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Functional equation and nontrivial zeros of the Zeta function

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Abstract

We obtain a more flexible functional equation of the Zeta function. Together with the Riemann functional equation, we prove that any non-trivial zeros s of the function ζ satisfies $2\Re(s) \geq 1$.

Keywords: Zeta function, Non trivial zeros, Riemann functional equation, Differential equations.

AMS subject classifications: 00A05

1 Main results

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \{u\} u^{-1-s} du, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}, \quad (1)$$

where $\{u\}$ is the fractional part of the real u . In order to simplify the notation, denote $B \subset \mathbb{C}$ the critical strip, defined as

$$B := \{s \in \mathbb{C} : \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}\}.$$

We prove the following theorems,

Theorem 1. Consider the function ζ given by the Equation (1). For every $s \in B$ we have

$$\begin{aligned} \frac{2\pi^2\zeta(s)}{1+s} &= \alpha(\zeta, 1-\bar{s})\beta_-(s) \\ &\quad + \alpha(\zeta, s)\left[\beta_+(s) - \beta_-(s)\right], \quad 2\Re(s) < 1, \\ \frac{2\pi^2\zeta(1-\bar{s})}{2-\bar{s}} &= \alpha(\zeta, 1-\bar{s})\beta_+(1-\bar{s}) \\ &\quad + \alpha(\zeta, s)\left[\beta_-(s) - \beta_-(1-\bar{s})\right], \quad 2\Re(s) > 1. \end{aligned}$$

where

$$\begin{aligned} \alpha(\zeta, s) &:= -\frac{\zeta(\bar{s})}{\bar{s}} - \frac{1}{1-\bar{s}}, \quad \forall s \in B, \\ \beta_-(s) &:= s(1-s) \int_0^1 u^{-2-s}(\cos(2\pi u) - 1)du, \quad \forall s \in B, \\ \beta_+(s) &:= s(1-s) \int_0^{+\infty} u^{-2-s}(\cos(2\pi u) - 1)du, \quad \forall s \in B \end{aligned}$$

Thanks to the Riemann functional equation, we obtain the following theorem,

Theorem 2. Let be $s \in B$ and suppose that s is a zero of the function ζ . Then $2\Re(s) \geq 1$.

2 Main Proposition

We prove in this section the following Proposition

Proposition 3. For every $s \in B$ and $t \in (0, 1)$ we have

$$\frac{2\pi^2}{1+s}\zeta(s)t^{-\frac{1}{2}+s} - \frac{2\pi^2}{2-\bar{s}}\zeta(1-\bar{s})t^{\frac{1}{2}-\bar{s}} = \delta(s, t) - \delta(1-\bar{s}, t),$$

where

$$\begin{aligned} \delta(s, t) &:= -\pi^2 \frac{s}{1-s} t^{\frac{1}{2}} + (1-s)t^{-\frac{3}{2}} \int_1^{+\infty} z^{-2-s}(\cos(2\pi tz) - 1)dz \\ &\quad + st^{-\frac{3}{2}} \int_1^{+\infty} \{z\} \frac{d}{dz} f_s(z, t) dz, \quad \forall s \in B, \quad \forall t \in (0, 1]. \end{aligned}$$

and where

$$s \in B, \forall z \in \mathbb{R}_+^*, \forall t \in (0, 1) :$$

$$f_s(z, t) = -z^{-1+s} \int_z^{+\infty} u^{-2-s} (\cos(2\pi ut) - 1) du$$

Notation 4. Denote

$$p(t) := \int_1^t \left(\{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 1,$$

where we recall that $\{u\}$ is the fractional part of the real u .

