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Non trivial zeros of the Zeta function using the differential equations

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Abstract

Using the differential equations, we obtain a more flexible expression for the Riemann Zeta function on the right half-plan except the point $s = 1$. Thanks to the Riemann functional equation, we obtain an approach to prove that the Zeta function does not vanish at any point s of the critical strip such that $1 - 2\Re(s) \neq 0$.

Keywords: Zeta function, Non trivial zeros, Riemann functional equation, Bernoulli numbers, Differential equations.

AMS subject classifications: 00A05

1 Main results

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} u^{-1-s} \{u\} du, \quad s \neq 1, \Re(s) > 0, \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{u\}$ is the fractional part of the real u . In order to simplify the notation, denote $B \subset \mathbb{C}$ the right half-plan except the point $s = 1$, defined as

$$B := \{s \in \mathbb{C} : s \neq 1, \Re(s) > 0, \Im(s) \in \mathbb{R}^*\},$$

In this manuscript, we denote $(B_k)_{k \in \mathbb{N}}$ the Bernoulli numbers. We prove the following theorem,

Theorem 1. Consider the Zeta function given by the Equation (1). For every $s \in B$ we have

$$\zeta(s) = -\frac{s}{1-s} + s \frac{\eta_s(n)}{\tilde{\eta}_s(n)}, \quad \forall n \geq 0,$$

where

$$\begin{aligned} \tilde{\eta}_s(n) &:= \sum_{k=1}^{n+1} \frac{2^{n+1-k}}{\prod_{j=k}^{n+1} (j+s)} \frac{1}{(k-1)!} - \frac{1}{n!(1+s)} \\ &\quad - \sum_{k=0}^{n-1} \frac{1}{\prod_{j=1}^{k+1} (n+2-j+s)} \sum_{j=1}^{n-k} \frac{2^{n-k-j}}{\prod_{m=j}^{n-k} (m+s)} \frac{1}{(j-1)!} \\ &\quad - 2^s \left[\frac{2^{n+1}}{\prod_{j=1}^{n+1} (j+s)} - \sum_{k=0}^{n-1} \frac{1}{\prod_{j=1}^{k+1} (n+2-j+s)} \frac{2^{n-k}}{\prod_{j=1}^{n-k} (j+s)} \right] \\ \eta_s(n) &:= \frac{1+s-2^s}{s(1-s)} \left[\frac{2^{n+1}}{\prod_{j=1}^{n+1} (j+s)} - \sum_{k=0}^{n-1} \frac{1}{\prod_{j=1}^{k+1} (n+2-j+s)} \frac{2^{n-k}}{\prod_{j=1}^{n-k} (j+s)} \right] \\ &\quad - \sum_{k=1}^{n+1} \frac{B_k}{k!} \left[\frac{1}{(n+2-k)!(k-1+s)} - \prod_{j=0}^{n+2-k} \frac{1}{n+1+s-j} \right] \\ &\quad - \sum_{k=0}^{n-1} \frac{1}{\prod_{j=1}^{k+1} (n+2-j+s)} \sum_{j=1}^{n-k} \frac{2^{n-k-j} q(j-1, 1)}{\prod_{m=j}^{n-k} (m+s)} \\ &\quad + \sum_{k=1}^{n+1} \frac{2^{n+1-k} q(k-1, 1)}{\prod_{j=k}^{n+1} (j+s)} \\ &\quad - \frac{1}{\prod_{j=1}^n (j+1+s)} \int_0^{+\infty} (u+1)^{-2-s} \psi_s(u+1) du. \end{aligned}$$

and where

$$\begin{aligned} q(n, u) &:= \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{u^{n+2-k}}{(n+2-k)!}, \quad \forall n \geq 0, \quad \forall u \geq 0, \\ \psi_s(u) &:= -(u+1)^s \int_{u+1}^{+\infty} v^{-1-s} \{v\} dv, \quad \forall u \geq 0. \end{aligned}$$

2 Main Proposition

For every $s \in B$, the Equation (1) is equivalent to,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s}\{u\}du.$$

