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# Non trivial zeros of the Zeta function using the differential equations

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## Abstract

Using the differential equations, we obtain a more flexible representation of the Zeta function in the critical strip. Thanks to the Riemann functional equation, we obtain an approach to prove that the Zeta function does not vanish at any point  $s$  of the critical strip such that  $1 - 2\Re(s) \neq 0$ .

**Keywords:** Zeta function, Non trivial zeros, Riemann functional equation, Bernoulli numbers, Differential equations.

**AMS subject classifications:** 00A05

## 1 Main results

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \{u\} u^{-1-s} du, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{u\}$  is the fractional part of the real  $u$ . In order to simplify the notation, denote  $B \subset \mathbb{C}$  the critical strip, defined as

$$B := \{s \in \mathbb{C} : \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*\},$$

In this manuscript, we denote  $(B_k)_{k \in \mathbb{N}}$  the Bernoulli numbers. We prove the following theorem,

**Theorem 1.** Consider the Zeta function given by the Equation (1). For every real sequence  $(t_n)_n \subset \mathbb{R}_+$  and for every  $s \in B$  we have

$$\lim_{n \rightarrow +\infty} \frac{t_n^n}{(n-1)!} \left[ V_s[q(n, u)](t_n) + \left( \frac{\zeta(s)}{s} + \frac{1}{1-s} \right) V_s[u^n](t_n) \right] = 0,$$

where

$$q(n, u) := - \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{n! u^{n-k+2}}{(n+2-k)!} \quad \forall n \geq 1, \quad \forall u \in \mathbb{R},$$

and where for every  $n \geq 1$  and every real polynomial  $u \in \mathbb{R} \mapsto g_n(u)$  of order less than  $n+1$ , the complex function  $t \in \mathbb{R}_+ \mapsto V_s[g_n(u)](t)$ , is defined as

$$\begin{aligned} V_s[g_n(u)](t) := & \int_t^{+\infty} \left( \frac{(t+1)^{1+s}}{(u+1)^{n+2+s}} - \frac{t^{1+s}}{u^{n+2+s}} \right) g_n(u) du \\ & - t^{1+s} \int_t^{+\infty} \frac{(u+1)^{n+s}}{u^{n+2+s}} \int_u^{+\infty} \frac{g_{n-1}(v)}{(v+1)^{n+1+s}} dv du. \end{aligned}$$

For the particular sequence  $(t_n = n)_{n \in \mathbb{N}^*}$ , the representation of the Zeta function given by the previous Theorem, can be written as

$$\lim_{n \rightarrow +\infty} n^{n+1} \left[ \frac{\eta_s(n)}{n!} \left( \frac{\zeta(s)}{s} + \frac{1}{1-s} \right) + \tilde{\eta}_s(n) \right] = 0,$$

where

$$\begin{aligned} \eta_s(n) := & \sum_{j=1}^n \binom{n}{j} \frac{(-1)^j (n+1)^{-j}}{j+1+s} \\ & - \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{j+1+s} \sum_{k=0}^{n-1-j} \binom{n-1-j}{k} \frac{n^{-j-k}}{j+k+2+s}, \\ \tilde{\eta}_s(n) = & - \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{n}{(n+2-k)!} \sum_{j=0}^{n-k+2} \binom{n-k+2}{j} \frac{(-1)^j (n+1)^{2-k-j}}{k+j-1+s} \\ & + \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{n^{2-k}}{(n+2-k)!(k-1+s)} \\ & + \sum_{k=1}^n \frac{B_k}{k!} \frac{1}{(n+1-k)!} \sum_{j=0}^{n-k+1} \frac{w_{n,k,j}}{k+j+1+s} \sum_{m=0}^{n-k-j} \frac{v_{n,k,j,m}}{k+j+m+1+s}, \\ w_{n,k,j} := & \binom{n-k+1}{j} \frac{(-1)^j}{k+j+1}, \quad v_{n,k,j,m} := \binom{n-k-j}{m} n^{-k-j-m}. \end{aligned}$$

In the conclusion of this manuscript, we present the approach to prove that the Zeta function does not vanish at any point  $s \in B$  such that  $1 - 2\Re(s) \neq 0$ .

