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Particular polynomials generating rational solutions to the KdV equation

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Abstract

We construct here rational solutions to the KdV equation by means of particular polynomials. We get solutions in terms of determinants of order n for any positive integer n and we call these solutions, solutions of order n . So we obtain a very efficient method to get rational solutions to the KdV equation and we can construct very easily explicit solutions. In the following, we present some solutions until order 10.

1 Introduction

We consider the KdV equation in the following normalization

$$4u_t = 6uu_x + u_{xxx}. \quad (1)$$

As usual, the subscripts x and t denote partial derivatives.

This equation appears in a footnote on page 360 of Boussinesq's massive 680-page memoir [4] written in 1872. Korteweg and de Vries [64] studied this equation (1) in a paper published in 1895 and from that moment this equation carried their names. This equation describes the propagation of waves with weak dispersion in various nonlinear media.

Gardner et al. [61] proposed a method of resolution in 1967. Zakharov and Faddeev [73] in 1971 proved that this equation is a complete integrable system. Hirota [62] constructed in 1971 solutions by using the bilinear method. A lot of works have been realized in the following years. We can mention for example, Its and Matveev [63] in 1975, Lax [66] in the same year, Airault et al. [3] in 1977, Adler and Moser [2] in 1978, Ablowitz and Cornille [1] in 1979, Freeman and Nimmo [5] in 1984, Matveev [71] in 1992, Ma [68] in 2004, Kovalyov [65] in 2005 and Ma [69] in 2015.

In the following, we consider certain polynomials and we construct rational solutions using determinants of order n . The proof of the result is based on the verification of the corresponding Hirota bilinear expression for the KdV equation.

So we get a very efficient method to construct rational solutions to the KdV

equation. We give explicit solutions in the simplest cases for orders $n = 1$ until 10.

2 Rational solutions to the KdV equation

We consider the polynomials p_n defined by

$$\begin{cases} p_n(x, t) = \sum_{k=0}^n \frac{x^k t^{\binom{n-k}{3}}}{k! \left(\frac{n-k}{3}\right)!} \left(1 - \left[\frac{1}{2}(n-k+1) - 3 \left[\frac{n-k}{3}\right]\right]\right), & k \geq 0, \\ p_n(x, t) = 0, & n < 0, \end{cases} \quad (2)$$

where $[x]$ denotes the the largest integer less than or equal to x .

We denote $A_n(x, t)$ the following determinant

$$A_n(x, t) = \det(p_{n+1-2i+j}(x, t))_{\{1 \leq i \leq n, 1 \leq j \leq n\}} \quad (3)$$

With these notations we have the following result

Theorem 2.1 *The function $v_n(x, t)$ defined by*

$$v_n(x, t) = 2\partial_x^2(\ln(A_n(x, t))) \quad (4)$$

is a rational solution to the (KdV) equation (1)

$$4u_t = 6uu_x + u_{xxx}. \quad (5)$$

Remark 2.1 *The presented rational solutions can be expressed as*

$$\begin{aligned} v(x, t) &= -2(\ln W(f_1, f_2, \dots, f_k))_{xx}(x, t) \\ &= -2(\ln W(f_k, \dots, f_2, f_1))_{xx}(x, t) \\ &= 2((\ln W(\phi_0, \phi_1, \dots, \phi_{k-1}))_{xx}(x, t))_{t \rightarrow -1/4t, x \rightarrow x}, \end{aligned}$$

where the ϕ_i 's are specific polynomials determined in Theorem 2.1. This corresponds to the subsequent discussion made in [68] by Ma and You. This is because f_1, f_2, \dots, f_k satisfy $f_{j,xx} = f_{j+1}$ and $f_{j,t} = f_{j,xxx}$.

A general solution to the KdV equation was analyzed and presented in [68]. Therefore, all rational solutions in the manuscript will become special cases of the ones in [68]. Actually, a combination of theorem 2.1 and theorem 3.1 in [68] presents a larger class of rational solutions to the KdV equation in wronskian form than the class given in the manuscript.

I want to thank the referee for that comment.

3 Explicit rational solutions to the KdV equation for the first orders

In this section, we use the previous method to construct explicitly rational solutions to the KdV equation.

In the following, we will call

$$v_k(x, t) = 2\partial_x^2(\ln(A_k(x, t)))$$

a rational solution to the KdV equation of order k .

We present some examples of these solutions for the first simplest orders.

