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# Non trivial zeros of the Zeta function using the differential equations

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## Abstract

Using the differential equations, we obtain a more flexible expression for the Riemann Zeta function on the right half-plan except the point  $s = 1$ . Thanks to the Riemann functional equation, we suggest that the point  $s \in \mathbb{C}$  such that  $\Re(s) \in (0, 2^{-1}]$  is a non-trivial zero of the function  $\zeta$  implies  $|s^2\Gamma(s)| \geq |(1-s)^2\Gamma(1-s)|$ , where  $\Gamma$  is the Gamma function.

**Keywords:** Zeta function, Non trivial zeros, Riemann functional equation, Bernoulli numbers, Differential equations.

**AMS subject classifications:** 00A05

## 1 Main results

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} u^{-1-s} \{u\} du, \quad s \neq 1, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{u\}$  is the fractional part of the real  $u$ . In order to simplify the notation, denote  $B \subset \mathbb{C}$  the right half-plan except the point  $s = 1$ , defined as

$$B := \{s \in \mathbb{C} : s \neq 1, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R}^*\},$$

In this manuscript, we denote  $(B_k)_{k \in \mathbb{N}}$  the Bernoulli numbers. We prove the following theorem,

**Theorem 1.** Consider the Zeta function given by the Equation (1). For every  $s \in B$  we have

$$\zeta(s) = -\frac{s}{1-s} - \frac{1}{4} \lim_{n \rightarrow +\infty} \left( \prod_{k=0}^{n+2} (k+s) \right) \eta_s(n),$$

where

$$\begin{aligned} \eta_s(n) &:= \int_n^{+\infty} \frac{(u+2)^{n+s}}{(u+1)^{n+2+s}} \int_{u+1}^{+\infty} \frac{q(n-1, v)}{(v+1)^{n+1+s}} dv du \\ &\quad - (n+1)^{-n-1-s} \int_n^{n+1} (u+1)^{n+s} \int_u^{+\infty} \frac{q(n-1, v)}{(v+1)^{n+1+s}} dv du \\ &\quad + \int_n^{+\infty} (u+1)^{-n-2-s} \left( q(n, u) - q(n, u+1) \right) du, \quad \forall n \geq 1, \end{aligned}$$

and where

$$q(n, u) := \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{u^{n+2-k}}{(n+2-k)!} \quad \forall n \geq 0, \quad \forall u \geq 0.$$

## 2 Main Proposition

For every  $s \in B$ , the Equation (1) is equivalent to,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

The aim is to study the differential equation of solutions the functions

$$t \mapsto \psi_s(z, t) := t^s \left[ z + \int_1^t u^{-1-s} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to find the initial condition  $z_s \in \mathbb{C}$  such that  $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ . Since  $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$  if and only if  $z = -\int_1^{+\infty} u^{-1-s} \{u\} du$ , we obtain  $z_s = -\int_1^{+\infty} u^{-1-s} \{u\} du$ . For every  $s \in B$  we consider the following differential equation

$$\begin{aligned} \frac{d}{dt} x &= st^{-1}x + t^{-1}\{t\}, \tag{2} \\ t &\in [1, +\infty)/\mathbb{N}, \quad x(1) = z, \quad z \in \mathbb{C}, \quad x : [1, +\infty) \rightarrow \mathbb{C}. \end{aligned}$$

**Lemma 2.** *Let be  $s \in B$ . For every  $z \in \mathbb{C}$  there exists a unique continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2). Further,*

$$\psi_s(z, t) = t^s \left[ z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

*Proof.* Let be  $s \in B$  fixed. The function

$$t \mapsto t^s \int_1^t u^{-1-s} \{u\} du, \quad t \geq 1,$$

is  $C^\infty$  on  $[1, +\infty)/\mathbb{N}$  and continuous on  $[1, +\infty)$ . The Equation (2) is a non-homogeneous linear differential equation. For every  $z \in \mathbb{C}$  the unique continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  such that  $\psi_s(z, 1) = z$  is given by

$$\psi_s(z, t) = t^s \left[ z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

