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Non trivial zeros of the Zeta function using the differential equations

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Abstract

We investigate a relationship between differential equations and the Zeta function. We find a more flexible form of the Zeta function in the critical strip. We prove that all the nontrivial zeros lie on the critical line.

Keywords: Zeta function, Non trivial zeros, Riemann functional equation, Differential equations.

AMS subject classifications: 00A05

1 Main results

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \{u\} u^{-1-s} du, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{u\}$ is the fractional part of the real u . In order to simplify the notation, denote $B \subset \mathbb{C}$ the critical strip, defined as

$$B := \{s \in \mathbb{C} : \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*\},$$

and define the critical line as $s \in B$ such that $1 - 2\Re(s) = 0$. We prove the following theorems,

Theorem 1. Consider the function ζ given by the Equation (1). For every $s \in B$ we have

$$\frac{\zeta(s)}{s(1+s)} = -\frac{1}{2(1-s)} + \lim_{n \rightarrow +\infty} \frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} B_k \binom{2n}{k} \frac{(2n+1)^{-k}}{k+s+1},$$

where $(B_k)_{k \in \mathbb{N}}$ are the Bernoulli numbers and $\binom{2n}{k}$ are the binomial coefficients.

We prove that $|\zeta(s)| \neq 0$ for every $s \in B$ such that $2\Re(s) < 1$. Thanks to the Riemann functional equation, we obtain the following theorem,

Theorem 2. All the zeros of the Zeta function on B , lie on the critical line.

2 Main Proposition

For every $s \in B$, the Equation (1) is equivalent to,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

The aim is to study the differential equation of solutions the functions

$$t \mapsto \psi_s(z, t) := t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to find the initial condition $z_s \in \mathbb{C}$ such that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$ if and only if $z = -\int_1^{+\infty} u^{-1-s} \{u\} du$, we obtain $z_s = -\int_1^{+\infty} u^{-1-s} \{u\} du$.

For every $s \in B$ we consider the following differential equation

$$\frac{d}{dt} x = st^{-1}x + t^{-1}\{t\}, \quad (2)$$

$$t \in [1, +\infty) \setminus \mathbb{N}, \quad x(1) = z, \quad z \in \mathbb{C}, \quad x : [1, +\infty) \rightarrow \mathbb{C}.$$

Lemma 3. Let be $s \in B$. For every $z \in \mathbb{C}$ there exists a unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Further,

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

Proof. Let be $s \in B$ fixed. The function

$$t \mapsto t^s \int_1^t u^{-1-s} \{u\} du, \quad t \geq 1,$$

is C^∞ on $[1, +\infty)/\mathbb{N}$ and continuous on $[1, +\infty)$. The Equation (2) is a non-homogeneous linear differential equation. For every $z \in \mathbb{C}$ the unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ such that $\psi_s(z, 1) = z$ is given by

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

□

Proposition 4. *Let be $z \in \mathbb{C}$ and consider the continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2). Then $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$ if and only if the initial condition $z \in \mathbb{C}$ satisfies*

$$z_s = \frac{1}{2} - (1+s) \lim_{n \rightarrow +\infty} \frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} B_k \binom{2n}{k} \frac{(2n+1)^{-k}}{k+s+1},$$

where $(B_k)_{k \in \mathbb{N}}$ are the Bernoulli numbers.

Lemma 5. *For every $s \in B$, there exists a unique $z_s \in \mathbb{C}$ such that the solution $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (2) satisfies $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Further,*

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Proof. Let be $s \in B$ fixed. There is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm. Let be $z \in \mathbb{C}$ and suppose that

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty.$$

Since $\Re(s) \in (0, 1)$, by the Lemma 3, we have

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty \implies \left| z + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

Then

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Prove that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since

$$\left| z_s + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

then

$$z_s + \int_1^t u^{-1-s} \{u\} du = - \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t > 1.$$

By the Lemma 3, we get

$$\psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Then $\sup_{t \geq 1} |\psi_s(z_s, t)| \leq \frac{1}{\Re(s)}$. □

Let us introduce the following notations,

Notation 6. Let $g : \mathbb{R}_+ \rightarrow \mathbb{C}$ be a continuous function, we denote the function $\Phi[g] : \mathbb{R}_+ \rightarrow \mathbb{C}$ as

$$\Phi[g](t) := \int_0^t g(u) du, \quad \forall t \geq 0,$$

Notation 7. For every $k \in \mathbb{N}$ define the function $p_k : \mathbb{R} \rightarrow \mathbb{R}$ as

$$p_{2k+1} := (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+3}} \sin(j2\pi t),$$

$$p_{2k} := (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}} \cos(j2\pi t),$$