3.1 Rational solutions of order 1 to the KdV equation

Example 3.1 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = -\frac{2}{x^2} \quad (6)$$

is a rational solution to the KdV equation.

3.2 Rational solutions of order 2 to the KdV equation

Example 3.2 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = -6 \frac{(x^3 + 6t)x}{(-x^3 + 3t)^2} \quad (7)$$

is a rational solution to the KdV equation.

3.3 Rational solutions of order 3 to the KdV equation

Example 3.3 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{6(-2x^9 - 675t^2x^3 + 1350t^3)x}{(-x^6 + 15tx^3 + 45t^2)^2} \quad (8)$$

is a rational solution to the KdV equation.

3.4 Rational solutions of order 4 to the KdV equation

Example 3.4 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{-20x^{18} + 720tx^{15} - 28350t^2x^{12} - 661500t^3x^9 + 11907000t^4x^6 - 44651250t^6}{(-x^9 + 45tx^6 + 4725t^3)^2 x^2} \quad (9)$$

is a rational solution to the KdV equation.

3.5 Rational solutions of order 5 to the KdV equation

Example 3.5 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (10)$$

with

$$n(x, t) = -30(x^{27} - 126tx^{24} + 7560t^2x^{21} + 5655825t^4x^{15} - 500094000t^5x^{12} + 4313310750t^6x^9 - 11252115000t^7x^6 + 295368018750t^8x^3 + 590736037500t^9)x$$

and

$$d(x, t) = (x^{15} - 105tx^{12} + 1575t^2x^9 - 33075t^3x^6 - 992250t^4x^3 + 1488375t^5)^2$$

is a rational solution to the KdV equation.

3.6 Rational solutions of order 6 to the KdV equation

Example 3.6 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (11)$$

with

$$n(x, t) = 42(-x^{39} + 300tx^{36} - 37800t^2x^{33} + 1890000t^3x^{30} - 90932625t^4x^{27} + 2062887750t^5x^{24} + 389269597500t^6x^{21} - 17947123425000t^7x^{18} + 263869129321875t^8x^{15} - 2859162421500000t^9x^{12} - 63795061529718750t^{10}x^9 - 96496731725625000t^{11}x^6 - 4221732012996093750t^{12}x^3 + 5066078415595312500t^{13})x$$

and

$$d(x, t) = (x^{21} - 210tx^{18} + 10395t^2x^{15} - 264600t^3x^{12} - 5457375t^4x^9 - 343814625t^5x^6 + 3438146250t^6x^3 + 5157219375t^7)^2$$

is a rational solution to the KdV equation.

3.7 Rational solutions of order 7 to the KdV equation

Example 3.7 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (12)$$

with

$$n(x, t) = -56x^{54} + 33264tx^{51} - 8334900t^2x^{48} + 1073217600t^3x^{45} - 88841699100t^4x^{42} + 4208291010000t^5x^{39} - 20822788948500t^6x^{36} + 16700107780125000t^7x^{33} - 2828512200341520000t^8x^{30} + 146885752493414550000t^9x^{27} - 3822896546264326781250t^{10}x^{24} + 10605936484617098625000t^{11}x^{21} - 1400650109245812737812500t^{12}x^{18} + 17562387611658575936250000t^{13}x^{15}$$

$$- 803657230405119631947656250t^{14}x^{12} - 4797023778387148190695312500t^{15}x^9$$

$$+ 86346428010968667432515625000t^{16}x^6 - 107933035013710834290644531250t^{18}$$

and

$$d(x, t) = (x^{27} - 378tx^{24} + 42525t^2x^{21} - 2182950t^3x^{18} + 19646550t^4x^{15} - 3094331625t^5x^{12} - 147496474125t^6x^9 + 6637341335625t^7x^6 + 232306946746875t^9)^2x^2$$

is a rational solution to the KdV equation.

