



**HAL**  
open science

## Non trivial zeros of the Zeta function

W Oukil

► **To cite this version:**

| W Oukil. Non trivial zeros of the Zeta function. 2023. hal-03849860v3

**HAL Id: hal-03849860**

**<https://hal.science/hal-03849860v3>**

Preprint submitted on 2 Feb 2023 (v3), last revised 8 Apr 2024 (v17)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Non trivial zeros of the Zeta function

W. Oukil

Faculty of Mathematics.

University of Science and Technology Houari Boumediene.

BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

February 2, 2023

## Abstract

We prove that  $|\zeta(r+i\tau)| \neq 0$  for every  $r \in (\frac{1}{2}, 1)$  and  $\tau \in \mathbb{R}^*$ , where  $\zeta$  is the Riemann Zeta function.

**Keywords:** Zeta function, non trivial zeros, Riemann hypothesis.

**AMS subject classifications:** 00A05

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . We prove the following Theorem.

**Theorem 1.** *Consider the function  $\zeta$  defined by the Equation (1). For every  $\tau \in \mathbb{R}^*$  and  $r \in (\frac{1}{2}, 1)$  we have*

$$|\zeta(r+i\tau)| \neq 0.$$

Thanks to the Riemann functional equation we deduce that any *non trivial zero* of the Zeta function has a real part equal to  $\frac{1}{2}$ , where the *non trivial zeros* of the function  $\zeta$  are defined in the following sense

**Definition 2.** Consider the function  $\zeta$  defined by the Equation (1). Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function  $\zeta$*  if

$$|\zeta(s)| = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

## 2 Main Proposition

**Notation 3.** In order to simplify the notation, for every  $r \in [\frac{1}{2}, 1)$  denote

$$f_r(u) := \{u\}u^{-r}, \quad u > 0.$$

By the Equation (1), remark that

$$\lim_{t \rightarrow +\infty} \left| \int_0^t u^{-i\tau-1} f_r(u) du \right| := \left| \frac{\zeta(r+i\tau)}{r+i\tau} \right|.$$

We have the following Main proposition,

**Proposition 4.** *Let be  $\tau \in \mathbb{R}^*$  and  $r \in [\frac{1}{2}, 1)$ . Suppose that  $|\zeta(r+i\tau)| = 0$ , then*

$$\lim_{n \rightarrow +\infty} (2-r)(1-r)^{2n+1} \int_0^{+\infty} \sum_{k=1}^{2n} \frac{(\ln(u))^k}{k!} u^{-1} f_r(u) du = 1.$$

**Notation 5.** Denote

$$p(t) := \int_0^t \left( \{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 0,$$

where we recall that  $\{u\}$  is the fractional part of the real  $u$ . Denote

$$\omega := 2 \max_{t \in [0,1]} |p(t)|.$$

**Lemma 6.** *The function  $p$  is a continuous 1-periodic function.*

*Proof.* The function  $u \mapsto \{u\}$  is 1-periodic, then there exists a continuous 1-periodic function  $p: \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t \frac{1}{2} + p(t), \quad \forall t \geq 0.$$

□

**Notation 7.** For every  $r \in [\frac{1}{2}, 1)$  and  $\tau \in \mathbb{R}^*$  denote

$$\Psi_{\tau,r}(k) = \int_1^{+\infty} \frac{(\ln(u))^k}{k!} u^{i\tau-1} \int_u^{+\infty} v^{-i\tau-1} f_r(v) dv du, \quad \forall k \geq 0.$$

**Lemma 8.** Let be  $\tau \in \mathbb{R}^*$  and  $r \in [\frac{1}{2}, 1)$ . Then

$$|\Psi_{\tau,r}(k) - \frac{1}{2} \frac{1}{r+i\tau} \frac{1}{r^{k+1}}| \leq \frac{\omega(|\tau|+2)}{(1+r)^{k+1}}, \quad \forall k \geq 0,$$

where  $\omega$  is given by the Notation 5

*Proof.* Let be  $\tau \in \mathbb{R}^*$  and  $r \in [\frac{1}{2}, 1)$ . Consider the function  $p$  given in the Notation 5. By Lemma 6, the function  $p$  is 1-periodic, then it is bounded. Use the integration part formula

$$\begin{aligned} & \left| \int_u^{+\infty} v^{-i\tau-1} \left( f_r(v) - \frac{1}{2} v^{-r} \right) dv \right| \\ &= \left| \int_u^{+\infty} v^{-i\tau-1} \left( \{v\} - \frac{1}{2} \right) v^{-r} dv \right| \\ &= \left| (i\tau + 1 + r) \int_u^{+\infty} v^{-i\tau-2-r} (p(v) - p(u)) dv \right|. \end{aligned}$$