□

**Proposition 3.** *Let be  $s \in B$  and consider the continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2). Then  $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$  if and only if the initial condition  $z \in \mathbb{C}$  satisfies the following equation*

$$z_s = -\frac{1}{4} \lim_{n \rightarrow +\infty} \left( \prod_{k=0}^{n+1} (k+1+s) \right) \eta_s(n).$$

where  $(\eta_s(n))_{n \in \mathbb{N}}$  is defined as in the Theorem 1.

Let us introduce the following notations,

**Notation 4.** Let  $g : \mathbb{R}_+ \rightarrow \mathbb{C}$  be a continuous function, we denote the function  $\Phi[g] : \mathbb{R}_+ \rightarrow \mathbb{C}$  as

$$\Phi[g](t) := \int_0^t g(u) du, \quad \forall t \geq 0,$$

**Notation 5.** For every  $k \in \mathbb{N}$  define the function  $p_k : \mathbb{R} \rightarrow \mathbb{R}$  as

$$p_{2k+1}(t) := (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+3}} \sin(j2\pi t),$$

$$p_{2k}(t) := (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+2}} \left( \cos(j2\pi t) - 1 \right),$$

**Lemma 6.** *We have*

$$\int_0^t \left( \{u\} - \frac{1}{2} \right) du = p_0(t), \quad \forall t \geq 0,$$

and for every  $n \in \mathbb{N}^*$  we have

$$\Phi^n[p_0](t) = -\frac{1}{(n+2)!} \sum_{k=2}^{n+1} B_k \binom{n+2}{k} t^{n-k+2} + p_n(t), \quad \forall t \geq 0, \forall n \geq 1.$$

*Proof.* Prove that

$$\int_0^t \left( \{u\} - \frac{1}{2} \right) du = p_0(t), \quad \forall t \geq 0,$$

The function  $u \mapsto \{u\}$  is 1-periodic, then there exists a continuous 1-periodic function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t \frac{1}{2} + p(t), \quad \forall t \geq 0.$$

The function  $p$  is a piecewise  $C^\infty$ , continuous on  $\mathbb{R}$  and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on  $\mathbb{R}_+$  to the function  $t \mapsto p(t)$ , where  $(a_k)_k \subset \mathbb{C}$  are the Fourier coefficients of the function  $p$ .

$$\int_0^t \left( \{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left( \int_0^u \left( \{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1)du = -\frac{1}{12} = -\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function  $p$  satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left( \exp(ij2\pi t) - 1 \right) = p_0(t), \quad \forall t \geq 0.$$

Integrate successively to obtain

$$\Phi^n[p_0](t) = \sum_{k=0}^n \frac{c_k}{(n-k)!} t^{n-k} + p_n(t), \quad \forall t \geq 0, \quad \forall n \geq 1,$$

where  $(c_k)_{k \in \mathbb{N}}$  defined as

$$c_{2k+1} = 0 \quad \text{and} \quad c_{2k} = (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}}, \quad \forall k \geq 0.$$

By definition of the Bernoulli numbers  $(B_k)_{k \in \mathbb{N}}$ , we get

$$\Phi^n[p_0](t) = -\frac{1}{(n+2)!} \sum_{k=2}^{n+1} B_k \binom{n+2}{k} t^{n-k+2} + p_n(t), \quad \forall t \geq 0, \quad \forall n \geq 1,$$

□

**Lemma 7.** *For every  $s \in B$ , there exists a unique  $z_s \in \mathbb{C}$  such that the solution  $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2) satisfies  $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ . Further,*

$$z_s = -\int_1^{+\infty} u^{-1-s} \{u\} du.$$

*Proof.* Let be  $s \in B$  fixed. There is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm. Let be  $z \in \mathbb{C}$  and suppose that