The aim is to studies the differential equation of solutions the functions

$$t \mapsto \psi_s(z, t) := t^s \left[z + \int_1^t u^{-1-s}\{u\}du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to find the initial condition $z_s \in \mathbb{C}$ such that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$ if and only if $z = -\int_1^{+\infty} u^{-1-s}\{u\}du$, we obtain $z_s = -\int_1^{+\infty} u^{-1-s}\{u\}du$. For every $s \in B$ we consider the following differential equation

$$\begin{aligned} \frac{d}{dt}x &= st^{-1}x + t^{-1}\{t\}, & (2) \\ t \in [1, +\infty)/\mathbb{N}, \quad x(1) &= z, \quad z \in \mathbb{C}, \quad x : [1, +\infty) \rightarrow \mathbb{C}. \end{aligned}$$

Lemma 2. *Let be $s \in B$. For every $z \in \mathbb{C}$ there exists a unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Further,*

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s}\{u\}du \right], \quad \forall t \geq 1.$$

Proof. Let be $s \in B$ fixed. The function

$$t \mapsto t^s \int_1^t u^{-1-s}\{u\}du, \quad t \geq 1,$$

is C^∞ on $[1, +\infty)/\mathbb{N}$ and continuous on $[1, +\infty)$. The Equation (2) is a non-homogeneous linear differential equation. For every $z \in \mathbb{C}$ the unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ such that $\psi_s(z, 1) = z$ is given by

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s}\{u\}du \right], \quad \forall t \geq 1.$$

□

Proposition 3. *Let be $s \in B$ and consider the continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Then $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$ if and only if the initial condition $z \in \mathbb{C}$ satisfies the following equation*

$$z = \frac{\eta_s(n)}{\tilde{\eta}_s(n)}, \quad \forall n \geq 0,$$

where $(\eta_s(n))_{n \in \mathbb{N}}$ and $(\tilde{\eta}_s(n))_{n \in \mathbb{N}}$ are defined as in the Theorem 1.

Let us introduce the following notations,

Notation 4. Let $g : \mathbb{R}_+ \rightarrow \mathbb{C}$ be a continuous function, we denote the function $\Phi[g] : \mathbb{R}_+ \rightarrow \mathbb{C}$ as

$$\Phi[g](t) := \int_0^t g(u) du, \quad \forall t \geq 0,$$

Notation 5. For every $k \in \mathbb{N}$ define the function $p_k : \mathbb{R} \rightarrow \mathbb{R}$ as

$$p_{2k+1}(t) := (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+3}} \sin(j2\pi t),$$

$$p_{2k}(t) := (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+2}} (\cos(j2\pi t) - 1),$$

Lemma 6. *We have*

$$\int_0^t \left(\{u\} - \frac{1}{2} \right) du = p_0(t), \quad \forall t \geq 0,$$

and for every $n \in \mathbb{N}^*$ we have

$$\Phi^n[p_0](t) = -\frac{1}{(n+2)!} \sum_{k=2}^{n+1} B_k \binom{n+2}{k} t^{n-k+2} + p_n(t), \quad \forall t \geq 0, \forall n \geq 1.$$

Proof. Prove that

$$\int_0^t \left(\{u\} - \frac{1}{2} \right) du = p_0(t), \quad \forall t \geq 0,$$

The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t \frac{1}{2} + p(t), \quad \forall t \geq 0.$$

The function p is a piecewise C^∞ , continuous on \mathbb{R} and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on \mathbb{R}_+ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p .

$$\int_0^t \left(\{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left(\int_0^u \left(\{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12} = -\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function p satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left(\exp(ij2\pi t) - 1 \right) = p_0(t), \quad \forall t \geq 0.$$

Integrate successively to obtain

$$\Phi^n[p_0](t) = \sum_{k=0}^n \frac{c_k}{(n-k)!} t^{n-k} + p_n(t), \quad \forall t \geq 0, \quad \forall n \geq 1,$$

where $(c_k)_{k \in \mathbb{N}}$ defined as

$$c_{2k+1} = 0 \quad \text{and} \quad c_{2k} = (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}}, \quad \forall k \geq 0.$$

