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AUTOMORPHISM GROUPS OF RIGID AFFINE SURFACES: THE IDENTITY COMPONENT

ALEXANDER PEREPECHKO AND MIKHAIL ZAIDENBERG

ABSTRACT. It is known that the identity component of the automorphism group of a projective algebraic variety is an algebraic group. This is not true in general for quasi-projective varieties. In this note we address the question: given an affine algebraic surface Y , as to when the identity component $\text{Aut}^\circ(Y)$ of the automorphism group $\text{Aut}(Y)$ is an algebraic group? We show that this happens if and only if Y admits no effective action of the additive group. In the latter case, $\text{Aut}^\circ(Y)$ is an algebraic torus of rank ≤ 2 .

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1. INTRODUCTION

Let \mathbb{K} be an algebraically closed field of characteristic zero and let \mathbb{G}_a and \mathbb{G}_m be the additive and the multiplicative groups of \mathbb{K} , respectively. If Y is an affine variety over \mathbb{K} , then the automorphism group $\text{Aut}(Y)$ has a canonical structure of an ind-group endowed with its Zariski ind-topology, see e.g. [18]. In this paper we concentrate on the identity component $\text{Aut}^\circ(Y)$ of $\text{Aut}(Y)$ with respect to the Zariski topology. Notice that, similarly to the case of a projective variety, the group of components $\text{Aut}(Y)/\text{Aut}^\circ(Y)$ is at most countable, see [18, Proposition 1.7.1]. We address the following conjecture. We say that Y is *rigid* if Y admits no effective

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\mathbb{G}_a -action or, in other words, $\text{Aut}^\circ(Y)$ contains no \mathbb{G}_a -subgroup, that is, no nontrivial unipotent element.

Conjecture 1.0.1. *If Y is a rigid affine algebraic variety over \mathbb{K} , then the group $\text{Aut}^\circ(Y)$ is an algebraic torus of rank $\leq \dim Y$.*

If Conjecture 1.0.1 is true, then the following one is true as well¹.

Conjecture 1.0.2. *If an affine variety Y admits no effective \mathbb{G}_a - and \mathbb{G}_m -actions, then $\text{Aut}(Y)$ is a discrete group.*

Some partial results on these conjectures can be found e.g. in [1, Theorem 2.1], [24, Proposition 5], [25, 26], [27, Theorems 4.4 and 4.7], [28, Theorem 1.3 and Section 7], and [36, 37]. Conjecture 1.0.1 holds, for instance, for toric affine varieties, see [4, Theorem 3]. In the present paper we establish Conjecture 1.0.1 in the case $\dim Y = 2$.

Theorem 1.0.3. *Let Y be a normal affine surface over \mathbb{K} . Then the following hold.*

- (a) *$\text{Aut}^\circ(Y)$ is an algebraic group if and only if Y admits no effective \mathbb{G}_a -action, if and only if $\text{Aut}^\circ(Y)$ is an algebraic torus of rank ≤ 2 .*
- (b) *Let (X, D) be a minimal completion of Y by a normal crossing divisor. Then $\text{Aut}^\circ(Y)$ is an algebraic group if and only if the dual graph $\Gamma(D)$ is birationally rigid.*

See Definition 3.2.1 for birational rigidity of a weighted graph. As a corollary, we establish the following effective combinatorial criterion of rigidity, cf. Corollary 4.3.5.

Corollary 1.0.4. *In the setup of Theorem 1.0.3 the surface Y is rigid if and only if $\Gamma(D)$ has no non-admissible extremal linear segment.*

An *extremal linear segment* of a weighted graph Γ is a maximal linear subgraph with no branching point of Γ which contains a tip of Γ . It is *admissible* if all its weights are ≤ -2 , see Definitions 2.7.1 and 3.1.2.

Using Corollary 1.0.4 we show that the affine surface $Y = \mathbb{P}^2 \setminus \text{supp } D$, where D is a reduced effective divisor on \mathbb{P}^2 with only nodes as singularities, is rigid if and only if $\deg(D) \geq 3$, see Example 4.3.6.

The first part of Theorem 1.0.3 was announced in [27, Proposition 4.7]. Our approach goes back to the work of Danilov and Gizatullin [8]. Namely, the automorphism group $\text{Aut}(Y)$ can be realized as a group of birational transformations between the NC-completions of Y , i.e., the completions (X, D) with a normal projective surface X and a normal crossing boundary divisor D contained in a smooth part of X .

Our main instrument is the birational transformations of weighted graphs. Different aspects of this subject were developed e.g. in [6, 7, 8, 9, 11, 13, 17, 20, 23, 34, 35, 39, 40], etc. We consider, more generally, weighted graphs with a distinguished subset of marked vertices called *rational*. In the case where our graph Γ is the dual graph of a finite collection of curves on a surface, the rational vertices of Γ correspond to rational curves.

Section 2 contains preliminaries on NC-completions, their dual graphs and birational transformations of weighted graphs. For the sake of completeness we provide thorough proofs of some known results.

In Section 3 we discuss birationally rigid weighted graphs. A weighted graph Γ is *minimal* if it has no rational vertex of degree ≤ 2 and of weight -1 , and *birationally*

¹We thank Hanspeter Kraft who suggested the second conjecture.

rigid if it is minimal and any birational transformation $\Gamma \dashrightarrow \Gamma'$ where Γ' is a minimal weighted graph is an isomorphism, see Definition 3.2.1.

The proof of Theorem 1.0.3 is done in Section 4, see Theorem 4.3.4. Let us indicate the main ingredients of the proof.