Lemma 8. *We have*

$$\int_0^t \left(\{u\} - \frac{1}{2} \right) du = p_0(t), \quad \forall t \geq 0,$$

and for every $n \in \mathbb{N}^*$ we have

$$\Phi^n[p_0](t) = \frac{1}{(n+1)!} t^{n+1} - \frac{1}{2} \frac{1}{n!} t^n - \frac{1}{(n+1)!} B_{n+1}(t) + p_n(t), \quad \forall t \geq 0.$$

where $B_n(t)$ is the Bernoulli polynomial defined as

$$B_n(t) := \sum_{k=0}^n B_k \binom{n}{k} t^{n-k}, \quad \forall t \geq 0, \quad \forall n \geq 1,$$

Proof. Prove that

$$\int_0^t \left(\{u\} - \frac{1}{2} \right) du = p_0(t), \quad \forall t \geq 0,$$

The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t \frac{1}{2} + p(t), \quad \forall t \geq 0.$$

The function p is a piecewise C^∞ , continuous on \mathbb{R} and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on \mathbb{R}_+ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p .

$$\int_0^t \left(\{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left(\int_0^u \left(\{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12} = - \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function p satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left(\exp(ij2\pi t) - 1 \right) = p_0(t), \quad \forall t \geq 0.$$

Integrate successively to obtain

$$\Phi^n[p_0](t) = \sum_{k=0}^{n-1} \frac{c_k}{(n-k)!} t^{n-k} + p_n(t), \quad \forall t \geq 0, \forall n \geq 1,$$

where $(c_k)_{k \in \mathbb{N}}$ defined as

$$c_{2k+1} = 0 \quad \text{and} \quad c_{2k} = (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}}, \quad \forall k \geq 0.$$

By definition of the Bernoulli numbers $(B_k)_{k \in \mathbb{N}}$, we get

$$\Phi^n[p_0](t) = -\frac{1}{(n+1)!} \sum_{k=2}^n B_k \binom{n+1}{k} t^{n-k+1} + p_n(t), \quad \forall t \geq 0, \forall n \geq 1,$$

then

$$\Phi^n[p_0](t) = \frac{B_0}{(n+1)!} t^{n+1} + \frac{B_1}{n!} t^n - \frac{1}{(n+1)!} B_{n+1}(t) + p_n(t), \quad \forall t \geq 0, \forall n \geq 1,$$

where the Bernoulli polynomial is defined as

$$B_n(t) := \sum_{k=0}^{n-1} B_k \binom{n}{k} t^{n-k}, \quad \forall t \geq 0, \forall n \geq 1,$$

We have $B_0 = 1$ and $B_1 = -\frac{1}{2}$. □

Proof of the Proposition 4. Let be $s \in B$ and consider the bounded continuous solution $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$ defined in the Lemma 5. In order to simplify the notation, denote

$$\tilde{\psi}_s(t) := \int_0^t \psi_s(z_s, u+1) du, \quad \forall t \geq 0.$$

Since $t \mapsto \psi_s(z_s, t)$ is solution of the differential equation (2), then the function $t \mapsto \tilde{\psi}_s(t)$ satisfies the following deferential equation

$$(t+1) \frac{d}{dt} \tilde{\psi}_s(t) = (1+s) \tilde{\psi}_s(t) + \frac{1}{2} t + z_s + p_0(t),$$

$$t \in \mathbb{R}_+/\mathbb{N}, \quad \tilde{\psi}_s(0) = 0.$$

where the function p_0 is defined by the Lemma 8. By definition of Φ in the Notation 6, we get

$$\begin{aligned} \frac{d}{dt}\Phi^{n-1}[\tilde{\psi}_s](t) &= (n+s)(t+1)^{-1}\Phi^{n-1}[\tilde{\psi}_s](t) \\ &\quad + (t+1)^{-1}h_s(n,t), \quad \forall t \geq 0, \quad \forall n \geq 1, \\ \Phi^{n-1}[\tilde{\psi}_s](0) &= 0, \quad \forall n \geq 1, \\ h_s(n,t) &:= \frac{1}{2} \frac{t^n}{n!} + z_s \frac{t^{n-1}}{(n-1)!} + \Phi^{n-1}[p_0](t), \quad \forall t \geq 0, \quad \forall n \geq 1, \end{aligned} \quad (3)$$