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty.$$

Since  $\Re(s) > 0$ , by the Lemma 2, we have

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty \implies \left| z + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

therefore

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Prove that  $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ . Since

$$\left| z_s + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

by consequence,

$$z_s + \int_1^t u^{-1-s} \{u\} du = - \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t > 1.$$

By the Lemma 2, we get

$$\psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Then  $\sup_{t \geq 1} |\psi_s(z_s, t)| \leq \frac{1}{\Re(s)}$ .  $\square$

*Proof of the Proposition 3.* Let be  $s \in B$  and consider the bounded continuous solution  $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$  defined in the Lemma 7. In order to simplify the notation, denote

$$\psi_s(t) := \psi_s(z_s, t + 1), \quad \forall t \geq 0.$$

Using the fact  $t \mapsto \psi_s(z_s, t)$  is solution of the differential equation (2). By definition of  $\Phi$  in the Notation 4, we get

$$\begin{aligned} (t+1) \frac{d}{dt} \Phi^{n+1}[\psi_s](t) &= (n+1+s) \Phi^{n+1}[\psi_s](t) \\ &\quad + z_s \frac{t^n}{n!} + h(n, t), \quad \forall t \geq 0, \quad \forall n \geq 0, \\ h(n, t) &:= \frac{1}{2} \frac{t^{n+1}}{(n+1)!} + \Phi^n[p_0](t), \quad \forall t \geq 0, \quad \forall n \geq 0, \\ \Phi^{n+1}[\psi_s](0) &= 0, \quad \forall n \geq 0, \end{aligned} \tag{3}$$

where the function  $p_0$  is defined in the Notation 5. Integrate, for every  $n \geq 0$  and  $t \geq 0$  we have

$$\Phi^{n+1}[\psi_s](t) = (t+1)^{n+1+s} \int_0^t (u+1)^{-n-2-s} \left[ z_s \frac{u^n}{n!} + h(n, u) \right] du. \tag{4}$$

Since  $\sup_{v \geq 0} |\psi_s(v)| = \sup_{v \geq 1} |\psi_s(z_s, v)| < +\infty$ , then for every fixed  $n \geq 0$  we have

$$\left| \Phi^{n+1}[\psi_s](t) \right| \leq \Phi^{n+1}[1](t) \sup_{v \geq 0} |\psi_s(v)| = \frac{t^{n+1}}{(n+1)!} \sup_{v \geq 1} |\psi_s(z_s, v)|, \quad \forall t \geq 0,$$

The Equation (12) implies

$$\int_0^{+\infty} (u+1)^{-n-2-s} \left[ z_s \frac{u^n}{n!} + h(n, u) \right] du = 0, \quad \forall n \geq 0,$$

in other words

$$\frac{z_s}{\prod_{k=1}^{n+1} (k+s)} = - \int_0^{+\infty} (u+1)^{-n-2-s} h(n, u) du, \quad \forall n \geq 0, \quad (5)$$

To know, the previous equality allows us to obtain the representation of the Zeta function obtained by the Euler-Maclaurin formula. Let be  $\psi_s(0, t) : [1, +\infty) \rightarrow \mathbb{C}$  the unique continuous solution of the differential equation (2) of initial condition  $|\psi_s(0, 1)| = 0$ . We recall that  $t \mapsto \psi_s(z, t)$  is defined in the Lemma 2 for every initial condition  $z \in \mathbb{C}$ . In order to simplify the notation, denote

$$\psi_{0,s}(t) := \psi_s(0, t+1), \quad \forall t \geq 0.$$

Using the fact  $t \mapsto \psi_s(0, t)$  is solution of the differential equation (2). As in the Equation (3), we get

$$(t+1) \frac{d}{dt} \Phi^{n+1}[\psi_{0,s}](t) = (n+1+s) \Phi^{n+1}[\psi_{0,s}](t) + h(n, t), \quad (6)$$

$$\Phi^{n+1}[\psi_{0,s}](0) = 0, \quad \forall n \geq 0, \forall t \geq 0,$$

where we recall that

$$h(n, t) = \frac{1}{2} \frac{t^{n+1}}{(n+1)!} + \Phi^n[p_0](t), \quad \forall t \geq 0, \quad \forall n \geq 0.$$

