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Nontrivial zeros of the Zeta function using the differential equations

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Abstract

We investigate a relation between the differential equations and the Zeta function. We find a more flexible form of the Zeta function in the critical strip. We prove that all the nontrivial zeros lie on the critical line.

Keywords: Zeta function, Non trivial zeros, Riemann functional equation, Differential equations.

AMS subject classifications: 00A05

1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}, \quad (1)$$

where $\{t\}$ is the fractional part of the real t . In order to simplify the notation, denote $B \subset \mathbb{C}$ the critical strip, defined as

$$B := \{s \in \mathbb{C} : \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}\},$$

and define the critical line as $s \in B$ such that $1 - 2\Re(s) = 0$. We prove the following theorems,

Theorem 1. Consider the function ζ given by the Equation (1). For every $s \in B$ we have

$$\begin{aligned} \frac{\zeta(s)}{s(1+s)}t^s &= -\frac{1}{2}\left(\frac{t}{1-s} + \frac{1}{s}\right) \\ &+ 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{2k-1-s}, \quad \forall t \in (0, 1]. \end{aligned} \quad (2)$$

Theorem 2. All the zeros of the Zeta function on B , lie on the critical line.

2 Main Proposition

For every $s \in B$, the Equation (1) is equivalent to,

$$\frac{\zeta(s)}{s} = -\frac{1}{1-s} - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

The aim is to studies the differential equation of solutions the functions

$$t \mapsto \psi_s(z, t) := t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad z \in \mathbb{C}, \quad t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to find the initial condition $z_s \in \mathbb{C}$ such that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$ if and only if $z = -\int_1^{+\infty} u^{-1-s} \{u\} du$, we obtain $-\int_1^{+\infty} u^{-1-s} \{u\} du = z_s$.

For every $s \in B$ we consider the following differential equation

$$\frac{d}{dt}x = st^{-1}x + t^{-1}\{t\}, \quad (3)$$

$$t \in [1, +\infty)/\mathbb{N}, \quad x(1) = z, \quad z \in \mathbb{C}, \quad x : [1, +\infty) \rightarrow \mathbb{C}.$$

Lemma 3. Let be $s \in B$. For every $z \in \mathbb{C}$ there exists a unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (3). Further,

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

Proof. Let be $s \in B$ fixed. The function

$$t \mapsto t^s \int_1^t u^{-1-s} \{u\} du, \quad t \geq 1,$$

is C^∞ on $[1, +\infty)/\mathbb{N}$ and continuous on $[1, +\infty)$. The Equation (3) is a non-homogeneous linear differential equation. For every $z \in \mathbb{C}$ the unique continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ such that $\psi_s(z, 1) = z$ is given by

$$\psi_s(z, t) = t^s \left[z + \int_1^t u^{-1-s} \{u\} du \right], \quad \forall t \geq 1.$$

□

Proposition 4. *Let be $s \in B$. Let be $z \in \mathbb{C}$ and consider the continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (3). Then $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$, if and only if*

$$z = -\frac{1}{2} \frac{1}{s} + 2(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}.$$

In other words,

$$\int_1^{+\infty} u^{-1-s} \{u\} du = \frac{1}{2} \frac{1}{s} - 2(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}.$$

Lemma 5. *For every $s \in B$, there exists a unique $z_s \in \mathbb{C}$ such that the solution $\psi_s(z_s, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (3) satisfies $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Further,*

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du, \quad \text{and} \quad \psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Proof. Let be $s \in B$ fixed. There is a unique bounded solution. All other solutions diverge to infinity in norm. Let be $z \in \mathbb{C}$ and suppose that

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty.$$

Since $\Re(s) \in (0, 1)$, by the Lemma 3, we have

$$\sup_{t \geq 1} |\psi_s(z, t)| < +\infty \implies \left| z + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

Then

$$z_s = - \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Prove that $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$. Since

$$\left| z_s + \int_1^{+\infty} u^{-1-s} \{u\} du \right| = 0,$$

then

$$z_s + \int_1^t u^{-1-s} \{u\} du = - \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t > 1.$$

By the Lemma 3, we get

$$\psi_s(z_s, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du, \quad \forall t \geq 1.$$

