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Non trivial zeros of the Zeta function using the differential equations

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Abstract

In this paper, we investigate a relation between the differential equations and the non trivial zeros of the Zeta function.

Keywords: Zeta function, Riemann hypothesis, differential equations.

AMS subject classifications: 00A05

1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{t\}$ is the fractional part of the real t . Define the *non trivial zeros* of the function ζ in the following sense

Definition 1. Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function* ζ if

$$|\zeta(s)| = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

Theorem 2. [Riemann hypothesis] Let $s \in \mathbb{C}$ be a non-trivial zero of the function ζ . Then $\Re(s) = \frac{1}{2}$.

2 Main Proposition

For every $r \in (0, 1)$ and $\tau \in \mathbb{R}^*$ the Equation (1) implies,

$$\frac{\zeta(r + i\tau)}{r + i\tau} = -\frac{1}{1 - r - i\tau} - \int_1^{+\infty} u^{-i\tau-1-r} \{u\} du.$$

The aim is to studies the differential equation of solution the functions

$$t \mapsto \psi_{\tau,r}(z, t) := t^{r+i\tau} \left[z + \int_1^t u^{-i\tau-1-r} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 2, is to prove that $\sup_{t \geq 1} |\psi_{\tau,r}(\frac{1}{1-r-i\tau}, t)| < +\infty$ implies $r = \frac{1}{2}$. In other words $|\frac{\zeta(r+i\tau)}{r+i\tau}| = 0$ implies $r = \frac{1}{2}$.

For every $\tau \in \mathbb{R}^*$ and $r \in (0, 1)$ we consider the following differential equation

$$\begin{aligned} \frac{d}{dt} x &= (r + i\tau) \frac{x}{t} + \frac{1}{t} \{t\}, & (2) \\ t \in \mathbb{R}_+/\mathbb{N}, \quad x(1) &= \frac{1}{1 - r - i\tau}, \quad x : [1, +\infty) \rightarrow \mathbb{C}. \end{aligned}$$

Lemma 3. *For every $\tau \in \mathbb{R}^*$ and $r \in (0, 1)$ there exists a unique continuous solution $\psi_{\tau,r}(t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Further,*

$$\psi_{\tau,r}(t) = t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du, \quad \forall t \geq 1.$$

Proof. Let be $r \in (0, 1)$ and $\tau \in \mathbb{R}^*$ fixed. Since $\{u\} = u$ for every $u \in (0, 1)$ then

$$t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du = \frac{t}{1 - r - i\tau}, \quad \forall t \in [0, 1].$$

Since $0 \leq \{u\} \leq 1$ for every $u \geq 1$ then the function

$$t \mapsto t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du,$$

is continuous and C^1 on \mathbb{R}_+/\mathbb{N} . The Equation (2) is a non-homogeneous linear differential equation. The unique continuous solution $\psi_{\tau,r}(t) : [1, +\infty) \rightarrow \mathbb{C}$ such that $\psi_{\tau,r}(1) = \frac{1}{1-r-i\tau}$ is given by

$$\psi_{\tau,r}(t) = t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du, \quad \forall t \geq 0.$$

□

Proposition 4. *Let be $\tau \in \mathbb{R}^*$ and $r \in (0, 1)$. Consider the continuous solution $\psi_{\tau,r}(t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Suppose that $\sup_{t \geq 1} |\psi_{\tau,r}(t)| < +\infty$, then $r = \frac{1}{2}$.*

Notation 5. Denote

$$p(t) := \int_0^t \left(\{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 0,$$

where we recall that $\{u\}$ is the fractional part of the real u .

Lemma 6. *The function p is a continuous 1-periodic function and $\max_{t \in [0,1]} |p(t)| \leq \frac{1}{2}$.*

Proof. The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t \frac{1}{2} + p(t), \quad \forall t \geq 0.$$

We have

$$\max_{t \in [0,1]} |p(t)| \leq \max_{t \in [0,1]} \left| \int_0^t \left(\{u\} - \frac{1}{2} \right) du \right| \leq \frac{1}{2}.$$

□

Lemma 7. *Let be $\tau \in \mathbb{R}^*$ and $r \in (0, 1)$. Let $\psi_{\tau,r}(t) : [1, +\infty) \rightarrow \mathbb{C}$ be the continuous solution of the differential equation (2). Suppose that $\sup_{t \geq 1} |\psi_{\tau,r}(t)| < +\infty$. Then*

$$\left| \psi_{\tau,r}(t) + \frac{1}{2} \frac{1}{r + i\tau} \right| < \frac{|1 + r + i\tau|}{1 + r} t^{-1}, \quad \forall t > 1.$$

Proof. Let be $r \in (0, 1)$ and $\tau \in \mathbb{R}^*$ fixed. By Lemma 3

$$\psi_{\tau,r}(t) = t^{r+i\tau} \int_0^t u^{-i\tau-1-r} \{u\} du, \quad \forall t > 0. \quad (3)$$

