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Non trivial zeros of the generalized Zeta function

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Abstract

We define the generalized Riemann Zeta function on the right half plane. We prove, in particular, that for every $\tau \in \mathbb{R}$ there exists at most one point $r \in (0, 1)$ such that $|\zeta(r + i\tau)| = 0$.

Keywords: Zeta function, Non trivial zeros.

AMS subject classifications: 00A05

1 Main results

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} u^{-1-s} \{u\} du, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R},$$

where $\{u\}$ is the fractional part of the real u . Using the integration by parts formula, that can be written as

$$\zeta(s) := -\frac{s}{1-s} - \frac{1}{2} + s(1+s) \int_1^{+\infty} u^{-2-s} \eta_*(u) du, \quad (1)$$

where by Dirichlet Theorem, the periodic function $\eta_* : [1, +\infty) \rightarrow \mathbb{R}$ is defined as

$$\eta_*(u) := \int_1^u \left(\frac{1}{2} - \{v\} \right) dv = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left(1 - \exp(ij2\pi u) \right), \quad \forall u \geq 1.$$

We denote by $L^\infty := L^\infty([1, +\infty), \mathbb{R})$ the usual set of essentially bounded measurable functions from $[1, +\infty)$ to \mathbb{R} . Define the generalized Zeta function in the following sens

Definition 1. For every $\eta \in L^\infty$, the generalized Zeta function is defined on the right half plane as

$$\zeta[\eta](s) := \int_1^{+\infty} u^{-2-s} \eta(u) du, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R}.$$

We shall prove the following results:

Theorem 2. Let be $\eta \in L^\infty$ and consider the function $\zeta[\eta]$ given by the Definition (1). Let be $(r_*, \tau) \in \mathbb{R}_+^* \times \mathbb{R}$ and suppose that $|\zeta[\eta](r_* + i\tau)| \neq 0$, then

$$\zeta[\eta](r + i\tau) \neq \zeta[\eta](r_* + i\tau), \quad \forall r \in (r_*, 1 + r_*).$$

Denote by $B \subset \mathbb{C}$ the critical strip, defined as

$$B := \left\{ s \in \mathbb{C} : \Re(s) \in (0, 1), \Im(s) \in \mathbb{R} \right\}.$$

Notice that for every $s \in B$ we have $|f_*(s)| \neq 0$ and $f_*(s) = f_*(1-s)$ where the function $f_* : B \rightarrow \mathbb{C}$ is defined as

$$f_*(s) := \frac{1}{1-s^2} + \frac{1}{2s(1+s)} = \frac{1}{2} \left(\frac{1}{1-s} + \frac{1}{s} \right), \quad \forall s \in B.$$

Using Equation (1) and the symmetry of the non trivial zeros of the Riemann Zeta function, the following Corollary prove in particular, that for every $\tau \in \mathbb{R}$ there exists at most one point $r \in (0, 1)$ such that $\left| (r + i\tau)^{-1} (1 + r + i\tau)^{-1} \zeta(r + i\tau) \right| = 0$.

Corollary 3. Let be $\eta \in L^\infty$ and $\tau \in \mathbb{R}$. Let $f : B \rightarrow \mathbb{C}$ be a continuous function such that $|f(s)| \neq 0$ and $f(s) = f(1-s)$ for every $s \in B$. Suppose that there exists $s_* \in B$ such that $\Re(s_*) \in (0, 2^{-1})$ and such that $\zeta[\eta](s_*) = f(s_*)$. Then $\zeta[\eta](1-s_*) \neq f(1-s_*)$.

2 Proof of the Main results

In the following definition, we define a particular Laplace transform,

Definition 4. Let $g : t \in [1, +\infty) \mapsto \mathbb{C}$ be an essentially bounded measurable function. We define the function $\mathcal{L}[g(t)] : \mathbb{R}_+^* \rightarrow \mathbb{C}$ as

$$\mathcal{L}[g(t)](x) := \int_1^{+\infty} \exp(-xt) g(t) dt, \quad \forall x > 0.$$

Proof of the Theorem 2 . Let be $\eta \in L^\infty$ and $\tau \in \mathbb{R}$ fixed. By definition of the generalized Zeta function in the Definition 1 we have

$$\zeta[\eta](\rho + i\tau) = \int_1^{+\infty} u^{-2-\rho-i\tau} \eta(u) du, \quad \forall \rho > 0. \quad (2)$$