Then $\sup_{t \geq 1} |\psi_s(z_s, t)| \leq \frac{1}{\Re(s)}$. □

Lemma 6. *Let be $s \in B$. For every $z \in \mathbb{C}$ there exists a unique $\tilde{z} \in \mathbb{C}$ such that the solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (3) satisfies*

$$\psi_s(z, t) = \psi_s(z_s, t) + t^s \tilde{z}, \quad \forall t \geq 1.$$

where $t \mapsto \psi_s(z_s, t)$ is the bounded solution given by the Lemma 5. In addition,

$$\tilde{z} = z + \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Proof. Consider

$$\tilde{z} = z + \int_1^{+\infty} u^{-1-s} \{u\} du,$$

By the Lemma 3, we have

$$\psi_s(z, t) = -t^s \int_t^{+\infty} u^{-1-s} \{u\} du + t^s \tilde{z}, \quad \forall t \geq 1.$$

By the Lemma 5, we get

$$\psi_s(z, t) = \psi_s(z_s, t) + t^s \tilde{z}, \quad \forall t \geq 1.$$

□

Notation 7. Denote

$$p(t) := \int_1^t \left(\{u\} - \frac{1}{2} \right) du, \quad \forall t \geq 1,$$

where we recall that $\{u\}$ is the fractional part of the real u .

Lemma 8. *The function p is a continuous 1-periodic function and satisfies*

$$p(t) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} (\cos(j2\pi t) - 1), \quad \forall t \geq 1.$$

Proof. The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_1^t \{u\} du = (t-1) \int_1^2 \{u\} du + p(t), \quad \forall t \geq 1.$$

Since

$$\int_1^2 \{u\} du = \int_0^1 \{u\} du = \int_0^1 u du = \frac{1}{2},$$

we get

$$\int_1^t \{u\} du = (t-1) \frac{1}{2} + p(t), \quad \forall t \geq 1.$$

The function p is a piecewise C^∞ , continuous on \mathbb{R} and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on \mathbb{R}_+ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p .

$$\int_0^t \left(\{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left(\int_0^u \left(\{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12} = - \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}.$$

The function p satisfies

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} (\exp(ij2\pi t) - 1), \quad \forall t \geq 0.$$

□

Proof of the Proposition 4. Let be $s \in B$. Let be $z \in \mathbb{C}$ and consider the continuous solution $\psi_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ of the differential equation (3). Define the function $\tilde{\psi}_s(z, t) : [1, +\infty) \rightarrow \mathbb{C}$ as

$$\tilde{\psi}_s(z, t) := \int_1^t \psi_s(z, u) du, \quad t \geq 1.$$

The Equation (3) implies

$$\frac{d}{dt} \tilde{\psi}_s(z, t) = t^{-1}(1+s)\tilde{\psi}_s(z, t) + t^{-1} \int_1^t \{u\} du + t^{-1}z, \quad \forall t \geq 1.$$

By the Lemma 8 we have

$$\begin{aligned} \int_1^t \left(\{u\} - \frac{1}{2} \right) du &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left(\cos(j2\pi t) - 1 \right), \\ &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} t^{2k}, \quad \forall t \geq 1. \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} \tilde{\psi}_s(z, t) &= t^{-1}(1+s)\tilde{\psi}_s(z, t) + \frac{1}{2}(1 - (1-2z)t^{-1}) \\ &\quad + 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} t^{2k-1}, \quad \forall t \geq 1. \end{aligned} \quad (4)$$

Use the fact $|\tilde{\psi}_s(z, 1)| = 0$, integrate, to obtain

$$\begin{aligned} \tilde{\psi}_s(z, t) &= \frac{1}{2} \left(-\frac{1}{s}t + \frac{1-2z}{1+s} \right) \\ &\quad + 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s} t^{2k} \\ &\quad - t^{1+s} \left[\frac{1}{2} \left(-\frac{1}{s} + \frac{1-2z}{1+s} \right) + 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s} \right]. \end{aligned} \quad (5)$$