Lemma 5. *The function p is a continuous 1-periodic function and satisfies*

$$p(t) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} (\cos(j2\pi t) - 1), \quad \forall t \geq 1.$$

Proof. The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_1^t \{u\} du = (t-1) \int_1^2 \{u\} du + p(t), \quad \forall t \geq 1.$$

Since

$$\int_1^2 \{u\} du = \int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_1^t \{u\} du = (t-1) \frac{1}{2} + p(t), \quad \forall t \geq 1.$$

The function p is a piecewise C^∞ , continuous on \mathbb{R} and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on \mathbb{R}_+ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p .

$$\int_0^t \left(\{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left(\int_0^u \left(\{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12} = -\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function p satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left(\exp(ij2\pi t) - 1 \right), \quad \forall t \geq 0.$$

□

Lemma 6. *Consider the function ζ given by the Equation (1). For every $s \in B$ we have*

$$\begin{aligned} 2\pi^2 \frac{\zeta(s)}{1+s} t^{1+s} &= -\pi^2 \left(\frac{s}{1-s} t^2 + t \right) \\ &\quad - s \sum_{j \in \mathbb{N}^*} j^{-1-s} \int_j^{+\infty} u^{-2-s} \left(\cos(2\pi ut) - 1 \right) du, \quad \forall t \in (0, 1). \end{aligned}$$

Proof. By the Equation (1). For every $s \in B$ we have

$$\begin{aligned} -\frac{\zeta(s)}{s} &= \frac{1}{1-s} t^{1-s} + \int_t^{+\infty} \{u\} u^{-1-s} du \\ &= \frac{1}{1-s} t^{1-s} + \frac{1}{2s} t^{-s} + \int_t^{+\infty} \left(\{u\} - \frac{1}{2} \right) u^{-1-s} du, \quad \forall t \in (0, 1). \end{aligned}$$

Use the integration by part formula

$$-\frac{\zeta(s)}{s} = \frac{1}{1-s} t^{1-s} + \frac{1}{2s} t^{-s} + (1+s) \int_t^{+\infty} p(u) u^{-2-s} du - p(t) t^{-1-s},$$

where the function $p(u)$ is given by the Lemma 5. Use the fact $p(t) = \frac{1}{2}(t^2 - t)$ for every $t \in (0, 1)$ we get

$$\begin{aligned} -\frac{\zeta(s)}{s} &= \frac{1}{1-s} t^{1-s} + \frac{1}{2s} t^{-s} + (1+s) \int_t^{+\infty} p(u) u^{-2-s} du \\ &\quad - \frac{1}{2} (t^2 - t) t^{-1-s}, \quad \forall t \in (0, 1). \end{aligned}$$

Since

$$p(t) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} (\cos(j2\pi t) - 1), \quad \forall t \in \mathbb{R}.$$

Use the fact

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left| \int_t^{+\infty} u^{-2-s} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} (\cos(j2\pi u) - 1) du \right| \\ \leq \int_t^{+\infty} u^{-2-r} du \lim_{n \rightarrow +\infty} \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} = 0, \end{aligned}$$

then

$$\begin{aligned} \zeta(s) &= -\frac{s}{2} \frac{1+s}{1-s} t^{1-s} - \frac{1}{2} (1+s) t^{-s} \\ &\quad - 2s(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \int_t^{+\infty} u^{-2-s} (\cos(j2\pi u) - 1) du \quad \forall t \in (0, 1). \end{aligned}$$

Use the change of variable $u \mapsto u = tj^{-1}v$ to obtain

$$\begin{aligned} \zeta(s) &= -\frac{s}{2} \frac{1+s}{1-s} t^{1-s} - \frac{1}{2} (1+s) t^{-s} \\ &\quad - \frac{s}{2\pi^2} (1+s) t^{-1-s} \sum_{j \in \mathbb{N}^*} j^{-1-s} \int_j^{+\infty} u^{-2-s} (\cos(2\pi ut) - 1) du. \end{aligned}$$

□

Proof of the Proposition 3. Let $s \in B$ fixed. In order to simplify the notation, denote

$$\forall t \in (0, 1] : \quad \varphi_s(n, t) := \sum_{j=1}^n f_s(j, t), \quad \forall n \in \mathbb{N}^*.$$

where

$$f_s(z, t) = -z^{-1+s} \int_z^{+\infty} u^{-2-s} (\cos(2\pi ut) - 1) du, \quad \forall z \in \mathbb{R}_+^*, \forall t \in (0, 1), \quad (2)$$

by the the Lemma 6, we get

$$\frac{2\pi^2 \zeta(s)}{1+s} t^{1+s} = s \lim_{n \rightarrow +\infty} \varphi_s(n, t) + \pi^2 \left(\frac{s}{1-s} t^2 + t \right), \quad \forall t \in (0, 1). \quad (3)$$