## 2 Main Proposition

For every  $s \in B$ , the Equation (1) is equivalent to,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

The aim is to study the differential equation of solutions the functions

$$t \mapsto \psi_s(z, t) := t^s \left[ z + \int_1^t u^{-1-s} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to find the initial condition  $z_s \in \mathbb{C}$  such that  $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ . Since  $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$  if and only if  $z = -\int_1^{+\infty} u^{-1-s} \{u\} du$ , we obtain  $z_s = -\int_1^{+\infty} u^{-1-s} \{u\} du$ .

For every  $s \in B$  we consider the following differential equation

$$\begin{aligned} \frac{d}{dt} x &= st^{-1}x + t^{-1}\{t\}, & (2) \\ t \in [1, +\infty)/\mathbb{N}, \quad x(1) &= z, \quad z \in \mathbb{C}, \quad x : [1, +\infty) \rightarrow \mathbb{C}. \end{aligned}$$

**Lemma 2.** *Let be  $s \in B$ . For every  $z \in \mathbb{C}$  there exists a unique continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2). Further,*

$$\psi_s(z, t) = t^s \left[ z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

*Proof.* Let be  $s \in B$  fixed. The function

$$t \mapsto t^s \int_1^t u^{-1-s} \{u\} du, \quad t \geq 1,$$

is  $C^\infty$  on  $[1, +\infty)/\mathbb{N}$  and continuous on  $[1, +\infty)$ . The Equation (2) is a non-homogeneous linear differential equation. For every  $z \in \mathbb{C}$  the unique continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  such that  $\psi_s(z, 1) = z$  is given by

$$\psi_s(z, t) = t^s \left[ z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

□

**Proposition 3.** *Let be  $s \in B$  and consider the continuous solution  $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2). Then  $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$  if and only if for every real sequence  $(t_n)_n \subset \mathbb{R}_+$  the initial condition  $z \in \mathbb{C}$  satisfies the following equation*

$$\lim_{n \rightarrow +\infty} \frac{t_n^n}{(n-1)!} \left( V_s[q(n, u)](t_n) + zV_s[u^n](t_n) \right) = 0,$$

where  $q(n, u)$  and  $V_s$  are defined as in the Theorem 1.

Let us introduce the following notations,

**Notation 4.** Let  $g : \mathbb{R}_+ \rightarrow \mathbb{C}$  be a continuous function, we denote the function  $\Phi[g] : \mathbb{R}_+ \rightarrow \mathbb{C}$  as

$$\Phi[g](t) := \int_0^t g(u) du, \quad \forall t \geq 0,$$

**Notation 5.** For every  $k \in \mathbb{N}$  define the function  $p_k : \mathbb{R} \rightarrow \mathbb{R}$  as

$$p_{2k+1}(t) := (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+3}} \sin(j2\pi t),$$

$$p_{2k}(t) := (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+2}} \left( \cos(j2\pi t) - 1 \right),$$

**Lemma 6.** *We have*

$$\int_0^t \left( \{u\} - \frac{1}{2} \right) du = \tilde{p}_0(t) := \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left( \exp(ij2\pi t) - 1 \right), \quad \forall t \geq 0,$$

and for every  $n \in \mathbb{N}^*$  we have

$$\Phi^n[\tilde{p}_0](t) = -\frac{1}{(n+2)!} \sum_{k=2}^{n+1} B_k \binom{n+2}{k} t^{n-k+2} + p_n(t), \quad \forall t \geq 0, \quad \forall n \geq 1.$$

*Proof.* Prove that

$$\int_0^t \left( \{u\} - \frac{1}{2} \right) du = \tilde{p}_0(t), \quad \forall t \geq 0,$$

The function  $u \mapsto \{u\}$  is 1-periodic, then there exists a continuous 1-periodic function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t \frac{1}{2} + p(t), \quad \forall t \geq 0.$$

The function  $p$  is a piecewise  $C^\infty$ , continuous on  $\mathbb{R}$  and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on  $\mathbb{R}_+$  to the function  $t \mapsto p(t)$ , where  $(a_k)_k \subset \mathbb{C}$  are the Fourier coefficients of the function  $p$ .

$$\int_0^t \left( \{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left( \int_0^u \left( \{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12} = -\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function  $p$  satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left( \exp(ij2\pi t) - 1 \right) = \tilde{p}_0(t), \quad \forall t \geq 0.$$