- It is well known that any birational transformation between smooth surfaces can be decomposed into a sequence of blowups and blowdowns. The theory of birational transformations of weighted graphs is parallel to that of birational transformations of NC-pairs.
- Namely, to every birational transformation of an NC-pair (X, D) there corresponds a birational transformation of the dual graph $\Gamma(D)$ and vice versa, see Proposition 2.7.8 and Lemma 2.7.11.
- The group $\text{Aut}(Y)$ of automorphisms of a normal affine surface Y contains a \mathbb{G}_a -subgroup if and only if Y admits a minimal NC-completion (X, D) whose dual graph $\Gamma(D)$ has an extremal linear segment with a tip of weight zero, see Proposition 4.1.2 and its proof.
- A minimal linear weighted graph is non-admissible if and only if it is birationally equivalent to a linear graph with a tip of weight zero, see e.g. [11, Examples 2.11].
- A minimal weighted graph is birationally rigid if and only if it contains no non-admissible extremal linear segment, see Proposition 3.2.2.

These facts imply that Y is rigid if and only if the minimal graph $\Gamma(D)$ is birational rigid. Now Corollary 1.0.4 follows immediately.

- A blowup of an NC-pair (X, D) is called *inner* if its center is a node of D . A blowdown is *inner* if so is the inverse blowup. A birational transformation is *inner* if it is composed of inner blowups, inner blowdowns and isomorphisms, see Definitions 2.2.1–2.2.2.
- Let $\text{Bir}(X, D)$ stand for the subgroup of $\text{Bir}(X)$ of birational transformations biregular on $Y = X \setminus D$. Then the inner birational transformations of (X, D) form a subgroup of $\text{Bir}(X, D)$ denoted $\text{Inn}(X, D)$.
- Similar definitions can be applied for birational transformations of weighted graphs, see Definitions 2.3.4 and 2.3.7. To every inner birational transformation of an NC-pair (X, D) there corresponds a unique inner birational transformation of the dual graph $\Gamma(D)$ and vice versa, see Proposition 2.7.8.
- If a minimal weighted graph Γ is birationally rigid then any birational transformation of Γ can be replaced by an inner birational transformation, see Proposition 3.2.3.
- Let (X, D) be a minimal NC-completion of a rigid affine surface Y . Then the ind-group $\text{Aut}^\circ(Y) =: \text{Bir}^0(X, D)$ coincides with its closed ind-subgroup $\text{Inn}^\circ(X, D)$ of inner automorphisms, see Definitions 2.2.1, 2.2.2 and Theorem 3.2.5.
- In turn, the identity component $\text{Inn}^\circ(X, D)$ coincides with the affine algebraic subgroup $\text{Aut}^\circ(X, D)$, see Proposition 4.2.3.
- The absence of nontrivial unipotent subgroups forces $\text{Aut}^\circ(Y) = \text{Aut}^\circ(X, D)$ to be an algebraic torus.

These facts imply Theorem 1.0.3. As a side result we obtain that for a rigid normal affine surface Y the group $\text{Aut}^\circ(X, D)$ does not depend on the choice of a minimal NC-completion (X, D) of Y .

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2. NC-PAIRS AND DUAL GRAPHS

We recall here the correspondence between compactifications of affine algebraic surfaces and weighted graphs following [11]. We use a slightly modified version of the terminology in [11].

2.1. Birational transformations of NC-pairs.

Definition 2.1.1. Let X be a normal projective surface over \mathbb{K} and D be a reduced effective divisor on X . The pair (X, D) is called an *NC-pair* if D is a normal crossing divisor (i.e., the only singularities of D are nodes) contained in the smooth locus of X . An NC-pair (X, D) is called an *SNC-pair* if D is a simple normal crossing divisor, i.e., each component of D is smooth and any two components of D are disjoint or intersect at a single point.

Clearly, an NC-pair (X, D) is not an SNC-pair if and only if D has either a nodal component, or a pair of components with more than one point in common. As an example, one can consider the pair (\mathbb{P}^2, C) , where $C \subset \mathbb{P}^2$ is a nodal cubic curve. Any NC-pair is dominated by an SNC-pair. For instance, in the above example such a domination is obtained after blowing up \mathbb{P}^2 in the node of C and one more blowing up in a node of the resulting curve, cf. Example 2.7.2.

Definition 2.1.2. Let (X, D) and (X', D') be NC-pairs. We denote by

- (1) $\text{Bir}((X, D), (X', D'))$ the group of birational transformations $X \dashrightarrow X'$ which restrict to biregular isomorphisms of the complements $X \setminus \text{supp } D \xrightarrow{\cong} X' \setminus \text{supp } D'$;
- (2) $\text{Bir}(X, D)$ the group $\text{Bir}((X, D), (X, D))$;
- (3) $\text{Aut}(X, D)$ the group of biregular automorphisms of X that preserve D ;
- (4) $\text{Aut}^{\natural}(X, D)$ the subgroup of $\text{Aut}(X, D)$ of automorphisms preserving each component and each node of D ;
- (5) $\text{Aut}^{\circ}(X, D)$ the identity component of $\text{Aut}(X, D)$.

Remark 2.1.3. The group $\text{Bir}(X, D)$ is naturally isomorphic to the automorphism group of the complement $Y = X \setminus \text{supp } D$. Indeed, for $g \in \text{Bir}(X, D)$ the restriction $g|_Y$ is an automorphism of Y , and any $f \in \text{Aut}(Y)$ extends to a unique birational transformation of X .

Thus, the group $\text{Bir}(X, D) \simeq \text{Aut}(Y)$ does not depend on the completion (X, D) of $Y = X \setminus \text{supp } D$. However, the group $\text{Aut}(X, D)$ generally depends on the chosen completion. For instance, for the completion of the affine plane $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ we have $\text{Aut}(\mathbb{P}^2, \mathbb{P}^1) \simeq \text{Aff}(\mathbb{A}^2)$, whereas for the completion $\mathbb{A}^2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ we have

$$\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{P}^1) \simeq \mathbb{Z}_2 \ltimes (\text{Aff}(\mathbb{A}^1))^2.$$

2.2. Inner transformations.

Definition 2.2.1. A blowup of an NC-pair (X, D) with center $x \in D$ gives again an NC-pair. Such a blowup is called *inner* if x is a node of D and *outer* otherwise. A blowdown of a component of D is called *inner* (resp. *outer*) if the inverse blowup is inner (resp. outer).

