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# Non trivial zeros of the Zeta function using the differential equations

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## Abstract

In this paper, we investigate a relation between the differential equations and the non trivial zeros of the Zeta function.

**Keywords:** Zeta function, Riemann hypothesis, Differential equations.

**AMS subject classifications:** 00A05

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . Define the *non trivial zeros* of the function  $\zeta$  in the following sense

**Definition 1.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function*  $\zeta$  if

$$|\zeta(s)| = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

**Theorem 2.** [Riemann hypothesis] Let  $s \in \mathbb{C}$  be a non-trivial zero of the function  $\zeta$ . Then  $\Re(s) = \frac{1}{2}$ .

## 2 Main Proposition

For every  $r \in (0, 1)$  and  $\tau \in \mathbb{R}^*$  the Equation (1) implies,

$$\frac{\zeta(r + i\tau)}{r + i\tau} = -\frac{1}{1 - r - i\tau} - \int_1^{+\infty} u^{-i\tau-1-r} \{u\} du.$$

The aim is to studies the differential equation of solutions the functions

$$t \mapsto \psi_{\tau,r}(z, t) := t^{r+i\tau} \left[ z + \int_1^t u^{-i\tau-1-r} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 2, is to prove that  $\sup_{t \geq 1} |\psi_{\tau,r}(\frac{1}{1-r-i\tau}, t)| < +\infty$  implies  $r = \frac{1}{2}$ . In other words  $|\frac{\zeta(r+i\tau)}{r+i\tau}| = 0$  implies  $r = \frac{1}{2}$ .

For every  $\tau \in \mathbb{R}^*$  and  $r \in (0, 1)$  we consider the following differential equation

$$\begin{aligned} \frac{d}{dt} x &= (r + i\tau) \frac{x}{t} + \frac{1}{t} \{t\}, \\ t \in \mathbb{R}_+/\mathbb{N}, \quad x(1) &= \frac{1}{1 - r - i\tau}, \quad x : [1, +\infty) \rightarrow \mathbb{C}. \end{aligned} \quad (2)$$

**Lemma 3.** *For every  $\tau \in \mathbb{R}^*$  and  $r \in (0, 1)$  there exists a unique continuous solution  $\psi_{\tau,r}(t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2). Further,*

$$\psi_{\tau,r}(t) = t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du, \quad \forall t \geq 1.$$

*Proof.* Let be  $r \in (0, 1)$  and  $\tau \in \mathbb{R}^*$  fixed. Since  $\{u\} = u$  for every  $u \in (0, 1)$  then

$$t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du = \frac{t}{1 - r - i\tau}, \quad \forall t \in [0, 1].$$

Since  $0 \leq \{u\} \leq 1$  for every  $u \geq 1$  then the function

$$t \mapsto t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du,$$

is continuous and  $C^1$  on  $\mathbb{R}_+/\mathbb{N}$ . The Equation (2) is a non-homogeneous linear differential equation. The unique continuous solution  $\psi_{\tau,r}(t) : [1, +\infty) \rightarrow \mathbb{C}$  such that  $\psi_{\tau,r}(1) = \frac{1}{1-r-i\tau}$  is given by

$$\psi_{\tau,r}(t) = t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du, \quad \forall t \geq 0.$$

□

**Proposition 4.** *Let be  $\tau \in \mathbb{R}^*$  and  $r \in (0, 1)$ . Consider the continuous solution  $\psi_{\tau, r}(t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2). Suppose that  $\sup_{t \geq 1} |\psi_{\tau, r}(t)| < +\infty$ , then  $r = \frac{1}{2}$ .*

**Notation 5.** Denote

$$p(t) := \int_0^t \left( \{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 0,$$

where we recall that  $\{u\}$  is the fractional part of the real  $u$ .

**Lemma 6.** *The function  $p$  is a continuous 1-periodic function and  $\max_{t \in [0, 1]} |p(t)| \leq \frac{1}{2}$ .*

*Proof.* The function  $u \mapsto \{u\}$  is 1-periodic, then there exists a continuous 1-periodic function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t \frac{1}{2} + p(t), \quad \forall t \geq 0.$$

We have

$$\max_{t \in [0, 1]} |p(t)| \leq \max_{t \in [0, 1]} \left| \int_0^t \left( \{u\} - \frac{1}{2} \right) du \right| \leq \frac{1}{2}.$$

More precisely, the function  $p$  is a piecewise  $C^\infty$ , continuous on  $\mathbb{R}$  and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on  $\mathbb{R}_+$  to the function  $t \mapsto p(t)$ , where  $(a_k)_k \subset \mathbb{C}$  are the Fourier coefficients of the function  $p$ .

