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# Non trivial zeros of the generalized Zeta function

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## Abstract

We define the generalized Riemann Zeta function on the right half plane. We prove, in particular, that for every  $\tau \in \mathbb{R}$  there exists at most one point  $r \in (0, 1)$  such that  $|\zeta(r + i\tau)| = 0$ .

**Keywords:** Zeta function, Non trivial zeros.

**AMS subject classifications:** 00A05

## 1 Main results

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} u^{-1-s} \{u\} du, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R},$$

where  $\{u\}$  is the fractional part of the real  $u$ . Using the integration by parts formula, that can be written as

$$\zeta(s) := -\frac{s}{1-s} - \frac{1}{2} + s(1+s) \int_1^{+\infty} u^{-2-s} \eta_*(u) du, \quad (1)$$

where by Dirichlet Theorem, the periodic function  $\eta_* : [1, +\infty) \rightarrow \mathbb{R}$  is defined as

$$\eta_*(u) := \int_1^u \left( \frac{1}{2} - \{v\} \right) dv = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left( 1 - \exp(ij2\pi u) \right), \quad \forall u \geq 1.$$

We denote by  $L^\infty := L^\infty([1, +\infty), \mathbb{R})$  the usual set of essentially bounded measurable functions from  $[1, +\infty)$  to  $\mathbb{R}$ . Define the generalized Zeta function in the following sens

**Definition 1.** For every  $\eta \in L^\infty$ , the generalized Zeta function is defined on the right half plane as

$$\zeta[\eta](s) := \int_1^{+\infty} u^{-2-s} \eta(u) du, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R}.$$

We shall prove the following results:

**Theorem 2.** Let be  $\eta \in L^\infty$  and consider the function  $\zeta[\eta]$  given by the Definition (1). Let be  $(r_*, \tau) \in \mathbb{R}_+^* \times \mathbb{R}$  and suppose that  $|\zeta[\eta](r_* + i\tau)| \neq 0$ , then

$$\zeta[\eta](r + i\tau) \neq \zeta[\eta](r_* + i\tau), \quad \forall r \in (r_*, 1 + r_*).$$

Denote by  $B \subset \mathbb{C}$  the critical strip, defined as

$$B := \left\{ s \in \mathbb{C} : \Re(s) \in (0, 1), \Im(s) \in \mathbb{R} \right\}.$$

Notice that for every  $s \in B$  we have  $|f_*(s)| \neq 0$  and  $f_*(s) = f_*(1-s)$  where the function  $f_* : B \rightarrow \mathbb{C}$  is defined as

$$f_*(s) := \frac{1}{1-s^2} + \frac{1}{2s(1+s)} = \frac{1}{2} \left( \frac{1}{1-s} + \frac{1}{s} \right), \quad \forall s \in B.$$

Using Equation (1) and the symmetry of the non trivial zeros of the Riemann Zeta function, the following Corollary prove in particular, that for every  $\tau \in \mathbb{R}$  there exists at most one point  $r \in (0, 1)$  such that  $\left| (r + i\tau)^{-1} (1 + r + i\tau)^{-1} \zeta(r + i\tau) \right| = 0$ .

**Corollary 3.** Let be  $\eta \in L^\infty$  and  $\tau \in \mathbb{R}$ . Let  $f : B \rightarrow \mathbb{C}$  be a continuous function such that  $|f(s)| \neq 0$  and  $f(s) = f(1-s)$  for every  $s \in B$ . Suppose that there exists  $s_* \in B$  such that  $\Re(s_*) \in (0, 2^{-1})$  and such that  $\zeta[\eta](s_*) = f(s_*)$ . Then  $\zeta[\eta](1-s_*) \neq f(1-s_*)$ .

## 2 Proof of the Main results

In the following definition, we define a particular Laplace transform,

**Definition 4.** Let  $g : t \in [1, +\infty) \mapsto \mathbb{C}$  be an essentially bounded measurable function. We define the function  $\mathcal{L}[g(t)] : \mathbb{R}_+^* \rightarrow \mathbb{C}$  as

$$\mathcal{L}[g(t)](x) := \int_1^{+\infty} \exp(-xt) g(t) dt, \quad \forall x > 0.$$

*Proof of the Theorem 2 .* Let be  $\eta \in L^\infty$  and  $\tau \in \mathbb{R}$  fixed. By definition of the generalized Zeta function in the Definition 1 we have

$$\zeta[\eta](\rho + i\tau) = \int_1^{+\infty} u^{-2-\rho-i\tau} \eta(u) du, \quad \forall \rho > 0. \quad (2)$$

