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Non trivial zeros of the Zeta function using the differential equations

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Abstract

We investigate a relationship between differential equations and the Zeta function. We obtain a more flexible representation of the Riemann Zeta function in the critical strip. This allows us to prove that the Zeta function does not vanish at any point $s \in \mathbb{C}$ such that $\Re(s) \in (0, 2^{-1})$.

Keywords: Zeta function, Non trivial zeros, Differential equations.

AMS subject classifications: 00A05

1 Main results

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \{u\} u^{-1-s} du, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{u\}$ is the fractional part of the real u . In order to simplify the notation, denote $B \subset \mathbb{C}$ the critical strip, defined as

$$B := \{s \in \mathbb{C} : \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*\}.$$

We prove the following theorems,

Theorem 1. Consider the Zeta function given by the Equation (1). For every $s \in B$ we have

$$\zeta(s) = \frac{1}{2} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} h_n(u) du.$$

where for every $n \geq 1$ the function $h_n : (0, 1) \rightarrow \mathbb{R}$ is defined as

$$h_n(u) = \frac{\sin(\pi(2n-1)u)}{\sin(\pi u)} - \frac{\sin(2\pi n u)}{\pi u}.$$

Theorem 2. The Zeta function does not vanish at any point $s \in B$ such that $\Re(s) \in (0, 2^{-1})$.

2 Main Proposition

For every $s \in B$, the Equation (1) is equivalent to,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

The aim is to study the differential equation of solutions the functions

$$t \mapsto \psi_s(z, t) := t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to find the initial condition $z_s \in \mathbb{C}$ such that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$ if and only if $z = -\int_1^{+\infty} u^{-1-s} \{u\} du$, we obtain $z_s = -\int_1^{+\infty} u^{-1-s} \{u\} du$. For every $s \in B$ we consider the following differential equation

$$\frac{d}{dt} x = st^{-1}x + t^{-1}\{t\}, \quad (2)$$

$$t \in [1, +\infty)/\mathbb{N}, \quad x(1) = z, \quad z \in \mathbb{C}, \quad x : [1, +\infty) \rightarrow \mathbb{C}.$$

Lemma 3. Let be $s \in B$. For every $z \in \mathbb{C}$ there exists a unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Further,

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

Proof. Let be $s \in B$ fixed. The function

$$t \mapsto t^s \int_1^t u^{-1-s} \{u\} du, \quad t \geq 1,$$

is C^∞ on $[1, +\infty)/\mathbb{N}$ and continuous on $[1, +\infty)$. The Equation (2) is a non-homogeneous linear differential equation. For every $z \in \mathbb{C}$ the unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ such that $\psi_s(z, 1) = z$ is given by

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

□

Proposition 4. *Let be $s \in B$. Let be $z \in \mathbb{C}$ and consider the continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Then $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$, if and only if*

$$\begin{aligned} sz_s = & -\frac{1}{2} + \frac{1}{1-s} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ & - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} h_n(u) du, \end{aligned}$$

where the sequence function $h_n : (0, 1) \rightarrow \mathbb{R}$ are defined as in the Theorem 1.

Lemma 5. *For every $s \in B$, there exists a unique $z_s \in \mathbb{C}$ such that the solution $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2) satisfies $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Further,*

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Proof. Let be $s \in B$ fixed. There is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm. Let be $z \in \mathbb{C}$ and suppose that

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty.$$

Since $\Re(s) \in (0, 1)$, by the Lemma 3, we have

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty \implies \left| z + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

Then

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Prove that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since

$$\left| z_s + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

then

$$z_s + \int_1^t u^{-1-s} \{u\} du = - \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t > 1.$$

By the Lemma 3, we get

$$\psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Then $\sup_{t \geq 1} |\psi_s(z_s, t)| \leq \frac{1}{\Re(s)}$. □

Notation 6. Denote

$$p(t) := \int_1^t \left(\{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 1,$$

where we recall that $\{u\}$ is the fractional part of the real u .