$$\int_0^t \left( \{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left( \int_0^u \left( \{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12}.$$

The function  $p$  satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left( \exp(ij2\pi t) - 1 \right), \quad \forall t \geq 0.$$

□

**Lemma 7.** *Let be  $\tau \in \mathbb{R}^*$  and  $r \in (0, 1)$ . Let  $\psi_{\tau, r}(t) : [1, +\infty) \rightarrow \mathbb{C}$  be the continuous solution of the differential equation (2). Suppose that  $\sup_{t \geq 1} |\psi_{\tau, r}(t)| < +\infty$ . Then*

$$\left| \psi_{\tau, r}(t) + \frac{1}{2} \frac{1}{r + i\tau} \right| < \frac{|1 + r + i\tau|}{1 + r} t^{-1}, \quad \forall t > 1.$$

*Proof.* Let be  $r \in (0, 1)$  and  $\tau \in \mathbb{R}^*$  fixed. By Lemma 3

$$\psi_{\tau, r}(t) = t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du, \quad \forall t > 0. \quad (3)$$

Suppose that

$$\sup_{t \geq 1} |\psi_{\tau, r}(t)| < +\infty.$$

Since

$$\sup_{t \geq 1} |\psi_{\tau, r}(t)| < +\infty \implies \left| \int_0^{+\infty} u^{-i\tau-1-r} \{u\} du \right| = 0,$$

then

$$\int_0^t u^{-i\tau-1-r} \{u\} du = - \int_t^{+\infty} u^{-i\tau-1-r} \{u\} du, \quad \forall t > 1.$$

Equation (3) can be written as

$$\psi_{\tau, r}(t) = -t^{r+i\tau} \int_t^{+\infty} u^{-i\tau-1-r} \{u\} du, \quad \forall t > 1. \quad (4)$$

Consider the function  $p$  given in the Notation 5. By Lemma 6, the function  $p$  is 1-periodic, then it is bounded. Use the integration part formula in Equation (4),

$$\begin{aligned} \psi_{\tau,r}(t) &= -\frac{1}{2} \frac{1}{r+i\tau} \\ &\quad - (i\tau + 1 + r)t^{i\tau+r} \int_t^{+\infty} u^{-i\tau-2-r} (p(u) - p(t)) du, \end{aligned}$$

Since

$$|p(u) - p(t)| < 1, \quad \forall u \geq t > 1,$$

we obtain

$$\begin{aligned} \left| \psi_{\tau,r}(t) + \frac{1}{2} \frac{1}{r+i\tau} \right| &< |1+r+i\tau| t^r \int_t^{+\infty} u^{-2-r} \\ &= \frac{|1+r+i\tau|}{1+r} t^{-1}. \end{aligned}$$

□

*Proof of the Proposition 4.* Let be  $r \in (0, 1)$  and  $\tau \in \mathbb{R}^*$ . For every  $n \geq 1$ , use the change of variable

$$\psi_{\tau,r,n}(t) := (r+i\tau)\psi_{\tau,r}(\exp(t)) - \frac{r+i\tau}{1-r-i\tau} \exp(t) - n, \quad \forall t \in [\ln(n), \ln(n+1)].$$

Use the fact  $\{u\} = u - n$  for every  $u \in [n, n+1)$ . The Equation (2) implies

$$\frac{d}{dt} \psi_{\tau,r,n}(t) = (r+i\tau)\psi_{\tau,r,n}(t), \quad \forall t \in [\ln(n), \ln(n+1)].$$

Integrate,

$$\psi_{\tau,r,n}(\ln(n+1)) = \left(\frac{n+1}{n}\right)^{r+i\tau} \psi_{\tau,r,n}(\ln(n)), \quad \forall n \geq 1.$$

Use the fact

$$\prod_{k=1}^n \left(\frac{k+1}{k}\right)^{r+i\tau} = (n+1)^{r+i\tau}, \quad \forall n \geq 1.$$

We obtain

$$\prod_{k=1}^n \psi_{\tau,r,k}(\ln(k+1)) = (n+1)^{r+i\tau} \prod_{k=1}^n \psi_{\tau,r,k}(\ln(k)), \quad \forall n \geq 1.$$

Then

$$\prod_{k=1}^n |\psi_{\tau,r,k}(\ln(k+1))|^2 = (n+1)^{2r} \prod_{k=1}^n |\psi_{\tau,r,k}(\ln(k))|^2, \quad \forall n \geq 1. \quad (5)$$