Suppose that

$$\sup_{t \geq 1} |\psi_{\tau,r}(t)| < +\infty.$$

Since

$$\sup_{t \geq 1} |\psi_{\tau,r}(t)| < +\infty \implies \left| \int_0^{+\infty} u^{-i\tau-1-r} \{u\} du \right| = 0,$$

then

$$\int_0^t u^{-i\tau-1-r} \{u\} du = - \int_t^{+\infty} u^{-i\tau-1-r} \{u\} du, \quad \forall t > 1.$$

Equation (3) can be written as

$$\psi_{\tau,r}(t) = -t^{r+i\tau} \int_t^{+\infty} u^{-i\tau-1-r} \{u\} du, \quad \forall t > 1. \quad (4)$$

Consider the function p given in the Notation 5. By Lemma 6, the function p is 1-periodic, then it is bounded. Use the integration part formula in Equation (4),

$$\begin{aligned} \psi_{\tau,r}(t) &= -\frac{1}{2} \frac{1}{r+i\tau} \\ &\quad + (i\tau+1+r)t^{i\tau+r} \int_t^{+\infty} u^{-i\tau-2-r} (p(u) - p(t)) du, \end{aligned}$$

Since

$$|p(u) - p(t)| < 1, \quad \forall u \geq t > 1,$$

we obtain

$$\begin{aligned} \left| \psi_{\tau,r}(t) + \frac{1}{2} \frac{1}{r+i\tau} \right| &< |1+r+i\tau| t^r \int_t^{+\infty} u^{-2-r} \\ &= \frac{|1+r+i\tau|}{1+r} t^{-1}. \end{aligned}$$

□

Proof of the Proposition 4. Let be $r \in (0, 1)$ and $\tau \in \mathbb{R}^*$. For every $n \geq 1$, use the change of variable

$$\psi_{\tau,r,n}(t) := (r+i\tau)\psi_{\tau,r}(\exp(t)) - \frac{r+i\tau}{1-r-i\tau} \exp(t) - n, \quad \forall t \in [\ln(n), \ln(n+1)).$$

Use the fact $\{u\} = u - n$ for every $u \in [n, n + 1)$. The Equation (2) implies

$$\frac{d}{dt}\psi_{\tau,r,n}(t) = (r + i\tau)\psi_{\tau,r,n}(t), \quad \forall t \in [\ln(n), \ln(n + 1)).$$

Integrate,

$$\psi_{\tau,r,n}(\ln(n + 1)) = \left(\frac{n + 1}{n}\right)^{r+i\tau} \psi_{\tau,r,n}(\ln(n)), \quad \forall n \geq 1.$$

Use the fact

$$\prod_{k=1}^n \left(\frac{k + 1}{k}\right)^{r+i\tau} = (n + 1)^{r+i\tau}, \quad \forall n \geq 1.$$

We obtain

$$\prod_{k=1}^n \psi_{\tau,r,k}(\ln(k + 1)) = (n + 1)^{r+i\tau} \prod_{k=1}^n \psi_{\tau,r,k}(\ln(k)), \quad \forall n \geq 1.$$

Then

$$\prod_{k=1}^n |\psi_{\tau,r,k}(\ln(k + 1))|^2 = (n + 1)^{2r} \prod_{k=1}^n |\psi_{\tau,r,k}(\ln(k))|^2, \quad \forall n \geq 1. \quad (5)$$

By definition of $\psi_{\tau,r,n}$, for every $n \geq 1$ we have

$$\begin{aligned} \psi_{\tau,r,k}(\ln(k + 1)) &= (r + i\tau)\psi_{\tau,r}(k + 1) - \frac{r + i\tau}{1 - r - i\tau}(k + 1) - k \\ &= (r + i\tau)\psi_{\tau,r}(k + 1) - \frac{r + i\tau}{1 - r - i\tau}(k + 1) - k \\ &= \psi_{\tau,r,k+1}(\ln(k + 1)) + 1, \quad \forall k \geq 1. \end{aligned}$$

The Equation (5), implies

$$\prod_{k=1}^{n-1} |\psi_{\tau,r,k+1}(\ln(k + 1)) + 1|^2 = n^{2r} \prod_{k=1}^{n-1} |\psi_{\tau,r,k}(\ln(k))|^2, \quad \forall n \geq 2,$$

in other words

$$\begin{aligned} \prod_{k=3}^{n+1} |\psi_{\tau,r,k}(\ln(k)) + 1|^2 \\ = n^{2r} |\psi_{\tau,r,1}(0)|^2 \prod_{k=2}^{n-1} |\psi_{\tau,r,k}(\ln(k))|^2, \quad \forall n \geq 3. \end{aligned} \quad (6)$$

Since

$$\psi_{\tau,r}(1) = \frac{1}{1 - r - i\tau},$$

and by the definition of $\psi_{\tau,r,n}$ we get

$$\psi_{\tau,r,1}(0) = -1$$

The Equation (6) implies

$$\prod_{k=2}^n |\psi_{\tau,r,k}(\ln(k)) + 1|^2 = n^{2r} \prod_{k=2}^{n-1} |\psi_{\tau,r,k}(\ln(k))|^2, \quad \forall n \geq 3.$$

