is to studies the differential equation of solutions the functions

$$t \mapsto \psi_{\tau,r}(z, t) := t^{r+i\tau} \left[z + \int_1^t u^{-i\tau-1-r} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to prove that $\sup_{t \geq 1} |\psi_{\tau,r}(\frac{1}{1-r-i\tau}, t)| < +\infty$ implies $2r \leq 1$. In other words $|\frac{\zeta(r+i\tau)}{r+i\tau}| = 0$ implies $2r \leq 1$.

For every $\tau \in \mathbb{R}^*$ and $r \in (0, 1)$ we consider the following differential equation

$$\frac{d}{dt} x = (r + i\tau)t^{-1}x + t^{-1}\{t\}, \quad (2)$$

$$t \in \mathbb{R}_+/\mathbb{N}, \quad x(1) = \frac{1}{1 - r - i\tau}, \quad x : [1, +\infty) \rightarrow \mathbb{C}.$$

In this paper we derive the functions only on \mathbb{R}_+/\mathbb{N} .

Lemma 3. *For every $\tau \in \mathbb{R}^*$ and $r \in (0, 1)$ there exists a unique continuous solution $\psi_{\tau,r}(t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Further,*

$$\psi_{\tau,r}(t) = t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du, \quad \forall t \geq 1.$$

Proof. Let be $r \in (0, 1)$ and $\tau \in \mathbb{R}^*$ fixed. Since $\{u\} = u$ for every $u \in (0, 1)$ then

$$t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du = \frac{t}{1 - r - i\tau}, \quad \forall t \in [0, 1].$$

Since $0 \leq \{u\} \leq 1$ for every $u \geq 1$ then the function

$$t \mapsto t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du,$$

is continuous and C^1 on \mathbb{R}_+/\mathbb{N} . The Equation (2) is a non-homogeneous linear differential equation. The unique continuous solution $\psi_{\tau,r}(t) : [1, +\infty) \rightarrow \mathbb{C}$ such that $\psi_{\tau,r}(1) = \frac{1}{1-r-i\tau}$ is given by

$$\psi_{\tau,r}(t) = t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du, \quad \forall t \geq 0.$$

□

Proposition 4. *Let be $\tau \in \mathbb{R}^*$ and $r \in (0, 1)$. Consider the continuous solution $\psi_{\tau, r}(t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Suppose that $\sup_{t \geq 1} |\psi_{\tau, r}(t)| < +\infty$, then $2r \leq 1$.*

Notation 5. Denote

$$p(t) := \frac{1}{12} + \int_0^t \left(\{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 0,$$

where we recall that $\{u\}$ is the fractional part of the real u .

Lemma 6. *The function p is a continuous 1-periodic function and satisfies*

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \exp(ij2\pi t), \quad \forall t \geq 0.$$

Proof. The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du - \frac{1}{12} + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t \frac{1}{2} - \frac{1}{12} + p(t), \quad \forall t \geq 0.$$

The function p is a piecewise C^∞ , continuous on \mathbb{R} and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on \mathbb{R}_+ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p .

$$\int_0^t \left(\{v\} - \frac{1}{2} \right) dv + \frac{1}{12} = \frac{1}{12} + \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left(\int_0^u \left(\{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12}.$$

The function p satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \exp(ij2\pi t), \quad \forall t \geq 0.$$

□

Lemma 7. *Let be $\tau \in \mathbb{R}^*$ and $r \in (0, 1)$. Let $\psi_{\tau, r}(t) : [1, +\infty) \rightarrow \mathbb{C}$ be the continuous solution of the differential equation (2). Suppose that $\sup_{t \geq 1} |\psi_{\tau, r}(t)| < +\infty$. Then*

$$\sup_{n \in \mathbb{N}^*} \left| n^3 \theta_{\tau, r}(n) \right| < +\infty \quad \text{and} \quad \sup_{\substack{k, n \in \mathbb{N}^* \\ k \geq n}} \left| n^{2(1+r)} \int_n^k u^{-2r-1} [u] \theta_{\tau, r}(u) du \right| < +\infty,$$

where

$$\theta_{\tau, r}(u) := \psi_{\tau, r}(u) + \frac{1}{2} \frac{1}{r + i\tau} - p(u)u^{-1}, \quad \forall u \geq 1,$$

and where $u \mapsto [u]$ is the floor function.