Let be  $\rho_1 > 0$  and  $\rho_2 \in (\rho_1, 1 + \rho_1)$  fixed. By definition of the transform  $\mathcal{L}$  and by using the change of variable  $u = t^{-1}v$  on the following integral

$$\int_0^{+\infty} u^{-1+\rho_2-\rho_1} \mathcal{L}[t^{-2-\rho_1-i\tau} \eta(t)](u) du,$$

we obtain

$$\begin{aligned} \int_0^{+\infty} u^{-1+\rho_2-\rho_1} \mathcal{L}[t^{-2-\rho_1-i\tau} \eta(t)](u) du \\ = \zeta[\eta](\rho_2 + i\tau) \int_0^{+\infty} u^{-1+\rho_2-\rho_1} \exp(-u) du. \end{aligned}$$

In order to simplify the notation, denote

$$\varphi_{\tau, \rho_2, \rho_1}(u) := \zeta[\eta](\rho_2 + i\tau) \exp(-u) - \mathcal{L}[t^{-2-\rho_1-i\tau} \eta(t)](u), \quad \forall u \geq 0. \quad (3)$$

Then

$$\int_0^{+\infty} u^{-1+\rho_2-\rho_1} \varphi_{\tau, \rho_2, \rho_1}(u) du = 0.$$

We recall that  $\rho_2 \in (\rho_1, 1 + \rho_1)$ . Use the integration by parts formula, we get

$$\int_0^{+\infty} u^{-2+\rho_2-\rho_1} \int_0^u \varphi_{\tau, \rho_2, \rho_1}(v) dv du = 0,$$

which can be written as

$$\begin{aligned} x^{1-\rho_2+\rho_1} \int_x^{+\infty} u^{-2+\rho_2-\rho_1} \int_0^u \varphi_{\tau, \rho_2, \rho_1}(v) dv du \\ = -x^{1-\rho_2+\rho_1} \int_0^x u^{-2+\rho_2-\rho_1} \int_0^u \varphi_{\tau, \rho_2, \rho_1}(v) dv du, \quad \forall x \geq 0. \end{aligned} \quad (4)$$

In order to simplify the notation, denote for every continuous function,  $g : v \in \mathbb{R}_+ \rightarrow \mathbb{C}$  the integral  $\Phi[g(v)] : \mathbb{R}_+ \rightarrow \mathbb{C}$  as

$$\Phi[g(v)](u) := \int_0^u g(v) dv, \quad \forall u \geq 0, \quad (5)$$

Since

$$\sup_{u \geq 0} \left| \int_0^u \varphi_{\tau, \rho_2, \rho_1}(v) dv \right| < +\infty,$$

Then

$$|\Phi^{n+1}[\varphi_{\tau, \rho_2, \rho_1}(v)](u)| \leq \frac{u^n}{n!} \sup_{u \geq 0} \left| \int_0^u \varphi_{\tau, \rho_2, \rho_1}(v) dv \right|, \quad \forall u > 0.$$

Using a successive integration by parts formula on the right term of Equation (4), we get

$$\begin{aligned} & x^{1-\rho_2+\rho_1} \int_x^{+\infty} u^{-2+\rho_2-\rho_1} \int_0^u \varphi_{\tau, \rho_2, \rho_1}(v) dv du \\ &= - \left( \Pi_{j=1}^{n+1}(k - \rho_2 + \rho_1) \right) x^{1-\rho_2+\rho_1} \int_0^x u^{-2-n+\rho_2-\rho_1} \Phi^{n+1}[\varphi_{\tau, \rho_2, \rho_1}(v)](u) du \\ & - \frac{1}{1 - \rho_2 + \rho_1} \sum_{k=1}^n \left( \Pi_{j=1}^k(k - \rho_2 + \rho_1) \right) x^{-k} \Phi^{k+1}[\varphi_{\tau, \rho_2, \rho_1}(v)](x). \end{aligned} \quad (6)$$

Since  $\sup_{u \geq 0} |\varphi_{\tau, \rho_2, \rho_1}(u)| < +\infty$ , then

$$|\Phi^{n+1}[\varphi_{\tau, \rho_2, \rho_1}(v)](u)| \leq \frac{u^{n+1}}{(n+1)!} \sup_{u \geq 0} |\varphi_{\tau, \rho_2, \rho_1}(u)|, \quad \forall u > 0.$$