Integrate successively to obtain

$$\Phi^n[\tilde{p}_0](t) = \sum_{k=0}^n \frac{c_k}{(n-k)!} t^{n-k} + p_n(t), \quad \forall t \geq 0, \quad \forall n \geq 1,$$

where  $(c_k)_{k \in \mathbb{N}}$  defined as

$$c_{2k+1} = 0 \quad \text{and} \quad c_{2k} = (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}}, \quad \forall k \geq 0.$$

By definition of the Bernoulli numbers  $(B_k)_{k \in \mathbb{N}}$ , we get

$$\Phi^n[\tilde{p}_0](t) = -\frac{1}{(n+2)!} \sum_{k=2}^{n+1} B_k \binom{n+2}{k} t^{n-k+2} + p_n(t), \quad \forall t \geq 0, \forall n \geq 1,$$

□

**Lemma 7.** *For every  $s \in B$ , there exists a unique  $z_s \in \mathbb{C}$  such that the solution  $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$  of the differential equation (2) satisfies  $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ . Further,*

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

*Proof.* Let be  $s \in B$  fixed. There is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm. Let be  $z \in \mathbb{C}$  and suppose that

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty.$$

Since  $\Re(s) \in (0, 1)$ , by the Lemma 2, we have

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty \implies \left| z + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

Then

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Prove that  $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ . Since

$$\left| z_s + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

by consequence,

$$z_s + \int_1^t u^{-1-s} \{u\} du = - \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t > 1.$$

By the Lemma 2, we get

$$\psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Then  $\sup_{t \geq 1} |\psi_s(z_s, t)| \leq \frac{1}{\Re(s)}$ . □

*Proof of the Proposition 3.* Let be  $s \in B$  and consider the bounded continuous solution  $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$  defined in the Lemma 7. In order to simplify the notation, denote

$$\tilde{\psi}_s(t) := \psi_s(z_s, t + 1), \quad \forall t \geq 0.$$

Using the fact  $t \mapsto \psi_s(z_s, t)$  is solution of the differential equation (2). By definition of  $\Phi$  in the Notation 4, we get

$$\begin{aligned} (t+1) \frac{d}{dt} \Phi^{n+1}[\tilde{\psi}_s](t) &= (n+1+s) \Phi^{n+1}[\tilde{\psi}_s](t) \\ &+ h_s(n, t), \quad \forall t \geq 0, \quad \forall n \geq 1, \\ \Phi^{n+1}[\tilde{\psi}_s](0) &= 0, \quad \forall n \geq 1, \\ h_s(n, t) &:= \frac{1}{2} \frac{t^{n+1}}{(n+1)!} + z_s \frac{t^n}{n!} + \Phi^n[\tilde{p}_0](t), \quad \forall t \geq 0, \quad \forall n \geq 1, \end{aligned} \quad (3)$$

where the function  $\tilde{p}_0$  is defined by the Lemma 6. The previous differential equation can be written as

$$\begin{aligned} \frac{d}{dt} \Phi^{n+1}[\tilde{\psi}_s](t) &= (n+1+s)t^{-1} \Phi^{n+1}[\tilde{\psi}_s](t) \\ &+ t^{-1} \left[ h_s(n, t) - \frac{d}{dt} \Phi^{n+1}[\tilde{\psi}_s](t) \right], \quad \forall t > 0, \quad \forall n \geq 1. \end{aligned} \quad (4)$$

Integrate the differential equation (3), for every  $n \geq 1$  and  $t \geq 0$  we have

$$\Phi^{n+1}[\tilde{\psi}_s](t) = (t+1)^{n+1+s} \int_0^t (u+1)^{-n-2-s} h_s(n, u) du. \quad (5)$$

Integrate the differential equation (4), for every  $n \geq 1$  and  $t \geq 1$  we get

$$\begin{aligned} \Phi^{n+1}[\tilde{\psi}_s](t) &= t^{n+1+s} \Phi^{n+1}[\tilde{\psi}_s](1) \\ &+ t^{n+1+s} \int_1^t u^{-n-2-s} \left[ h_s(n, u) - \frac{d}{dt} \Phi^{n+1}[\tilde{\psi}_s](u) \right] du. \end{aligned} \quad (6)$$