Proof. Let be $r \in (0, 1)$ and $\tau \in \mathbb{R}^*$ fixed. By Lemma 3

$$\psi_{\tau, r}(t) = t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du, \quad \forall t > 0. \quad (3)$$

Suppose that

$$\sup_{t \geq 1} |\psi_{\tau, r}(t)| < +\infty.$$

Since

$$\sup_{t \geq 1} |\psi_{\tau, r}(t)| < +\infty \implies \left| \int_0^{+\infty} u^{-i\tau-1-r} \{u\} du \right| = 0,$$

then

$$\int_0^t u^{-i\tau-1-r}\{u\}du = -\int_t^{+\infty} u^{-i\tau-1-r}\{u\}du, \quad \forall t > 1.$$

Equation (3) can be written as

$$\psi_{\tau,r}(t) = -t^{r+i\tau} \int_t^{+\infty} u^{-i\tau-1-r}\{u\}du, \quad \forall t > 1. \quad (4)$$

Consider the function p given in the Notation 5. By Lemma 6, the function p is 1-periodic, then it is bounded. Use the integration part formula in Equation (4),

$$\begin{aligned} \psi_{\tau,r}(t) &= -\frac{1}{2} \frac{1}{r+i\tau} + p(t)t^{-1} \\ &\quad - \left(\prod_{j=1}^2 (i\tau + j + r) \right) t^{i\tau+r} \int_t^{+\infty} u^{-i\tau-3-r} \int_t^u p(v)dvdu, \end{aligned}$$

or by the notation of $\theta_{\tau,r}$ of the present Lemma, we have

$$\theta_{\tau,r}(t) = -\left(\prod_{j=1}^2 (i\tau + j + r) \right) t^{i\tau+r} \int_t^{+\infty} u^{-i\tau-3-r} \int_t^u p(v)dvdu. \quad (5)$$

By Lemma 6, we have

$$p(u) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \exp(ij2\pi u), \quad \forall u \geq 1.$$

Since $\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^3} = 0$, then

$$\left| \int_s^u \int_n^v p(\mu)d\mu dv \right| \leq 2 \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^4} < 1, \quad \forall u \geq s, \forall n \in \mathbb{N}. \quad (6)$$

Item 1.

Using the integration part formula in the Equation (5), for every $\forall n \in \mathbb{N}$ we obtain

$$|\theta_{\tau,r}(n)| \leq \prod_{j=1}^3 |j + r + i\tau| \left| n^{i\tau+r} \int_n^{+\infty} u^{-i\tau-4-r} \int_n^u \int_n^v p(\mu)d\mu dvdu \right|.$$

By the Equation (6), we get

$$\frac{|\theta_{\tau,r}(n)|}{|3 + r + i\tau|^3} < n^r \int_n^{+\infty} u^{-4-r} < n^{-3}, \quad \forall n \in \mathbb{N}^*.$$

Item 2.

In order to simplify the notation, for every $k, n \in \mathbb{N}^*$, denote

$$V_{\tau,r}(k, n) := \int_n^k u^{-1} [u] u^{i\tau-r} \int_u^{+\infty} v^{-i\tau-3-r} \int_u^v p(\mu) d\mu dv du.$$

For every $k, n \in \mathbb{N}^*$, we have

$$\begin{aligned} V_{\tau,r}(k, n) &= \int_n^k u^{-1} [u] u^{i\tau-r} \int_u^{+\infty} v^{-i\tau-3-r} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^3} \exp(ij2\pi v) dv \\ &\quad - \frac{1}{i\tau + 2 + r} \int_n^k [u] u^{-3-2r} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^3} \exp(ij2\pi u) du. \end{aligned}$$