By definition of the Bernoulli numbers $(B_k)_{k \in \mathbb{N}}$, we get

$$\Phi^n[p_0](t) = -\frac{1}{(n+2)!} \sum_{k=2}^{n+1} B_k \binom{n+2}{k} t^{n-k+2} + p_n(t), \quad \forall t \geq 0, \forall n \geq 1,$$

□

Lemma 7. *For every $s \in B$, there exists a unique $z_s \in \mathbb{C}$ such that the solution $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2) satisfies $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Further,*

$$z_s = -\int_1^{+\infty} u^{-1-s} \{u\} du,$$

and

$$\psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Proof. Let be $s \in B$ fixed. There is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm. Let be $z \in \mathbb{C}$ and suppose that

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty.$$

Since $\Re(s) > 0$, by the Lemma 2, we have

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty \implies \left| z + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

Then

$$z_s = -\int_1^{+\infty} u^{-1-s} \{u\} du.$$

Prove that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since

$$\left| z_s + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

by consequence,

$$z_s + \int_1^t u^{-1-s} \{u\} du = -\int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t > 1.$$

By the Lemma 2, we get

$$\psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Then $\sup_{t \geq 1} |\psi_s(z_s, t)| \leq \frac{1}{\Re(s)}$. □

Proof of the Proposition 3. Let be $s \in B$ and consider the bounded continuous solution $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$ defined in the Lemma 7. In order to simplify the notation, denote

$$\psi_s(t) := \psi_s(z_s, t + 1), \quad \forall t \geq 0.$$

Using the fact $t \mapsto \psi_s(z_s, t)$ is solution of the differential equation (2). By definition of Φ in the Notation 4, we get

$$(t + 1) \frac{d}{dt} \Phi^{n+1}[\psi_s](t) = (n + 1 + s) \Phi^{n+1}[\psi_s](t) \quad (3)$$

$$+ h_s(n, t), \quad \forall t \geq 0, \quad \forall n \geq 1,$$

$$\Phi^{n+1}[\psi_s](0) = 0, \quad \forall n \geq 1,$$

$$h_s(n, t) := \frac{1}{2} \frac{t^{n+1}}{(n+1)!} + z_s \frac{t^n}{n!} + \Phi^n[p_0](t), \quad \forall t \geq 0, \quad \forall n \geq 0,$$

where the function p_0 is defined in the Notation 5. By the Lemma 6, we have

$$h_s(n, u) = p_n(u) + z_s \frac{u^n}{n!} - q(n, u). \quad (4)$$

where p_n are given by the Notation 5 and where in order to simplify the notation, we denoted $q(n, u)$ the following polynomial

$$q(n, u) := -\frac{1}{2} \frac{u^{n+1}}{(n+1)!} + \frac{1}{(n+2)!} \sum_{k=2}^{n+1} B_k \binom{n+2}{k} u^{n-k+2}.$$

or since $B_1 = -\frac{1}{2}$, the function q_s can be written as

$$q(n, u) := \frac{1}{(n+2)!} \sum_{k=1}^{n+1} B_k \binom{n+2}{k} u^{n-k+2} = \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{u^{n+2-k}}{(n+2-k)!}. \quad (5)$$

In order to simplify the notation, denote

$$\delta_s(n, t) := \Phi^{n+1}[\psi_s](t + 1) - \Phi^{n+1}[\psi_s](t), \quad \forall n \geq 0, \quad \forall t \geq 0.$$

From the Equation (3), the periodicity of the function p_n implies

$$\begin{aligned} (t + 1) \frac{d}{dt} \delta_s(n, t) &= (n + 1 + s) \delta_s(n, t) + \frac{z_s}{n!} ((t + 1)^n - t^n) \\ &\quad - q(n, t + 1) + q(n, t) \\ &\quad - \frac{d}{dt} \Phi^{n+1}[\psi_s](t + 1), \quad \forall t \geq 0, \quad \forall n \geq 1, \\ \delta_s(n, 0) &= \Phi^{n+1}[\psi_s](1), \quad \forall n \geq 1. \end{aligned}$$