Definition 2.2.2. A birational transformation $(X, D) \dashrightarrow (X', D')$ is called *inner* if it can be decomposed into a sequence of inner blowups, inner blowdowns and isomorphisms of NC-pairs. The subset of inner birational transformations $\text{Inn}(X, D) \subset \text{Bir}(X, D)$ forms a subgroup.

Lemma 2.2.3. *We have*

$$\text{Aut}^{\circ}(X, D) \subset \text{Aut}^{\natural}(X, D) \subset \text{Aut}(X, D) \subset \text{Inn}(X, D) \subset \text{Bir}(X, D).$$

Proof. This is immediate from our definitions. \square

Lemma 2.2.4. $\text{Aut}^\circ(X, D) = (\text{Aut}^{\natural}(X, D))^\circ$ is a connected algebraic group.

Proof. It is well known that $\text{Aut}^\circ(X)$ is a connected algebraic group, see [31, 32, 38], and so is its closed connected subgroup $\text{Aut}^\circ(X, D)$. \square

Lemma 2.2.5. Every inner birational transformation $\phi: (X, D) \dashrightarrow (X', D')$ satisfies the relation

$$\phi \circ \text{Aut}^{\natural}(X, D) = \text{Aut}^{\natural}(X', D') \circ \phi.$$

In particular, $\text{Aut}^{\natural}(X, D)$ is a normal subgroup of $\text{Inn}(X, D)$.

Proof. It is enough to check the statement for an inner blowdown $\sigma: (X_1, D_1) \rightarrow (X_2, D_2)$. Indeed, σ induces an isomorphism

$$\sigma_*: \text{Aut}^{\natural}(X_1, D_1) \xrightarrow{\simeq} \text{Aut}^{\natural}(X_2, D_2)$$

where $\pi_*: \alpha \mapsto \sigma \circ \alpha \circ \sigma^{-1}$. \square

Remark 2.2.6. The group $\text{Aut}^{\natural}(X, D)$ is the kernel of the natural homomorphism $\text{Aut}(X, D) \rightarrow \text{Aut}(\Gamma(D))$, where $\Gamma(D)$ is the dual graph of D , see Definition 2.7.1.

Proposition 2.2.7. The group $\text{Inn}(X, D)/\text{Aut}^{\natural}(X, D)$ is countable.

Proof. Since $\text{Aut}(X, D)/\text{Aut}^{\natural}(X, D)$ is a finite group, it suffices to show that $\text{Inn}(X, D)/\text{Aut}(X, D)$ is countable. We claim that any $\phi \in \text{Inn}(X, D)$ can be written as

$$(1) \quad \phi = \alpha \sigma_n^{\pm 1} \cdots \sigma_1^{\pm 1} = \sigma'_n{}^{\pm 1} \cdots \sigma'_1{}^{\pm 1} \beta$$

where $\alpha, \beta \in \text{Aut}(X, D)$ and the σ_i, σ'_i are inner blowdowns. Indeed, consider a commutative diagram

$$\begin{array}{ccc} (X_1, D_1) & \overset{\sigma^{\pm 1}}{\dashrightarrow} & (X_2, D_2) \\ \alpha_1 \downarrow \simeq & & \simeq \downarrow \alpha_2 \\ (X'_1, D'_1) & \overset{\sigma'{}^{\pm 1}}{\dashrightarrow} & (X'_2, D'_2) \end{array}$$

where σ and σ' are inner blowdowns. Given a pair (σ', α_1) there exists a unique pair (σ, α_2) fitting in the diagram, and vice versa. Now the claim follows.

Therefore, any left coset in $\text{Inn}(X, D)/\text{Aut}(X, D)$ has the form $\psi \cdot \text{Aut}(X, D)$ for some $\psi = \sigma'_n{}^{\pm 1} \cdots \sigma'_1{}^{\pm 1} \in \text{Inn}(X, D)$. These cosets form a countable set. \square

2.3. Weighted graphs. The birational geometry of NC-pairs has its combinatorial counterpart, namely, the birational geometry of weighted graphs, see the references in the Introduction.

Definition 2.3.1. Let Γ be a (nonempty) finite weighted graph with the set of vertices $\text{Vert}(\Gamma)$ where vertices are weighted by integer numbers, and let $\text{Ratio}(\Gamma) \subseteq \text{Vert}(\Gamma)$ be a distinguished subset of vertices which we call *rational vertices*. A rational vertex of weight -1 (resp. of weight 0) is called *(-1) -vertex* (resp. *(0) -vertex*). The graph Γ may be disconnected and may contain loops, cycles and multiple edges, that is, several edges joining the same pair of vertices.

Definition 2.3.2.

- Given a weighted graph Γ , the *degree* $\deg_{\Gamma}(v)$ of a vertex $v \in \text{Vert}(\Gamma)$ is the number of incident edges of Γ at v , where a loop $[v, v]$ is counted twice. We simply write $\deg(v)$ when the graph Γ has been fixed.

- A rational vertex v of Γ is called an *end vertex* if $\deg(v) = 1$, *linear* if $\deg(v) = 2$ and *at most linear* if $\deg(v) \leq 2$ and no loop of Γ is incident with v . We let $\text{End}(\Gamma)$ be the set of end vertices of Γ .
- A vertex v is called *branching* if either v is non-rational or $\deg(v) \geq 3$. We let $\text{Br}(\Gamma)$ be the set of all branching vertices of Γ . Thus, $\text{Vert}(\Gamma) \setminus \text{Ratio}(\Gamma) \subseteq \text{Br}(\Gamma)$.
- Given a vertex $v \in \Gamma$, the connected components of $\Gamma \setminus \{v\}$ linked to v are the *branches* of Γ at v . A branch linked to v via exactly one edge is called *simple*.