Implies

$$\begin{aligned} V_{\tau,r}(k, n) &= \int_n^k u^{-1} [u] u^{i\tau-r} \int_u^{+\infty} v^{-i\tau-3-r} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^3} \exp(ij2\pi v) dv \\ &\quad + \frac{1}{i\tau + 2 + r} \int_n^k \{u\} u^{-3-2r} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^3} \exp(ij2\pi u) du \\ &\quad - \frac{1}{i\tau + 2 + r} \int_n^k u^{-2(1+r)} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^3} \exp(ij2\pi u) du. \end{aligned}$$

Use the fact

$$\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^3} = 0 \quad \text{and} \quad \int_n^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^3} \exp(ij2\pi v) dv = 0,$$

and use the integration part formula to obtain

$$\begin{aligned} &V_{\tau,r}(k, n) \\ &= (i\tau + 3 + r) \int_n^k u^{-1} [u] u^{i\tau-r} \int_u^{+\infty} v^{-i\tau-4-r} \int_u^v \int_n^s p(\mu) d\mu ds dv du \\ &\quad + \frac{1}{i\tau + 2 + r} \int_n^k \{u\} u^{-3-2r} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^3} \exp(ij2\pi u) du \\ &\quad - 2 \frac{1+r}{i\tau + 2 + r} \int_n^k u^{-3-2r} \int_n^u \int_n^v p(\mu) d\mu dv du. \end{aligned} \tag{7}$$

Using the Equation (6); the Equation (5) and (7) implies

$$\begin{aligned} \left| \int_n^k u^{-2r-1} [u] \theta_{\tau,r}(u) du \right| &= \prod_{j=1}^2 |i\tau + j + r| |V_{\tau,r}(k, n)| \\ &< |3 + r + i\tau|^3 \int_n^k u^{-r} \int_u^{+\infty} v^{-4-r} dv du \\ &+ 5|1 + r + i\tau| \int_n^k u^{-3-2r} du < 6|3 + r + i\tau|^3 n^{-2(r+1)}. \end{aligned}$$

□

Proof of the Proposition 4. Let be $r \in (0, 1)$ and $\tau \in \mathbb{R}^*$. Use the change of variable

$$\tilde{\psi}_{\tau,r}(t) := -\frac{t}{1-r-i\tau} + (r+i\tau)\psi_{\tau,r}(t) + \{t\}, \quad t \geq 1.$$

The Equation (2) can be written as

$$\frac{d}{dt} \tilde{\psi}_{\tau,r}(t) = t^{-1}(r+i\tau)\tilde{\psi}_{\tau,r}(t), \quad \forall t \in \mathbb{R}_+/\mathbb{N}, \quad t \geq 1. \quad (8)$$

For the particular initial condition of $\psi_{\tau,r}$, we have

$$\psi_{\tau,r}(1) = \frac{1}{1-r-i\tau},$$

by the definition of $\tilde{\psi}_{\tau,r}$ we get

$$\tilde{\psi}_{\tau,r}(1) = -1.$$

The Equation (8) implies the following new differential equation,

$$t^{-1} \tilde{\psi}_{\tau,r}(t) = (r+i\tau)t^{-1} \int_1^t u^{-1} \tilde{\psi}_{\tau,r}(u) du - t^{-1} [t], \quad \forall t \geq 1,$$

where we recall that $u \mapsto [u]$ is the floor function. The wronksien is given by

$$\frac{d}{dt} \Psi_{\tau,r}(t) = 2rt^{-1} \Psi_{\tau,r}(t) - 2t^{-1} [t] w_{\tau,r}(t), \quad (9)$$

where in order to simplify the notation, we denoted

$$\Psi_{\tau,r}(t) := \left| \int_1^t u^{-1} \tilde{\psi}_{\tau,r}(u) du \right|^2 \quad \text{and} \quad w_{\tau,r}(t) := \int_1^t u^{-1} \Re(\tilde{\psi}_{\tau,r}(u)) du.$$