Use the change of variable

$$\tilde{\Phi}_s(n,t) := (n-1)! \int_0^t (u+1)^{-n} \Phi^{n-1}[\tilde{\psi}_s](u) du, \quad \forall t \geq 0, \quad \forall n \geq 1.$$

Then

$$\begin{aligned} \frac{d}{dt}\tilde{\Phi}_s(n,t) &= (1+s)(t+1)^{-1}\tilde{\Phi}_s(n,t) \\ &\quad + (n-1)! (t+1)^{-1} \int_0^t (u+1)^{-n} h_s(n,u) du, \quad \forall n \geq 1. \end{aligned}$$

Integrate, for every $n \geq 1$ we get

$$\begin{aligned} \tilde{\Phi}_s(n,t) & \\ &= (n-1)! (t+1)^{1+s} \int_0^t (u+1)^{-2-s} \int_0^u (v+1)^{-n} h_s(n,v) dv du. \end{aligned} \quad (4)$$

Since $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$, then for every fixed $n \geq 1$ we have

$$\begin{aligned} \left| \tilde{\Phi}_s(n,t) \right| &= \left| (n-1)! \int_0^t (u+1)^{-n} \Phi^{n-1}[\tilde{\psi}_s](u) du \right| \\ &\leq \sup_{u \geq 0} |\psi_s(z_s, u+1)| \frac{1}{n} \int_0^t (u+1)^{-n} u^n du \\ &\leq \sup_{u \geq 1} |\psi_s(z_s, u)| \frac{1}{n} \int_0^t 1 du \leq \frac{t}{n} \sup_{u \geq 1} |\psi_s(z_s, u)|, \quad \forall t \geq 0. \end{aligned} \quad (5)$$

Since $\Re(s) \in (0, 1)$ the Equation (4) implies

$$\int_0^{+\infty} (u+1)^{-2-s} \int_0^u (n-1)! (v+1)^{-n} h_s(n,v) dv du = 0,$$

and the sequence $\tilde{\Phi}_s(n, n)$ can be written as

$$\begin{aligned} \tilde{\Phi}_s(n, n) & \\ &= -n^{1+s} \int_n^{+\infty} (u+1)^{-2-s} \int_0^u (n-1)! (v+1)^{-n} h_s(n, v) dv du. \end{aligned} \quad (6)$$

By the notation of h_s and by the Lemma 8, we have

$$(n-1)! h_s(n, v) = \frac{3}{2} \frac{v^n}{n} + (z_s - \frac{1}{2}) v^{n-1} - \frac{1}{n} B_n(v) + (n-1)! p_{n-1}(v)$$

Implies

$$\begin{aligned} & n^{1+s} \int_n^{+\infty} (u+1)^{-2-s} \int_0^u (n-1)! (v+1)^{-n} (h_s(n, v) - p_{n-1}(v)) dv du \\ & \approx \frac{3}{2} n^s \int_n^{+\infty} u^{-1-s} du + (z_s - \frac{1}{2}) n^{1+s} \int_n^{+\infty} u^{-2-s} \ln(u) du \\ & - n^{-1+s} \int_n^{+\infty} u^{-2-s} \int_0^u (v+1)^{-n} B_n(v) du \\ & = \frac{3}{2} n^{1+s} \int_n^{+\infty} u^{-1-s} du + (z_s + \frac{1}{2}) n^{1+s} \int_n^{+\infty} u^{-2-s} \ln(u) du \\ & - n^s \int_n^{+\infty} u^{-2-s} \int_0^u \sum_{k=1}^n B_k \binom{n}{k} (v+1)^{-1-k} du \\ & - n^s \int_n^{+\infty} u^{-2-s} \int_0^u (B_n(v) - B_n(v+1)) (v+1)^{-n} du, \end{aligned}$$

Use the fact

$$B_n(v) - B_n(v+1) = -nv^{n-1},$$

Then

$$\begin{aligned} & n^{1+s} \int_n^{+\infty} (u+1)^{-2-s} \int_0^u (n-1)! (v+1)^{-n} (h_s(n, v) - \tilde{p}_{n-1}(v)) dv du \\ & \approx \frac{3}{2} n^{1+s} \int_n^{+\infty} u^{-1-s} du + (z_s + \frac{1}{2}) n^{1+s} \int_n^{+\infty} u^{-2-s} \ln(u) du \\ & + n^s \int_n^{+\infty} u^{-2-s} \sum_{k=1}^{n-2} \frac{B_k}{k} \binom{n}{k} (1 - (u+1)^{-k}) du \\ & - \frac{1}{1+s} \ln(n) \end{aligned}$$

where

$$\begin{aligned}\tilde{p}_{2k+1} &:= (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2k+3}} \sin(j2\pi t), \\ \tilde{p}_{2k} &:= (-1)^k \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}} (\cos(j2\pi t) - 1),\end{aligned}\quad (7)$$

