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Non trivial zeros of the Zeta function using the differential equations

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Abstract

Using the differential equations, we obtain a more flexible representation of the Zeta function in the critical strip. Thanks to the Riemann functional equation, this representation allows us to prove that the Zeta function does not vanish at any point s of the critical strip such that $1 - 2\Re(s) \neq 0$.

Keywords: Zeta function, Non trivial zeros, Differential equations.

AMS subject classifications: 00A05

1 Main results

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \{u\} u^{-1-s} du, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{u\}$ is the fractional part of the real u . In order to simplify the notation, denote $B \subset \mathbb{C}$ the critical strip, defined as

$$B := \{s \in \mathbb{C} : \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*\}.$$

We prove the following results,

Theorem 1. Consider the Zeta function given by the Equation (1). For every $s \in B$ we have

$$\zeta(s) = \frac{1}{2} - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \sin(2\pi nu) \left(\frac{1}{\pi u} + \frac{1}{\tan(\pi u)} \right) du.$$

Theorem 2. For every $s \in B$ we have

$$\zeta(s) = -\frac{2}{\pi} \lim_{n \rightarrow +\infty} \int_0^1 u^{-1-s} \sin(2\pi nu) du.$$

Theorem 3. For every $s \in B$ such that $1 - 2\Re(s) \neq 0$, we have

$$\left| \frac{\zeta(s)}{1+s} - \frac{\zeta(1-\bar{s})}{2-\bar{s}} \right| \neq 0.$$

In other words, the Zeta function does not vanish at any point $s \in B$ such that $1 - 2\Re(s) \neq 0$.

2 Main Proposition

For every $s \in B$, the Equation (1) is equivalent to,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

The aim is to studies the differential equation of solutions the functions

$$t \mapsto \psi_s(z, t) := t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to find the initial condition $z_s \in \mathbb{C}$ such that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$ if and only if $z = -\int_1^{+\infty} u^{-1-s} \{u\} du$, we obtain $z_s = -\int_1^{+\infty} u^{-1-s} \{u\} du$. For every $s \in B$ we consider the following differential equation

$$\begin{aligned} \frac{d}{dt} x &= st^{-1}x + t^{-1}\{t\}, & (2) \\ t \in [1, +\infty)/\mathbb{N}, \quad x(1) &= z, \quad z \in \mathbb{C}, \quad x : [1, +\infty) \rightarrow \mathbb{C}. \end{aligned}$$

Lemma 4. *Let be $s \in B$. For every $z \in \mathbb{C}$ there exists a unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Further,*

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

Proof. Let be $s \in B$ fixed. The function

$$t \mapsto t^s \int_1^t u^{-1-s} \{u\} du, \quad t \geq 1,$$

is C^∞ on $[1, +\infty)/\mathbb{N}$ and continuous on $[1, +\infty)$. The Equation (2) is a non-homogeneous linear differential equation. For every $z \in \mathbb{C}$ the unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ such that $\psi_s(z, 1) = z$ is given by

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

□

Proposition 5. *Let be $s \in B$. Let be $z \in \mathbb{C}$ and consider the continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Then $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$, if and only if*

$$sz = -\frac{1}{2} + \frac{1}{1-s} - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \sin(2\pi nu) \left(\frac{1}{\pi u} + \frac{1}{\tan(\pi u)} \right) du,$$

and

$$\begin{aligned} sz = & -\frac{1}{2} + \frac{1}{1-s} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ & + \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \sin(2\pi nu) \left(\frac{1}{\pi u} - \frac{1}{\tan(\pi u)} \right) du, \end{aligned}$$

Lemma 6. *For every $s \in B$, there exists a unique $z_s \in \mathbb{C}$ such that the solution $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2) satisfies $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Further,*

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Proof. Let be $s \in B$ fixed. There is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm. Let be $z \in \mathbb{C}$ and suppose that

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty.$$

Since $\Re(s) \in (0, 1)$, by the Lemma 4, we have

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty \implies \left| z + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

Then

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Prove that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since

$$\left| z_s + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

then

$$z_s + \int_1^t u^{-1-s} \{u\} du = - \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t > 1.$$

By the Lemma 4, we get

$$\psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Then $\sup_{t \geq 1} |\psi_s(z_s, t)| \leq \frac{1}{\Re(s)}$. □

Notation 7. Denote

$$p(t) := \int_1^t \left(\{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 1,$$

where we recall that $\{u\}$ is the fractional part of the real u .