By definition of $\tilde{\psi}_{\tau,r}$ we have

$$\begin{aligned} \int_1^t u^{-1} \tilde{\psi}_{\tau,r}(u) du &= -\frac{(t-1)}{1-r-i\tau} + \int_1^t u^{-1} \left[(r+i\tau)\psi_{\tau,r}(u) + \{u\} \right] du \\ &= -\frac{(t-1)}{1-r-i\tau} + \int_1^t \frac{d}{du} \psi_{\tau,r}(u) du \\ &= -\frac{t}{1-r-i\tau} + \psi_{\tau,r}(t) \end{aligned}$$

Then

$$\begin{cases} \Psi_{\tau,r}(t) &= \frac{1}{(1-r)^2 + \tau^2} t^2 + |\psi_{\tau,r}(t)|^2 \\ &\quad - 2 \frac{(1-r)\Re(\psi_{\tau,r}(t)) + \tau\Im(\psi_{\tau,r}(t))}{(1-r)^2 + \tau^2} t, \quad \forall t \geq 1, \\ w_{\tau,r}(t) &= -\frac{(1-r)}{(1-r)^2 + \tau^2} t + \Re(\psi_{\tau,r}(t)), \quad \forall t \geq 1. \end{cases} \quad (10)$$

Prove the Proposition by contradiction. Suppose that $2r > 1$ and $\sup_{t \geq 1} |\psi_{\tau,r}(t)| < +\infty$. By The Lemma 7, we have

$$\sup_{n \in \mathbb{N}^*} \left| n^3 \psi_{\tau,r}(n) + \frac{1}{2} \frac{1}{r+i\tau} n^3 - n^2 p(n) \right| < +\infty. \quad (11)$$

Using the Equation (10), we get

$$\sup_{n \in \mathbb{N}^*} n^2 \left| \alpha_{\tau,r} \Psi_{\tau,r}(n) - \Phi_{\tau,r}(n) \right| < +\infty, \quad (12)$$

where

$$\Phi_{\tau,r}(n) := n^2 - \frac{\beta_{\tau,r} - r}{\beta_{\tau,r}} n + \frac{\alpha_{\tau,r}}{4\beta_{\tau,r}} - 2(1-r)p(n),$$

and where

$$\alpha_{\tau,r} := (1-r)^2 + \tau^2 \quad \text{and} \quad \beta_{\tau,r} := r^2 + \tau^2.$$

Integrate the equation (9) as a non-homogeneous linear differential equation, we obtain

$$\begin{aligned} \Psi_{\tau,r}(t) &= t^{2r} h_{\tau,r}(t, s), \quad \forall t \geq s \geq 1, \\ h_{\tau,r}(t, s) &:= s^{-2r} \Psi_{\tau,r}(s) - 2 \int_s^t u^{-2r-1} [u] w_{\tau,r}(u) du. \end{aligned} \quad (13)$$

By hypothesis $2r > 1$. From the Equation (12), remark that

$$\lim_{k \rightarrow +\infty} k^{-2r} \left[k^2 - \alpha_{\tau,r} \Psi_{\tau,r}(k) \right] = 0.$$

The strategy to prove the present Proposition is to find a constant time $n_{\tau,r} \gg 1$ such that

$$\lim_{k \rightarrow +\infty} \left| k^{2(1-r)} - \alpha_{\tau,r} h_{\tau,r}(k, n_{\tau,r}) \right| > 0,$$

which gives a contradiction. In other words, the aim is to study the asymptotic behavior of the function (solution) $t \mapsto t^2 - \alpha_{\tau,r} \Psi_{\tau,r}(t)$ and find this function, when $2r > 1$, can not approximate a polynomial function of first order. More precisely, the polynomial approximated by this function is of $2r$ order. We have

$$\begin{aligned} k^{2(1-r)} - \alpha_{\tau,r} h_{\tau,r}(k, n) &= k^{2(1-r)} - \alpha_{\tau,r} n^{-2r} \Psi_{\tau,r}(n) \\ &\quad + 2\alpha_{\tau,r} \int_n^k u^{-2r-1} [u] w_{\tau,r}(u) du, \quad \forall k, n \in \mathbb{N}^*. \end{aligned}$$