By definition of  $\psi_{\tau,r,n}$ , for every  $n \geq 1$  we have

$$\begin{aligned} \psi_{\tau,r,k}(\ln(k+1)) &= (r+i\tau)\psi_{\tau,r}(k+1) - \frac{r+i\tau}{1-r-i\tau}(k+1) - k \\ &= (r+i\tau)\psi_{\tau,r}(k+1) - \frac{r+i\tau}{1-r-i\tau}(k+1) - k \\ &= \psi_{\tau,r,k+1}(\ln(k+1)) + 1, \quad \forall k \geq 1. \end{aligned}$$

The Equation (5), implies

$$\prod_{k=1}^{n-1} |\psi_{\tau,r,k+1}(\ln(k+1)) + 1|^2 = n^{2r} \prod_{k=1}^{n-1} |\psi_{\tau,r,k}(\ln(k))|^2, \quad \forall n \geq 2,$$

in other words

$$\begin{aligned} \prod_{k=2}^n |\psi_{\tau,r,k}(\ln(k)) + 1|^2 & \\ &= n^{2r} |\psi_{\tau,r,1}(0)|^2 \prod_{k=2}^{n-1} |\psi_{\tau,r,k}(\ln(k))|^2, \quad \forall n \geq 3. \end{aligned} \quad (6)$$

Since

$$\psi_{\tau,r}(1) = \frac{1}{1-r-i\tau},$$

and by the definition of  $\psi_{\tau,r,n}$  we get

$$\psi_{\tau,r,1}(0) = -1$$

The Equation (6) implies

$$\prod_{k=2}^n |\psi_{\tau,r,k}(\ln(k)) + 1|^2 = n^{2r} \prod_{k=2}^{n-1} |\psi_{\tau,r,k}(\ln(k))|^2, \quad \forall n \geq 3,$$

equivalent to

$$\prod_{k=2}^n k^{-2r} |\psi_{\tau,r,k}(\ln(k)) + 1|^2 = \prod_{k=2}^{n-1} k^{-2r} |\psi_{\tau,r,k}(\ln(k))|^2, \quad \forall n \geq 3. \quad (7)$$

Then for every  $m > n \geq 3$  we get

$$\prod_{k=n+1}^m k^{-2r} |\psi_{\tau,r,k}(\ln(k)) + 1|^2 = \prod_{k=n}^{m-1} k^{-2r} |\psi_{\tau,r,k}(\ln(k))|^2.$$

For every  $m > n \geq 3$  we have obtained

$$\prod_{k=n}^{m-1} \left( \frac{k}{k+1} \right)^{2r} |\psi_{\tau,r,k+1}(\ln(k+1)) + 1|^2 = \prod_{k=n}^{m-1} |\psi_{\tau,r,k}(\ln(k))|^2. \quad (8)$$

Suppose that  $\sup_{t \geq 1} |\psi_{\tau,r}(t)| < +\infty$ . By The Lemma 7 we have

$$\left| (r + i\tau)\psi_{\tau,r}(t) + \frac{1}{2} \right| < \frac{|1 + r + i\tau|^2}{1 + r} t^{-1}, \quad \forall t > 1.$$

By the definition of  $\psi_{\tau,r,k}$ , we obtain

$$\left| \psi_{\tau,r,k}(\ln(k)) + \frac{1}{2} + \frac{k}{1 - r - i\tau} \right| < \frac{|1 + r + i\tau|^2}{1 + r} \frac{1}{k}, \quad \forall k \geq 1,$$

Which can be written as

$$\psi_{\tau,r,k}(\ln(k)) = -\frac{1}{2} - \frac{k}{1 - r - i\tau} + \frac{1}{k}\theta_{\tau,r,k}, \quad \forall k \geq 1,$$

where

$$\theta_{\tau,r,k} := k\psi_{\tau,r,k}(\ln(k)) + \frac{k}{2} + \frac{k^2}{1 - r - i\tau}, \quad \forall k \geq 1,$$

and satisfies

$$|\theta_{\tau,r,k}| < \frac{|1 + r + i\tau|^2}{1 + r}, \quad \forall k \geq 1.$$

For every  $m > n \geq 3$ , the Equation (8) implies

$$\begin{aligned} \prod_{k=n}^{m-1} \left( \frac{k}{k+1} \right)^{2r} \left| \frac{k+1}{1-r-i\tau} - \frac{1}{2} - \frac{\theta_{\tau,r,k+1}}{k+1} \right|^2 \\ = \prod_{k=n}^{m-1} \left| \frac{k}{1-r-i\tau} + \frac{1}{2} - \frac{\theta_{\tau,r,k}}{k} \right|^2. \end{aligned}$$