Lemma 8. *The function p is a continuous 1-periodic function and satisfies*

$$p(t) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left(\cos(2\pi jt) - 1 \right), \quad \forall t \geq 1.$$

Proof. The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_1^t \{u\} du = (t-1) \int_1^2 \{u\} du + p(t), \quad \forall t \geq 1.$$

Since

$$\int_1^2 \{u\} du = \int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_1^t \{u\} du = (t-1)\frac{1}{2} + p(t), \quad \forall t \geq 1.$$

The function p is a piecewise C^∞ , continuous on \mathbb{R} and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on \mathbb{R}_+ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p .

$$\int_0^t \left(\{v\} - \frac{1}{2}\right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left(\int_0^u \left(\{v\} - \frac{1}{2}\right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12} = -\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function p satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left(\exp(i2\pi jt) - 1 \right), \quad \forall t \geq 0.$$

□

Proof of the Proposition 5. Let be $s \in B$. Let be $z \in \mathbb{C}$ and consider the unique continuous and bounded solution $\psi_s(t) := \psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2) given by the Lemma 6. Define the function $\tilde{\psi}_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ as

$$\tilde{\psi}_s(t) := \int_1^t \psi_s(u) du, \quad t \geq 1.$$

The Equation (2) implies

$$\frac{d}{dt}\tilde{\psi}_s(t) = t^{-1}(1+s)\tilde{\psi}_s(t) + t^{-1}\int_1^t \{u\}du + t^{-1}z, \quad \tilde{\psi}(1) = 0, \quad \forall t \geq 1.$$

By the Lemma 8 we have

$$\int_1^t \left(\{u\} - \frac{1}{2}\right)du = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left(\cos(2\pi jt) - 1\right),$$

Then

$$\begin{aligned} \frac{d}{dt}\tilde{\psi}_s(t) &= t^{-1}(1+s)\tilde{\psi}_s(t) + \frac{1}{2}(1 - (1 - 2z_s)t^{-1}) \\ &+ 2t^{-1} \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left(\cos(2\pi jt) - 1\right), \quad \tilde{\psi}(1) = 0, \quad \forall t \geq 1. \end{aligned} \quad (3)$$

We will use the approximation method. Let be $t \mapsto \tilde{\psi}_{s,n}(t)$ the unique solution of the following differential linear equation

$$\begin{aligned} \frac{d}{dt}\tilde{\psi}_{s,n}(t) &= t^{-1}(1+s)\tilde{\psi}_{s,n}(t) + \frac{1}{2}(1 - (1 - 2z_s)t^{-1}) \\ &+ 2t^{-1} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} \left(\cos(2\pi jt) - 1\right), \quad \tilde{\psi}_{s,n}(1) = 0, \quad \forall t \geq 1. \end{aligned} \quad (4)$$

In order to simplify the notation, denote

$$\delta_{s,n}(t) := \tilde{\psi}_s(t) - \tilde{\psi}_{s,n}(t), \quad \forall t \geq 1. \quad (5)$$

The Equations (7) and (4) implies

$$\begin{aligned} \frac{d}{dt}\delta_{s,n}(t) &= t^{-1}(1+s)\delta_{s,n}(t) \\ &+ 2t^{-1} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} \left(\cos(2\pi jt) - 1\right), \quad \delta_{s,n}(1) = 0, \quad \forall t \geq 1. \end{aligned}$$

Integrate

$$\delta_{s,n}(t) = 2t^{1+s} \int_1^t u^{-2-s} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} \left(\cos(2\pi ju) - 1\right)du, \quad \forall t \geq 1.$$

Implies

$$|\delta_{s,n}(t)| \leq 2 \frac{t^{1+\Re(s)}}{1+\Re(s)} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2}, \quad \forall t \geq 1, \quad \forall n \geq 2. \quad (6)$$

For every fixed $n \in \mathbb{N}^*$, use the change of variable,

$$\varphi_{s,n}(u) = n\tilde{\psi}_{s,n}\left(\frac{u}{n}\right), \quad \forall u \geq n.$$