Since

$$\frac{d}{dt}\Phi^{n+1}[\psi_s](t) = \Phi^n[\psi_s](t), \quad \forall t > 0, \quad \forall n \geq 0,$$

the previous differential Equation can be written as

$$\begin{aligned} \frac{d}{dt}\delta_s(n, t) &= (t+1)^{-1} \left[(n+1+s)\delta_s(n, t) + \tilde{h}_s(n, t) - \Phi^n[\psi_s](t+1) \right], \\ \tilde{h}_s(n, t) &:= \frac{z_s}{n!} ((t+1)^n - t^n) - q(n, t+1) + q(n, t), \quad \forall t \geq 0, \quad \forall n \geq 1, \\ \delta_s(n, 0) &= \Phi^{n+1}[\psi_s](1), \quad \forall n \geq 1. \end{aligned} \quad (6)$$

Integrate , for every $n \geq 1$ and $t \geq 0$ we have

$$\delta_s(n, t) = (t+1)^{n+1+s} \left[\Phi^{n+1}[\psi_s](1) + \int_0^t \frac{\tilde{h}_s(n, u) - \Phi^n[\psi_s](u+1)}{(u+1)^{n+2+s}} du \right]. \quad (7)$$

Since $\sup_{v \geq 0} |\psi_s(v)| = \sup_{v \geq 1} |\psi_s(z_s, v)| < +\infty$, then for every fixed $n \geq 1$ we have

$$\left| \Phi^{n+1}[\psi_s](t) \right| \leq \Phi^{n+1}[1](t) \sup_{t \geq 0} |\psi_s(t)| = \frac{t^{n+1}}{(n+1)!} \sup_{v \geq 0} |\psi_s(v)|, \quad \forall t \geq 0,$$

Using the last inequality and the notation of $\delta_s(n, t)$ we get

$$\begin{aligned} |\delta_s(n, t)| &\leq \int_t^{t+1} \Phi^n[1](u) du \left(\sup_{v \geq 0} |\psi_s(v)| \right) \\ &= \frac{(t+1)^{n+1} - t^{n+1}}{(n+1)!} \sup_{v \geq 0} |\psi_s(v)|, \quad \forall t \geq 0, \quad \forall n \geq 0. \end{aligned}$$

Since $\Re(s) > 0$ then

$$\lim_{t \rightarrow +\infty} \left| (t+1)^{-n-1-s} \delta_s(n, t) \right| = 0, \quad \forall n \geq 0.$$

The Equations (7) implies

$$\Phi^{n+1}[\psi_s](1) + \int_0^{+\infty} \frac{\tilde{h}_s(n, u) - \Phi^n[\psi_s](u+1)}{(u+1)^{n+2+s}} du = 0, \quad \forall n \geq 0. \quad (8)$$

Use a successive integration by parts formula, for every $n \geq 0$ we have

$$\begin{aligned} \int_0^{+\infty} \frac{\Phi^n[\psi_s](u+1)}{(u+1)^{n+2+s}} du &= \sum_{k=0}^{n-1} \frac{\Phi^{n-k}[\psi_s](1)}{\prod_{j=1}^{k+1} (n+2-j+s)} \\ &\quad + \frac{1}{\prod_{j=1}^n (j+1+s)} \int_0^{+\infty} (u+1)^{-2-s} \psi_s(u+1) du. \end{aligned} \quad (9)$$

The Equation (3) implies

$$2\Phi^n[\psi_s](1) = (n + 1 + s)\Phi^{n+1}[\psi_s](1) + h_s(n, 1), \quad \forall n \geq 0,$$

then

$$\Phi^{n+1}[\psi_s](1) = \frac{2^{n+1}}{\prod_{j=1}^{n+1}(j+s)}\psi_s(1) - \sum_{k=1}^{n+1} \frac{2^{n+1-k}}{\prod_{j=k}^{n+1}(j+s)}h_s(k-1, 1). \quad (10)$$