For every fixed $t \in (0, 1]$, use the Euler-Maclaurin summation formula, given by

$$\begin{aligned}\varphi_s(n, t) &= \frac{1}{2}[f_s(1, t) - f_s(n, t)] + \int_1^n f_s(z, t)dz \\ &\quad + \int_1^n \left(\{z\} - \frac{1}{2}\right) \frac{d}{dz} f_s(z, t) dz, \quad \forall n \in \mathbb{N}^*,\end{aligned}$$

or,

$$\begin{aligned}\varphi_s(n, t) &= f_s(1, t) - \frac{1}{2}f_s(n, t) + \int_1^n f_s(z, t)dz \\ &\quad + \int_1^n \{z\} \frac{d}{dz} f_s(z, t) dz, \quad \forall n \in \mathbb{N}^*.\end{aligned}$$

Using the Equation (2) and the integration by parts formula, we obtain

$$\begin{aligned}\int_1^n f_s(z, t) dz &= \frac{1}{s} n f_s(n, t) + \frac{1}{s} \int_1^{+\infty} z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^n u^{-2} (\cos(2\pi tu) - 1) du, \quad \forall n \in \mathbb{N}^*.\end{aligned}$$

Then

$$\begin{aligned}\varphi_s(n, t) &= f_s(1, t) - \frac{1}{2}f_s(n, t) \\ &\quad + \frac{1}{s} n f_s(n, t) + \frac{1}{s} \int_1^{+\infty} z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^n u^{-2} (\cos(2\pi tu) - 1) du \\ &\quad + \int_1^n \{z\} \frac{d}{dz} f_s(z, t) dz, \quad \forall n \in \mathbb{N}^*.\end{aligned}$$

The Equation (2) implies $\lim_{z \rightarrow +\infty} z f_s(z, t) = 0$. By consequence,

$$\begin{aligned}\lim_{n \rightarrow +\infty} \varphi_s(n, t) &= \frac{1-s}{s} \int_1^{+\infty} z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^{+\infty} z^{-2} (\cos(2\pi tz) - 1) dz \\ &\quad + \int_1^{+\infty} \{z\} \frac{d}{dz} f_s(z, t) dz\end{aligned}$$

The Equation (3), implies

$$\frac{2\pi^2\zeta(s)}{1+s}t^{1+s} = -\pi^2t - \int_1^{+\infty} z^{-2}(\cos(2\pi tz) - 1)dz + \delta(s, t), \quad \forall t \in (0, 1].$$

where

$$\begin{aligned} \delta(s, t) &:= -\pi^2 \frac{t^2s}{1-s} + (1-s) \int_1^{+\infty} z^{-2-s}(\cos(2\pi tz) - 1)dz \\ &\quad + s \int_1^{+\infty} \{z\} \frac{d}{dz} f_s(z, t) dz, \quad \forall t \in (0, 1]. \end{aligned}$$

□

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $s \in B$. Thanks to the Proposition 3 we have

$$\frac{2\pi^2}{1+s}\zeta(s)t^{-\frac{1}{2}+s} - \frac{2\pi^2}{2-\bar{s}}\zeta(1-\bar{s})t^{\frac{1}{2}-\bar{s}} = \delta(s, t) - \delta(1-\bar{s}, t), \quad (4)$$

where

$$\begin{aligned} \delta(s, t) &:= -\pi^2 \frac{s}{1-s} t^{\frac{1}{2}} + (1-s)t^{-\frac{3}{2}} \int_1^{+\infty} z^{-2-s}(\cos(2\pi tz) - 1)dz \\ &\quad + st^{-\frac{3}{2}} \int_1^{+\infty} \{z\} \frac{d}{dz} f_s(z, t) dz, \quad \forall t \in (0, 1]. \end{aligned} \quad (5)$$

and where

$$\begin{aligned} s \in B, \quad \forall z \in \mathbb{R}_+^*, \quad \forall t \in (0, 1) : \\ f_s(z, t) = -z^{-1+s} \int_z^{+\infty} u^{-2-s}(\cos(2\pi ut) - 1)du \end{aligned} \quad (6)$$