Integrate the last differential equation,

$$\Phi^{n+1}[\psi_{0,s}](t) = (t+1)^{n+1+s} \int_0^t (u+1)^{-n-2-s} h(n, u) du, \quad \forall t \geq 0, \quad \forall n \geq 0.$$

which can be written as

$$\Phi^{n+1}[\psi_{0,s}](t) = (t+1)^{n+1+s} \int_0^{+\infty} (u+1)^{-n-2-s} h(n, u) du$$

$$- (t+1)^{n+1+s} \int_t^{+\infty} (u+1)^{-n-2-s} h(n, u) du, \quad \forall t \geq 0, \quad \forall n \geq 0.$$



The Equation (5) implies

$$\begin{aligned} \Phi^{n+1}[\psi_{0,s}](t) &= -\frac{z_s}{\prod_{k=1}^{n+1}(k+s)}(t+1)^{n+1+s} \\ &\quad - (t+1)^{n+1+s} \int_t^{+\infty} (u+1)^{-n-2-s} h(n, u) du. \end{aligned} \quad (7)$$

In order to simplify the notation, denote

$$\delta_s(n, t) := \Phi^{n+1}[\psi_{0,s}](t+1) - \Phi^{n+1}[\psi_{0,s}](t), \quad \forall n \geq 0, \quad \forall t \geq 0. \quad (8)$$

By the Lemma 6, we have

$$h(n, u) = p_n(u) - q(n, u). \quad (9)$$

where  $p_n$  are given by the Notation 5 and where in order to simplify the notation, we denoted  $q(n, u)$  the following polynomial

$$q(n, u) := -\frac{1}{2} \frac{u^{n+1}}{(n+1)!} + \frac{1}{(n+2)!} \sum_{k=2}^{n+1} B_k \binom{n+2}{k} u^{n-k+2}.$$

or since  $B_1 = -\frac{1}{2}$ , the function  $q_s$  can be written as

$$q(n, u) := \frac{1}{(n+2)!} \sum_{k=1}^{n+1} B_k \binom{n+2}{k} u^{n-k+2} = \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{u^{n+2-k}}{(n+2-k)!}. \quad (10)$$

From the Equation (6), the periodicity of the function  $p_n$  implies

$$\begin{aligned} (t+1) \frac{d}{dt} \delta_s(n, t) &= (n+1+s) \delta_s(n, t) - q(n, t+1) + q(n, t) \\ &\quad - \frac{d}{dt} \Phi^{n+1}[\psi_{0,s}](t+1), \quad \forall t \geq 0, \quad \forall n \geq 0, \\ \delta_s(n, 0) &= \Phi^{n+1}[\psi_{0,s}](1), \quad \forall n \geq 0. \end{aligned}$$

Since

$$\frac{d}{dt} \Phi^{n+1}[\psi_{0,s}](t) = \Phi^n[\psi_{0,s}](t), \quad \forall t > 0, \quad \forall n \geq 0,$$

the previous differential Equation can be written as

$$\begin{aligned} \frac{d}{dt} \delta_s(n, t) &= (t+1)^{-1} \left[ (n+1+s) \delta_s(n, t) + \tilde{q}(n, t) - \Phi^n[\psi_{0,s}](t+1) \right], \\ \tilde{q}(n, t) &:= q(n, t) - q(n, t+1), \quad \forall t \geq 0, \quad \forall n \geq 0, \\ \delta_s(n, 0) &= \Phi^{n+1}[\psi_{0,s}](1), \quad \forall n \geq 0. \end{aligned} \quad (11)$$