In term of this new variable, the differential equation (4) can be written as

$$\begin{aligned} \frac{d}{du}\varphi_{s,n}(u) &= u^{-1}(1+s)\varphi_{s,n}(u) + \frac{1}{2}(1-n(1-2z_s)u^{-1}) \\ &\quad + 2nu^{-1} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} \left(\cos(2\pi jn^{-1}u) - 1 \right), \\ \varphi_{s,n}(n) &= 0, \quad \forall u \geq n. \end{aligned} \quad (7)$$

Integrate

$$\begin{aligned} \varphi_{s,n}(u) &= u^{1+s} \int_n^u \mu^{-1-s} \frac{1}{2} (1-n(1-2z_s)\mu^{-1}) d\mu \\ &\quad + 2nu^{1+s} \int_n^u \mu^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} \left(\cos(2\pi jn^{-1}u) - 1 \right) d\mu, \quad \forall u \geq n. \end{aligned}$$

In other words, we have

$$\begin{aligned} \varphi_{s,n}(u) &= \frac{1}{2s} \left(u^{1+s} n^{-s} - u \right) - n \frac{1-2z_s}{2(1+s)} \left(u^{1+s} n^{-1-s} - 1 \right) \\ &\quad - V_s(n, u) + W_s(n, u), \quad \forall u \geq n. \end{aligned} \quad (8)$$

where in order to simplify the notation, for every $u \geq n$ we denoted

$$\begin{aligned} V_s(n, u) &= 2nu^{1+s} \int_0^n \mu^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} \left(\cos(2\pi jn^{-1}\mu) - 1 \right) d\mu, \quad (9) \\ W_s(n, u) &= 2nu^{1+s} \int_0^u \mu^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} \left(\cos(2\pi jn^{-1}\mu) - 1 \right) d\mu. \end{aligned}$$

We recall that

$$\varphi_{s,n}(u) = n\tilde{\psi}_{s,n}\left(\frac{u}{n}\right), \quad \forall u \geq n.$$

Which can be written as

$$\varphi_{s,n}(u) = n\delta_{s,n}\left(\frac{u}{n}\right) + n\tilde{\psi}_s\left(\frac{u}{n}\right), \quad \forall u \geq n.$$

where $\delta_{s,n}$ is given by the Equation (5). Replcae the last quantity of $\varphi_{s,n}$ in the Equation (8). By the Equations (6), we obtain

$$\begin{aligned} & \left| \frac{1}{2s} \left(u^{1+s} n^{-s} - u \right) - n \frac{1-2z_s}{2(1+s)} \left(u^{1+s} n^{-1-s} - 1 \right) - V_s(u, n) + W_s(n, u) \right| \\ & \leq 2 \frac{n^{-\Re(s)} u^{1+\Re(s)}}{1+\Re(s)} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} + n \left(\frac{u}{n} - 1 \right) \sup_{u \geq 1} |\psi_s(u)|, \quad \forall u > n. \end{aligned}$$

We deduce that

$$\lim_{n \rightarrow +\infty} \lim_{u \rightarrow +\infty} \left| \frac{1}{2s} - \frac{1-2z_s}{2(1+s)} - u^{-1-s} n^s \left(V_s(u, n) - W_s(n, u) \right) \right| = 0.$$

In other words,

$$z_s = -\frac{1}{2s} + (1+s) \lim_{n \rightarrow +\infty} \lim_{u \rightarrow +\infty} u^{-1-s} n^s \left(V_s(u, n) - W_s(n, u) \right). \quad (10)$$

Now, estimate the limits of the previous equation. For V_s , we have

$$\begin{aligned} \lim_{u \rightarrow +\infty} u^{-1-s} n^s V_s(u, n) &= 2n^{1+s} \int_0^n \mu^{-2-s} \sum_{j=1}^{n-1} \frac{\cos(2\pi j n^{-1} \mu) - 1}{(j2\pi)^2} d\mu \\ &= 2 \int_0^1 v^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(2\pi j)^2} \left(\cos(2\pi j v) - 1 \right) dv, \end{aligned}$$

Use the fact

$$\frac{\sin(\pi(2n-1)u)}{\sin(\pi u)} = 1 + 2 \sum_{j=1}^{n-1} \cos(2\pi j u),$$