Since

$$\lim_{t \rightarrow 0} t^{-\frac{3}{2}} \left[\pi^2 \frac{t^2s}{1-s} + (1-s) \int_0^1 u^{-2-s}(\cos(2\pi tu) - 1)du \right] = 0$$

then by definition of the function δ in the Equation (5) and the Equation (4), we get

$$\begin{aligned} 2\pi^2 \lim_{t \rightarrow 0} \left[\frac{\zeta(s)}{1+s} t^{-\frac{1}{2}+s} - \frac{\zeta(1-\bar{s})}{2-\bar{s}} t^{\frac{1}{2}-\bar{s}} \right] \\ = \lim_{t \rightarrow 0} t^{-\frac{3}{2}} \int_1^{+\infty} \{z\} \left[s \frac{d}{dz} f_s(z, t) - (1-\bar{s}) \frac{d}{dz} f_{1-\bar{s}}(z, t) \right] dz. \end{aligned} \quad (7)$$

From the Equation (6) , we have

$$f_s(z, t) = -t^{1+s}z^{-1+s} \int_{zt}^{+\infty} v^{-2-s}(\cos(2\pi v) - 1)dv, \quad \forall z \in \mathbb{R}_+^*, \forall t \in (0, 1),$$

Then

$$\begin{aligned} \frac{d}{dz}f_s(z, t) &= t^{1+s}(1-s)z^{-2+s} \int_{zt}^{+\infty} v^{-2-s}(\cos(2\pi v) - 1)dv \\ &\quad + z^{-3}(\cos(2\pi zt) - 1), \quad \forall z \in \mathbb{R}_+^*, \forall t \in (0, 1), \end{aligned}$$

In order to simplify the notation, denote

$$\alpha_s(v) := \int_v^{+\infty} \{u\}u^{-1-\bar{s}}du, \quad v \geq 1, \quad s \in B, \quad (8)$$

and denote

$$\begin{aligned} \beta(s) &:= s(1-s) \int_0^{+\infty} v^{-2-s}(\cos(2\pi v) - 1)dv \\ \eta_s(u) &:= -s(1-s) \int_0^u v^{-2-s}(\cos(2\pi v) - 1)dv, \quad u > 0, \quad s \in B, \quad (9) \end{aligned}$$

The Equation (7), implies that

$$\begin{aligned} &2\pi^2 \lim_{t \rightarrow 0} t^{-\frac{1}{2}+s} \left[\frac{\zeta(s)}{1+s} - \beta(s)\alpha_{1-\bar{s}}(1) - \int_1^{+\infty} \{z\}z^{-2+s}\eta_s(zt)dz \right] \\ &= \lim_{t \rightarrow 0} t^{\frac{1}{2}-\bar{s}} \left[\frac{\zeta(1-\bar{s})}{2-\bar{s}} - \beta(1-\bar{s})\alpha_s(1) - \int_1^{+\infty} \{z\}z^{-1-\bar{s}}\eta_{1-\bar{s}}(zt)dz \right] \\ &\quad + (2\Re(s) - 1)t^{-\frac{3}{2}} \int_1^{+\infty} \{z\}z^{-3}(\cos(2\pi zt) - 1)dz. \end{aligned}$$