Since

$$|p(v) - p(u)| < \omega, \quad \forall u \geq 0, \quad \forall v \geq 0,$$

where  $\omega$  is given by the Notation 5, we get

$$\begin{aligned} \left| \int_u^{+\infty} v^{-i\tau-1} \left( f_r(v) - \frac{1}{2} v^{-r} \right) dv \right| &\leq \omega(|\tau|+2) \int_u^{+\infty} v^{-2-r} dv \\ &\leq \omega(|\tau|+2) u^{-1-r}. \end{aligned}$$

Then

$$\begin{aligned} & \left| \int_1^{+\infty} \frac{(\ln(u))^k}{k!} u^{i\tau-1} \int_u^{+\infty} v^{-i\tau-1} \left( f_r(v) - \frac{1}{2} v^{-r} \right) dv du \right| \\ &\leq \omega(|\tau|+2) \int_1^{+\infty} \frac{(\ln(u))^k}{k!} u^{-2-r} du \\ &= \frac{\omega(|\tau|+2)}{(1+r)^{k+1}}, \quad \forall k \geq 0. \end{aligned}$$

Since

$$u^{-i\tau} \int_u^{+\infty} v^{-i\tau-1-r} dv = \frac{1}{r+i\tau} u^{-r}, \quad \forall u > 0.$$

Then

$$|\Psi_{\tau,r}(k) - \frac{1}{2} \frac{1}{r+i\tau} \frac{1}{r^{k+1}}| \leq \frac{\omega(|\tau|+2)}{(1+r)^{k+1}}, \quad \forall k \geq 0.$$

□

*Proof of the Proposition 4.* Let be  $\tau \in \mathbb{R}^*$  and  $r \in [\frac{1}{2}, 1)$ . Suppose that

$$|\zeta(r+i\tau)| = 0.$$

By the Equation (1) we have

$$\left| \int_0^{+\infty} u^{-i\tau-1} f_r(u) du \right| = \left| \frac{\zeta(r+i\tau)}{r+i\tau} \right| = 0.$$

Implies

$$\int_0^t u^{-i\tau-1} f_r(u) du = - \int_t^{+\infty} u^{-i\tau-1} f_r(u) du, \quad \forall t > 0. \quad (2)$$

Since  $f_r(u) = u^{1-r}$  for every  $u \in (0, 1)$  then for every  $k \geq 0$  we have

$$\begin{aligned} & \int_0^1 \frac{(\ln(u))^k}{k!} u^{i\tau-1} \int_u^{+\infty} v^{-i\tau-1} f_r(v) dv du \\ &= - \int_0^1 \frac{(\ln(u))^k}{k!} u^{i\tau-1} \int_0^u v^{-i\tau-1} f_r(v) dv du \\ &= - \frac{1}{1-r-i\tau} \int_0^1 \frac{(\ln(u))^k}{k!} u^{-r} du \\ &= - \frac{1}{1-r-i\tau} \frac{1}{(r-1)^{k+1}}, \quad \forall k \geq 0. \end{aligned}$$

Implies

$$\begin{aligned} & \lim_{n \rightarrow +\infty} (r-1)^{n+1} \int_0^1 \sum_{k=0}^n \frac{(\ln(u))^k}{k!} u^{i\tau-1} \int_u^{+\infty} v^{-i\tau-1} f_r(v) dv du \\ &= - \frac{1}{1-r-i\tau} \frac{1}{2-r}. \end{aligned} \quad (3)$$