Definition 2.3.3. Let Γ be a weighted graph with vertices v_1, \dots, v_n . The *intersection form* $I(\Gamma)$ is a bilinear form on \mathbb{R}^n given by the symmetric square matrix $A(\Gamma) = (a_{i,j})$ where $a_{i,i}$ is the weight of v_i and for $j \neq i$, $a_{i,j}$ equals the number of edges between v_i and v_j . The *discriminant* of Γ is the determinant $\det(-A(\Gamma))$. We let $i_+(\Gamma)$, $i_-(\Gamma)$ and $i_0(\Gamma)$ be the inertia indices of the quadratic form $I(\Gamma)$.

Definition 2.3.4. Let $[v_1, v_2]$ (where possibly $v_1 = v_2$) be an edge of Γ . The *inner blowup* in $[v_1, v_2]$ consists in adding a new (-1) -vertex v , replacing the edge $[v_1, v_2]$ by two new edges $[v_1, v]$ and $[v, v_2]$ and decreasing the weights of v_1 and v_2 by 1 if $v_1 \neq v_2$ and by 4 if $v_1 = v_2$ (i.e. $[v_1, v_2]$ is a loop).

The *outer blowup* at a vertex v in Γ consists in introducing a new (-1) -vertex v' along with a new edge $[v, v']$ and decreasing the weight of v by 1.

The modifications inverse to inner and outer blowups are called *inner* and *outer blowdowns*, respectively.

Proposition 2.3.5 (cf. [34, Proposition 1.1], [40, [Proposition 1.14], [11, Lemma 4.6]). *A blowup of a weighted graph Γ adds 1 to $i_-(\Gamma)$ while $i_+(\Gamma)$ and $i_0(\Gamma)$ remain unchanged.*

Proof. Letting $\text{Vert}(\Gamma) = \{v_1, \dots, v_n\}$ consider the \mathbb{R} -vector space $V = \bigoplus_i \mathbb{R}v_i$ with the dot product given by $I(\Gamma)$. The symmetric matrix M_n of $I(\Gamma)$ is diagonal in a suitable orthogonal basis $\{e_1, \dots, e_n\}$ of V .

A blowup $\delta: \Gamma' \dashrightarrow \Gamma$ creating a vertex v_{n+1} induces a map $\delta^*: V \hookrightarrow V' = \bigoplus_{i=1}^{n+1} \mathbb{R}v_i$ defined by

$$\delta^*: v \mapsto v + I(\Gamma')(v, v_{n+1})v_{n+1}.$$

The vector v_{n+1} is an eigenvector of the matrix M_{n+1} of $I(\Gamma')$ with eigenvalue -1 . One can check that $I(\Gamma')$ restricted on $\delta^*(V)$ coincides with $\delta^*I(\Gamma)$ and v_{n+1} is orthogonal to $\delta^*(V)$. The orthogonal basis in V' of the eigenvectors

$$\delta^*(e_1), \dots, \delta^*(e_n), v_{n+1}$$

is diagonalizing for M_{n+1} , which proves the claim. \square

Remark 2.3.6. In the case where $\Gamma = \Gamma(D)$ is the dual graph of an NC-divisor D on a smooth surface X and $\delta: X' \rightarrow X$ is the blowup of a point on D , the homomorphism δ^* sends any divisor N supported on $\text{supp } D$ to its total transform $\delta^*(N)$ on X' , and v_{n+1} represents the exceptional (-1) -curve of δ . The properties of δ^* used in the proof follow in this case from the Projection Formula.

Definition 2.3.7. A *birational transformation of weighted graphs* $\phi: \Gamma \dashrightarrow \Gamma'$ is a finite sequence of blowups, blowdowns and isomorphisms. If all these blowups and blowdowns are inner, then ϕ is called *inner*. Two weighted graphs are called *birationally equivalent* if one can be transformed into the other by means of a birational transformation.

From Proposition 2.3.5 we deduce the following

Corollary 2.3.8. *The inertia indices $i_+(\Gamma)$ and $i_0(\Gamma)$ are birational invariants.*

Remarks 2.3.9. 1. The definition of inner birational transformations of weighted graphs in [11, Definitions 2.3 and 2.8] has a broader meaning. Indeed, the inverse of an outer blowup is considered in *loc.cit.* to be inner. With our more restrictive definition the inverse of an inner transformation is inner, while the inverse to an outer blowup is not. Cf. also Remark 2.7.2 below.

2. Given a weighted graph Γ and a sequence ϕ of blowups, blowdowns and isomorphisms, the final graph Γ' appears usually as the result of applying ϕ to Γ . If, besides isomorphisms, ϕ includes also blowups or blowdowns, then one can ignore the isomorphisms which participate in ϕ by simply renaming the created graphs, including the final one. If $\Gamma' = \Gamma$, then one can reduce to a single isomorphism either before or after the sequence of blowups and blowdowns, see (1).

3. A birational transformation of weighted graphs $\phi: \Gamma \dashrightarrow \Gamma'$ is not a mapping of graphs: in general, it is not well defined neither on the sets of vertices nor on the sets of edges. Nonetheless, given a second birational transformation $\phi': \Gamma' \dashrightarrow \Gamma''$ the composition $\phi' \circ \phi: \Gamma \dashrightarrow \Gamma''$ is a well defined birational transformation of weighted graphs. The identical map and the inverse $\phi^{-1}: \Gamma' \dashrightarrow \Gamma$ are well defined too.

2.4. Contractible graphs.

Definition 2.4.1. A birational transformation ϕ is a *birational morphism* (or *domination*) if no blowup participates in ϕ . In this case we write $\phi: \Gamma \rightarrow \Gamma'$. Notice that a birational morphism of graphs is surjective. If, moreover, ϕ is composed only of blowdowns then we say that ϕ is a *contraction*. The contractions form a proper subset of the set of birational morphisms.

Remark 2.4.2. The composition of birational morphisms (resp., contractions) of weighted graphs is a birational morphism (resp., a contraction).

Definition 2.4.3. A connected weighted graph Γ is said to be *contractible* if either Γ consists of a single (-1) -vertex with no loop, or there exists a contraction of Γ to a graph consisting of a single (-1) -vertex with no loop.