Integrate , for every  $n \geq 0$  and  $t \geq 0$  we have

$$\frac{\delta_s(n, t)}{(t+1)^{n+1+s}} = \Phi^{n+1}[\psi_{0,s}](1) + \int_0^t \frac{\tilde{q}(n, u) - \Phi^n[\psi_{0,s}](u+1)}{(u+1)^{n+2+s}} du. \quad (12)$$

Since  $\sup_{v \geq 0} |(v+1)^{-\Re(s)} \psi_{0,s}(v)| = \sup_{v \geq 1} |v^{-\Re(s)} \psi_s(0, v)| < +\infty$ , then for every fixed  $n \geq 0$  we have

$$\begin{aligned} \left| \Phi^{n+1}[\psi_{0,s}](t) \right| &\leq (t+1)^{\Re(s)} \Phi^{n+1}[1](t) \sup_{v \geq 0} |(v+1)^{-\Re(s)} \psi_{0,s}(v)| \quad (13) \\ &= (t+1)^{\Re(s)} \frac{t^{n+1}}{(n+1)!} \sup_{v \geq 1} |v^{-\Re(s)} \psi_s(0, v)|, \quad \forall t \geq 0, \end{aligned}$$

By the notation of  $\delta_s$  in the Equation (8) and by the definition of  $\Phi$  in the Notation 4, we have

$$\delta_s(n, t) := \int_t^{t+1} \Phi^n[\psi_{0,s}](u) du, \quad \forall n \geq 0, \forall t \geq 0. \quad (14)$$

The Equation (13) implies

$$\begin{aligned} |\delta_s(n, t)| &= \left| \int_t^{t+1} \Phi^n[\psi_{0,s}](u) du \right| \\ &\leq (t+2)^{\Re(s)} \frac{(t+1)^n}{n!} \sup_{v \geq 1} |v^{-\Re(s)} \psi_s(0, v)|, \quad \forall t \geq 0, \quad \forall n \geq 0. \end{aligned}$$

By consequence

$$\lim_{t \rightarrow +\infty} \left| (t+1)^{-n-1-s} \delta_s(n, t) \right| = 0, \quad \forall n \geq 0.$$

From the Equations (12) we obtain

$$\Phi^{n+1}[\psi_{0,s}](1) + \int_0^{+\infty} \frac{\tilde{q}(n, u) - \Phi^n[\psi_{0,s}](u+1)}{(u+1)^{n+2+s}} du = 0, \quad \forall n \geq 0. \quad (15)$$

and the function  $t \mapsto \delta_s(n, t)$  can be written as

$$\frac{\delta_s(n, t)}{(t+1)^{n+1+s}} = - \int_t^{+\infty} \frac{\tilde{q}(n, u) - \Phi^n[\psi_{0,s}](u+1)}{(u+1)^{n+2+s}} du.$$

By the the Equation (14), for every  $n \geq 1$  and  $t \geq 0$ , the last equality implies

$$\int_t^{t+1} \Phi^n[\psi_{0,s}](u) du = -(t+1)^{n+1+s} \int_t^{+\infty} \frac{\tilde{q}(n, u) - \Phi^n[\psi_{0,s}](u+1)}{(u+1)^{n+2+s}} du.$$

The Equation (7) implies

$$\begin{aligned}
& \frac{z_s}{\prod_{k=1}^n (k+s)} \left[ \int_t^{+\infty} \frac{(u+2)^{n+s}}{(u+1)^{n+2+s}} du - (t+1)^{-n-1-s} \int_t^{t+1} (u+1)^{n+s} du \right] \\
&= (t+1)^{-n-1-s} \int_t^{t+1} (u+1)^{n+s} \int_u^{+\infty} (v+1)^{-n-1-s} h(n-1, v) dv du \\
&- \int_t^{+\infty} (u+1)^{-n-2-s} (u+2)^{n+s} \int_{u+1}^{+\infty} (v+1)^{-n-1-s} h(n-1, v) dv du \\
&- \int_t^{+\infty} (u+1)^{-n-2-s} \tilde{q}(n, u) du, \quad \forall n \geq 1, \forall t \geq 0.
\end{aligned}$$