Use the second Item of the Lemma 7 and the Equation (10). Use the Equations (12), we get

$$\sup_{\substack{k, n \in \mathbb{N}^* \\ k \geq n}} \left| n^{2(1+r)} \left[k^{2(1-r)} - \alpha_{\tau,r} h_{\tau,r}(k, n) - \Gamma_{\tau,r}(k, n) \right] \right| < +\infty, \quad (14)$$

where

$$\begin{aligned} \Gamma_{\tau,r}(k, n) &:= k^{2(1-r)} - n^{-2} \Phi_{\tau,r}(n) \\ &\quad - \int_n^k u^{-2r-1} [u] \left[2(1-r)u + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} - 2\alpha_{\tau,r} p(u)u^{-1} \right] du, \end{aligned}$$

or by the notation of $\Phi_{\tau,r}$,

$$\begin{aligned} \Gamma_{\tau,r}(k, n) &= k^{2(1-r)} - n^{2(1-r)} + \frac{\beta_{\tau,r} - r}{\beta_{\tau,r}} n^{1-2r} \\ &\quad - \left(\frac{\alpha_{\tau,r}}{4\beta_{\tau,r}} - 2(1-r)p(n) \right) n^{-2r} \\ &\quad - \int_n^k u^{-2r-1} [u] \left[2(1-r)u + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} - 2\alpha_{\tau,r} p(u)u^{-1} \right] du. \end{aligned}$$

As in the Proof of the Lemma 7, we use the integration part formula. Using the fact

$$\frac{\beta_{\tau,r} - r}{\beta_{\tau,r}} + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}(1-2r)} - \frac{1-r}{1-2r} = 0,$$

Since $2r > 1$, we have

$$\begin{aligned} \lim_{k \rightarrow +\infty} \Gamma_{\tau,r}(k, n) &= 2(1-r)p(n)n^{-2r} + 2\alpha_{\tau,r} \int_n^{+\infty} u^{-2r-1}p(u)du \\ &+ 2(1-r) \int_n^{+\infty} u^{-2r}(\{u\} - \frac{1}{2})du + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} \int_n^{+\infty} u^{-2r-1}(\{u\} - \frac{1}{2})du \\ &- 2\alpha_{\tau,r} \int_n^{+\infty} u^{-2r-2}\{u\}p(u)du. \end{aligned}$$

Implies

$$\begin{aligned} \lim_{k \rightarrow +\infty} \Gamma_{\tau,r}(k, n) &= 2\alpha_{\tau,r} \int_n^{+\infty} u^{-2r-1}p(u)du \\ &+ 4r(1-r) \int_n^{+\infty} u^{-2r-1}p(u)du + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} \int_n^{+\infty} u^{-2r-1}(\{u\} - \frac{1}{2})du \\ &- \alpha_{\tau,r} \int_n^{+\infty} u^{-2r-2}p(u)du - 2\alpha_{\tau,r} \int_n^{+\infty} u^{-2r-2}(\{u\} - \frac{1}{2})p(u)du. \end{aligned}$$

$$\begin{aligned} \lim_{k \rightarrow +\infty} \Gamma_{\tau,r}(k, n) &= -\frac{r\alpha_{\tau,r}}{\beta_{\tau,r}}n^{-2r-1}p(n) \\ &+ 2(r+1)(2r+1)\frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} \int_n^{+\infty} u^{-2r-3} \int_n^u p(v)dvdu \\ &+ 8r(1-r^2)(2r+1) \int_n^{+\infty} u^{-2r-3} \int_n^u \int_n^v p(\mu)d\mu dvu \\ &+ 4(2r+1)(r+1)\alpha_{\tau,r} \int_n^{+\infty} u^{-2r-3} \int_n^u \int_n^v p(\mu)d\mu dvu \\ &- 2(r+1)\alpha_{\tau,r} \int_n^{+\infty} u^{-2r-3} \int_n^u p(v)dvdu \\ &- 2(r+1)\alpha_{\tau,r} \int_n^{+\infty} u^{-2r-3} \left((p(u))^2 - (p(n))^2 \right) du. \end{aligned}$$