The following lemma is well known.

Lemma 2.4.4. *Let Γ be a weighted graph.*

- (a) *If Γ is contractible, then Γ is a tree with only rational vertices. Furthermore, the intersection form $I(\Gamma)$ is negative definite of discriminant 1.*
- (b) *Conversely, if Γ is a tree with only rational vertices and negative definite intersection form $I(\Gamma)$ of discriminant 1, then Γ is contractible.*
- (c) *Contracting at most linear (-1) -vertex of a contractible graph with at least two vertices yields a contractible graph and so does a blowup of a contractible graph. Thus, the contractibility is a birational invariant.*

Proof. To show the first assertion in (a) it suffices to look at the reconstruction process of blowups starting from the graph which consists of a single isolated (-1) -vertex. The second assertion follows from [23, Section 3, Proposition]. We address [40, Proposition 1.20] and [11, Remark 4.8, formula (17)] for statement (b). Statement (c) follows e.g. from (b) due to [11, Remark 4.8, formula (17)]; cf. also [23, Section 3, Proposition]. \square

The next corollary is immediate.

Corollary 2.4.5. *For a contractible weighted graph Γ the following hold.*

- (a) *The weight of every vertex v of Γ is negative.*
- (b) *No two (-1) -vertices of Γ are neighbors.*
- (c) *Every (-1) -vertex v of Γ is at most linear.*

- (d) If Γ contains at least two vertices, then every (-1) -vertex of Γ is contracted in any process of contraction of Γ to a graph which consists of a single isolated (-1) -vertex.

2.5. Contraction of a subgraph. More generally, we have the following facts. Given a birational morphism $\phi: \Gamma \rightarrow \Gamma'$ and a vertex v of Γ we let ϕ^v be the maximal subsequence of blowdowns and isomorphisms in ϕ preserving v . Thus, $\phi^v = \phi$ if and only if v is not contracted by ϕ .

Lemma 2.5.1 (see [11, Lemma 2.4(a)]). *If a branching vertex $v \in \text{Br}(\Gamma)$ is blown down under a contraction $p: \Gamma \rightarrow \Gamma'$, then v is rational and has no incident loop. Moreover, at least $\deg(v) - 2$ branches of Γ at v are simple and contracted by p^v .*

Proof. Consider the contraction $p^v: \Gamma \rightarrow \Gamma^v$ and let $\sigma_v: \Gamma^v \rightarrow \Gamma'$ be the blowdown of v . Observe that $p^v(v)$ is at most linear (-1) -vertex of Γ^v blown down by σ_v . By Lemma 2.4.4(a) the maximal subgraph E of Γ contracted by the subsequence $\sigma_v \circ p^v$ of p is a forest. In particular, the connected component of E which contains v is a tree. Thus all branches of Γ at v contracted by p^v are trees, are simple, there are at least $\deg(v) - 2$ branches, and v is rational. \square

Definition 2.5.2. Given a blowup $\delta: \Gamma \dashrightarrow \Gamma'$ we define the image of a vertex $v \in \text{Vert}(\Gamma)$ as being the vertex $\delta(v)$ and the image of an edge $e = [u, v]$ where $u, v \in \text{Vert}(\Gamma)$ as being the edge $[\delta(u), \delta(v)]$ except in the case where δ is the inner blowup at e ; in the latter case $\delta(e)$ is the path of length 2 joining $\delta(u)$ and $\delta(v)$ which contains the vertex introduced by δ . We define the image $\delta(\Delta)$ of a subgraph $\Delta \subset \Gamma$ as being the subgraph of Γ' whose vertices and edges are images of the vertices and edges of Δ including the blown-up vertices.

Given a blowdown $\sigma: \Gamma' \rightarrow \Gamma$ and a subgraph $\Delta \subset \Gamma$ we define the preimage $\sigma^{-1}(\Delta)$ as being the image of Δ under the blowup $\delta = \sigma^{-1}$. We extend this notion inductively to any birational morphism $\phi: \Gamma_1 \rightarrow \Gamma_2$.

Recall that a subgraph Δ of Γ is *induced* if Δ contains every edge of Γ that connects vertices of Δ .

Lemma 2.5.3. *Consider a birational morphism $\phi: \Gamma' \rightarrow \Gamma$.*

- (a) *The preimage $\phi^{-1}(\gamma)$ of a simple path γ connecting vertices $u, v \in \text{Vert}(\Gamma)$ is a simple path connecting $\phi^{-1}(u)$ and $\phi^{-1}(v)$.*
- (b) *The preimage $\phi^{-1}(\Delta)$ of a subgraph $\Delta \subset \Gamma$ is a subgraph of Γ' homeomorphic to Δ , i.e. the associated simplicial 1-complexes are homeomorphic as topological spaces. If $\Delta \subset \Gamma$ is an induced subgraph then so is $\phi^{-1}(\Delta) \subset \Gamma'$.*

Proof. These statements can be easily checked for a single blowdown. Therefore, they hold for an arbitrary ϕ . \square

Let us introduce the following notions.

Definition 2.5.4. Given a weighted graph Γ we say that two birational morphisms $p_i: \Gamma \rightarrow \Gamma_i$, $i = 1, 2$, are *equivalent* if there is a (uniquely defined) isomorphism of weighted graphs $\iota: \Gamma_1 \rightarrow \Gamma_2$ such that $p_1^{-1}(\Delta) = p_2^{-1}(\iota(\Delta))$ for any subgraph $\Delta \subset \Gamma_1$. If this holds for $\Gamma_1 = \Gamma_2$ with $\iota = \text{id}$ then we say that p_1 and p_2 are *equal* and we write $p_1 = p_2$. We say that a diagram of weighted graphs

$$\begin{array}{ccc}
 & \Gamma & \\
 p_1 \swarrow & & \searrow p_2 \\
 \Gamma_1 & \longrightarrow & \Gamma_2 \\
 & p_3 & \\
 & 8 &
 \end{array}$$

where the p_i are birational morphisms of graphs, commutes if $p_3 \circ p_1 = p_2$. In the latter case the diagram

$$\begin{array}{ccc} & \Gamma & \\ p_2 \swarrow & & \searrow p_1 \\ \Gamma_2 & \dashrightarrow & \Gamma_1 \\ & p_3^{-1} & \end{array}$$

is also called commutative. Square commutative diagrams of birational morphisms and more complicated commutative diagrams are defined in a similar way.