Lemma 7. *The function p is a continuous 1-periodic function and satisfies*

$$p(t) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left(\cos(j2\pi t) - 1 \right), \quad \forall t \geq 1.$$

Proof. The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_1^t \{u\} du = (t-1) \int_1^2 \{u\} du + p(t), \quad \forall t \geq 1.$$

Since

$$\int_1^2 \{u\} du = \int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_1^t \{u\} du = (t-1) \frac{1}{2} + p(t), \quad \forall t \geq 1.$$

The function p is a piecewise C^∞ , continuous on \mathbb{R} and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on \mathbb{R}_+ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p .

$$\int_0^t \left(\{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left(\int_0^u \left(\{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12} = -\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function p satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left(\exp(ij2\pi t) - 1 \right), \quad \forall t \geq 0.$$

□

Proof of the Proposition 4. Let be $s \in B$. Let be $z \in \mathbb{C}$ and consider the unique continuous and bounded solution $\psi_s(t) := \psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2) given by the Lemma 5. Define the function $\tilde{\psi}_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ as

$$\tilde{\psi}_s(t) := \int_1^t \psi_s(u) du, \quad t \geq 1.$$

The Equation (2) implies

$$\frac{d}{dt} \tilde{\psi}_s(t) = t^{-1}(1+s)\tilde{\psi}_s(t) + t^{-1} \int_1^t \{u\} du + t^{-1}z, \quad \tilde{\psi}_s(1) = 0, \quad \forall t \geq 1.$$

By the Lemma 7 we have

$$\int_1^t \left(\{u\} - \frac{1}{2} \right) du = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left(\cos(j2\pi t) - 1 \right),$$

Then

$$\begin{aligned} \frac{d}{dt}\tilde{\psi}_s(t) &= t^{-1}(1+s)\tilde{\psi}_s(t) + \frac{1}{2}(1 - (1 - 2z_s)t^{-1}) \\ &+ 2t^{-1} \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} (\cos(j2\pi t) - 1), \quad \tilde{\psi}(1) = 0, \quad \forall t \geq 1. \end{aligned} \quad (3)$$

We will use the approximation method. Let be $t \mapsto \tilde{\psi}_{s,n}(t)$ the unique solution of the following differential linear equation

$$\begin{aligned} \frac{d}{dt}\tilde{\psi}_{s,n}(t) &= t^{-1}(1+s)\tilde{\psi}_{s,n}(t) + \frac{1}{2}(1 - (1 - 2z_s)t^{-1}) \\ &+ 2t^{-1} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} (\cos(j2\pi t) - 1), \quad \tilde{\psi}_{s,n}(1) = 0, \quad \forall t \geq 1. \end{aligned} \quad (4)$$

In order to simplify the notation, denote

$$\delta_{s,n}(t) := \tilde{\psi}_s(t) - \tilde{\psi}_{s,n}(t), \quad \forall t \geq 1. \quad (5)$$

The Equations (7) and (4) implies

$$\begin{aligned} \frac{d}{dt}\delta_{s,n}(t) &= t^{-1}(1+s)\delta_{s,n}(t) \\ &+ 2t^{-1} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} (\cos(j2\pi t) - 1), \quad \delta_{s,n}(1) = 0, \quad \forall t \geq 1. \end{aligned}$$

Integrate

$$\delta_{s,n}(t) = 2t^{1+s} \int_1^t u^{-2-s} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} (\cos(j2\pi u) - 1) du, \quad \forall t \geq 1.$$

Implies

$$|\delta_{s,n}(t)| \leq 2 \frac{t^{1+\Re(s)}}{1+\Re(s)} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2}, \quad \forall t \geq 1, \quad \forall n \geq 2. \quad (6)$$

For every fixed $n \in \mathbb{N}^*$, use the change of variable,

$$\varphi_{s,n}(u) = n\tilde{\psi}_{s,n}\left(\frac{u}{n}\right), \quad \forall u \geq n.$$