By Lemma 6 and the definition of $\tilde{\psi}_s(z, t)$ we have

$$\begin{aligned} \tilde{\psi}_s(z, t) &= \int_1^t \psi_s(z, u) du \\ &= \int_1^t \psi_s(z_s, u) du - \frac{\tilde{z}}{1+s} + t^{1+s} \frac{\tilde{z}}{1+s}, \quad \forall t \geq 1. \end{aligned} \quad (6)$$

where $t \mapsto \psi_s(z_s, t)$ is the bounded solution given by the Lemma 5 and where

$$\tilde{z} = z + \int_1^{+\infty} u^{-1-s} \{u\} du. \quad (7)$$

Equations (5) and (6) implies

$$\begin{aligned} & 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s} t^{2k} \\ &= t^{1+s} \omega_s(z, \tilde{z}) - \frac{1}{2} \left(-\frac{1}{s} t + \frac{1-2z}{1+s} \right) + \int_1^t \psi_s(z_s, u) du - \frac{\tilde{z}}{1+s}. \end{aligned} \quad (8)$$

where

$$\begin{aligned} \omega_s(z, \tilde{z}) &:= \frac{\tilde{z}}{1+s} + \frac{1}{2} \left(-\frac{1}{s} + \frac{1-2z}{1+s} \right) \\ &+ 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}. \end{aligned}$$

The aim is to prove that $|\omega_s(z, \tilde{z})| = 0$. The convergence of the above series can be deduced by the approximation in the differential Equation (4), as

$$\left| \sum_{j=n}^{+\infty} \frac{1}{(j2\pi)^2} (\cos(j2\pi t) - 1) \right| \leq \frac{1}{4\pi^2} \sum_{j=n}^{+\infty} j^{-2}, \quad \forall n \geq 1.$$

In order to simplify the notation, denote

$$\theta(t) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s} t^{2k}.$$

We have

$$\begin{aligned} & t \frac{d}{dt} \Re(\theta(t)) - (1 + \Re(s)) \Re(\theta(t)) + \Im(s) \Im(\theta(t)) \\ &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} t^{2k} \\ &= \int_1^t \left(\{u\} - \frac{1}{2} \right) du, \end{aligned} \quad (9)$$

By Equation (8), we have

$$\theta(t) = t^{1+s} \omega_s(z, \tilde{z}) - \frac{1}{2} \left(-\frac{1}{s} t + \frac{1-2z}{1+s} \right) + \int_1^t \psi_s(z_s, u) du - \frac{\tilde{z}}{1+s}. \quad (10)$$

Since $\sup_{t \geq 1} |\psi_s(z_s, t)| < +\infty$ and $\Re(s) \in (0, 1)$, the Equations (9) and (10) implies $|\omega_s(z, \tilde{z})| = 0$. In other words,

$$\begin{aligned} \frac{\tilde{z}}{1+s} &= -\frac{1}{2} \left(-\frac{1}{s} + \frac{1-2z}{1+s} \right) \\ &\quad - 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}, \end{aligned}$$

Using the Equation (7), we obtain

$$\begin{aligned} \int_1^{+\infty} u^{-1-s} \{u\} du &= \frac{1}{2} \frac{1}{s} \\ &\quad - 2(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}. \end{aligned}$$

By the Lemma 5, we have $\sup_{t \geq 1} |\psi_s(z, t)| < +\infty$, if and only if

$$\begin{aligned} z = z_s &= - \int_1^{+\infty} u^{-1-s} \{u\} du = -\frac{1}{2} \frac{1}{s} \\ &\quad + 2(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}. \end{aligned}$$

□

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $s \in B$. Recall the Equation (1),

$$\zeta(s) = -\frac{s}{1-s} - s \int_1^{+\infty} u^{-1-s} \{u\} du.$$

Thanks to Main Proposition 4,

$$\begin{aligned} \zeta(s) &= -\frac{s}{1-s} - \frac{1}{2}s \\ &\quad + 2s(1+s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k}}{(2k)!} \frac{1}{2k-1-s}. \end{aligned} \tag{11}$$

In one hand, for every $t \in (0, 1]$ we have

$$\begin{aligned} \int_t^1 u^{-2-s} \left[\int_0^u \left(v - \frac{1}{2} \right) dv \right] du &= \frac{1}{2} \int_t^1 u^{-2-s} (u^2 - u) du \\ &= \frac{1}{2s(1-s)} - \frac{1}{2} t^{-s} \left(\frac{t}{1-s} + \frac{1}{s} \right). \end{aligned} \tag{12}$$

We recall that the function $p : \mathbb{R} \rightarrow \mathbb{R}$, defined as

$$p(t) := \int_0^t \left(\{u\} - \frac{1}{2} \right) du, \quad \forall t \in \mathbb{R}$$

is a continuous 1-periodic function and satisfies

$$\begin{aligned} p(t) &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left(\cos(j2\pi t) - 1 \right) \\ &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k} t^{2k}}{(2k)!}, \quad \forall t \in \mathbb{R}. \end{aligned}$$