Then

$$\begin{aligned} s(1+s) \lim_{u \rightarrow +\infty} u^{-1-s} n^s V_s(u, n) &= -2 \int_0^1 v^{-s} \sum_{j=1}^{n-1} \cos(2\pi j v) dv \quad (11) \\ &= \int_0^1 v^{-s} dv - \int_0^1 v^{-s} \frac{\sin(\pi(2n-1)v)}{\sin(\pi v)} dv \\ &= \frac{1}{1-s} - \int_0^1 v^{-s} \frac{\sin(\pi(2n-1)v)}{\sin(\pi v)} dv \\ &= \frac{1}{1-s} - \int_0^1 v^{-s} \left[\frac{\sin(2n\pi v)}{\tan(\pi v)} - \cos(2\pi n v) \right] dv. \end{aligned}$$

For W_s we have

$$\begin{aligned}
\lim_{u \rightarrow +\infty} u^{-1-s} n^s W_s(n, u) &= 2n^{1+s} \int_0^{+\infty} \mu^{-2-s} \sum_{j=1}^{n-1} \frac{\cos(j2\pi n^{-1}\mu) - 1}{(j2\pi)^2} d\mu \quad (12) \\
&= \frac{1}{2\pi^2} \int_0^{+\infty} u^{-2-s} (\cos(2\pi u) - 1) du \sum_{j=1}^{n-1} \frac{1}{j^{1-s}} \\
&= -\frac{1}{\pi(1+s)} \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \sum_{j=1}^{n-1} \frac{1}{j^{1-s}} \\
&= -\frac{n^{-s}}{\pi(1+s)} \int_0^{+\infty} v^{-1-s} \sin(2\pi n v) dv \sum_{j=1}^{n-1} \frac{1}{j^{1-s}}.
\end{aligned}$$

Using the fact

$$\begin{aligned}
\lim_{n \rightarrow +\infty} \left| \int_0^1 v^{-s} \cos(2\pi n v) dv \right| &= \lim_{n \rightarrow +\infty} \left| n^{-1+s} \int_0^n v^{-s} \cos(2\pi v) dv \right| \\
&= \lim_{n \rightarrow +\infty} \left| n^{-1+s} \int_1^n v^{-s} \cos(2\pi v) dv \right| \\
&= \frac{|s|}{2\pi} \lim_{n \rightarrow +\infty} \left| n^{-1+s} \int_1^n v^{-1-s} \sin(2\pi v) dv \right| \\
&\leq \frac{|s|}{2\pi \Re(s)} \lim_{n \rightarrow +\infty} n^{-1+\Re(s)} (1 - n^{-\Re(s)}) = 0,
\end{aligned}$$

Replace, in the Equation (10) the quantities of V_s and W_s given by the Equations (11) and (12), we obtain

$$s z_s = -\frac{1}{2} + \frac{1}{1-s} - \lim_{n \rightarrow +\infty} \left[U_s(n) + \int_0^1 v^{-s} \frac{\sin(2\pi n v)}{\tan(\pi v)} dv \right], \quad (13)$$

where in order to simplify the notation, for every $u \geq n$ we denoted

$$U_s(n) := \frac{s}{\pi} \int_0^{+\infty} v^{-1-s} \sin(2\pi n v) dv \left[n^{-s} \sum_{j=1}^{n-1} \frac{1}{j^{1-s}} \right],$$

By the Abel summation formula, we have

$$\lim_{n \rightarrow +\infty} \left[\sum_{j=1}^{n-1} \frac{1}{j^{1-s}} - \frac{n^s}{s} \right] = \zeta(1-s). \quad (14)$$

Item 1.

By the last equality; we have

$$\lim_{n \rightarrow +\infty} n^{-s} \sum_{j=1}^{n-1} \frac{1}{j^{1-s}} = \frac{1}{s}$$

Since

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_1^{+\infty} u^{-1-s} \sin(2\pi nu) du \\ &= \lim_{n \rightarrow +\infty} n^s \int_n^{+\infty} u^{-1-s} \sin(2\pi u) du \\ &= -\frac{1+s}{2\pi} \lim_{n \rightarrow +\infty} n^s \int_n^{+\infty} u^{-2-s} (\cos(2\pi u) - 1) du \\ &= 0. \end{aligned}$$

By the notation of U_s , the Equation (13), implies

$$sz_s = -\frac{1}{2} + \frac{1}{1-s} - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \sin(2\pi nu) \left(\frac{1}{\pi u} + \frac{1}{\tan(\pi u)} \right) du.$$

Item 2.