Use the fact

$$\begin{aligned} &\lim_{t \rightarrow 0} t^{-\frac{3}{2}} \left| \int_1^{+\infty} \{z\}z^{-3}(\cos(2\pi zt) - 1)dz \right| \\ &= \lim_{t \rightarrow 0} t^{\frac{1}{2}} \left| \int_t^{+\infty} \left\{ \frac{z}{t} \right\} z^{-3}(\cos(2\pi z) - 1)dz \right| \\ &\leq \lim_{t \rightarrow 0} t^{\frac{1}{2}} \left| \int_1^{+\infty} \left\{ \frac{z}{t} \right\} z^{-3}(\cos(2\pi z) - 1)dz \right| \\ &\quad + \lim_{t \rightarrow 0} t^{\frac{1}{2}} \left| \int_t^1 \left\{ \frac{z}{t} \right\} z^{-3}(\cos(2\pi z) - 1)dz \right| \\ &\leq 2\pi \lim_{t \rightarrow 0} t^{\frac{1}{2}} \left[\int_1^{+\infty} z^{-3}dz + 2\pi \int_t^1 z^{-1}dz \right] \\ &= 0 \end{aligned}$$

then

$$\begin{aligned} & 2\pi^2 \lim_{t \rightarrow 0} t^{-\frac{1}{2}+s} \left[\frac{\zeta(s)}{1+s} - \beta(s)\alpha_{1-\bar{s}}(1) - \int_1^{+\infty} \{z\} z^{-2+s} \eta_s(zt) dz \right] \\ &= \lim_{t \rightarrow 0} t^{\frac{1}{2}-\bar{s}} \left[\frac{\zeta(1-\bar{s})}{2-\bar{s}} - \beta(1-\bar{s})\alpha_s(1) - \int_1^{+\infty} \{z\} z^{-1-\bar{s}} \eta_{1-\bar{s}}(zt) dz \right]. \end{aligned}$$

For every $s \in B$ such that $2\Re(s) < 1$ we obtain

$$\frac{\zeta(s)}{1+s} - \beta(s)\alpha_{1-\bar{s}}(1) = \lim_{t \rightarrow 0} t^{-\frac{1}{2}+s} \int_1^{+\infty} \{z\} z^{-2+s} \eta_s(zt) dz. \quad (10)$$

By definition of the function η_s in the Equation (14), we have

$$\begin{aligned} & t^{-\frac{1}{2}+s} \int_1^{+\infty} \{z\} z^{-2+s} \eta_s(zt) dz \quad (11) \\ &= -s(1-s)t^{-\frac{1}{2}+s} \int_1^{+\infty} \{z\} z^{-2+s} \int_0^{zt} v^{-2-s} (\cos(2\pi v) - 1) dv dz \\ &= s(1-s)t^{-\frac{1}{2}+s} \int_1^{+\infty} \{z\} z^{-2+s} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (2\pi)^2 k (zt)^{2k-1-s}}{2k-1-s} dz \\ &= s(1-s)t^{\frac{1}{2}} \int_1^{+\infty} \{z\} z^{-2+s} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (2\pi)^2 k z^{2k-3} t^{2k-2}}{2k-1-s} dz. \end{aligned}$$

Since

$$\begin{aligned} & s(1-s)z^{-2+s} \int_0^{zt} v^{-2-s} (\cos(2\pi v) - 1) dv \\ &= s(1-s) \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k z^{2k-3} (2\pi t)^{2k}}{2k-1-s} \\ &= \sum_{k \in \mathbb{N}^*} \frac{(2k-2)(2k-1)}{(2k)!} \frac{(-1)^k z^{2k-3} (2\pi t)^{2k}}{2k-1-s}. \end{aligned}$$

In order to simplify the notation, denote

$$X(s) := 2\pi^2 \frac{\zeta(s)}{1+s} - \beta(s)\alpha_s(1), \quad \forall s \in B. \quad (12)$$

The Equations (7) and (11), implies that

$$\begin{aligned} & \lim_{t \rightarrow 0} \left[X(s)t^{-\frac{1}{2}+s} - X(1-\bar{s})t^{\frac{1}{2}-\bar{s}} \right] \quad (13) \\ &= (1-2\Re(s)) \lim_{t \rightarrow 0} t^{\frac{1}{2}-\bar{s}} \left[\int_1^{+\infty} \{z\} z^{-1-\bar{s}} \zeta_s(zt) dz \right]. \end{aligned}$$

where

$$\xi_x(\mu) := \sum_{k \geq 2} \frac{(2k-1)(2k-2)}{(2k)!} \frac{(-1)^k (2\pi)^{2k} \mu^{2k-2+\bar{x}}}{(2k-2+\bar{x})(2k-1-s)}, \quad \forall x \in B. \quad (14)$$