Since

$$\frac{d}{dt} \Phi^{n+1}[\tilde{\psi}_s](t) = \Phi^n[\tilde{\psi}_s](t), \quad \forall n \geq 1, \quad \forall t > 0,$$

the Equation (6) is equivalent to

$$\begin{aligned} \Phi^{n+1}[\tilde{\psi}_s](t) &= t^{n+1+s} \Phi^{n+1}[\tilde{\psi}_s](1) \\ &+ t^{n+1+s} \int_1^t u^{-n-2-s} \left[ h_s(n, u) - \Phi^n[\tilde{\psi}_s](u) \right] du. \end{aligned} \quad (7)$$



Since  $\sup_{t \geq 0} |\tilde{\psi}_s(t)| = \sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ , then for every fixed  $n \geq 1$  we have

$$\left| \Phi^{n+1}[\tilde{\psi}_s](t) \right| \leq \Phi^{n+1}[1](t) \sup_{t \geq 0} |\tilde{\psi}_s(t)| = \frac{t^{n+1}}{(n+1)!} \sup_{t \geq 0} |\tilde{\psi}_s(t)|, \quad \forall t \geq 0, \quad (8)$$

Since  $\Re(s) \in (0, 1)$  the Equations (5) and (7) implies

$$\begin{aligned} \int_0^{+\infty} (u+1)^{-n-2-s} h_s(n, u) du &= 0, & \forall n \geq 1 \\ \Phi^{n+1}[\tilde{\psi}_s](1) + \int_1^{+\infty} u^{-n-2-s} \left[ h_s(n, u) - \Phi^n[\tilde{\psi}_s](u) \right] du &= 0, & \forall n \geq 1, \end{aligned}$$

by consequence, for every  $n \geq 1$  and  $t \geq 1$ , the function  $t \mapsto \Phi^{n+1}[\tilde{\psi}_s](t)$  given by the Equations (5) and (7) can be written as

$$\begin{aligned} \Phi^{n+1}[\tilde{\psi}_s](t) &= -(t+1)^{n+1+s} \int_t^{+\infty} (u+1)^{-n-2-s} h_s(n, u) du, & (9) \\ \Phi^{n+1}[\tilde{\psi}_s](t) &= -t^{n+1+s} \int_t^{+\infty} u^{-n-2-s} \left[ h_s(n, u) - \Phi^n[\tilde{\psi}_s](u) \right] du. \end{aligned}$$

Implies

$$\left(\frac{t+1}{t}\right)^{1+s} \int_t^{+\infty} \frac{h_s(n, u)}{(u+1)^{n+2+s}} du = \int_t^{+\infty} \frac{h_s(n, u) - \Phi^n[\tilde{\psi}_s](u)}{u^{n+2+s}} du.$$

Use the Equation (9), to obtain

$$\begin{aligned} \left(\frac{t+1}{t}\right)^{1+s} \int_t^{+\infty} \frac{h_s(n, u)}{(u+1)^{n+2+s}} du &= \int_t^{+\infty} \frac{h_s(n, u)}{u^{n+2+s}} du & (10) \\ &+ \int_t^{+\infty} \frac{(u+1)^{n+s}}{u^{n+2+s}} \int_u^{+\infty} \frac{h_s(n-1, v)}{(v+1)^{n+1+s}} dv du. \end{aligned}$$

By the notation of  $h_s$  and by the Lemma 6, we have

$$h_s(n, u) = p_n(u) + \frac{1}{n!} \left[ z_s u^n + q(n, u) \right]. \quad (11)$$

where in order to simplify the notation, we denoted  $q(n, u)$  the following polynomial

$$q(n, u) := \frac{1}{2} \frac{n! u^{n+1}}{(n+1)!} - \frac{n!}{(n+2)!} \sum_{k=2}^{n+1} B_k \binom{n+2}{k} u^{n-k+2}.$$

or since  $B_1 = -\frac{1}{2}$ , the function  $q_s$  can be written as

$$q(n, u) := -\frac{n!}{(n+2)!} \sum_{k=1}^{n+1} B_k \binom{n+2}{k} u^{n-k+2} = -\sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{n! u^{n-k+2}}{(n+2-k)!}.$$