Use the fact  $m > n \approx +\infty$  implies

$$\begin{aligned} \frac{\prod_{k=n}^{m-1} \left| k+1 - \frac{1}{2}(1-r-i\tau) - \frac{1}{k+1}\theta_{\tau,r,k+1}(1-r-i\tau) \right|^2}{\prod_{k=n}^{m-1} u_{r,k}} \approx 1, \\ \frac{\prod_{k=n}^{m-1} \left| k + \frac{1}{2}(1-r-i\tau) - \frac{1}{k}\theta_{\tau,r,k}(1-r-i\tau) \right|^2}{\prod_{k=n}^{m-1} v_{r,k}} \approx 1, \end{aligned}$$

where

$$\begin{aligned} u_{r,k} &:= (k+1)^2 - (1-r)(k+1), \\ v_{r,k} &:= k^2 + (1-r)k, \quad \forall k \geq 3. \end{aligned}$$

We get

$$m > n \approx +\infty \implies \prod_{k=n}^{m-1} \left( \frac{k}{k+1} \right)^{2r} u_{r,k} \approx \prod_{k=n}^{m-1} v_{r,k}$$



Which is satisfied if

$$\sup_{m>n>>1} \left| \prod_{k=n}^{m-1} \left( \frac{k}{k+1} \right)^{2r} u_{r,k} - \prod_{k=n}^{m-1} v_{r,k} \right| = 0.$$

We deduce that  $r = \frac{1}{2}$ . □

**Remark 8.** In the previous Proof, by the notation

$$S_{\tau,r}(n) = \prod_{k=2}^n \frac{\psi_{\tau,r,k}(\ln(k))}{\psi_{\tau,r,k}(\ln(k)) + 1}, \quad n \geq 2.$$

remark that

$$\psi_{\tau,r,n+1}(\ln(n+1)) + 1 = \psi_{\tau,r,n}(\ln(n+1)) = -(n+1)^{r+i\tau} S_{\tau,r}(n).$$

We get

$$S_{\tau,r}(n+1) = \frac{(n+1)^{r+i\tau} S_{\tau,r}(n) + 1}{(n+1)^{r+i\tau} S_{\tau,r}(n)} S_{\tau,r}(n).$$

In other words,

$$S_{\tau,r}(n+1) = \frac{1}{(n+1)^{r+i\tau}} + S_{\tau,r}(n),$$

we obtain the Riemann series

$$S_{\tau,r}(n) = S_{\tau,r}(2) - 1 - \frac{1}{2^{r+i\tau}} + \sum_{k=1}^n \frac{1}{k^{r+i\tau}}.$$

### 3 Proof of the Theorem 2

*Proof of the Theorem 2.* Let be  $x \in (0, 1)$  and  $\tau \in \mathbb{R}^*$ . Suppose that  $|\zeta(x+i\tau)| = 0$ . By the Equation (1) we have,

$$\frac{\zeta(x+i\tau)}{x+i\tau} = - \int_0^{+\infty} u^{-i\tau-1-x} \{u\} du.$$

Then

$$\left| \int_0^{+\infty} u^{-i\tau-1-x} \{u\} du \right| = 0.$$

Implies

$$\int_0^t u^{-i\tau-1-x} \{u\} du = - \int_t^{+\infty} u^{-i\tau-1-x} \{u\} du, \quad \forall t \geq 1.$$

$$\begin{aligned} \sup_{t \geq 1} \left| t^{x+i\tau} \int_0^t u^{-i\tau-1-x} \{u\} du \right| &= \sup_{t \geq 1} \left| t^{x+i\tau} \int_t^{+\infty} u^{-i\tau-1-x} \{u\} du \right| \\ &\leq \sup_{t \geq 1} \left[ t^x \int_t^{+\infty} u^{-1-x} du \right] \leq \frac{1}{x}. \end{aligned} \quad (9)$$

Let  $\psi_{\tau,x}(t) : [1, +\infty) \rightarrow \mathbb{C}$  be the unique continuous solution of the differential equation (2). By Lemma 3 and the Equation (9) we have,

$$\sup_{t \geq 1} |\psi_{\tau,x}(t)| < +\infty,$$

Thanks to the Main Proposition 4 we get  $x = \frac{1}{2}$ . □

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).