Which implies

$$\begin{aligned} & \left| x^{1-\rho_2+\rho_1} \int_0^x u^{-2-n+\rho_2-\rho_1} \Phi^{n+1}[\varphi_{\tau, \rho_2, \rho_1}(v)](u) du \right| \\ & \leq \frac{x}{(n+1)!} \frac{1}{\rho_2 - \rho_1}, \quad \forall x > 0, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Since  $\rho_2 \in (\rho_1, 1 + \rho_1)$  then

$$\lim_{n \rightarrow +\infty} \frac{\Pi_{j=1}^n(k+1 - \rho_2 + \rho_1)}{(n+1)!} = 0.$$

For every fixed  $x > 0$ , Equation (6) implies

$$\begin{aligned} & (1 - \rho_2 + \rho_1) x^{1-\rho_2+\rho_1} \int_x^{+\infty} u^{-2+\rho_2-\rho_1} \int_0^u \varphi_{\tau, \rho_2, \rho_1}(v) dv du \\ &= - \sum_{k=1}^{+\infty} \left( \Pi_{j=1}^k(k - \rho_2 + \rho_1) \right) x^{-k} \Phi^{k+1}[\varphi_{\tau, \rho_2, \rho_1}(v)](x). \end{aligned}$$

By the definition of  $\varphi_{\tau, \rho_2, \rho_1}$  on Equation (3), we get

$$\begin{aligned}
& \zeta[\eta](\rho_2 + i\tau)(1 - \rho_2 + \rho_1)x^{-\rho_2 + \rho_1} \int_x^{+\infty} u^{-2 + \rho_2 - \rho_1} \int_0^u \exp(-v) dv du \\
& + \zeta[\eta](\rho_2 + i\tau) \sum_{k=1}^{+\infty} \left( \Pi_{j=1}^k(k - \rho_2 + \rho_1) \right) x^{-1-k} \Phi^{k+1}[\exp(-v)](x) \\
& = (1 - r + \rho_1)x^{-\rho_2 + \rho_1} \int_x^{+\infty} u^{-2 + \rho_2 - \rho_1} \int_0^u \mathcal{L}[t^{-2 - \rho_1 - i\tau} \eta(t)](v) dv du \\
& + \sum_{k=1}^{+\infty} \left( \Pi_{j=1}^k(k - \rho_2 + \rho_1) \right) x^{-1-k} \Phi^{k+1}[\mathcal{L}[t^{-2 - \rho_1 - i\tau} \eta(t)](v)](x)
\end{aligned}$$

In other words,

$$\begin{aligned}
& \zeta[\eta](\rho_2 + i\tau) \Delta(\rho_2 - \rho_1, x, 1) \\
& = (1 - \rho_2 + \rho_1)x^{-\rho_2 + \rho_1} \int_x^{+\infty} u^{-2 + \rho_2 - \rho_1} \int_0^u \mathcal{L}[t^{-2 - \rho_1 - i\tau} \eta(t)](v) dv du \\
& + \sum_{k=1}^{+\infty} \left( \Pi_{j=1}^k(k - \rho_2 + \rho_1) \right) x^{-1-k} \Phi^{k+1}[\mathcal{L}[t^{-2 - \rho_1 - i\tau} \eta(t)](v)](x)
\end{aligned} \tag{7}$$

where in order to simplify the notation, we defined the real function  $\Delta : (0, 1) \times \mathbb{R}_+^* \times [1, +\infty) \rightarrow \mathbb{R}$  as

$$\begin{aligned}
\Delta(\mu, x, t) & := (1 - \mu)x^{-\mu} \int_x^{+\infty} u^{-2 + \mu} \int_0^u \exp(-tv) dv du \\
& + \sum_{k=1}^{+\infty} \left( \Pi_{j=1}^k(k - \mu) \right) x^{-1-k} \Phi^{k+1}[\exp(-tv)](x).
\end{aligned} \tag{8}$$

By definition of the transform  $\mathcal{L}$ , Equation (7) is equivalent to the following equality

$$\zeta[\eta](\rho_2 + i\tau) \Delta(\rho_2 - \rho_1, x, 1) = \int_1^{+\infty} \Delta(\rho_2 - \rho_1, x, t) t^{-2 - \rho_1 - i\tau} \eta(t) dt, \quad \forall x \geq 0.$$