In other words

$$\left(\frac{n}{n+1}\right)^{2r} \left| \frac{\psi_{\tau,r,n+1}(\ln(n+1)) + 1}{\psi_{\tau,r,n}(\ln(n))} \right|^2 = 1, \quad \forall n \geq 3. \quad (7)$$

Suppose that $\sup_{t \geq 1} |\psi_{\tau,r}(t)| < +\infty$. By The Lemma 7 we have

$$\left| (r + i\tau)\psi_{\tau,r}(t) + \frac{1}{2} \right| < \frac{|1 + r + i\tau|^2}{1 + r} t^{-1}, \quad \forall t > 1.$$

By the definition of $\psi_{\tau,r,k}$, we obtain

$$\left| \psi_{\tau,r,k}(\ln(k)) + \frac{1}{2} + \frac{k}{1 - r - i\tau} \right| < \frac{|1 + r + i\tau|^2}{1 + r} \frac{1}{k}, \quad \forall k \geq 1,$$

Which can be written as

$$\psi_{\tau,r,k}(\ln(k)) = -\frac{1}{2} - \frac{k}{1 - r - i\tau} + \theta_{\tau,r,k} \frac{1}{k}, \quad \forall k \geq 1,$$

where

$$\theta_{\tau,r,k} := k\psi_{\tau,r,k}(\ln(k)) + \frac{k}{2} + \frac{k^2}{1 - r - i\tau}, \quad \forall k \geq 1,$$

and satisfies

$$|\theta_{\tau,r,k}| < \frac{|1 + r + i\tau|^2}{1 + r}, \quad \forall k \geq 1.$$

The Equation (7) implies

$$\left(\frac{n}{n+1}\right)^{2r} \frac{\left| \frac{n+1}{1-r-i\tau} - \frac{1}{2} - \theta_{\tau,r,n+1} \frac{1}{n+1} \right|^2}{\left| \frac{n}{1-r-i\tau} + \frac{1}{2} - \theta_{\tau,r,n} \frac{1}{n} \right|^2} = 1, \quad \forall n \geq 3,$$

then $r = \frac{1}{2}$. □

Remark 8. In the previous Proof, by the notation

$$S_{\tau,r}(n) = \prod_{k=2}^n \frac{\psi_{\tau,r,k}(\ln(k))}{\psi_{\tau,r,k}(\ln(k)) + 1}, \quad n \geq 2.$$

remark that

$$\psi_{\tau,r,n+1}(\ln(n+1)) + 1 = \psi_{\tau,r,n}(\ln(n+1)) = -(n+1)^{r+i\tau} S_{\tau,r}(n).$$

We get

$$S_{\tau,r}(n+1) = \frac{(n+1)^{r+i\tau} S_{\tau,r}(n) + 1}{(n+1)^{r+i\tau} S_{\tau,r}(n)} S_{\tau,r}(n).$$

In other words,

$$S_{\tau,r}(n+1) = -\frac{1}{(n+1)^{r+i\tau}} + S_{\tau,r}(n),$$

we obtain the Riemann series

$$S_{\tau,r}(n) = S_{\tau,r}(2) - 1 - \frac{1}{2^{r+i\tau}} + \sum_{k=1}^n \frac{1}{k^{r+i\tau}}.$$

3 Proof of the Theorem 2

Proof of the Theorem 2. Let be $x \in (0, 1)$ and $\tau \in \mathbb{R}^*$. Suppose that $|\zeta(x+i\tau)| = 0$. By the Equation (1) we have,

$$\frac{\zeta(x+i\tau)}{x+i\tau} = -\int_0^{+\infty} u^{-i\tau-1-x}\{u\}du.$$

Then

$$\left| \int_0^{+\infty} u^{-i\tau-1-x}\{u\}du \right| = 0.$$

Implies

$$\int_0^t u^{-i\tau-1-x}\{u\}du = -\int_t^{+\infty} u^{-i\tau-1-x}\{u\}du, \quad \forall t \geq 1.$$

$$\begin{aligned} \sup_{t \geq 1} \left| t^{x+i\tau} \int_0^t u^{-i\tau-1-x}\{u\}du \right| &= \sup_{t \geq 1} \left| t^{x+i\tau} \int_t^{+\infty} u^{-i\tau-1-x}\{u\}du \right| \\ &\leq \sup_{t \geq 1} \left[t^x \int_t^{+\infty} u^{-1-x} du \right] \leq \frac{1}{x}. \end{aligned} \quad (8)$$

Let $\psi_{\tau,x}(t) : [1, +\infty) \rightarrow \mathbb{C}$ be the unique continuous solution of the differential equation (2). By Lemma 3 and the Equation (8) we have,

$$\sup_{t \geq 1} |\psi_{\tau,x}(t)| < +\infty,$$

Thanks to the Main Proposition 4 we get $x = \frac{1}{2}$. □

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).