Using the fact

$$\frac{d^2}{du^2} \left[u^{-1} (\cos(2\pi u) - 1) \right] = \sum_{k \geq 2} \frac{(2k-1)(2k-2)}{(2k)!} (-1)^k (2\pi)^{2k} u^{2k-3},$$

then ξ_s can be written as,

$$\xi_s(v) = \int_0^v z^{-2+2\Re(s)} \int_0^z u^{1-s} \frac{d^2}{du^2} \left[u^{-1} (\cos(2\pi u) - 1) \right] dudz, \quad (15)$$

For a fixed $t \in (0, 1)$ we have

$$\begin{aligned} \int_1^{+\infty} \{z\} z^{-1-\bar{s}} \xi_s(z) dz &= \alpha_s(1) \xi_s(1) \\ &+ \int_1^{+\infty} \{z\} z^{-1-\bar{s}} (\xi_s(z) - \xi_s(1)) dz. \end{aligned}$$

Use the integration part formula for the last term of the right side of the last equality, we get

$$\int_1^{+\infty} \{z\} z^{-1-\bar{s}} \xi_s(z) dz = \alpha_s(1) \xi_s(1) - t \int_1^{+\infty} \alpha_s(z) \xi'_s(z) dz. \quad (16)$$

By the Equation (15), we have

$$\xi'_s(v) = v^{-2+2\Re(s)} \int_0^v u^{1-s} \frac{d^2}{du^2} \left[u^{-1} (\cos(2\pi u) - 1) \right] du.$$

In order to simplify the notation, denote

$$\theta(u) := \frac{d^2}{du^2} \left[u^{-1} (\cos(2\pi u) - 1) \right], \quad \forall u > 0, \quad (17)$$

and denote

$$\beta_x(v) := \int_0^v u^{1-x} \theta(u) du, \quad v > 0, \quad x \in B. \quad (18)$$

Then

$$\xi'_s(v) = v^{-2+2\Re(s)} \beta_s(v).$$

In order to simplify the notation, denote

$$\tilde{\alpha}_x(v) := - \int_v^{+\infty} \alpha_x(u) u^{-2+2\Re(x)} du \quad v \geq 1, \quad x \in B.$$

The Equation (16) implies

$$\begin{aligned} \int_1^{+\infty} \{z\} z^{-1-\bar{s}} \xi_s(z) dz &= \alpha_s(1) \xi_s(1) + \tilde{\alpha}_s(1) \beta_s(1) t^{-1+2\Re(s)} \\ &+ t^{-1+2\Re(s)} \int_1^{+\infty} \alpha_s(z) z^{-2+2\Re(s)} (\beta_s(z) - \beta_s(1)) dz. \end{aligned}$$

Use the integration part formula for the last term of the right side of the last equality, we obtain

$$\begin{aligned} \int_1^{+\infty} \{z\} z^{-1-\bar{s}} \xi_s(z) dz &= \alpha_s(1) \xi_s(1) + \tilde{\alpha}_s(1) \beta_s(1) t^{-1+2\Re(s)} \\ &+ t^{1+\bar{s}} \int_1^{+\infty} \tilde{\alpha}_s(z) z^{1-s} \theta(z) dz. \end{aligned} \quad (19)$$

By definition of the function θ in the Equation (17), we have

$$\begin{aligned} \theta(u) &= \frac{d^2}{du^2} [u^{-1} (\cos(2\pi u) - 1)] \\ &= 2u^{-3} (\cos(2\pi u) - 1) - 4\pi^2 u^{-1} \cos(2\pi u), \end{aligned}$$