By the Equation (5), for every we have

$$\lim_{n \rightarrow +\infty} \frac{1}{\ln(n)} \left| \tilde{\Phi}_s(n, n) \right| = 0,$$

The Equation (6) implies

$$\lim_{n \rightarrow +\infty} \left[\frac{z_s - \frac{1}{2}}{1+s} + \frac{1}{\ln(n)} \sum_{k=1}^{n-2} B_k \binom{n-1}{k} \frac{n^{-k}}{k+s+1} + \epsilon_s(n) \right] = 0,$$

where

$$\epsilon_s(n) := \frac{n^{1+s}}{\ln(n)} \int_n^{+\infty} (u+1)^{-2-s} \int_0^u (n-1)! (v+1)^{-n} \tilde{p}_{n-1}(v) dv, \quad \forall n \geq 2.$$

In particular,

$$\frac{z_s - \frac{1}{2}}{1+s} + \lim_{n \rightarrow +\infty} \left[\frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} B_k \binom{2n}{k} \frac{(2n+1)^{-k}}{k+s+1} + \epsilon_s(2n+1) \right] = 0, \quad (8)$$

where we prove now that $\lim_{n \rightarrow +\infty} \epsilon_s(\alpha, 2n+1) = 0$. By the notation (7), for every $u \geq 0$ and $n \geq 1$ we have

$$\begin{aligned}\left| \int_0^u (v+1)^{-2n-1} \tilde{p}_{2n}(v) dv \right| &= \int_0^u (v+1)^{-2n-1} \sum_{j \in \mathbb{Z}^*} \frac{1 - (\cos(j2\pi v))}{(j2\pi)^{2(n+1)}} dv \\ &\leq \int_0^{+\infty} (v+1)^{-2n-1} \sum_{j \in \mathbb{Z}^*} \frac{1 - (\cos(j2\pi v))}{(j2\pi)^{2(n+1)}} dv \\ &= \left| \int_0^{+\infty} (v+1)^{-2n-1} \tilde{p}_{2n}(v) dv \right|.\end{aligned}\quad (9)$$

Integrate by parts successively, to obtain

$$\begin{aligned}(2n)! \int_0^{+\infty} (v+1)^{-2n-1} \tilde{p}_{2n}(v) dv &= \sum_{k=1}^{2n-1} (2n-k)! \frac{d^{k-1}}{dv^{k-1}} \tilde{p}_{2n}(0) \\ &\quad + \int_0^{+\infty} (v+1)^{-2} \frac{d^{2n-1}}{dv^{2n-1}} \tilde{p}_{2n}(v) dv.\end{aligned}$$

By the notation (7), we have

$$\tilde{p}_{2n}(v) = (-1)^n \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(n+1)}} (\cos(j2\pi v) - 1),$$

then

$$\left| \frac{d^{2n-1}}{dv^{2n-1}} \tilde{p}_{2n}(v) \right| \leq \sum_{j \in \mathbb{Z}^*} \left| \frac{1}{(j2\pi)^3} \sin(j2\pi v) \right| \leq 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^3}.$$

We get

$$\begin{aligned} (2n)! \left| \int_0^{+\infty} (v+1)^{-2n-1} \tilde{p}_{2n}(v) dv \right| &\leq \left| \sum_{k=1}^{2n-1} (2n-k)! \frac{d^{k-1}}{dv^{k-1}} \tilde{p}_{2n}(0) \right| \\ &\quad + 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^3}. \end{aligned}$$

By the Lemma 8, we have

$$\tilde{p}_{2n}(v) = \Phi^{2n}[p_0](v) + \sum_{k=2}^{n-1} B_k \binom{n}{k} t^{n-k}, \quad \forall v \geq 0,$$

Then

$$(2n)! \left| \int_0^{+\infty} (v+1)^{-2n-1} p_{2n}(v) dv \right| \leq \left| \sum_{k=2}^{2n-1} \frac{B_k}{k!} \right| + 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^3}.$$

Since

$$\lim_{n \rightarrow +\infty} \left| \sum_{k=2}^{2n-1} \frac{B_k}{k!} \right| = \frac{1}{\exp(1) - 1} + \frac{1}{2}.$$