By the notation of U_s we have

$$\begin{aligned} U_s(n) &= \frac{s}{\pi} \int_0^{+\infty} v^{-1-s} \sin(2\pi nv) dv \left[n^{-s} \sum_{j=1}^{n-1} \frac{1}{j^{1-s}} \right] \\ &= \frac{s}{\pi} \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \sum_{j=1}^{n-1} \frac{1}{j^{1-s}}, \end{aligned}$$

which can be written as

$$\begin{aligned} U_s(n) &= \frac{s}{\pi} \left[\sum_{j=1}^{n-1} \frac{1}{j^{1-s}} - \frac{n^s}{s} \right] \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ &\quad + \frac{n^s}{\pi} \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du, \end{aligned}$$

By the Equation (14), we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} U_s(n) &= \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ &\quad + \frac{n^s}{\pi} \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du. \end{aligned}$$

The Equation (13), implies

$$sz_s = -\frac{1}{2} + \frac{1}{1-s} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \frac{\sin(\pi(2n-1)u)}{\sin(\pi u)} du + \frac{1}{\pi} \int_0^{+\infty} u^{-1-s} \sin(2\pi n u) du.$$

We have

$$\lim_{n \rightarrow +\infty} \int_1^{+\infty} u^{-1-s} \sin(2\pi n u) du = \lim_{n \rightarrow +\infty} n^s \int_n^{+\infty} u^{-1-s} \sin(2\pi u) du \\ = -\frac{1+s}{2\pi} \lim_{n \rightarrow +\infty} n^s \int_n^{+\infty} u^{-2-s} (\cos(2\pi u) - \cos(2\pi n)) du \\ = 0.$$

By consequence,

$$sz_s = -\frac{1}{2} + \frac{1}{1-s} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ + \int_0^1 u^{-s} \sin(2\pi n u) \left(\frac{1}{\pi u} - \frac{1}{\tan(\pi u)} \right),$$

□

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $s \in B$. By the Equation (1) we have,

$$-s \int_1^{+\infty} u^{-1-s} \{u\} du = \zeta(s) + \frac{s}{1-s}.$$

Thanks to the Main Proposition 5 and the Lemma 6 we have

$$\zeta(s) + \frac{s}{1-s} = sz_s = -\frac{1}{2} + \frac{1}{1-s} \\ - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \sin(2\pi n u) \left(\frac{1}{\pi u} + \frac{1}{\tan(\pi u)} \right) du.$$

Implies

$$\zeta(s) = \frac{1}{2} - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \sin(2\pi n u) \left(\frac{1}{\pi u} + \frac{1}{\tan(\pi u)} \right) du.$$

□

4 Proof of the Theorem 2

Lemma 9. Consider the function ζ given by the Equation (1). For every $s \in B$ we have

$$\begin{aligned} \frac{2\pi^2\zeta(s)}{s(1+s)}t^{1+s} &= -\pi^2\left(\frac{t^2}{1-s} + \frac{t}{s}\right) \\ &\quad - \sum_{j \in \mathbb{N}^*} j^{-1+s} \int_j^{+\infty} u^{-2-s} (\cos(2\pi ut) - 1) du, \quad \forall t \in (0, 1). \end{aligned}$$

Proof. By the Equation (1). For every $s \in B$ we have

$$\begin{aligned} -\frac{\zeta(s)}{s} &= \frac{1}{1-s}t^{1-s} + \int_t^{+\infty} \{u\}u^{-1-s}du \\ &= \frac{1}{1-s}t^{1-s} + \frac{1}{2s}t^{-s} + \int_t^{+\infty} (\{u\} - \frac{1}{2})u^{-1-s}du, \quad \forall t \in (0, 1]. \end{aligned}$$

Use the integration by part formula

$$-\frac{\zeta(s)}{s} = \frac{1}{1-s}t^{1-s} + \frac{1}{2s}t^{-s} + (1+s) \int_t^{+\infty} p(u)u^{-2-s} du - p(t)t^{-1-s},$$

where the function $p(u)$ is given by the Lemma 8. Use the fact $p(t) = \frac{1}{2}(t^2 - t)$ for every $t \in (0, 1]$ we get

$$\begin{aligned} -\frac{\zeta(s)}{s} &= \frac{1}{1-s}t^{1-s} + \frac{1}{2s}t^{-s} + (1+s) \int_t^{+\infty} p(u)u^{-2-s} du \\ &\quad - \frac{1}{2}(t^2 - t)t^{-1-s}, \quad \forall t \in (0, 1). \end{aligned}$$