By the Notation 5, we have $p_k(1) = 0$ for every $k \in \mathbb{N}$, then

$$h_s(k, 1) = z_s \frac{1}{k!} - q(k, 1), \quad \forall k \in \mathbb{N}.$$

By the notation of the function h_s , the Equation (10), can be written as

$$\begin{aligned} \Phi^{n+1}[\psi_s](1) &= \frac{2^{n+1}}{\prod_{j=1}^{n+1}(j+s)}\psi_s(1) \\ &\quad - \sum_{k=1}^{n+1} \frac{2^{n+1-k}}{\prod_{j=k}^{n+1}(j+s)} \left[z_s \frac{1}{(k-1)!} - q(k-1, 1) \right]. \end{aligned}$$

By consequence, for every $n \geq 0$, the Equations (8) and (9), implies

$$\begin{aligned} &\psi_s(1) \left[\frac{2^{n+1}}{\prod_{j=1}^{n+1}(j+s)} - \sum_{k=0}^{n-1} \frac{1}{\prod_{j=1}^{k+1}(n+2-j+s)} \frac{2^{n-k}}{\prod_{j=1}^{n-k}(j+s)} \right] \\ &- \sum_{k=1}^{n+1} \frac{2^{n+1-k}}{\prod_{j=k}^{n+1}(j+s)} \left[z_s \frac{1}{(k-1)!} - q(k-1, 1) \right] + \int_0^{+\infty} \frac{\tilde{h}_s(n, u)}{(u+1)^{n+2+s}} du \\ &+ \sum_{k=0}^{n-1} \frac{1}{\prod_{j=1}^{k+1}(n+2-j+s)} \sum_{j=1}^{n-k} \frac{2^{n-k-j}}{\prod_{m=j}^{n-k}(m+s)} \left[z_s \frac{1}{(j-1)!} - q(j-1, 1) \right] \\ &- \frac{1}{\prod_{j=1}^n(j+1+s)} \int_0^{+\infty} (u+1)^{-2-s} \psi_s(u+1) du \\ &= 0. \end{aligned}$$

By the notation of the function \tilde{h}_s in the Equation (6) and the notation of

the function q in the Equation (5), for every $n \geq 0$ we get

$$\begin{aligned}
& \psi_s(1) \left[\frac{2^{n+1}}{\prod_{j=1}^{n+1} (j+s)} - \sum_{k=0}^{n-1} \frac{1}{\prod_{j=1}^{k+1} (n+2-j+s)} \frac{2^{n-k}}{\prod_{j=1}^{n-k} (j+s)} \right] \\
& - \sum_{k=1}^{n+1} \frac{2^{n+1-k}}{\prod_{j=k}^{n+1} (j+s)} \left[z_s \frac{1}{(k-1)!} - q(k-1, 1) \right] \\
& + z_s \frac{1}{n!(1+s)} - \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{B_k}{(n+2-k)!(k-1+s)} \\
& + \sum_{k=1}^{n+1} \frac{B_k}{k!} \prod_{j=0}^{n+2-k} \frac{1}{n+1+s-j} \\
& + \sum_{k=0}^{n-1} \frac{1}{\prod_{j=1}^{k+1} (n+2-j+s)} \sum_{j=1}^{n-k} \frac{2^{n-k-j}}{\prod_{m=j}^{n-k} (m+s)} \left[z_s \frac{1}{(j-1)!} - q(j-1, 1) \right] \\
& - \frac{1}{\prod_{j=1}^n (j+1+s)} \int_0^{+\infty} (u+1)^{-2-s} \psi_s(u+1) du \\
& = 0.
\end{aligned}$$

By the Lemma 2 and the notation of the function ψ_s , we have

$$\psi_s(1) = \psi(z_s, 2) = 2^s \left[z_s + \int_1^2 u^{-1-s} \{u\} du \right] = z_s 2^s + \frac{1+s-2^s}{s(1-s)}.$$