In term of this new variable, the differential equation (4) can be written as

$$\begin{aligned} \frac{d}{du}\varphi_{s,n}(u) &= u^{-1}(1+s)\varphi_{s,n}(u) + \frac{1}{2}(1-n(1-2z_s)u^{-1}) \\ &\quad + 2nu^{-1}\sum_{j=1}^{n-1}\frac{1}{(j2\pi)^2}\left(\cos(j2\pi n^{-1}u) - 1\right), \\ \varphi_{s,n}(n) &= 0, \quad \forall u \geq n. \end{aligned} \quad (7)$$

Integrate

$$\begin{aligned} \varphi_{s,n}(u) &= u^{1+s} \int_n^u \mu^{-1-s} \frac{1}{2}(1-n(1-2z_s)\mu^{-1})d\mu \\ &\quad + 2nu^{1+s} \int_n^u \mu^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} \left(\cos(j2\pi n^{-1}u) - 1\right)d\mu, \quad \forall u \geq n. \end{aligned}$$

In other words, we have

$$\begin{aligned} \varphi_{s,n}(u) &= \frac{1}{2s} \left(u^{1+s}n^{-s} - u\right) - n\frac{1-2z_s}{2(1+s)} \left(u^{1+s}n^{-1-s} - 1\right) \\ &\quad - V_s(n, u) + W_s(n, u), \quad \forall u \geq n. \end{aligned} \quad (8)$$

where in order to simplify the notation, for every $u \geq n$ we denoted

$$\begin{aligned} V_s(n, u) &= 2nu^{1+s} \int_0^n \mu^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} \left(\cos(j2\pi n^{-1}\mu) - 1\right)d\mu, \quad (9) \\ W_s(n, u) &= 2nu^{1+s} \int_0^u \mu^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} \left(\cos(j2\pi n^{-1}\mu) - 1\right)d\mu. \end{aligned}$$

We recall that

$$\varphi_{s,n}(u) = n\tilde{\psi}_{s,n}\left(\frac{u}{n}\right), \quad \forall u \geq n.$$

Which can be written as

$$\varphi_{s,n}(u) = n\delta_{s,n}\left(\frac{u}{n}\right) + n\tilde{\psi}_s\left(\frac{u}{n}\right), \quad \forall u \geq n.$$

where $\delta_{s,n}$ is given by the Equation (5). Replcae the last quantity of $\varphi_{s,n}$ in the Equation (8). By the Equations (6), we obtain

$$\begin{aligned} &\left| \frac{1}{2s} \left(u^{1+s}n^{-s} - u\right) - n\frac{1-2z_s}{2(1+s)} \left(u^{1+s}n^{-1-s} - 1\right) - V_s(u, n) + W_s(n, u) \right| \\ &\leq 2\frac{n^{-\Re(s)}u^{1+\Re(s)}}{1+\Re(s)} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} + n \sup_{u \geq 1} |\tilde{\psi}_s(u)|, \quad \forall u > n. \end{aligned}$$

We deduce that

$$\lim_{n \rightarrow +\infty} \lim_{u \rightarrow +\infty} \left| \frac{1}{2s} - \frac{1-2z_s}{2(1+s)} - u^{-1-s} n^s (V_s(u, n) - W_s(n, u)) \right| = 0.$$

In other words,

$$z_s = -\frac{1}{2s} + (1+s) \lim_{n \rightarrow +\infty} \lim_{u \rightarrow +\infty} u^{-1-s} n^s (V_s(u, n) - W_s(n, u)). \quad (10)$$

Now, estimate the limits of the previous equation. For V_s , we have

$$\begin{aligned} \lim_{u \rightarrow +\infty} u^{-1-s} n^s V_s(u, n) &= 2n^{1+s} \int_0^n \mu^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} (\cos(j2\pi n^{-1}\mu) - 1) d\mu \\ &= \int_0^1 v^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(2\pi j)^2} (\cos(2\pi jv) - 1) dv, \end{aligned}$$