By Equation (2), for every  $x > 0$  the last Equation can be written as

$$\begin{aligned}
& \Delta(\rho_2 - \rho_1, x, 1) \left( \zeta[\eta](\rho_2 + i\tau) - \zeta[\eta](\rho_1 + i\tau) \right) \\
& = \int_1^{+\infty} \left( \Delta(\rho_2 - \rho_1, x, 1) - \Delta(\rho_2 - \rho_1, x, t) \right) t^{-2 - \rho_1 - i\tau} \eta(t) dt,
\end{aligned} \tag{9}$$

Using the integration by parts formula on the integral that defines the function  $\Delta$  in Equation (8), for every  $\mu \in (0, 1)$ , we obtain

$$\begin{aligned}\Delta(\mu, x, t) &= x^{-1} \int_0^x \exp(-tu) du + x^{-\mu} \int_x^{+\infty} u^{-1+\mu} \exp(-tu) du \\ &\quad + \sum_{k=1}^{+\infty} \left( \Pi_{j=1}^k(k-\mu) \right) x^{-1-k} \Phi^{k+1}[\exp(-tv)](x) \\ &= \int_0^1 \exp(-txu) du + \int_1^{+\infty} u^{-1+\mu} \exp(-txu) du \\ &\quad + \sum_{k=1}^{+\infty} \left( \Pi_{j=1}^k(k-\mu) \right) x^{-1-k} \Phi^{k+1}[\exp(-tv)](x)\end{aligned}$$

By definition of  $\Phi$  in Equation (5) and by the Taylor formula, we have

$$\Phi^{k+1}[\exp(-tv)](x) = \frac{x^{k+1}}{k!} \int_0^1 (1-z)^k \exp(-txz) dz$$

Then every  $(\mu, x, t) \in (0, 1) \times \mathbb{R}_+^* \times [1, +\infty) \rightarrow \mathbb{R}$  we have

$$\Delta(\mu, x, t) = \tilde{\Delta}(\mu, xt),$$

where the real function  $\tilde{\Delta} : (0, 1) \times \mathbb{R}_+^* \rightarrow \mathbb{R}$  is defined as

$$\begin{aligned}\tilde{\Delta}(\mu, y) &= \int_0^1 \exp(-yu) du + \int_1^{+\infty} u^{-1+\mu} \exp(-yu) du \\ &\quad + \sum_{k=1}^{+\infty} \frac{1}{k!} \left( \Pi_{j=1}^k(k-\mu) \right) \int_0^1 \exp(-yz) (1-z)^k dz.\end{aligned}\tag{10}$$

For every  $x > 0$  Equation (9) is then equivalent to

$$\begin{aligned}\tilde{\Delta}(\rho_2 - \rho_1, x) \left( \zeta[\eta](\rho_2 + i\tau) - \zeta[\eta](\rho_1 + i\tau) \right) \\ = \int_1^{+\infty} \left( \tilde{\Delta}(\rho_2 - \rho_1, x) - \tilde{\Delta}(\rho_2 - \rho_1, tx) \right) t^{-2-\rho_1-i\tau} \eta(t) dt.\end{aligned}\tag{11}$$

Prove the present Theorem by contradiction. Let be  $r_* > 0$  and  $r \in (r_*, 1 + r_*)$ . Assuming that  $\zeta[\eta](r_* + i\tau) \neq 0$  and  $\zeta[\eta](r + i\tau) = \zeta[\eta](r_* + i\tau)$ . Equation (11) implies

$$\int_1^{+\infty} \left( \tilde{\Delta}(r - r_*, x) - \tilde{\Delta}(r - r_*, tx) \right) t^{-2-r_*-i\tau} \eta(t) dt = 0, \quad \forall x > 0.$$

For every  $x > 0$  we get

$$\int_x^{+\infty} \int_1^{+\infty} \left( \tilde{\Delta}(r - r_*, y) - \tilde{\Delta}(r - r_*, ty) \right) t^{-2-r_*-i\tau} \eta(t) dt dy = 0.$$

Which implies

$$\lim_{x \rightarrow 0} \int_x^{+\infty} \int_1^{+\infty} \left( \tilde{\Delta}(r - r_*, y) - \tilde{\Delta}(r - r_*, ty) \right) t^{-2-r_*-i\tau} \eta(t) dt dy = 0.$$