We get

$$\begin{aligned} \lim_{n \rightarrow +\infty} (2n)! \left| \int_0^{+\infty} (v+1)^{-2n-1} \tilde{p}_{2n}(v) dv \right| &\leq \alpha, \\ \alpha &:= \frac{1}{\exp(1) - 1} + \frac{1}{2} + 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^3}. \end{aligned}$$

The Equation (9) implies

$$\sup_{u > 0} \lim_{n \rightarrow +\infty} (2n)! \left| \int_0^u (v+1)^{-2n-1} \tilde{p}_{2n}(v) dv \right| \leq \alpha.$$

By the notation of $\epsilon_s(2n+1)$ we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} |\epsilon_s(2n+1)| &\leq \lim_{n \rightarrow +\infty} \frac{\alpha}{\ln(n)} (2n+1)^{1+\Re(s)} \int_{2n+1}^{+\infty} (u+1)^{-2-\Re(s)} du \\ &\leq \lim_{n \rightarrow +\infty} \frac{\alpha}{1+\Re(s)} \frac{1}{\ln(n)} = 0. \end{aligned}$$

The Equation (8), implies

$$z_s = \frac{1}{2} - (1+s) \lim_{n \rightarrow +\infty} \frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} B_k \binom{2n}{k} \frac{(2n+1)^{-k}}{k+s+1}.$$

□

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $s \in B$. By the Equation (1) we have,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Thanks to the Main Proposition 4 and the Lemma 5 we have

$$\begin{aligned} \int_1^{+\infty} u^{-1-s} \{u\} du &= -\frac{1}{2} \\ &+ (1+s) \lim_{n \rightarrow +\infty} \frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} B_k \frac{(2n)!}{k!(2n-k)!} \frac{(2n+1)^{-k}}{k(k+s+1)}. \end{aligned}$$

Then

$$\begin{aligned} \zeta(s) &= \frac{s}{2} - \frac{s}{1-s} \\ &+ s(1+s) \lim_{n \rightarrow +\infty} \frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} B_k \frac{(2n)!}{k!(2n-k)!} \frac{(2n+1)^{-k}}{k(k+s+1)}. \end{aligned}$$

□

4 Proof of the Theorem 2

Proof of the Theorem 2. Prove the present Theorem by contradiction. Suppose that there exists $s \in B$ such that $2\Re(s) < 1$ and $\zeta(s) = 0$. By the

Theorem 1, we get

$$\frac{1}{2(1-s)} + \lim_{n \rightarrow +\infty} \frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} c_k \frac{(2n)! (2n+1)^{-k}}{(2n-k)!(k+s+1)} = 0.$$

where the numbers $(c_k)_{k \in \mathbb{N}}$ are defined as

$$c_{2k+1} = 0 \quad \text{and} \quad c_{2k} = (-1)^{k+1} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}}, \quad \forall k \geq 0. \quad (10)$$

Since $1 - \bar{s}$ is also a zero of the function ζ , then

$$\frac{1}{2\bar{s}} + \lim_{n \rightarrow +\infty} \frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} c_k \frac{(2n)! (2n+1)^{-k}}{(2n-k)!(k-\bar{s}+2)} = 0.$$

The imaginary parts, satisfies

$$0 = \alpha_s := \frac{1}{2} \left[\frac{1}{|1-s|^2} - \frac{1}{|s|^2} \right] - \lim_{n \rightarrow +\infty} \frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} c_k v_s(n, k), \quad (11)$$

where

$$v_s(n, k) := \frac{(2n)! (2n+1)^{-k}}{(2n-k)!} \left[\frac{1}{|k+s+1|^2} - \frac{1}{|k-s+2|^2} \right]$$

By definition of $(c_{2k})_k$ in the Equation (10) and by the hypothesis $2\Re(s) < 1$, the sequence

$$\left(|c_{2k}| v_s(n, 2k) \right)_{k \geq 0}$$

is positive and strictly decreasing, then

$$\lim_{n \rightarrow +\infty} \frac{1}{\ln(2n+1)} \sum_{k=1}^{2n-1} c_k v_s(n, k) \geq 0,$$

Implies

$$\alpha_s \leq \frac{1}{2} \left[\frac{1}{|1-s|^2} - \frac{1}{|s|^2} \right].$$

Since $2\Re(s) < 1$, then $\alpha_s < 0$. Contradiction with the Equation (11). \square

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).