By the notation of the function  $h_s$  and the Equation (11), for every real sequence  $(t_n)_n \subset \mathbb{R}_+$  the equation (10) can be written as

$$\begin{aligned} \frac{(t_n+1)^{1+s}}{n!} \int_{t_n}^{+\infty} \frac{z_s u^n + q(n, u)}{(u+1)^{n+2+s}} du &= \frac{t_n^{1+s}}{n!} \int_{t_n}^{+\infty} \frac{z_s u^n + q(n, u)}{u^{n+2+s}} du \quad (12) \\ + \epsilon_s(n) + \frac{t_n^{1+s}}{(n-1)!} \int_{t_n}^{+\infty} \frac{(u+1)^{n+s}}{u^{n+2+s}} \int_u^{+\infty} \frac{z_s v^{n-1} + q(n-1, v)}{(v+1)^{n+1+s}} dv du. \end{aligned}$$

where in order to simplify the notation, for every  $n \geq 1$  we denoted

$$\begin{aligned} \epsilon_s(n) &:= -(t_n+1)^{1+s} \int_{t_n}^{+\infty} (u+1)^{-n-2-s} p_n(u) du \\ &\quad + t_n^{1+s} \int_{t_n}^{+\infty} u^{-n-2-s} p_n(u) du \\ &\quad + t_n^{1+s} \int_{t_n}^{+\infty} \frac{(u+1)^{n+s}}{u^{n+2+s}} \int_u^{+\infty} (v+1)^{-n-1-s} p_{n-1}(v) dv du. \end{aligned}$$

We have

$$\begin{aligned} |\epsilon_s(n)| &\leq (t_n+1)^{1+\Re(s)} \left( 2 + \frac{1}{n+\Re(s)} \right) \int_{t_n}^{+\infty} u^{-n-2-\Re(s)} du \left[ \max_{v \in [0,1]} |p_n(v)| \right] \\ &= \frac{t_n^{-n}}{n+1+\Re(s)} \left( 1 + \frac{1}{n} \right)^{1+\Re(s)} \left( 2 + \frac{1}{n+\Re(s)} \right) \max_{v \in [0,1]} |p_n(v)|, \quad \forall n \geq 1. \end{aligned}$$

By the Notation 5, we have  $\lim_{n \rightarrow +\infty} \max_{v \in [0,1]} |p_n(v)| = 0$ , we deduce that

$$\lim_{n \rightarrow +\infty} |n t_n^n \epsilon_s(n)| = 0.$$

Using the last limit, the Equation (12) implies

$$\lim_{n \rightarrow +\infty} \frac{t_n^n}{(n-1)!} \left( z_s V_s[u^n](t_n) + V_s[q(n, u)](t_n) \right) = 0,$$

where for every  $n \geq 1$  and every real polynomial  $u \in \mathbb{R} \mapsto g_n(u)$  of order less than  $n+1$ , the function  $t \mapsto V_s[g_n(u)](t)$ , is defined as

$$\begin{aligned} V_s[g_n(u)](t) &:= \int_t^{+\infty} \left( \frac{(t+1)^{1+s}}{(u+1)^{n+2+s}} - \frac{t^{1+s}}{u^{n+2+s}} \right) g_n(u) du \\ &\quad - t^{1+s} \int_t^{+\infty} \frac{(u+1)^{n+s}}{u^{n+2+s}} \int_u^{+\infty} \frac{g_{n-1}(v)}{(v+1)^{n+1+s}} dv du. \end{aligned}$$

□

### 3 Proof of the Theorem 1

*Proof of the Theorem 1.* Let be  $s \in B$ . By the Equation (1) we have,

$$\frac{\zeta(s)}{s} + \frac{1}{1-s} = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Thanks to the Main Proposition 3 and the Lemma 7 we get

$$\lim_{n \rightarrow +\infty} \frac{t_n^n}{(n-1)!} \left[ V_s[q(n, u)](t_n) + \left( \frac{\zeta(s)}{s} + \frac{1}{1-s} \right) V_s[u^n](t_n) \right] = 0,$$

where we recall that

$$q(n, u) := - \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{n! u^{n-k+2}}{(n+2-k)!} \quad \forall n \geq 1, \quad \forall u \in \mathbb{R}, \quad (13)$$

and where for every  $n \geq 1$  and every real polynomial  $u \in \mathbb{R} \mapsto g_n(u)$  of order less than  $n+1$ , the function  $t \mapsto V_s[g_n(u)](t) \in \mathbb{C}$ , is defined as

$$\begin{aligned} V_s[g_n(u)](t) &:= \int_t^{+\infty} \left( \frac{(t+1)^{1+s}}{(u+1)^{n+2+s}} - \frac{t^{1+s}}{u^{n+2+s}} \right) g_n(u) du \\ &\quad - t^{1+s} \int_t^{+\infty} \frac{(u+1)^{n+s}}{u^{n+2+s}} \int_u^{+\infty} \frac{g_{n-1}(v)}{(v+1)^{n+1+s}} dv du. \end{aligned}$$