Implies

$$\begin{aligned} s(1+s) \lim_{u \rightarrow +\infty} u^{-1-s} n^s V_s(u, n) &= -2 \int_0^1 v^{-s} \sum_{j=1}^{n-1} \cos(2\pi jv) dv \\ &= \int_0^1 v^{-s} dv - \int_0^1 v^{-s} \frac{\sin(\pi(2n-1)v)}{\sin(\pi v)} dv \\ &= \int_0^1 v^{-s} dv - \int_0^1 v^{-s} \sin(\pi(2n-1)v) \left[\frac{1}{\sin(\pi v)} - \frac{1}{\pi v} \right] dv \\ &\quad + \int_0^1 \frac{v^{-1-s}}{\pi} \sin(\pi(2n-1)v) dv. \end{aligned} \quad (11)$$

For W_s we have

$$\begin{aligned} \lim_{u \rightarrow +\infty} u^{-1-s} n^s W_s(n, u) &= 2n^{1+s} \int_0^{+\infty} \mu^{-2-s} \sum_{j=1}^{n-1} \frac{1}{(j2\pi)^2} (\cos(j2\pi n^{-1}\mu) - 1) d\mu \\ &= \frac{1}{2\pi^2} \int_0^{+\infty} u^{-2-s} (\cos(2\pi uv) - 1) du \sum_{j=1}^{n-1} \frac{1}{j^{1-s}} \\ &= -\frac{1}{\pi(1+s)} \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \sum_{j=1}^{n-1} \frac{1}{j^{1-s}}. \end{aligned} \quad (12)$$

Replace, in the Equation (10) the quantities of V_s and W_s given by the Equations (11) and (12), we obtain

$$sz_s = \lim_{n \rightarrow +\infty} U_s(n) - \frac{1}{2} + \frac{1}{1-s} \quad (13)$$

$$- \lim_{n \rightarrow +\infty} \int_0^1 v^{-s} \sin(\pi(2n-1)v) \left[\frac{1}{\sin(\pi v)} - \frac{1}{\pi v} \right] dv,$$

where in order to simplify the notation, for every $u \geq n$ we denoted

$$U_s(n) := -\frac{1}{\pi} \int_0^1 v^{-1-s} \sin(\pi(2n-1)v) dv$$

$$+ \frac{s}{\pi} \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \sum_{j=1}^{n-1} \frac{1}{j^{1-s}}$$

We have

$$U_s(n) = -\frac{n^s}{\pi} \int_0^n u^{-1-s} \sin(\pi(2-n^{-1})u) du$$

$$+ \frac{s}{\pi} \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \sum_{j=1}^{n-1} \frac{1}{j^{1-s}},$$

which can be written as

$$U_s(n) = \frac{s}{\pi} \left[\frac{n^s}{s} - \sum_{j=1}^{n-1} \frac{1}{j^{1-s}} \right] \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du$$

$$+ \frac{1}{\pi} n^s \left[\int_0^{+\infty} u^{-1-s} \sin(2\pi u) du - \int_0^n u^{-1-s} \sin(\pi(2-n^{-1})u) du \right],$$

or

$$U_s(n) = \frac{s}{\pi} \left[\frac{n^s}{s} - \sum_{j=1}^{n-1} \frac{1}{j^{1-s}} \right] \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du$$

$$+ \frac{n^s}{\pi} \int_0^{+\infty} u^{-1-s} \left[\sin(2\pi u) - \sin(\pi(2-n^{-1})u) \right] du$$

$$+ \frac{n^s}{\pi} \int_n^{+\infty} u^{-1-s} \sin(\pi(2-n^{-1})u) du$$

By the Abel summation formula, we have

$$\lim_{n \rightarrow +\infty} \left[\frac{n^s}{s} - \sum_{j=1}^{n-1} \frac{1}{j^{1-s}} \right] = \zeta(1-s).$$