Let be $\rho_1 > 0$ and $\rho_2 \in (\rho_1, 1 + \rho_1)$ fixed. By definition of the transform \mathcal{L} and by using the change of variable $u = t^{-1}v$ on the following integral

$$\int_0^{+\infty} u^{-1+\rho_2-\rho_1} \mathcal{L}[t^{-2-\rho_1-i\tau} \eta(t)](u) du,$$

we obtain

$$\begin{aligned} \int_0^{+\infty} u^{-1+\rho_2-\rho_1} \mathcal{L}[t^{-2-\rho_1-i\tau} \eta(t)](u) du \\ = \zeta[\eta](\rho_2 + i\tau) \int_0^{+\infty} u^{-1+\rho_2-\rho_1} \exp(-u) du. \end{aligned}$$

For every $x > 0$ and by using the change of variable $u \mapsto xv$ we get

$$\begin{aligned} \int_0^{+\infty} v^{-1+\rho_2-\rho_1} \mathcal{L}[t^{-2-\rho_1-i\tau} \eta(t)](xv) dv \\ = \zeta[\eta](\rho_2 + i\tau) \int_0^{+\infty} v^{-1+\rho_2-\rho_1} \exp(-xv) dv. \end{aligned}$$

By definition of the transform \mathcal{L} , The last Equation is equivalent to the following equality

$$\zeta[\eta](\rho_2 + i\tau) \Delta(\rho_2 - \rho_1, x) = \int_1^{+\infty} \Delta(\rho_2 - \rho_1, xt) t^{-2-\rho_1-i\tau} \eta(t) dt, \quad \forall x \geq 0,$$

where in order to simplify the notation, we defined the real function $\Delta : (0, 1) \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ as

$$\Delta(\mu, y) := \int_0^{+\infty} u^{-1+\mu} \exp(-yu) du \quad (3)$$

By Equation (2), for every $x > 0$ the last Equation can be written as

$$\begin{aligned} \Delta(\rho_2 - \rho_1, x) \left(\zeta[\eta](\rho_2 + i\tau) - \zeta[\eta](\rho_1 + i\tau) \right) \\ = \int_1^{+\infty} \left(\Delta(\rho_2 - \rho_1, x) - \Delta(\rho_2 - \rho_1, xt) \right) t^{-2-\rho_1-i\tau} \eta(t) dt, \end{aligned} \quad (4)$$

Prove the present Theorem by contradiction. Let be $r_* > 0$ and $r \in (r_*, 1 + r_*)$. Assuming that $\zeta[\eta](r_* + i\tau) \neq 0$ and $\zeta[\eta](r + i\tau) = \zeta[\eta](r_* + i\tau)$. Equation (4) implies

$$\int_1^{+\infty} \left(\Delta(r - r_*, x) - \Delta(r - r_*, tx) \right) t^{-2-r_*-i\tau} \eta(t) dt = 0, \quad \forall x > 0.$$

For every $x > 0$ we get

$$\int_x^{+\infty} \int_1^{+\infty} \left(\Delta(r - r_*, y) - \Delta(r - r_*, ty) \right) t^{-2-r_*-i\tau} \eta(t) dt dy = 0.$$

Which implies

$$\lim_{x \rightarrow 0} \int_x^{+\infty} \int_1^{+\infty} \left(\Delta(r - r_*, y) - \Delta(r - r_*, ty) \right) t^{-2-r_*-i\tau} \eta(t) dt dy = 0. \quad (5)$$

By definition of the function Δ in Equation (3), for every $t \geq 1$ and $\mu \in (0, 1)$, we have

$$\begin{aligned} & \lim_{x \rightarrow 0} \left| \int_x^{+\infty} \Delta(\mu, ty) - t^{-1} \Delta(\mu, y) dy \right| \\ & \leq \lim_{x \rightarrow 0} \left| \int_x^{+\infty} \int_1^{+\infty} u^{-1+\mu} \left(\exp(-tyu) - t^{-1} \exp(-yu) \right) du dy \right| \\ & + \lim_{x \rightarrow 0} \left| \int_x^{+\infty} \int_0^1 u^{-1+\mu} \left(\exp(-tyu) - t^{-1} \exp(-yu) \right) du dy \right| \\ & = \lim_{x \rightarrow 0} \left| \int_1^{+\infty} u^{-1+\mu} \int_x^{+\infty} \left(\exp(-tyu) - t^{-1} \exp(-yu) \right) dy du \right| \\ & + \lim_{x \rightarrow 0} \left| \int_0^1 u^{-1+\mu} \int_x^{+\infty} \left(\exp(-tyu) - t^{-1} \exp(-yu) \right) dy du \right| \\ & = t^{-1} \lim_{x \rightarrow 0} \int_1^{+\infty} u^{-2+\mu} \left| \exp(-txu) - \exp(-xu) \right| du \\ & + t^{-1} \lim_{x \rightarrow 0} \int_0^1 u^{-2+\mu} \left| \exp(-txu) - \exp(-xu) \right| du \\ & \leq t^{-1} \lim_{x \rightarrow 0} \int_1^{+\infty} u^{-2+\mu} \left| \exp(-txu) - \exp(-xu) \right| du \\ & + (t-1)t^{-1} \lim_{x \rightarrow 0} x \int_0^1 u^{-1+\mu} du \\ & = 0 \end{aligned}$$