Lemma 2.5.5.

- (a) *Given a birational morphism $\phi: \Gamma \rightarrow \Gamma'$ there is a contraction $p: \Gamma \rightarrow \Gamma''$ equivalent to ϕ .*
- (b) *Consider a composition $\Gamma \xrightarrow{\phi} \Gamma_1 \xrightarrow{\sigma} \Gamma_2$ of a birational morphism ϕ and a blow-down σ of a vertex v which is not adjacent in Γ to any vertex contracted by ϕ . Then there exists a birational morphism $\phi': \Gamma \rightarrow \Gamma'_2$ equivalent to ϕ and starting with the blowdown of v .*
- (c) *Consider a birational morphism $\phi: \Gamma \rightarrow \Gamma_1$ and a (-1) -vertex $v \in \text{Vert}(\Gamma)$ contracted by ϕ . Then there exists a birational morphism $\phi': \Gamma \rightarrow \Gamma_2$ equivalent to ϕ and starting with the blowdown σ_v of v .*

Proof. (a) We obtain the desired contraction by blowing down the vertices of Γ blown down by ϕ in the same order as this is done under ϕ .

(b) The weight of v is -1 both in Γ and Γ_1 , and the weight of any vertex contracted by ϕ is the same in Γ and after the blowdown of v in Γ . Thus, we obtain ϕ' by contracting v first and then contracting all the vertices contracted by ϕ in the same order. The weights of the non-contracted vertices are the same under $\sigma \circ \phi$ and ϕ' . Now the assertion follows.

(c) The vertex v does not change its weight under ϕ^v , hence it is adjacent to no vertex contracted by ϕ^v . The claim follows now from (b) applied to ϕ^v and σ_v . \square

Proposition 2.5.6. *Two birational morphisms $\phi_i: \Gamma \rightarrow \Gamma_i$, $i = 1, 2$ are equivalent if and only if the subsets of vertices of Γ contracted by ϕ_1 and ϕ_2 coincide.*

Proof. The ‘only if’ part is clear. To show the ‘if’ part assume that each vertex contracted by ϕ_1 is contracted by ϕ_2 and vice versa. If there are no contracted vertices in Γ , then both ϕ_1 and ϕ_2 are isomorphisms and it suffices to set $\iota = \phi_2 \circ \phi_1^{-1}$, see Definition 2.5.4. Let now $v \in \text{Vert}(\Gamma)$ be a (-1) -vertex contracted by the ϕ_i . By Lemma 2.5.5(c), up to equivalence we may suppose that both ϕ_1 and ϕ_2 start with the blowdown σ_v of v . This reduces the assertion to the one for the birational morphisms $\phi'_i: \sigma_v(\Gamma) \rightarrow \Gamma_i$. Now the induction by the number of contracted vertices ends the proof. \square

2.6. Relatively minimal graphs. Recall the following well known fact.

Lemma 2.6.1. *Any birational transformation of NC-pairs $\Phi: (X_1, D_1) \dashrightarrow (X_2, D_2)$ fits in a commutative diagram*

$$(2) \quad \begin{array}{ccc} & (\tilde{X}, \tilde{D}) & \\ P_1 \swarrow & & \searrow P_2 \\ (X_1, D_1) & \dashrightarrow & (X_2, D_2) \\ & \Phi & \end{array}$$

where every P_i is a composition of an isomorphism and blowdowns of smooth rational (-1) -components of boundary divisors and all the intermediate pairs are NC-pairs.

Proof. Any birational transformation of smooth projective surfaces $\Phi: X_1 \rightarrow X_2$ fits in a commutative diagram

$$\begin{array}{ccc} & \tilde{X} & \\ P_1 \swarrow & & \searrow P_2 \\ X_1 & \dashrightarrow & X_2 \\ & \Phi & \end{array}$$

where P_i is a composition of an isomorphism and blowdowns of smooth rational (-1) -curves, see e.g. [33, Corollaries (8.10)-(8.11)] or [3, Corollary II.12]. This can be applied as well in our setting after a simultaneous resolution of singularities of X_1 and X_2 . Indeed, $\Phi|_{X_1 \setminus \text{supp } D_1}: X_1 \setminus \text{supp } D_1 \rightarrow X_2 \setminus \text{supp } D_2$ is a biregular isomorphism and X_i is smooth near D_i .

Notice that P_i^{-1} is either isomorphism or a composition of blowups with smooth centers. A blowup of an NC-pair (X, D) in a point of D yields an NC-pair. Hence all the intermediate pairs in the decomposition

$$P_i: (\tilde{X}, \tilde{D}) \xrightarrow{\sigma_{i,n_i}} (X_{i,n_i-1}, D_{i,n_i-1}) \rightarrow \cdots \rightarrow (X_{i,2}, D_{i,2}) \xrightarrow{\sigma_{i,2}} (X_{i,1}, D_{i,1}) = (X_i, D_i)$$

are NC-pairs. \square

To formulate an analog of this lemma for birational transformations of weighted graphs one needs to define the meaning of the corresponding commutative diagram.