By definition of the function  $\tilde{\Delta}$  in Equation (10) and by using, at the end of the following equality, the fact that  $|\exp(-xtz) - \exp(-xz)| \leq xz(t-1)$  for every  $t \geq 1$ ,  $x > 0$  and  $z \in (0, 1)$ , we obtain for every  $\mu \in (0, 1)$  the following

$$\begin{aligned} \lim_{x \rightarrow 0} \int_x^{+\infty} \int_1^{+\infty} \left( \tilde{\Delta}(\mu, ty) - t^{-1} \tilde{\Delta}(\mu, y) \right) t^{-2-r_*-i\tau} \eta(t) dt \Big] (x) \\ = \lim_{x \rightarrow 0} \int_1^{+\infty} \left( \int_x^{+\infty} \tilde{\Delta}(\mu, ty) - t^{-1} \tilde{\Delta}(\mu, y) dy \right) \frac{\eta(t)}{t^{2+r_*+i\tau}} dt \\ = \int_1^{+\infty} \left( \lim_{x \rightarrow 0} \int_x^{+\infty} \tilde{\Delta}(\mu, ty) - t^{-1} \tilde{\Delta}(\mu, y) dy \right) \frac{\eta(t)}{t^{2+r_*+i\tau}} dt \\ = 0. \end{aligned}$$

The addition of the last two Equations, gives

$$\begin{aligned} \lim_{x \rightarrow 0} \int_x^{+\infty} \int_1^{+\infty} \tilde{\Delta}(r - r_*, y) (1 - t^{-1}) t^{-2-r_*-i\tau} \eta(t) dt dy \\ = \lim_{x \rightarrow 0} \int_x^{+\infty} \tilde{\Delta}(r - r_*, y) \int_1^{+\infty} (1 - t^{-1}) t^{-2-r_*-i\tau} \eta(t) dt dy = 0. \end{aligned} \quad (12)$$

By definition of the function  $\tilde{\Delta}$  in Equation (10) we have  $\tilde{\Delta}(\mu, y) > 0$  for all  $(\mu, y) \in (0, 1) \times \mathbb{R}_+^*$ . Equation (12) implies

$$\int_1^{+\infty} (1 - t^{-1}) t^{-2-r_*-i\tau} \eta(t) dt = 0.$$

In other words, and by Equation (2), we have obtained

$$\zeta[\eta](r_* + i\tau) = \zeta[\eta](1 + r_* + i\tau).$$

By hypothesis we have  $\zeta[\eta](r + i\tau) = \zeta[\eta](r_* + i\tau)$ . Then

$$\zeta[\eta](1 + r_* + i\tau) = \zeta[\eta](r + i\tau).$$



Since  $1 + r_* \in (r, r + 1)$  and using the same previous method, we obtain

$$\zeta[\eta](r + i\tau) = \zeta[\eta](1 + r + i\tau).$$

Which implies

$$\zeta[\eta](1 + r + i\tau) = \zeta[\eta](1 + r_* + i\tau).$$

By induction, we get

$$\zeta[\eta](r_* + i\tau) = \zeta[\eta](k + r_* + i\tau), \quad \forall k \in \mathbb{N}.$$

By hypothesis we have  $\zeta[\eta](r_* + i\tau) \neq 0$ . We then obtain a contradiction since  $\lim_{k \rightarrow +\infty} |\zeta[\eta](k + r_* + i\tau)| = 0$ .  $\square$

*Proof of the Corollary 3 .* Let be  $\eta \in L^\infty$  fixed. Let  $f : B \rightarrow \mathbb{C}$  be a continuous function such that  $|f(s)| \neq 0$  and  $f(s) = f(1 - s)$  for every  $s \in B$ . Suppose that there exists  $s_* \in B$  such that  $\Re(s_*) \in (0, 2^{-1})$  and such that  $\zeta[\eta](s_*) = f(s_*)$ . Then  $|\zeta[\eta](s_*)| = |f(s_*)| \neq 0$ . By the Theorem 2,

$$\begin{aligned} \zeta[\eta](1 - s_*) - f(1 - s_*) &= \zeta[\eta](1 - s_*) - f(1 - s_*) - \zeta[\eta](s_*) + f(s_*) \\ &= \zeta[\eta](1 - s_*) - \zeta[\eta](s_*) \neq 0. \end{aligned}$$

$\square$

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).