Then

$$\begin{aligned}
& \lim_{x \rightarrow 0} \int_x^{+\infty} \int_1^{+\infty} \left(\Delta(\mu, ty) - t^{-1} \Delta(\mu, y) \right) t^{-2-r_*-i\tau} \eta(t) dt \Big] (x) \\
&= \lim_{x \rightarrow 0} \int_1^{+\infty} \left(\int_x^{+\infty} \Delta(\mu, ty) - t^{-1} \Delta(\mu, y) dy \right) \frac{\eta(t)}{t^{2+r_*+i\tau}} dt \\
&= \int_1^{+\infty} \left(\lim_{x \rightarrow 0} \int_x^{+\infty} \Delta(\mu, ty) - t^{-1} \Delta(\mu, y) dy \right) \frac{\eta(t)}{t^{2+r_*+i\tau}} dt \\
&= 0.
\end{aligned} \tag{6}$$

The addition of the Equations (5) and (6), gives

$$\begin{aligned}
& \lim_{x \rightarrow 0} \int_x^{+\infty} \int_1^{+\infty} \Delta(r - r_*, y) (1 - t^{-1}) t^{-2-r_*-i\tau} \eta(t) dt dy \\
&= \lim_{x \rightarrow 0} \int_x^{+\infty} \Delta(r - r_*, y) \int_1^{+\infty} (1 - t^{-1}) t^{-2-r_*-i\tau} \eta(t) dt dy = 0.
\end{aligned} \tag{7}$$

By definition of the function Δ in Equation (3) we have $\Delta(\mu, y) > 0$ for all $(\mu, y) \in (0, 1) \times \mathbb{R}_+^*$. Equation (7) implies

$$\int_1^{+\infty} (1 - t^{-1}) t^{-2-r_*-i\tau} \eta(t) dt = 0.$$

In other words, and by Equation (2), we have obtained

$$\zeta[\eta](r_* + i\tau) = \zeta[\eta](1 + r_* + i\tau).$$

By hypothesis we have $\zeta[\eta](r + i\tau) = \zeta[\eta](r_* + i\tau)$. Then

$$\zeta[\eta](1 + r_* + i\tau) = \zeta[\eta](r + i\tau).$$

Since $1 + r_* \in (r, r + 1)$ and using the same previous method, we obtain

$$\zeta[\eta](r + i\tau) = \zeta[\eta](1 + r + i\tau).$$

Which implies

$$\zeta[\eta](1 + r + i\tau) = \zeta[\eta](1 + r_* + i\tau).$$

By induction, we get

$$\zeta[\eta](r_* + i\tau) = \zeta[\eta](k + r_* + i\tau), \quad \forall k \in \mathbb{N}.$$

By hypothesis we have $\zeta[\eta](r_* + i\tau) \neq 0$. We then obtain a contradiction since $\lim_{k \rightarrow +\infty} |\zeta[\eta](k + r_* + i\tau)| = 0$. \square

Proof of the Corollary 3 . Let be $\eta \in L^\infty$ fixed. Let $f : B \rightarrow \mathbb{C}$ be a continuous function such that $|f(s)| \neq 0$ and $f(s) = f(1-s)$ for every $s \in B$. Suppose that there exists $s_* \in B$ such that $\Re(s_*) \in (0, 2^{-1})$ and such that $\zeta[\eta](s_*) = f(s_*)$. Then $|\zeta[\eta](s_*)| = |f(s_*)| \neq 0$. By the Theorem 2,

$$\begin{aligned} \zeta[\eta](1-s_*) - f(1-s_*) &= \zeta[\eta](1-s_*) - f(1-s_*) - \zeta[\eta](s_*) + f(s_*) \\ &= \zeta[\eta](1-s_*) - \zeta[\eta](s_*) \neq 0. \end{aligned}$$

□

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).