Definition 2.6.2. Consider a diagram of weighted graphs

$$(3) \quad \begin{array}{ccc} & \Gamma & \\ p_1 \swarrow & & \searrow p_2 \\ \Gamma_1 & & \Gamma_2 \end{array}$$

where the p_i are birational morphisms of graphs. Consider also a birational transformation $\phi: \Gamma_1 \dashrightarrow \Gamma_2$ written as

$$(4) \quad \phi: \Gamma_1 = \Delta_1 \xrightarrow{\psi_1} \Delta_2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{n-2}} \Delta_{n-1} \xrightarrow{\psi_{n-1}} \Delta_n = \Gamma_2$$

where the Δ_i are weighted graphs and for $n > 1$ exactly one of the ψ_i and ψ_{i+1} is a birational morphism and the other one is the inverse of a birational morphism. We say that ϕ fits in (3) and that (3) dominates ϕ if (4) is included in a sequence of commutative diagrams

$$(5) \quad \begin{array}{ccc} & \Gamma & \\ \eta_i \swarrow & & \searrow \eta_{i+1} \\ \Delta_i & \dashrightarrow & \Delta_{i+1} \\ & \psi_i & \end{array}$$

where $\eta_1 = p_1$ and $\eta_n = p_2$; see Definition 2.5.4.

Remark 2.6.3. The graphs Γ_1 and Γ_2 in diagram (3) are birationally equivalent via the birational transformation $\phi' = p_2 \circ p_1^{-1}: \Gamma_1 \dashrightarrow \Gamma_2$. Thus, starting with a diagram (3) we come to a birational transformation ϕ' . In the next proposition we show that, conversely, given a birational transformation of weighted graphs one can construct a corresponding diagram (3).

Proposition 2.6.4 (see [13, Remark A1(1)]). *Let $\phi: \Gamma_1 \dashrightarrow \Gamma_2$ be a birational transformation of weighted graphs. Then ϕ fits in a diagram (3) for some Γ, p_1, p_2 .*

Proof. The assertion is evidently true if ϕ is an isomorphism. Assuming this is not the case, up to an isomorphism of Γ_2 , we can write ϕ as in (4) with every ψ_i being

a product of blowdowns or the inverse of such a product, see Remark 2.3.9.2. For $n = 1$ our assertion is obvious.

Assume now that $n = 2$ and $\phi = \sigma_2^{-1}\sigma_1$ where the $\sigma_i : \Gamma_i \rightarrow \Gamma_0$ are blowdowns. If both σ_1^{-1} and σ_2^{-1} are inner blowups of the same edge of Γ_0 , then the graph $\Gamma = \Gamma_1 = \Gamma_2$ dominates both Γ_1 and Γ_2 via isomorphisms. Otherwise, applying simultaneously both of them one transforms Γ_0 into a graph Γ with two distinct vertices v_1 and v_2 such that Γ dominates Γ_1 (resp., Γ_2) via the contraction p_1 of v_2 (resp., p_2 of v_1). This produces a desired diagram (3).

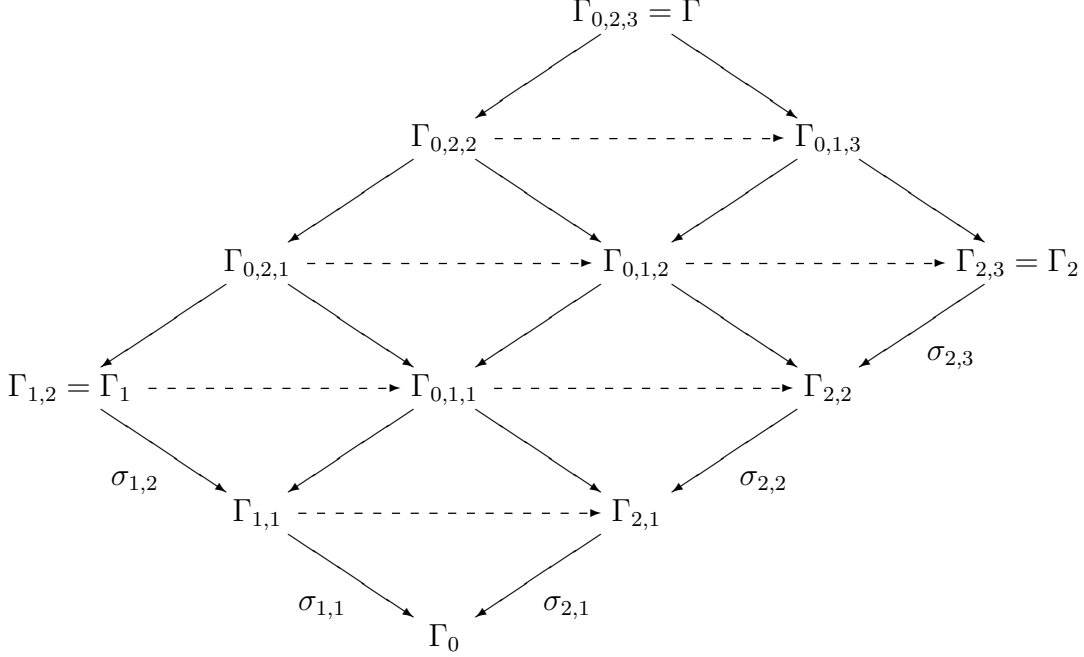


FIGURE 1. Construction of diagram (3) for $m_1 = 2$ and $m_2 = 3$.

Let further $n = 2$ and $\phi = \psi_2^{-1}\psi_1$, where this time the ψ_i stand for birational morphisms. We can write

$$\psi_i = \sigma_{i,1} \cdots \sigma_{i,m_i} : \Gamma_i = \Gamma_{i,m_i} \rightarrow \Gamma_0 = \Gamma_{i,0}$$

where $\sigma_{i,j} : \Gamma_{i,j} \rightarrow \Gamma_{i,j-1}$ is a blowdown, $i = 1, 2$. Applying the preceding case to the composition $\sigma_{2,1}^{-1}\sigma_{1,1} : \Gamma_{1,1} \dashrightarrow \Gamma_{2,1}$ we can find a weighted graph $\Gamma_{0,1,1}$ which dominates both $\Gamma_{1,1}$ and $\Gamma_{2,1}$ making the square diagram commutative, see Figure 1. The same procedure applied to the pairs $\{\Gamma_{2,2}, \Gamma_{0,1,1}\}$ and $\{\Gamma_{0,1,1}, \Gamma_{1,2}\}$ yields two new graphs $\Gamma_{0,1,2}$ and $\Gamma_{0,2,1}$ which dominate the corresponding pairs. Continuing in this way we fill in a commutative diagram in the form of a lattice parallelogram consisting of $(m_1 + 1)(m_2 + 1)$ weighted graphs and their morphisms. This parallelogram has the pairs of opposite vertices $\{\Gamma_1, \Gamma_2\}$ and $\{\Gamma_0, \Gamma\}$, where $\Gamma = \Gamma_{0,m_1,m_2}$ dominates both Γ_1 and Γ_2 via sequences of blowdowns inducing ϕ , see Figure 1 for $m_1 = 2$ and $m_2 = 3$. This yields the assertion for $n = 2$.