On the other hand, for every $t \in (0, 1]$ we have

$$\begin{aligned} &\int_t^1 u^{-2-s} \left[\int_0^u \left(v - \frac{1}{2} \right) dv \right] du \\ &= 2 \int_t^1 u^{-2-s} \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k} u^{2k}}{(2k)!} du \\ &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k}}{2k-1-s} \\ &\quad - 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1-s}}{2k-1-s}. \end{aligned} \quad (13)$$

Using the Equation (11), the Equations (12) and (13) implies that

$$\begin{aligned} \frac{\zeta(s)}{s(1+s)} &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1-s}}{2k-1-s} \\ &\quad - \frac{1}{2} \left(\frac{t^{1-s}}{1-s} + \frac{t^{-s}}{s} \right), \quad \forall t \in (0, 1]. \end{aligned}$$

□

4 Proof of the Theorem 2

Proof of the Theorem 2. Let $s \in B$. In order to simplify the notation, denote

$$\forall t \in (0, 1] : \quad \varphi_s(n, t) := \sum_{j=1}^n f_s(j, t), \quad \forall n \in \mathbb{N}^*.$$

where

$$f_s(z, t) := z^{-1+s} \int_0^z u^{-2-s} (\cos(2\pi tu) - 1) du, \quad \forall z \in \mathbb{R}_+^*, \forall t \in (0, 1]. \quad (14)$$

Since

$$f_s(z, t) = \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k z^{2k-2} (2\pi t)^{2k}}{2k-1-s}, \quad \forall z \in \mathbb{R}_+^*, \forall t \in (0, 1],$$

by the Theorem 1, we get

$$t^{1+s} 2\pi^2 \frac{\zeta(s)}{s(1+s)} = \lim_{n \rightarrow +\infty} \varphi_s(n, t) - \pi^2 \left(\frac{t^2}{1-s} + \frac{t}{s} \right), \quad \forall t \in (0, 1]. \quad (15)$$

For every fixed $t \in (0, 1]$, use the Euler-Maclaurin summation formula, given by

$$\begin{aligned} \varphi_s(n, t) &= \frac{1}{2} [f_s(1, t) - f_s(n, t)] + \int_1^n f_s(z, t) dz \\ &\quad + \int_1^n \left(\{z\} - \frac{1}{2} \right) \frac{d}{dz} f_s(z, t) dz, \quad \forall n \in \mathbb{N}^*. \end{aligned}$$

By the integration part formula, we have

$$\begin{aligned} \int_1^n f_s(z, t) dz &= \frac{1}{s} n f_s(n, t) - \frac{1}{s} \int_0^1 z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^n u^{-2} (\cos(2\pi tu) - 1) du, \quad \forall n \in \mathbb{N}^*. \end{aligned}$$

Then

$$\begin{aligned} \varphi_s(n, t) &= \frac{1}{2} [f_s(1, t) - f_s(n, t)] \\ &\quad + \frac{1}{s} n f_s(n, t) - \frac{1}{s} \int_0^1 z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^n u^{-2} (\cos(2\pi tu) - 1) du \\ &\quad + \int_1^n \left(\{z\} - \frac{1}{2} \right) \frac{d}{dz} f_s(z, t) dz, \quad \forall n \in \mathbb{N}^*. \end{aligned} \quad (16)$$

Prove that $\lim_{z \rightarrow +\infty} z f_s(z, t) = 0$. In order to simplify the notation, denote

$$\begin{aligned} \phi_s(t, z) &:= -\frac{1}{2\pi t} \left[(1+s) z f_s(z, t) + z^{-1} (\cos(2\pi tz) - 1) \right] \\ &= z^s \int_0^z u^{-1-s} \sin(2\pi tu) du, \quad \forall z \in [1, +\infty). \end{aligned} \quad (17)$$

We have

$$\phi_s(t, z) := \sum_{k \in \mathbb{N}^*} \frac{1}{(2k-1)!} \frac{(-1)^k z^{2k-1} (2\pi t)^{2k-1}}{2k-1-s}.$$

