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Non trivial zeros of the Zeta function

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Abstract

We prove that $|\zeta(\frac{1}{2} + r + i\tau)| \neq 0$ for every $\tau \in \mathbb{R}^*$ and $r \in (-\frac{1}{2}, 0)$, where ζ is the Riemann Zeta function.

Keywords: Zeta function, non trivial zeros, Riemann hypothesis.

AMS subject classifications: 00A05

1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{t\}$ is the fractional part of the real t . We prove the following Theorem.

Theorem 1. *Consider the function ζ defined by the Equation (1). For every $\tau \in \mathbb{R}^*$ and $r \in (-\frac{1}{2}, 0)$ we have*

$$|\zeta(\frac{1}{2} + r + i\tau)| \neq 0.$$

By the Riemann functional equation we deduce that any *non trivial zero* of the Zeta function has a real part equal to $\frac{1}{2}$, where the *non trivial zeros* of the function ζ are defined in the following sense

Definition 2. Consider the function ζ defined by the Equation (1). Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function ζ* if

$$|\zeta(s)| = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

2 Proof of the Theorem 1

Notation 3. In order to simplify the notation, for every $r \in (-\frac{1}{2}, 0)$ denote

$$f_r(u) := \{u\}u^{-\frac{1}{2}-r}, \quad u > 0.$$

By the Equation (1), remark that

$$\lim_{t \rightarrow +\infty} \left| \int_0^t u^{-i\tau-1} f_r(u) du \right| := \left| \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} \right|.$$

Notation 4. Denote

$$p(t) := \int_0^t \left(\{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 0,$$

where we recall that $\{u\}$ is the fractional part of the real u . Denote

$$\omega := 2 \max_{t \in [0,1]} |p(t)|.$$

Lemma 5. *The function p is a continuous 1-periodic function.*

Proof. The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t \frac{1}{2} + p(t), \quad \forall t \geq 0.$$

□

Notation 6. Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a continuous function. For every $t > s > 0$ we denoted

$$\int_{s,k}^t g(u) du = \underbrace{\int_s^t \frac{1}{u_1} \int_s^{u_1} \frac{1}{u_2} \dots \int_s^{u_{k-1}} \frac{1}{u_k} g(u_k) du_k du_{k-1} \dots du_1}_{k \text{ times}}, \quad \forall k \in \mathbb{N}^*.$$

For every $r \in (-\frac{1}{2}, 0)$ and $\tau \in \mathbb{R}^*$ denote

$$\begin{aligned}\underline{\Psi}_{\tau,r}(k, t) &= \int_{\frac{1}{t}, k}^t u^{i\tau} \int_0^u v^{-i\tau-1} f_r(v) dv du, & \forall t > 1, \forall k \geq 1, \\ \overline{\Psi}_{\tau,r}(k, t) &= \int_{\frac{1}{t}, k}^t u^{i\tau} \int_u^{+\infty} v^{-i\tau-1} f_r(v) dv du, & \forall t > 1, \forall k \geq 1.\end{aligned}$$

Proof of the Theorem 1. Let be $\tau \in \mathbb{R}^*$ and $r \in (-\frac{1}{2}, 0)$. Prove the Theorem by contradiction. Suppose that

$$\left| \zeta\left(\frac{1}{2} + r + i\tau\right) \right| = 0.$$

By the Equation (1) we have

$$\left| \frac{\zeta\left(\frac{1}{2} + r + i\tau\right)}{\frac{1}{2} + r + i\tau} \right| = \left| \int_0^{+\infty} u^{-i\tau-1} f_r(u) du \right|.$$

Then

$$\left| \int_0^{+\infty} u^{-i\tau-1} f_r(u) du \right| = 0,$$

Implies

$$\int_0^t u^{-i\tau-1} f_r(u) du = - \int_t^{+\infty} u^{-i\tau-1} f_r(u) du, \quad \forall t > 0.$$

By the notation of $\overline{\Psi}_{\tau,r}$ and $\underline{\Psi}_{\tau,r}$ in the Notation 6 we get

$$|\overline{\Psi}_{\tau,r}(k, t)| = |\underline{\Psi}_{\tau,r}(k, t)|, \quad \forall t > 1, \quad \forall k \geq 1. \quad (2)$$