Since

$$p(t) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} (\cos(j2\pi t) - 1), \quad \forall t \in \mathbb{R}.$$

Use the fact

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left| \int_t^{+\infty} u^{-2-s} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} (\cos(j2\pi u) - 1) du \right| \\ \leq \int_t^{+\infty} u^{-2-\Re(s)} du \lim_{n \rightarrow +\infty} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} = 0, \end{aligned}$$

then

$$\begin{aligned}\zeta(s) &= -\frac{s}{2} \frac{1+s}{1-s} t^{1-s} - \frac{1}{2} (1+s) t^{-s} \\ &\quad - 2s(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \int_t^{+\infty} u^{-2-s} (\cos(j2\pi u) - 1) du.\end{aligned}$$

Use the change of variable $u \mapsto u = tj^{-1}v$ to obtain

$$\begin{aligned}\zeta(s) &= -\frac{s}{2} \frac{1+s}{1-s} t^{1-s} - \frac{1}{2} (1+s) t^{-s} \\ &\quad - \frac{s}{2\pi^2} (1+s) t^{-1-s} \sum_{j \in \mathbb{N}^*} j^{-1+s} \int_j^{+\infty} u^{-2-s} (\cos(2\pi ut) - 1) du.\end{aligned}$$

□

Lemma 10. *For every $s \in B$, we have*

$$\zeta(s) = \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} v^{-1-x} \sin(2\pi v) dv.$$

Proof. Let $s \in B$ fixed. In order to simplify the notation, denote

$$\forall t \in (0, 1] : \quad \varphi_s(n, t) := \sum_{j=1}^n f_s(j, t), \quad \forall n \in \mathbb{N}^*.$$

where

$$f_s(z, t) = -z^{-1+s} \int_z^{+\infty} u^{-2-s} (\cos(2\pi ut) - 1) du, \quad \forall z \in \mathbb{R}_+, \quad \forall t \in (0, 1], \quad (15)$$

By the the Lemma 9, we get

$$\frac{2\pi^2 \zeta(s)}{s(1+s)} t^{1+s} = \lim_{n \rightarrow +\infty} \varphi_s(n, t) + \pi^2 \left(\frac{t^2}{1-s} + \frac{t}{s} \right), \quad \forall t \in (0, 1]. \quad (16)$$

For every fixed $t \in (0, 1]$, use the Euler-Maclaurin summation formula, given by

$$\begin{aligned}\varphi_s(n, t) &= \frac{1}{2} [f_s(1, t) - f_s(n, t)] + \int_1^n f_s(z, t) dz \\ &\quad + \int_1^n \left(\{z\} - \frac{1}{2} \right) \frac{d}{dz} f_s(z, t) dz, \quad \forall n \in \mathbb{N}^*,\end{aligned}$$

Using the Equation (15) and the integration by parts formula, we obtain

$$\begin{aligned} \int_1^n f_s(z, t) dz &= \frac{1}{s} n f_s(n, t) + \frac{1}{s} \int_1^{+\infty} z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^n u^{-2} (\cos(2\pi tu) - 1) du, \quad \forall n \in \mathbb{N}^*. \end{aligned}$$

Then

$$\begin{aligned} \varphi_s(n, t) &= \frac{1}{2} [f_s(1, t) - f_s(n, t)] \\ &\quad + \frac{1}{s} n f_s(n, t) + \frac{1}{s} \int_1^{+\infty} z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^n u^{-2} (\cos(2\pi tu) - 1) du \\ &\quad + \int_1^n \left(\{z\} - \frac{1}{2} \right) \frac{d}{dz} f_s(z, t) dz, \quad \forall n \in \mathbb{N}^*. \end{aligned}$$

The Equation (15) implies $\lim_{z \rightarrow +\infty} z f_s(z, t) = 0$. By consequence,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \varphi_s(n, t) &= \frac{2-s}{2s} \int_1^{+\infty} z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^{+\infty} z^{-2} (\cos(2\pi tz) - 1) dz \\ &\quad + \int_1^{+\infty} \left(\{z\} - \frac{1}{2} \right) \frac{d}{dz} f_s(z, t) dz \end{aligned}$$