The function $z \mapsto \phi_s(t, z)$ is the unique solution of the following deferential equation

$$\begin{aligned} \frac{d}{dz} \phi_s(t, z) &= sz^{-1} \phi_s(t, z) + z^{-1} \sin(2\pi tz), \\ \phi_s(t, 1) &= \sum_{k \in \mathbb{N}^*} \frac{1}{(2k-1)!} \frac{(-1)^k (2\pi t)^{2k-1}}{2k-1-s}, \quad \forall z \in [1, +\infty). \end{aligned}$$

Using the same method as in the proof of the Proposition 4, we prove that

$$\int_1^{+\infty} u^{-1-s} \sin(2\pi tu) du = - \sum_{k \in \mathbb{N}^*} \frac{1}{(2k-1)!} \frac{(-1)^k (2\pi t)^{2k-1}}{2k-1-s} = -\phi_s(t, 1),$$

and the solution, $z \mapsto \phi_s(t, z)$ can be written as

$$\begin{aligned} \phi_s(t, z) &= -z^s \int_z^{+\infty} u^{-1-s} \sin(2\pi tu) du \\ &= \frac{(1+s)}{2\pi t} z^s \int_z^{+\infty} u^{-2-s} (\cos(2\pi tu) - \cos(2\pi tz)) du, \quad \forall z \geq 1. \end{aligned}$$

Then we have $\lim_{z \rightarrow +\infty} |\phi_s(t, z)| = 0$. By the Equation (17) we get $\lim_{z \rightarrow +\infty} |zf_s(z, t)| = 0$. The Equation (16), implies

$$\begin{aligned} \lim_{n \rightarrow +\infty} \varphi_s(n, t) &= \frac{s-2}{2s} \int_0^1 z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^{+\infty} u^{-2} (\cos(2\pi tu) - 1) du \\ &\quad + \int_1^{+\infty} \left(\{z\} - \frac{1}{2} \right) \frac{d}{dz} f_s(z, t) dz \end{aligned}$$

The Equation (15), implies

$$\begin{aligned} t^{1+s} 2\pi^2 \frac{\zeta(s)}{s(1+s)} &= \frac{s-2}{2s} \int_0^1 z^{-2-s} (\cos(2\pi tz) - 1) dz \\ &\quad - \frac{1}{s} \int_1^{+\infty} u^{-2} (\cos(2\pi tu) - 1) du \\ &\quad + \int_1^{+\infty} \left(\{z\} - \frac{1}{2} \right) \frac{d}{dz} f_s(z, t) dz \\ &\quad - \pi^2 \left(\frac{t^2}{1-s} + \frac{t}{s} \right), \quad \forall t \in (0, 1]. \end{aligned}$$

Suppose that $s \in B$ is a zero of the function ζ . The above Equation implies that

$$\pi^2 t + \int_1^{+\infty} u^{-2} (\cos(2\pi t u) - 1) du = \delta(s, t), \quad \forall t \in (0, 1]. \quad (18)$$

where

$$\begin{aligned} \delta(s, t) := & -\pi^2 \frac{t^2 s}{1-s} + \frac{s-2}{2} \int_0^1 u^{-2-s} (\cos(2\pi t u) - 1) du \\ & + s \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) \frac{d}{dz} f_s(z, t) dz, \quad \forall t \in (0, 1]. \end{aligned} \quad (19)$$

Now we develop the last term of the right side of the previous equality. The aim is to find the following limit

$$\lim_{t \rightarrow 0} t^{-2} \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) \frac{d}{dz} f_s(z, t) dz.$$

By the Equation (14), we have

$$\begin{aligned} \frac{d}{dz} f_s(z, t) & := -(1-s)z^{-1} f_s(z, t) + z^{-3} (\cos(2\pi t z) - 1), \\ \frac{d}{dz} [z^{-1} f_s(z, t)] & = -(2-s)z^{-2} f_s(z, t) + z^{-4} (\cos(2\pi t z) - 1). \end{aligned}$$

Then

$$\begin{aligned} \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) \frac{d}{dz} f_s(z, t) dz & = -(1-s) \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) z^{-1} f_s(z, t) dz \\ & \quad + \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) z^{-3} (\cos(2\pi t z) - 1) dz. \end{aligned}$$

Use the integration part part for the first term of the right member.

$$\begin{aligned} \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) \frac{d}{dz} f_s(z, t) dz & = -(1-s)(2-s) \int_1^{+\infty} p(z) z^{-2} f_s(z, t) dz \\ & \quad + (1-s) \int_1^{+\infty} p(z) z^{-4} (\cos(2\pi t z) - 1) dz \\ & \quad + \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) z^{-3} (\cos(2\pi t z) - 1) dz. \end{aligned}$$

where

$$p(z) = \int_1^z \left(\{u\} - \frac{1}{2}\right) du = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left(\cos(j2\pi z) - 1\right).$$