Since $0 \leq f_r(u) \leq u^{\frac{1}{2}-r}$ for every $u > 0$ then for every $k \geq 1$ we have

$$\begin{aligned}|\underline{\Psi}_{\tau,r}(k, t)| &= \left| \int_{\frac{1}{t}, k}^t u^{i\tau} \int_0^u v^{-i\tau-1} f_r(v) dv du \right| \leq \int_{\frac{1}{t}, k}^t \int_0^u v^{-1} f_r(v) dv du \\ &\leq \frac{1}{\frac{1}{2} - r} \int_{\frac{1}{t}, k}^t u^{\frac{1}{2}-r} du, \quad \forall k \geq 1, \quad \forall t > 1.\end{aligned} \quad (3)$$

Consider the function p given in the Notation 4. By Lemma 5, the function p is 1-periodic, then it is bounded. Use the integration part formula

$$\begin{aligned}\left| \int_u^{+\infty} v^{-i\tau-1} \left(f_r(v) - \frac{1}{2} v^{-\frac{1}{2}-r} \right) dv \right| \\ &= \left| \int_u^{+\infty} v^{-i\tau-1} \left(\{v\} - \frac{1}{2} \right) v^{-\frac{1}{2}-r} dv \right| \\ &= \left| \left(i\tau + \frac{3}{2} + r \right) \int_u^{+\infty} v^{-i\tau-\frac{5}{2}-r} (p(v) - p(u)) dv \right|.\end{aligned}$$

Since

$$|p(v) - p(u)| < \omega, \quad \forall u > 0, \forall v > 0,$$

where ω is given by the Notation 4, we get

$$\begin{aligned} \left| \int_u^{+\infty} v^{-i\tau-1} \left(f_r(v) - \frac{1}{2} v^{-\frac{1}{2}-r} \right) dv \right| &\leq \omega(|\tau| + 2) \int_u^{+\infty} v^{-\frac{5}{2}-r} dv \\ &\leq \omega(|\tau| + 2) u^{-\frac{3}{2}-r}, \quad \forall u > 0. \end{aligned}$$

Then

$$\begin{aligned} \left| \int_{\frac{1}{t}, k}^t u^{i\tau} \int_u^{+\infty} v^{-i\tau-1} \left(f_r(v) - \frac{1}{2} v^{-\frac{1}{2}-r} \right) dv \right| du \\ \leq \omega(|\tau| + 2) \int_{\frac{1}{t}, k}^t u^{-\frac{3}{2}-r} du, \quad \forall t > 1, \forall k \geq 1. \end{aligned}$$

Since

$$u^{-i\tau} \int_u^{+\infty} v^{-i\tau-\frac{3}{2}-r} dv = \frac{1}{\frac{1}{2} + r - i\tau} u^{-\frac{1}{2}-r}, \quad \forall u > 0,$$

Then

$$|\bar{\Psi}_{\tau, r}(k)| \geq \frac{1}{2} \frac{1}{|\frac{1}{2} + r - i\tau|} \int_{\frac{1}{t}, k}^t u^{-\frac{1}{2}-r} du - \omega(|\tau| + 2) \int_{\frac{1}{t}, k}^t u^{-\frac{3}{2}-r} du. \quad (4)$$

Since $r \in (-\frac{1}{2}, 0)$, then there exists $t > 1$ such that

$$\lim_{k \rightarrow +\infty} \frac{\int_{\frac{1}{t}, k}^t u^{-\frac{1}{2}-r} du}{\int_{\frac{1}{t}, k}^t u^{\frac{1}{2}-r} du} = +\infty \quad \text{and} \quad \lim_{k \rightarrow +\infty} \frac{\int_{\frac{1}{t}, k}^t u^{-\frac{3}{2}-r} du}{\int_{\frac{1}{t}, k}^t u^{\frac{1}{2}-r} du} = 0.$$

The Equations (3) and (4) implies that

$$\lim_{k \rightarrow +\infty} \left| \frac{\bar{\Psi}_{\tau, r}(k, t)}{\underline{\Psi}_{\tau, r}(k, t)} \right| \geq \frac{1}{2} \frac{\frac{1}{2} - r}{|\frac{1}{2} + r - i\tau|} \lim_{k \rightarrow +\infty} \frac{\int_{\frac{1}{t}, k}^t u^{-\frac{1}{2}-r} du}{\int_{\frac{1}{t}, k}^t u^{\frac{1}{2}-r} du} = +\infty.$$

Contradiction with the Equation (2). □

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).