Since

$$\left| \int_n^u p(v)dv \right| \leq 1 \quad \text{and} \quad \left| \int_n^u \int_n^v p(\mu)d\mu dv \right| \leq 1, \quad \forall u \geq n, \quad n \in \mathbb{N},$$

Then, there exists $c_{\tau,r} > 0$ such that for all $n \in \mathbb{N}^*$ we have

$$\left| \lim_{k \rightarrow +\infty} \Gamma_{\tau,r}(k, n) + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}}p(n)n^{-2r-1} \right| < c_{\tau,r}n^{-2(1+r)}.$$

Since

$$p(n) = \frac{1}{12}, \quad \forall n \in \mathbb{N}$$

Then

$$\left| \lim_{k \rightarrow +\infty} \Gamma_{\tau,r}(k, n) + \frac{1}{12} \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} n^{-2r-1} \right| < c_{\tau,r} n^{-2(1+r)}.$$

From the Equation (14), there exist $n_{\tau,r} \in \mathbb{N}^*$ such that

$$\lim_{k \rightarrow +\infty} \left| k^{2(1-r)} - \alpha_{\tau,r} h_{\tau,r}(k, n_{\tau,r}) \right| > \frac{1}{24} \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} n_{\tau,r}^{-2r-1} > 0.$$

The Equation (13) implies

$$\lim_{k \rightarrow +\infty} k^{-2r} \left| k^2 - \alpha_{\tau,r} \Psi_{\tau,r}(k) \right| > \frac{1}{24} \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} n_{\tau,r}^{-2r-1}.$$

But since $r \in (\frac{1}{2}, 1)$, by the Equation (12) we have

$$\lim_{k \rightarrow +\infty} k^{-2r} \left[k^2 - \alpha_{\tau,r} \Psi_{\tau,r}(k) \right] = 0.$$

Contradiction. □

Remark 8. In the previous Proof, using the fact

$$\tilde{\psi}_{\tau,r}(1) = -1.$$

remark that the sequence $\left(n^{-r-i\tau} \tilde{\psi}(n) \right)_{n \in \mathbb{N}^*}$ is given by the Riemann series;

$$n^{-r-i\tau} \tilde{\psi}(n) = - \sum_{k=1}^n \frac{1}{k^{r+i\tau}}, \quad \forall n \in \mathbb{N}^*.$$

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $x \in (0, 1)$ and $\tau \in \mathbb{R}^*$. Suppose that $|\zeta(x + i\tau)| = 0$. By the Equation (1) we have,

$$\frac{\zeta(x + i\tau)}{x + i\tau} = - \int_0^{+\infty} u^{-i\tau-1-x} \{u\} du.$$

Then

$$\left| \int_0^{+\infty} u^{-i\tau-1-x} \{u\} du \right| = 0.$$

Implies

$$\int_0^t u^{-i\tau-1-x}\{u\}du = -\int_t^{+\infty} u^{-i\tau-1-x}\{u\}du, \quad \forall t \geq 1.$$

$$\sup_{t \geq 1} \left| t^{x+i\tau} \int_0^t u^{-i\tau-1-x}\{u\}du \right| = \sup_{t \geq 1} \left| t^{x+i\tau} \int_t^{+\infty} u^{-i\tau-1-x}\{u\}du \right|$$

$$\leq \sup_{t \geq 1} \left[t^x \int_t^{+\infty} u^{-1-x} du \right] \leq \frac{1}{x}. \quad (15)$$

Let $\psi_{\tau,x}(t) : [1, +\infty) \rightarrow \mathbb{C}$ be the unique continuous solution of the differential equation (2). By Lemma 3 and the Equation (15) we have,

$$\sup_{t \geq 1} |\psi_{\tau,x}(t)| < +\infty,$$

Thanks to the Main Proposition 4 we get $2x \leq 1$. □

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).