We deduce that

$$z_s = \frac{\eta_s(n)}{\tilde{\eta}_s(n)}, \quad \forall n \geq 0,$$

where

$$\begin{aligned}
\tilde{\eta}_s(n) &:= \left[\sum_{k=1}^{n+1} \frac{2^{n+1-k}}{\Pi_{j=k}^{n+1}(j+s)} \frac{1}{(k-1)!} - \frac{1}{n!(1+s)} \right] \\
&\quad - \sum_{k=0}^{n-1} \frac{1}{\Pi_{j=1}^{k+1}(n+2-j+s)} \sum_{j=1}^{n-k} \frac{2^{n-k-j}}{\Pi_{m=j}^{n-k}(m+s)} \frac{1}{(j-1)!} \\
&\quad - 2^s \left[\frac{2^{n+1}}{\Pi_{j=1}^{n+1}(j+s)} - \sum_{k=0}^{n-1} \frac{1}{\Pi_{j=1}^{k+1}(n+2-j+s)} \frac{2^{n-k}}{\Pi_{j=1}^{n-k}(j+s)} \right] \\
\eta_s(n) &:= \frac{1+s-2^s}{s(1-s)} \left[\frac{2^{n+1}}{\Pi_{j=1}^{n+1}(j+s)} - \sum_{k=0}^{n-1} \frac{1}{\Pi_{j=1}^{k+1}(n+2-j+s)} \frac{2^{n-k}}{\Pi_{j=1}^{n-k}(j+s)} \right] \\
&\quad - \sum_{k=1}^{n+1} \frac{B_k}{k!} \left[\frac{1}{(n+2-k)!(k-1+s)} - \Pi_{j=0}^{n+2-k} \frac{1}{n+1+s-j} \right] \\
&\quad - \sum_{k=0}^{n-1} \frac{1}{\Pi_{j=1}^{k+1}(n+2-j+s)} \sum_{j=1}^{n-k} \frac{2^{n-k-j} q(j-1,1)}{\Pi_{m=j}^{n-k}(m+s)} \\
&\quad + \sum_{k=1}^{n+1} \frac{2^{n+1-k} q(k-1,1)}{\Pi_{j=k}^{n+1}(j+s)} \\
&\quad - \frac{1}{\Pi_{j=1}^n(j+1+s)} \int_0^{+\infty} (u+1)^{-2-s} \psi_s(u+1) du.
\end{aligned}$$

By the Lemma 7 and the notation of the function ψ_s , we have

$$\psi_s(u) = \psi(z_s, u+1) = -(u+1)^s \int_{u+1}^{+\infty} v^{-1-s} \{v\} dv.$$

□

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $s \in B$. By the Equation (1) we have,

$$\frac{\zeta(s)}{s} + \frac{1}{1-s} = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Thanks to the Main Proposition 3 and the Lemma 7 we get

$$\frac{\zeta(s)}{s} + \frac{1}{1-s} = \frac{\eta_s(n)}{\tilde{\eta}_s(n)}, \quad \forall n \geq 0.$$

□

4 Conclusion

In this conclusion, we present the approach to prove that the Zeta function does not vanish at any point s of the critical strip such that $1 - 2\Re(s) \neq 0$. Let be $s \in B$ such that $\Re(s) \in (0, 1)$ and suppose that $|\zeta(s)| = 0$. By Theorem 1, we have

$$(1 - s) \frac{\eta_s(n)}{\tilde{\eta}_s(n)} = 1, \quad \forall n \geq 0,$$

Thanks to the Riemann functional equation, the point $1 - s$ is also a zero of the function ζ , then

$$s \frac{\eta_{1-s}(n)}{\tilde{\eta}_{1-s}(n)} = 1, \quad \forall n \geq 0,$$

For $1 - 2\Re(s) \neq 0$, we suggest that

$$\lim_{n \rightarrow +\infty} \left((1 - s) \frac{\eta_s(n)}{\tilde{\eta}_s(n)} - s \frac{\eta_{1-s}(n)}{\tilde{\eta}_{1-s}(n)} \right) \neq 0,$$

in other words, we suggest that if $\lim_{n \rightarrow +\infty} \left| \frac{\eta_s(n)}{\tilde{\eta}_s(n)} \right| \neq 0$, then

$$\lim_{n \rightarrow +\infty} \frac{s}{(1 - s)} \frac{\tilde{\eta}_s(n)}{\tilde{\eta}_{1-s}(n)} \frac{(n + 1)! \eta_{1-s}(n)}{(n + 1)! \eta_s(n)} \neq 1.$$

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).