3.8 Rational solutions of order 8 to the KdV equation

Example 3.8 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (13)$$

with

$$\begin{aligned} n(x, t) = & -18(4x^{69} - 4200tx^{66} + 1905750t^2x^{63} - 483159600t^3x^{60} + 78138695250t^4x^{57} - 8340851118600t^5x^{54} + \\ & 595545378986250t^6x^{51} - 39921395686672500t^7x^{48} + 1536199377447735000t^8x^{45} + 368199076771500975000t^9x^{42} - \\ & 65873050788783963099375t^{10}x^{39} + 4801800716006149418062500t^{11}x^{36} - 168255232705748229729187500t^{12}x^{33} + \\ & 4958157366226547182074375000t^{13}x^{30} - 51649152550511657393517890625t^{14}x^{27} - 1919124744346010431730656406250t^{15}x^{24} + \\ & 269892618860418044895397743750000t^{16}x^{21} - 691462195511837088799585125000000t^{17}x^{18} \\ & - 12773529308160633363628050210937500t^{18}x^{15} - 3145288661834468957677112837343750000t^{19}x^{12} + \\ & 19883967800655314009338194685957031250t^{20}x^9 - 33371694210890037498189977095312500000t^{21}x^6 + \\ & 54750435814741467704679311719970703125t^{22}x^3 + 657005229776897613245615174063964843750t^{23}x \end{aligned}$$

and

$$\begin{aligned} d(x, t) = & (-x^{36} + 630tx^{33} - 135135t^2x^{30} + 13825350t^3x^{27} - 638512875t^4x^{24} + 29499294825t^5x^{21} + \\ & 1179971793000t^6x^{18} + 54426198952125t^7x^{15} - 12683959292379375t^8x^{12} + 132879573539212500t^9x^9 - \\ & 2092853283242596875t^{10}x^6 - 31392799248638953125t^{11}x^3 + 31392799248638953125t^{12})^2 \end{aligned}$$

is a rational solution to the KdV equation.

3.9 Rational solutions of order 9 to the KdV equation

Example 3.9 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (14)$$

with

$$\begin{aligned} n(x, t) = & 90(-x^{87} + 1716tx^{84} - 1302840t^2x^{81} + 576298800t^3x^{78} - 167162670675t^4x^{75} + 33671337950250t^5x^{72} - \\ & 4883313265330500t^6x^{69} + 532930720393071000t^7x^{66} - 41982352119903195000t^8x^{63} + 1899274320510672105000t^9x^{60} - \\ & 207739128318600056928750t^{10}x^{57} + 77654423659228702621335000t^{11}x^{54} - 14074843198552667114468259375t^{12}x^{51} + \\ & 1353653968207045875528376968750t^{13}x^{48} - 83687105267491295623979425500000t^{14}x^{45} + 3232758461572985661403193106000000t^{15}x^{42} - \\ & 71106128573876049954454195744828125t^{16}x^{39} + 1019021984277855554839081692265312500t^{17}x^{36} \\ & - 336338305543556803028411633143849218750t^{18}x^{33} + 14179187914284365065814278734770343750000t^{19}x^{30} \\ & - 323539217779692473178173668096113772265625t^{20}x^{27} + 2132534440692278750097305649817876711718750t^{21}x^{24} + \\ & 357318888294083367780587267263211428546875000t^{22}x^{21} - 23068386039288974574146266938440837242968750000t^{23}x^{18} + \\ & 269958765098763940588462542502482645328945312500t^{24}x^{15} - 2537131474727900949349747901076993099485156250000t^{25}x^{12} - \\ & 29060008323400399465974418313549345719588476562500t^{26}x^9 - 35224252513212605413302325228544661478289062500000t^{27}x^6 - \\ & 809057049912852030586787782593135193329451904296875t^{28}x^3 + 693477471353873169074389527936973022853815917968750t^{29}x \end{aligned}$$

and

$$\begin{aligned} d(x, t) = & (-x^{45} + 990tx^{42} - 363825t^2x^{39} + 67567500t^3x^{36} - 6810804000t^4x^{33} + 445298879025t^5x^{30} - 8210637059625t^6x^{27} + \\ & 1121710685720625t^7x^{24} - 48406130360713125t^8x^{21} - 23885103343673446875t^9x^{18} + 1037875841064050113125t^{10}x^{15} - \\ & 23588549355427309378125t^{11}x^{12} - 305263579893765180187500t^{12}x^9 - 9615802766653603175906250t^{13}x^6 + 72118520749902023819296875t^{14}x^3 + \\ & 72118520749902023819296875t^{15})^2 \end{aligned}$$

is a rational solution to the KdV equation.