That implies

$$\begin{aligned} &\lim_{t \rightarrow 0} t^{\frac{3}{2}} \left| \int_1^{+\infty} \tilde{\alpha}_s(z) z^{1-s} \theta(z) dz \right| \\ &\leq 2 \lim_{t \rightarrow 0} t^{\frac{3}{2}} \left| \int_1^{+\infty} \tilde{\alpha}_s(z) z^{-2-s} (\cos(2\pi u) - 1) dz \right| \\ &\quad + 4\pi^2 \lim_{t \rightarrow 0} t^{\frac{3}{2}} \left| \int_1^{+\infty} \tilde{\alpha}_s(z) z^{-s} \cos(2\pi u) dz \right| \\ &= 2 \lim_{t \rightarrow 0} t^{\frac{3}{2}} \left| \int_1^{+\infty} \tilde{\alpha}_s(z) z^{-2-s} (\cos(2\pi u) - 1) dz \right| \\ &\quad + |s(1+s)| \lim_{t \rightarrow 0} t^{\frac{3}{2}} \left| \int_1^{+\infty} \tilde{\alpha}_s(z) z^{-2-s} (\cos(2\pi u) - 1) dz \right| \\ &\leq (2 + |s(1+s)|) \lim_{t \rightarrow 0} t^{\frac{1}{2}} \int_1^{+\infty} |\tilde{\alpha}_s(z)| z^{-2-r} dz = 0. \end{aligned}$$

The Equation (19), implies

$$\lim_{t \rightarrow 0} \left[t^{\frac{1}{2}-\bar{s}} \int_1^{+\infty} \{z\} z^{-1-\bar{s}} \xi_s(zt) dz - V(s, t) \right] = 0.$$

where

$$V(s, t) := \alpha_s(1) \xi_s(1) t^{\frac{1}{2}-\bar{s}} + \tilde{\alpha}_s(1) \beta_s(1) t^{-\frac{1}{2}+s}.$$

The Equation (13) and by the notation of X in the Equation (12), we get

$$\begin{aligned} 2\pi^2 \frac{\zeta(s)}{1+s} &= \beta(s) \alpha_s(1) + (1 - 2\Re(s)) \tilde{\alpha}_s(1) \beta_s(1), \quad 2\Re(s) < 1, \\ 2\pi^2 \frac{\zeta(1-\bar{s})}{2-\bar{s}} &= \beta(1-\bar{s}) \alpha_{1-\bar{s}}(1) + (1 - 2\Re(s)) \alpha_s(1) \xi_s(1), \quad 2\Re(s) > 1. \end{aligned}$$

By definition of the function ζ in the Equation (1) and the function α_s in the Equation (8), we have

$$\begin{aligned} \alpha_s(1) &= \int_1^{+\infty} \{u\} u^{-1-\bar{s}} du = -\frac{\zeta(\bar{s})}{\bar{s}} - \int_0^1 \{u\} u^{-1-\bar{s}} du \\ &= -\frac{\zeta(\bar{s})}{\bar{s}} - \frac{1}{1-\bar{s}}, \quad \forall x \in B, \end{aligned}$$

same,

$$(1 - 2\Re(s)) \tilde{\alpha}_x(1) = \alpha_{1-\bar{s}}(1) - \alpha_s(1)$$

Recall that, the function β_s is given by the Equation (18), and we have

$$\beta_s(1) := \int_0^1 u^{1-s} \theta(u) du = s(1-s) \int_0^1 u^{-2-s} (\cos(2\pi u) - 1) du, \quad \forall s \in B.$$

Finally recall that ξ_s is given by the Equation (15), using the integration by part formula, we obtain

$$\begin{aligned} (1 - 2\Re(s)) \xi_s(1) &= \int_0^1 u^{1-s} \theta(u) du + \int_0^1 z^{\bar{s}} \theta(u) du \\ &= \beta_s(1) - \beta_{1-\bar{s}}(1), \quad \forall s \in B. \end{aligned}$$