Use the integration part part for the last term of the right member

$$\begin{aligned} \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) \frac{d}{dz} f_s(z, t) dz &= -(1-s)(2-s) \int_1^{+\infty} p(z) z^{-2} f_s(z, t) z \\ &\quad - (2+s) \int_1^{+\infty} p(z) z^{-4} (\cos(2\pi tz) - 1) dz \\ &\quad - 2\pi t \int_1^{+\infty} p(z) z^{-3} \sin(2\pi tz) dz. \end{aligned}$$

By Lebesgue Theorem, we get

$$\begin{aligned} \lim_{t \rightarrow 0} t^{-2} \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) \frac{d}{dz} f_s(z, t) dz \\ &= -(1-s)(2-s) \int_1^{+\infty} p(z) z^{-2} \lim_{t \rightarrow 0} t^{-2} f_s(z, t) z \\ &\quad - (2+s) \int_1^{+\infty} p(z) z^{-4} \lim_{t \rightarrow 0} t^{-2} (\cos(2\pi tz) - 1) dz \\ &\quad - 2\pi \int_1^{+\infty} p(z) z^{-3} \lim_{t \rightarrow 0} t^{-1} \sin(2\pi tz) dz. \end{aligned}$$

We get

$$\begin{aligned} \lim_{t \rightarrow 0} t^{-2} \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) \frac{d}{dz} f_s(z, t) dz \\ &= -2\pi^2(1-s)(2-s) \int_1^{+\infty} p(z) z^{-3+s} \int_0^z u^{-s} du z \\ &\quad - 2(2+s)\pi^2 \int_1^{+\infty} p(z) z^{-2} dz \\ &\quad - 4\pi^2 \int_1^{+\infty} p(z) z^{-2} dz. \end{aligned}$$

Equivalent to

$$\lim_{t \rightarrow 0} t^{-2} \int_1^{+\infty} \left(\{z\} - \frac{1}{2}\right) \frac{d}{dz} f_s(z, t) dz = 4\pi^2 \int_1^{+\infty} p(z) z^{-2} du z. \quad (20)$$

Thanks to the Riemann functional equation, the point $1 - \bar{s}$ is a zero of the function ζ . The Equation (18), implies

$$\pi^2 t + \int_1^{+\infty} u^{-2} (\cos(2\pi tu) - 1) du = \delta(1 - \bar{s}, t), \quad \forall t \in (0, 1].$$

Then

$$|\delta(s, t) - \delta(1 - \bar{s}, t)| = 0, \quad \forall t \in (0, 1].$$

Implies

$$\lim_{t \rightarrow 0} t^{-2} |\delta(s, t) - \delta(1 - \bar{s}, t)| = 0, \quad \forall t \in (0, 1].$$

Since

$$\lim_{t \rightarrow 0} t^{-2} \left[-\pi^2 \frac{t^2 s}{1-s} + \frac{s-2}{2} \int_0^1 u^{-2-s} (\cos(2\pi t u) - 1) du \right] = 2\pi^2,$$

which does not depend on s . By definition of the function δ in the Equation (19), we get

$$\begin{aligned} 0 &= \lim_{t \rightarrow 0} t^{-2} |\delta(s, t) - \delta(1 - \bar{s}, t)| \\ &= \lim_{t \rightarrow 0} t^{-2} \left| \int_0^{+\infty} \left(\{z\} - \frac{1}{2} \right) \left[s \frac{d}{dz} f_s(z, t) - (1 - \bar{s}) \frac{d}{dz} f_{1-\bar{s}}(z, t) \right] dz \right| \end{aligned}$$

By the Equation (20), we obtain

$$\begin{aligned} 0 &= \lim_{t \rightarrow 0} t^{-2} |\delta(s, t) - \delta(1 - \bar{s}, t)| \\ &= 4\pi^2 \int_1^{+\infty} p(z) z^{-2} duz \left| s - (1 - \bar{s}) \right| \\ &= -4\pi^2 |1 - 2\Re(s)| \int_1^{+\infty} p(z) z^{-2} duz. \end{aligned}$$

□

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).