Since

$$\begin{aligned} & \lim_{n \rightarrow +\infty} n^s \int_n^{+\infty} u^{-1-s} \sin(\pi(2 - n^{-1})u) du \\ &= (1 + s) \lim_{n \rightarrow +\infty} n^s \int_n^{+\infty} u^{-2-s} \int_n^u \sin(\pi(2 - n^{-1})v) dv du \\ &= 0, \end{aligned}$$

Then

$$\begin{aligned} \lim_{n \rightarrow +\infty} U_s(n) &= \frac{s}{\pi} \zeta(1 - s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ &\quad + \frac{n^s}{\pi} \int_0^{+\infty} u^{-1-s} \left[\sin(2\pi u) - \sin(\pi(2 - n^{-1})u) \right] du. \end{aligned}$$

The Equation (13), implies

$$\begin{aligned} sz_s &= -\frac{1}{2} + \frac{1}{1-s} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ &\quad - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} \sin(\pi(2n-1)u) \left[\frac{1}{\sin(\pi u)} - \frac{1}{\pi u} \right] du \\ &\quad + \frac{1}{\pi} \int_0^{+\infty} u^{-1-s} \left[\sin(2\pi n u) - \sin(\pi(2n-1)u) \right] du. \end{aligned}$$

We have

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_1^{+\infty} u^{-1-s} \left[\sin(2\pi n u) - \sin(\pi(2n-1)u) \right] du \\ &= \lim_{n \rightarrow +\infty} n^s \int_n^{+\infty} u^{-1-s} \left[\sin(2\pi u) - \sin(\pi(2 - n^{-1})u) \right] du \\ &= (1 + s) \lim_{n \rightarrow +\infty} n^s \int_n^{+\infty} u^{-2-s} \int_n^u \left(\sin(2\pi v) - \sin(\pi(2 - n^{-1})v) \right) dv du \\ &= 0 \end{aligned}$$

By consequence,

$$\begin{aligned} sz_s &= -\frac{1}{2} + \frac{1}{1-s} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ &\quad - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} h_n(u) du, \end{aligned}$$

where

$$h_n(u) = \sin(\pi(2n-1)u) \left[\frac{1}{\sin(\pi u)} - \frac{1}{\pi u} \right] - \frac{u^{-1}}{\pi} \left[\sin(2\pi n u) - \sin(\pi(2n-1)u) \right].$$

□

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $s \in B$. By the Equation (1) we have,

$$-s \int_1^{+\infty} u^{-1-s} \{u\} du = \zeta(s) + \frac{s}{1-s}.$$

Thanks to the Main Proposition 4 and the Lemma 5 we have

$$\begin{aligned} \zeta(s) + \frac{s}{1-s} &= sz_s \\ &= -\frac{1}{2} + \frac{1}{1-s} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du \\ &\quad - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} h_n(u) du, \end{aligned}$$

Implies

$$\zeta(s) = \frac{1}{2} + \frac{s}{\pi} \zeta(1-s) \int_0^{+\infty} u^{-1-s} \sin(2\pi u) du - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} h_n(u) du.$$

□

4 Proof of the Theorem 2

Proof of the Theorem 2. Let be $s \in B$. Suppose that $s \in B$ is a zero of the function ζ . Thanks to the Riemann functional equation, the point $1-s$ is a zero. By the Theorem 1, we get

$$\frac{1}{2} - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} h_n(u) du = 0,$$

where we recall that

$$h_n(u) = \frac{\sin(\pi(2n-1)u)}{\sin(\pi u)} - \frac{\sin(2\pi n u)}{\pi u},$$

Since

$$\sup_{u \in (0,1)} \lim_{n \rightarrow +\infty} |h_n(u)| \leq 1,$$

then

$$\begin{aligned} \left| \frac{1}{2} - \lim_{n \rightarrow +\infty} \int_0^1 u^{-s} h_n(u) du \right| &\geq \frac{1}{2} - \int_0^1 u^{-\Re(s)} du \\ &= \frac{1}{2} - \frac{1}{1 - \Re(s)}. \end{aligned}$$

□

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