□

4 Proof of the Theorem 2

Proof of the Theorem 2. Let be $x \in B$ such that $2\Re(x) < 1$ and suppose that x is a zero of the function ζ . Thanks to the Riemann functional equation the point $1 - \bar{x}$ is also a zero. By the first item of the Theorem 1, we obtain

$$\frac{1 - 2\Re(x)}{x} \beta_-(x) = -\beta_+(x),$$

where we recall

$$\beta_-(s) := s(1-s) \int_0^1 u^{-2-s} (\cos(2\pi u) - 1) du, \quad \forall s \in B,$$

$$\beta_+(s) := s(1-s) \int_0^{+\infty} u^{-2-s} (\cos(2\pi u) - 1) du, \quad \forall s \in B$$

Then

$$\frac{1 - \bar{x}}{x} \int_0^1 u^{-2-x} (\cos(2\pi u) - 1) du = - \int_1^{+\infty} u^{-2-x} (\cos(2\pi u) - 1) du.$$

We apply the second item to the point $1 - \bar{x}$ we get

$$\begin{aligned} - \int_1^{+\infty} u^{-2-x} (\cos(2\pi u) - 1) du &= \frac{\bar{x}(1-\bar{x})^2}{x^2(1-x)} \int_0^1 u^{-3+\bar{x}} (\cos(2\pi u) - 1) du \\ &\quad + \frac{1 - 2\Re(x)}{x} \int_0^1 u^{-2-x} (\cos(2\pi u) - 1) du. \end{aligned}$$

Implies

$$\begin{aligned} x^2(1-x) \int_0^1 u^{-2-x} (\cos(2\pi u) - 1) du \\ = \bar{x}(1-\bar{x})^2 \int_0^1 u^{-3+\bar{x}} (\cos(2\pi u) - 1) du \end{aligned}$$

wich can be written as

$$x^2(1-x) \sum_{k \in \mathbb{N}^*} \frac{(-1)^k}{(2k)!} \frac{(2\pi)^{2k}}{2k-1-x} = \bar{x}(1-\bar{x})^2 \sum_{k \in \mathbb{N}^*} \frac{(-1)^k}{(2k)!} \frac{(2\pi)^{2k}}{2k-2+\bar{x}},$$

or again

$$\begin{aligned} \sum_{k \in \mathbb{N}^*} \frac{(-1)^k}{(2k)!} \frac{(2\pi)^{2k} (2k-2)(2k-1)^2}{2k-1-s} \\ = \sum_{k \in \mathbb{N}^*} \frac{(-1)^k}{(2k)!} \frac{(2\pi)^{2k} (2k-2)(2k-1)^2}{2k-2+\bar{s}}. \end{aligned}$$

Implies

$$(1 - 2\Re(s)) \sum_{k \in \mathbb{N}^*} \frac{(-1)^k (2\pi)^{2k} (2k-2)(2k-1)^2}{(2k)! (2k-1-s)(2k-2+\bar{s})} = 0.$$

Since $2\Re(s) < 1$ then

$$\sum_{k \in \mathbb{N}^*} \frac{(-1)^k (2\pi)^{2k} (2k-2)(2k-1)^2}{(2k)! (2k-1-s)(2k-2+\bar{s})} = 0.$$

For the imaginary part, we obtain

$$\sum_{k \in \mathbb{N}^*} (-1)^k u_{s,k} = 0, \quad u_{s,k} := \frac{(2\pi)^{2k} (4k-3)(2k-2)(2k-1)^2}{(2k)! |(2k-1-s)(2k-2+\bar{s})|^2}. \quad (20)$$

Since

$$\forall k \geq 12 : \quad u_{s,k} > u_{s,k+1}.$$

Then

$$\sum_{k=12}^{+\infty} (-1)^k u_{s,k} > 0.$$

Since

$$u_{s,2k} - u_{s,2(k+1)} > u_{s,13-2k} - u_{s,12-2k} \quad \forall k \in \llbracket 1, 2 \rrbracket \quad \text{and} \quad u_{s,6} > u_{s,7},$$

then $\sum_{k=2}^{11} (-1)^k u_{s,k} > 0$. We deduce that $\sum_{k \in \mathbb{N}^*} (-1)^k u_{s,k} > 0$. Contradiction with the Equation (20). \square

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).