The Equation (16), implies

$$\frac{2\pi^2 \zeta(s)}{s(1+s)} t^{1+s} = -\frac{\pi^2 t}{s} - \frac{1}{s} \int_1^{+\infty} z^{-2} (\cos(2\pi tz) - 1) dz + \delta(s, t), \quad \forall t \in (0, 1].$$

where

$$\begin{aligned} \delta(s, t) &:= -\pi^2 \frac{t^2}{1-s} + \frac{2-s}{2s} \int_1^{+\infty} z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad + \int_1^{+\infty} \left(\{z\} - \frac{1}{2} \right) \frac{d}{dz} f_s(z, t) dz, \quad \forall t \in (0, 1]. \end{aligned}$$

The Dirichlet integral, implies

$$-\pi^2 t - \int_1^{+\infty} z^{-2} (\cos(2\pi tz) - 1) dz = t \int_0^t z^{-2} (\cos(2\pi z) - 1) dz$$

Then

$$\frac{2\pi^2\zeta(s)t^{1+s}}{s(1+s)} = \frac{t}{s} \int_0^t z^{-2}(\cos(2\pi z) - 1)dz + \delta(s, t), \quad \forall t \in (0, 1]. \quad (17)$$

The function δ can be written as

$$\begin{aligned} \delta(s, t) &= \frac{2-s}{2s}\eta(s)t^{1+s} - \pi^2 \frac{t^2}{1-s} - \frac{2-s}{2s} \int_0^1 z^{-2-s}(\cos(2\pi tz) - 1)dz \\ &\quad + \int_1^{+\infty} (\{z\} - \frac{1}{2}) \frac{d}{dz} f_s(z, t) dz, \quad \forall t \in (0, 1]. \end{aligned} \quad (18)$$

where in order to simplify the notation, we denoted

$$\eta(x) := \int_0^{+\infty} v^{-2-x}(\cos(2\pi v) - 1)dv, \quad x \in B. \quad (19)$$

Since

$$\lim_{t \rightarrow 0} t^{-2} \left| \pi^2 \frac{t^2}{1-s} + \frac{2-s}{2s} \int_0^1 z^{-2-s}(\cos(2\pi tz) - 1)dz \right| < +\infty,$$

then by definition of the function δ in the Equation (18) and the Equation (17), we get

$$\lim_{t \rightarrow 0} t^{-2} \left| \left(\frac{2\pi^2\zeta(s)}{s(1+s)} - \frac{2-s}{2s}\eta(s) \right) t^{1+s} - V(s, t) \right| < +\infty, \quad (20)$$

where in order to simplify the notation, we denoted

$$V(x, t) := \int_1^{+\infty} (\{z\} - \frac{1}{2}) \frac{d}{dz} f_x(z, t) dz, \quad \forall x \in B.$$

From the Equation (15), for every $x \in B$ we have

$$\begin{aligned} \frac{d}{dz} f_x(z, t) &= (1-x)z^{-2+x} \int_z^{+\infty} u^{-2-x}(\cos(2\pi ut) - 1)du \\ &\quad + z^{-3}(\cos(2\pi zt) - 1). \end{aligned}$$

By the notation of $V(s, t)$ we get

$$\begin{aligned} V(s, t) &= \int_1^{+\infty} (\{z\} - \frac{1}{2}) z^{-3}(\cos(2\pi zt) - 1) dz \\ &\quad + (1-s) \int_1^{+\infty} (\{z\} - \frac{1}{2}) z^{-2+s} \int_z^{+\infty} u^{-2-s}(\cos(2\pi ut) - 1) du dz, \end{aligned}$$

which can be written as

$$\begin{aligned} V(s, t) &= t^{1+s} \eta(s) (1-s) \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) z^{-2+s} dz \\ &+ \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) z^{-3} (\cos(2\pi zt) - 1) dz \\ &- (1-s) \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) z^{-2+s} \int_0^z u^{-2-s} (\cos(2\pi ut) - 1) du dz, \end{aligned}$$

where we recall that the function $x \rightarrow \eta(x)$ is given by the Equation (19). By definition of the function ζ in the Equation (1), we have

$$\int_1^{+\infty} \left(\{u\} - \frac{1}{2}\right) u^{-2+s} du = -\frac{\zeta(1-s)}{1-s} - \frac{2-s}{2s(1-s)}.$$