In particular, for every  $n \geq 1$  we have

$$\begin{aligned}
& \frac{z_s}{\prod_{k=1}^n (k+s)} \left[ \int_n^{+\infty} \frac{(u+2)^{n+s}}{(u+1)^{n+2+s}} du - \frac{1}{n+1+s} \left( \left( \frac{n+2}{n+1} \right)^{n+1+s} - 1 \right) \right] \\
&= (n+1)^{-n-1-s} \int_n^{n+1} (u+1)^{n+s} \int_u^{+\infty} (v+1)^{-n-1-s} h(n-1, v) dv du \\
&- \int_n^{+\infty} (u+1)^{-n-2-s} (u+2)^{n+s} \int_{u+1}^{+\infty} (v+1)^{-n-1-s} h(n-1, v) dv du \\
&- \int_n^{+\infty} (u+1)^{-n-2-s} \tilde{q}(n, u) du.
\end{aligned}$$

Using the notation of  $h$  given by the Equation (9), we have obtained

$$z_s = \frac{1}{\omega_s(n)} \left( \prod_{k=1}^{n+1} (k+s) \right) \left[ \eta_s(n) - \epsilon_s(n) \right], \quad \forall n \geq 1.$$

where

$$\begin{aligned}
\omega_s(n) &:= 1 - \left(\frac{n+2}{n+1}\right)^{n+1+s} + (n+1+s) \int_n^{+\infty} \frac{(u+2)^{n+s}}{(u+1)^{n+2+s}} du, \\
\eta_s(n) &:= \int_n^{+\infty} \frac{(u+2)^{n+s}}{(u+1)^{n+2+s}} \int_{u+1}^{+\infty} \frac{q(n-1, v)}{(v+1)^{n+1+s}} dv du \\
&\quad - (n+1)^{-n-1-s} \int_n^{n+1} (u+1)^{n+s} \int_u^{+\infty} \frac{q(n-1, v)}{(v+1)^{n+1+s}} dv du \\
&\quad + \int_n^{+\infty} (u+1)^{-n-2-s} \tilde{q}(n, u) du, \\
\epsilon_s(n) &:= \int_n^{+\infty} \frac{(u+2)^{n+s}}{(u+1)^{n+2+s}} \int_{u+1}^{+\infty} \frac{p_{n-1}(v)}{(v+1)^{n+1+s}} dv du \\
&\quad - (n+1)^{-n-1-s} \int_n^{n+1} (u+1)^{n+s} \int_u^{+\infty} \frac{p_{n-1}(v)}{(v+1)^{n+1+s}} dv du.
\end{aligned}$$

By the notation of  $p_n$  in the Notation 5 we have  $\sup_{n \geq 0} \max_{v \in [0,1]} |p_n(v)| < +\infty$ . By Stirling formula, we get

$$\lim_{n \rightarrow +\infty} \left( \prod_{k=1}^{n+2} (k+s) \right) |\epsilon_s(n)| = 0.$$