Since  $r \in (\frac{1}{2}, 1)$ , by the Lemma 8, we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} (r-1)^{n+1} \sum_{k=0}^n \Psi_{\tau,r}(k) &= \frac{1}{2r} \frac{1}{r+i\tau} \lim_{n \rightarrow +\infty} (r-1)^{n+1} \sum_{k=0}^n \frac{1}{r^k} \\ &= \frac{1}{2(1-r)} \frac{1}{r+i\tau} \lim_{n \rightarrow +\infty} \left(\frac{r-1}{r}\right)^{n+1} = 0. \end{aligned} \quad (4)$$

and for  $r = \frac{1}{2}$  we get

$$\lim_{n \rightarrow +\infty} \frac{1}{2^{n+1}} \sum_{k=0}^n \Psi_{\tau,r}(k) = \frac{1}{\frac{1}{2} + i\tau}. \quad (5)$$

By definition of  $\Psi_{\tau,r}$  in the Notation 7, the addition of the Equations (3) and (4) gives

$$\lim_{n \rightarrow +\infty} (r-1)^{n+1} \Phi_{\tau,r}(n) = -\frac{1}{1-r-i\tau} \frac{1}{2-r}, \quad (6)$$

where

$$\Phi_{\tau,r}(n) := \int_0^{+\infty} \sum_{k=0}^n \frac{(\ln(u))^k}{k!} u^{i\tau-1} \int_u^{+\infty} v^{-i\tau-1} f_r(v) dv du.$$

the addition of the Equations (3) and (5) gives

$$\lim_{n \rightarrow +\infty} \frac{1}{2^{2n}} \Phi_{\tau, \frac{1}{2}}(2n-1) = -\frac{1}{\frac{1}{2} - i\tau} + \frac{2}{3} \frac{1}{\frac{1}{2} + i\tau}. \quad (7)$$

Use the Equation (2) and the integration part formula to obtain

$$\begin{aligned} \Phi_{\tau,r}(n) &= -i\tau \int_0^{+\infty} \sum_{k=1}^{n+1} \frac{(\ln(u))^k}{k!} u^{i\tau-1} \int_u^{+\infty} v^{-i\tau-1} f_r(v) dv du \\ &\quad + \int_0^{+\infty} \sum_{k=1}^{n+1} \frac{(\ln(u))^k}{k!} u^{-1} f_r(u) du. \end{aligned}$$

We get

$$\begin{aligned} \lim_{n \rightarrow +\infty} (r-1)^{n+1} \Phi_{\tau,r}(n) &= -\frac{i\tau}{r-1} \lim_{n \rightarrow +\infty} (r-1)^{n+2} \Phi_{\tau,r}(n+1) \\ &\quad + \lim_{n \rightarrow +\infty} (r-1)^{n+1} \int_0^{+\infty} \sum_{k=1}^{n+1} \frac{(\ln(u))^k}{k!} u^{-1} f_r(u) du. \end{aligned}$$

By the Equations (6) and (7), we obtain

$$\lim_{n \rightarrow +\infty} (2-r)(1-r)^{2n+1} \int_0^{+\infty} \sum_{k=1}^{2n} \frac{(\ln(u))^k}{k!} u^{-1} f_r(u) du = 1.$$

□

### 3 Proof of the Theorem 1

*Proof of the Theorem 1.* In order to simplify the notation, for all  $n \geq 1$  and  $x \in (\frac{1}{2}, 1)$  we denote

$$g(n, x) := (2-x)(1-x)^{2n+1} \int_0^{+\infty} \sum_{k=1}^{2n} \frac{(\ln(u))^k}{k!} u^{-1} f_x(u) du.$$

Since there exists at least one non trivial zero of the Zeta function of real part equal to  $\frac{1}{2}$  [1], in other words there exists  $\tau_* \in \mathbb{R}^*$  such that  $|\zeta(\frac{1}{2} + i\tau_*)| = 0$ . Thanks to the Lemma 4 we get  $\lim_{n \rightarrow +\infty} g(n, \frac{1}{2}) = 1$ . Since for every  $x \in (\frac{1}{2}, 1)$  we have

$$\lim_{n \rightarrow +\infty} \frac{d}{dx} g(n, x) < 0,$$

then

$$\lim_{n \rightarrow +\infty} g(n, x) < 1, \quad \forall x \in (\frac{1}{2}, 1).$$

The Lemma 4 implies

$$|\zeta(x + i\tau)| \neq 0, \quad \forall x \in (\frac{1}{2}, 1).$$

□

### References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).