Then

$$\begin{aligned} V(s, t) &= -t^{1+s} \eta(s) \zeta(1-s) - \frac{2-s}{2s} \eta(s) t^{1+s} \\ &+ \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) z^{-3} (\cos(2\pi zt) - 1) dz \\ &- (1-s) \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) z^{-2+s} \int_0^z u^{-2-s} (\cos(2\pi ut) - 1) du dz, \end{aligned}$$

Using the integration by parts formula on η , we obtain

$$\begin{aligned} V(s, t) &= -\left(\eta(s) \zeta(1-s) + \frac{2-s}{2s} \eta(s)\right) t^{1+s} \\ &- \int_1^{+\infty} p(z) \frac{d}{dz} \left[z^{-3} (\cos(2\pi zt) - 1) \right] dz \\ &+ (1-s) \int_1^{+\infty} p(z) \frac{d}{dz} \left[z^{-2+s} \int_0^z u^{-2-s} (\cos(2\pi ut) - 1) du \right] dz. \end{aligned}$$

where the 1-periodic function p is given by the Lemma 8. By the Lebesgue Theorem, we get

$$\lim_{t \rightarrow 0} t^{-2} \left| V(s, t) + \left(\eta(s) \zeta(1-s) + \frac{2-s}{2s} \eta(s)\right) t^{1+s} \right| < +\infty.$$

The Equation (20), implies

$$\frac{2\pi^2 \zeta(s)}{s(1+s)} = -\eta(s) \zeta(1-s). \quad (21)$$

By the notation of η in the Equation (19), we have

$$\eta(x) = \int_0^{+\infty} v^{-2-x}(\cos(2\pi v) - 1)dv = -\frac{2\pi}{1+x} \int_0^{+\infty} v^{-1-x} \sin(2\pi v)dv.$$

□

Proof of the Theorem 2 . By the Equation (1) we have,

$$-s \int_1^{+\infty} u^{-1-s}\{u\}du = \zeta(s) + \frac{s}{1-s}.$$

Thanks to the Main Proposition 5 and the Lemma 6 we have

$$\begin{aligned} \zeta(s) + \frac{s}{1-s} = sz_s &= -\frac{1}{2} + \frac{1}{1-s} \\ &+ \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u)du \\ &- \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} h_n(u)du, \end{aligned}$$

Implies

$$\zeta(s) = \frac{1}{2} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u)du - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} h_n(u)du.$$

By the Lemma 10, we have

$$\zeta(s) = \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} v^{-1-x} \sin(2\pi v)dv.$$

Then

$$\zeta(s) = \frac{1}{2} + \zeta(s) + \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \sin(2\pi nu) \left(\frac{1}{\pi u} - \frac{1}{\tan(\pi u)} \right) du,$$

In other words

$$\lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \frac{\sin(2\pi nu)}{\tan(\pi u)} du = \frac{1}{2} + \frac{1}{\pi} \lim_{n \rightarrow +\infty} \int_0^1 u^{-1-s} \sin(2\pi nu) du,$$

By the Theorem 1, we obtain

$$\zeta(s) = -\frac{2}{\pi} \lim_{n \rightarrow +\infty} \int_0^1 u^{-1-s} \sin(2\pi nu) du.$$

□

5 Proof of the Theorem 3

Proof of the Theorem 3. By Theorem 2, we have

$$\begin{aligned}\zeta(s) &= -\frac{2}{\pi} \lim_{n \rightarrow +\infty} \int_0^1 u^{-1-s} \sin(2\pi nu) du \\ &= -\frac{1+s}{\pi^2} \lim_{n \rightarrow +\infty} n^{-1} \int_0^1 u^{-2-s} (1 - \cos(2\pi nu)) du.\end{aligned}$$

Then

$$\begin{aligned}& \pi^2 \left(\frac{\zeta(s)}{1+s} - \frac{\zeta(1-\bar{s})}{2-\bar{s}} \right) \\ &= \lim_{n \rightarrow +\infty} n^{-1} \int_0^1 \left(u^{-3+\bar{s}} - u^{-2-s} \right) \left(1 - \cos(2\pi nu) \right) du \\ &= (1 - 2\Re(s)) \lim_{n \rightarrow +\infty} n^{-1} \sum_{k=1}^{+\infty} \frac{(-1)^k (2\pi n)^{2k}}{(2k)!(2k-2+\bar{s})(2k-1-s)}.\end{aligned}$$

□

References

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