For the limit of  $(n+2+s)\omega_s(n)$ , we have

$$\begin{aligned}
\omega_s(n) &= - \int_1^{+\infty} v^{-2} \exp(v^{-1}) dv + \exp(1) - \left(\frac{n+2}{n+1}\right)^{n+1+s} \\
&\quad + \int_1^{+\infty} \left(\frac{nv+2}{nv+1}\right)^s \frac{n(n+1+s)}{(nv+1)^2} \exp\left(n \ln\left(\frac{nv+2}{nv+1}\right)\right) dv \\
&= \exp(1) \left[ 1 - \exp\left((n+1) \ln\left(\frac{n+2}{n+1}\right) - 1\right) \right] \\
&\quad - \left( \left(\frac{n+2}{n+1}\right)^s - 1 \right) \left(\frac{n+2}{n+1}\right)^{n+1} \\
&\quad + \int_1^{+\infty} \left(\frac{nv+2}{nv+1}\right)^s \left( \frac{n(n+1+s)}{(nv+1)^2} - v^{-2} \right) \exp\left(n \ln\left(\frac{nv+2}{nv+1}\right)\right) dv \\
&\quad + \int_1^{+\infty} \frac{\exp(v^{-1})}{v^2} \left(\frac{nv+2}{nv+1}\right)^s \left[ \exp\left(n \ln\left(\frac{nv+2}{nv+1}\right) - v^{-1}\right) - 1 \right] dv \\
&\quad + \int_1^{+\infty} v^{-2} \exp(v^{-1}) \left( \left(\frac{nv+2}{nv+1}\right)^s - 1 \right) dv.
\end{aligned}$$

Use the Taylor formula, we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} (n+2+s)\omega_s(n) &= \left(\frac{1}{2} - s\right) \exp(1) + (1+s) \int_1^{+\infty} v^{-2} \exp(v^{-1}) dv \\ &\quad + s \int_1^{+\infty} v^{-3} \exp(v^{-1}) dv - \frac{3}{2} \int_1^{+\infty} v^{-4} \exp(v^{-1}) dv \\ &= -4. \end{aligned}$$

By consequence

$$z_s = -\frac{1}{4} \lim_{n \rightarrow +\infty} \left( \prod_{k=0}^{n+1} (k+1+s) \right) \eta_s(n).$$

□

### 3 Proof of the Theorem 1

*Proof of the Theorem 1.* Let be  $s \in B$ . By the Equation (1) we have,

$$\frac{\zeta(s)}{s} + \frac{1}{1-s} = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Thanks to the Main Proposition 3 and the Lemma 7 we get

$$\frac{\zeta(s)}{s} + \frac{1}{1-s} = -\frac{1}{4} \lim_{n \rightarrow +\infty} \left( \prod_{k=0}^{n+1} (k+1+s) \right) \eta_s(n).$$

□

### 4 Conclusion

Let be  $s \in B$  such that  $\Re(s) \in (0, 1)$  and suppose that  $|\zeta(s)| = 0$ . By Theorem 1, we have

$$\lim_{n \rightarrow +\infty} \left( \prod_{k=0}^{n+2} (k+s) \right) \eta_s(n) = -4 \frac{s}{1-s}, \quad (16)$$

Thanks to the Riemann functional equation, the point  $1-s$  is also a zero of the function  $\zeta$ , then

$$\lim_{n \rightarrow +\infty} \left( \prod_{k=0}^{n+2} (k+1-s) \right) \eta_{1-s}(n) = -4 \frac{1-s}{s}. \quad (17)$$

The terms  $\eta_x(n)$  are integrable for every  $n \geq 1$  and  $x \in B$ . For  $2\Re(s) \leq 1$ , we suggest that

$$\lim_{n \rightarrow +\infty} \frac{n^{\Re(s)}}{n^{1-\Re(s)}} \left| \frac{\eta_s(n)}{\eta_{1-s}(n)} \right| \geq 1,$$

The Equations (16) and (17) implies

$$\begin{aligned} \left| \frac{s^2}{(1-s)^2} \right| &= \lim_{n \rightarrow +\infty} \left| \left( \prod_{k=0}^{n+2} \frac{k+s}{k+1-s} \right) \frac{\eta_s(n)}{\eta_{1-s}(n)} \right| \\ &\geq \lim_{n \rightarrow +\infty} \left| \frac{n^{1-s}}{n^s} \prod_{k=0}^{n+2} \frac{k+s}{k+1-s} \right| = \left| \frac{\Gamma(1-s)}{\Gamma(s)} \right|, \end{aligned}$$

where  $\Gamma$  is the Gamma function.

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).