We can apply following integration for every arbitrary sequence  $(t_n)_n \subset \mathbb{R}_+$ . Consider the particular case where  $t_n = n$  for every  $n \geq 1$ . By change of variable, we have

$$\begin{aligned} V_s[g_n(u)](n) &= (n+1)^{1+s} \int_{n+1}^{+\infty} u^{-n-2-s} g_n(u-1) du \\ &\quad - n^{1+s} \int_n^{+\infty} u^{-n-2-s} g_n(u) du \\ &\quad - n^{1+s} \int_n^{+\infty} \frac{(u+1)^{n+s}}{u^{n+2+s}} \int_{u+1}^{+\infty} \frac{g_{n-1}(v-1)}{v^{n+1+s}} dv du \end{aligned}$$

Thanks to the Binomial theorem, we have

$$(u-1)^n = \sum_{j=0}^n (-1)^j \binom{n}{j} u^{n-j},$$

Then

$$\begin{aligned} V_s[u^n](n) &= \sum_{j=1}^n (-1)^j \binom{n}{j} \frac{(n+1)^{-j}}{j+1+s} \\ &\quad - \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{1}{j+1+s} \sum_{k=0}^{n-1-j} \binom{n-1-j}{k} \frac{n^{-j-k}}{j+k+2+s} \end{aligned}$$

and by the notation of  $q(n, u)$  given by the Equation (13)

$$\begin{aligned} V_s[q(n, u)](n) &= - \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{n}{(n+2-k)!} \sum_{j=0}^{n-k+2} (-1)^j \binom{n-k+2}{j} \frac{(n+1)^{2-k-j}}{k+j-1+s} \\ &\quad + \sum_{k=1}^{n+1} \frac{B_k}{k!} \frac{n^{2-k}}{(n+2-k)!(k-1+s)} \\ &\quad + \sum_{k=1}^n \frac{B_k}{k!} \frac{1}{(n+1-k)!} \sum_{j=0}^{n-k+1} \frac{w_{n,k,j}}{k+j+1+s} \sum_{m=0}^{n-k-j} \frac{v_{n,k,j,m}}{k+j+m+1+s} \end{aligned}$$

where

$$w_{n,k,j} := \binom{n-k+1}{j} \frac{(-1)^j}{k+j+1}, \quad v_{n,k,j,m} := \binom{n-k-j}{m} n^{-k-j-m}.$$

□

## 4 Conclusion

In this conclusion, we present the approach to prove that the Zeta function does not vanish at any point  $s \in B$  such that  $1 - 2\Re(s) \neq 0$ . Let be  $s \in B$  and suppose that  $|\zeta(s)| = 0$ . By Theorem 1, we have

$$\lim_{n \rightarrow +\infty} n^{n+1} \left[ \frac{\eta_s(n)}{n!} + (1-s)\tilde{\eta}_s(n) \right] = 0,$$

Thanks to the Riemann functional equation, the point  $1 - \bar{s}$  is also a zero of the function  $\zeta$ , then

$$\lim_{n \rightarrow +\infty} n^{n+1} \left[ \frac{\eta_{1-\bar{s}}(n)}{n!} + \bar{s}\tilde{\eta}_{1-\bar{s}}(n) \right] = 0,$$

For  $\Im(s) \neq 0$ , by taking the imaginary part of the subtraction, can be written as

$$(1 - 2\Re(s)) \lim_{n \rightarrow +\infty} n^{n+1} f_s(n) = 0,$$

where by using the definition of the Bernoulli numbers, the quantity of  $f_s(n)$  is defined as an alternating series. Using the monotonicity of the absolute value of the general term, we obtain  $\lim_{n \rightarrow +\infty} n^{n+1} f_s(n) \neq 0$ .

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).