3.10 Rational solutions of order 10 to the KdV equation

Example 3.10 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (15)$$

with

$$\begin{aligned} n(x, t) = & -110x^{108} + 291060tx^{105} - 347490000t^2x^{102} + 248154192000t^3x^{99} - 119078820110250t^4x^{96} + 40841520834114000t^5x^{93} - \\ & 10403875101641521500t^6x^{90} + 2022895713150642330000t^7x^{87} - 304310886232135639106250t^8x^{84} + 35771706849366129904762500t^9x^{81} - \\ & 3519939926423867815118250000t^{10}x^{78} + 296754986088067433541615000000t^{11}x^{75} + 4582775043052251449589779343750t^{12}x^{72} - \\ & 11407514722448205667096537621875000t^{13}x^{69} + 2793940550549247930228715261632187500t^{14}x^{66} \\ & - 389645507719688878112158874570400000000t^{15}x^{63} + 35750508746600149120830492075248360156250t^{16}x^{60} \\ & - 2267111177718461993872185235868925998437500t^{17}x^{57} + 105602095963586360431772913336582758203125000t^{18}x^{54} - \\ & 4746298210770147629947127050066564488468750000t^{19}x^{51} + 17914009152786314817603679852500052180664062500t^{20}x^{48} + \\ & 21517822607755427423214335600663223777394687500000t^{21}x^{45} - 1866304090193719979598090753176960710397522343750000t^{22}x^{42} + \\ & 124826969096794011744700655966112731665388906250000000t^{23}x^{39} - 1077606584700530778623127008158745262625547976269531250t^{24}x^{36} - \\ & 57629984931404039334696126368371105570079198658398437500t^{25}x^{33} \\ & - 8955758855718045099509061569538284662355447568285937500000t^{26}x^{30} \\ & + 535955408879595853282117469855285428376104165104867187500000t^{27}x^{27} \\ & - 13703065967666242029463436434276849693966219703293088378906250t^{28}x^{24} \\ & + 19427222931444129856882917267638979035967641555006220703125000t^{29}x^{21} \\ & - 2081507844816089416510573395441463169975510274693804902343750000t^{30}x^{18} \\ & + 13247544443667951100242271719618269946158265169097291718750000000t^{31}x^{15} \\ & - 624573785325233253763889340202068598695109656368707018286132812500t^{32}x^{12} \\ & - 1761618368866042510616098139031475534781078517963019795166015625000t^{33}x^9 \\ & + 31709130639588765191089766502566559626059413323334356312988281250000t^{34}x^6 \\ & - 19818206649742978244431104064104099766287133327083972695617675781250t^{36} \end{aligned}$$

and

$$\begin{aligned} d(x, t) = & (-x^{54} + 1485tx^{51} - 868725t^2x^{48} + 269594325t^3x^{45} - 49165491375t^4x^{42} + 5714434826100t^5x^{39} - \\ & 396365167823625t^6x^{36} + 24418778773764375t^7x^{33} - 435655173246418125t^8x^{30} - 194269936514328675000t^9x^{27} - \\ & 40358223939201682263750t^{10}x^{24} + 5744505548909944754437500t^{11}x^{21} - 290076716794050362473171875t^{12}x^{18} + \\ & 1331101840126763068207593750t^{13}x^{15} - 209648539819965183242696015625t^{14}x^{12} - 3997298825900669493827404031250t^{15}x^9 + \\ & 179878447165530127222233181406250t^{16}x^6 + 314787282539677226389080674609375t^{18})^2x^2 \end{aligned}$$

is a rational solution to the KdV equation.

We could go on and present more explicit rational solutions, but they become very complicated. For example, in the case of order 10 the numerator is a polynomial of degree 108 in x , 36 in t containing 36 terms; the denominator is a polynomial of degree 110 in x , 36 in t containing 36 terms.

In the case of order 20 the numerator is a polynomial of degree 418 in x , 139 in t containing 140 terms; the denominator is a polynomial of degree 420 in x , 140 in t containing 141 terms.

It will be interesting to study in detail the structure of these solutions.

4 Conclusion

Solutions to the KdV equation as the second derivative with respect to x of a logarithm of a determinant involving particular polynomials have been given. We get rational solutions as the quotient of two polynomials in x and t ; the numerator is a polynomial of degree $n(n+1) - 2$ in x and the denominator is a polynomial of degree $n(n+1)$ in x . The structure in t is more complex and should be studied in more details. This representation gives a very efficient method to construct rational solutions to the KdV equation. Some explicit expressions are given for some orders.

More recent work can be cited as [70], [72] or [67].

It can be noted that the solutions presented in this paper are different from these presented in a previous work of the author [52].

It will be relevant to study in detail the structure of polynomial entering in these last solutions.

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