Proceeding by induction on n we assume that $n \geq 3$ and we choose $i \geq 2$ such that ψ_i^{-1} and ψ_{i-1} in (4) are morphisms. By the preceding, the product $\psi_i\psi_{i-1}$ can be replaced by $\rho_i\rho_{i-1}$ where this time ρ_i and ρ_{i-1}^{-1} are morphisms. Replacing now $\psi_{i+1}\rho_i$ by ψ'_{i+1} provided $i \leq n - 1$ and $\rho_{i-1}\psi_{i-2}$ by ψ'_{i-2} provided $i \geq 3$ we transform (4) into a shorter decomposition of the same type. This gives the inductive step.

The graph Γ in diagram (3) constructed in the proof dominates every pair of intermediate graphs obtained under our procedure. In particular, ϕ fits in this diagram, see Definition 2.6.2. \square

Remark 2.6.5. Given a birational map $\phi: \Gamma_1 \dashrightarrow \Gamma_2$ fitting in (3), the graph Γ is not uniquely defined, in general; see Figure 2 for a simple example. In this example Γ_1 and Γ_2 are isomorphic linear graphs, the σ_i are blowdowns of (-1) -vertices and $\phi = \sigma_2^{-1} \circ \sigma_1: \Gamma_1 \dashrightarrow \Gamma_2$ is a birational map. The morphisms p_1 and p_2 in the first diagram on Figure 2 are isomorphisms, while p'_1 and p'_2 in the second diagram are blowdowns of distinct (-1) -vertices of Γ' . The vertical arrows in both diagrams correspond to the left arrow in diagram (5) with $i = 2$.

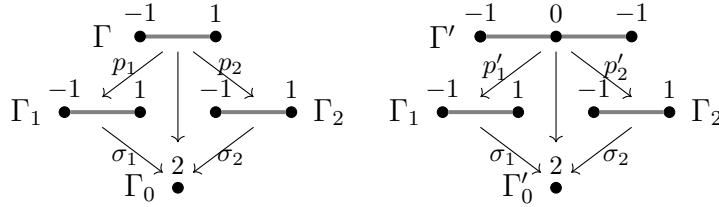


FIGURE 2. Two different dominations of the same weighted graphs.

Definition 2.6.6. A weighted graph Γ is called *minimal* if any birational morphism $\Gamma \rightarrow \Gamma'$ is in fact an isomorphism. A graph Γ is minimal if and only if it has no at most linear (-1) -vertex. Clearly, any weighted graph Γ dominates a minimal one called a *minimal model* of Γ .

A simple example of a minimal weighted graph is a graph consisting of a single vertex v and a loop, cf. Example 2.7.2. Indeed, recall that an at most linear (-1) -vertex has no incident loop, see Definition 2.3.2.

Definition 2.6.7. A diagram (3) with two minimal weighted graphs Γ_1 and Γ_2 is called *relatively minimal* if no (-1) -vertex of Γ is contracted in both Γ_1 and Γ_2 .

According to the following lemma, for any pair of birationally equivalent minimal weighted graphs Γ_1 and Γ_2 there is a weighted graph Γ and birational morphisms $p_i: \Gamma \rightarrow \Gamma_i$ that form a relatively minimal diagram (3).

Lemma 2.6.8 (cf. [13, Remark A1(1)]). *Given a diagram (3) with minimal weighted graphs Γ_1 and Γ_2 there exist birational morphisms*

$$\Gamma \xrightarrow{\psi} \Gamma' \xrightarrow{p'_i} \Gamma_i, \quad i = 1, 2$$

such that p_i is equivalent to $p'_i \circ \psi$ for $i = 1, 2$ and Γ', p'_1, p'_2 form a relatively minimal diagram (3).

Proof. Let $v \in \text{Vert}(\Gamma)$ be a (-1) -vertex contracted by both p_1 and p_2 . Replacing the p_i by equivalent birational morphisms $\Gamma \rightarrow \Gamma_i$ as in Lemma 2.5.5(c) we may suppose that both p_1 and p_2 start with the same blowdown $\sigma_v: \Gamma \rightarrow \Gamma_v$ of v and there are the factorizations

$$p_i: \Gamma \xrightarrow{\sigma_v} \Gamma_v \xrightarrow{p'_{v,i}} \Gamma_i$$

where $\Gamma_v, p'_{v,1}$ and $p'_{v,2}$ form a diagram (3). Now the recursion on the number of vertices of Γ ends the proof. \square

Remark 2.6.9. Notice that under the assumptions of Lemma 2.6.8 the resulting relatively minimal diagram (3) is not unique, in general, see Example 3.1.5 and [11, Example 2.6]. For the both examples, in the corresponding relatively minimal diagram (3) we have $\Gamma_1 = \Gamma_2 \not\cong \Gamma$. So in both cases there is a second relatively minimal diagram (3) with $\Gamma = \Gamma_1 = \Gamma_2$.

2.7. Dual graphs. Let us recall the correspondence between NC-pairs and weighted graphs.

Definition 2.7.1. Given an NC-pair (X, D) the *dual graph* $\Gamma(D)$ is a weighted graph whose vertices are in bijection with the irreducible components of D and edges are in bijection with the nodes of D . The loops of $\Gamma(D)$ at a vertex C are in bijection with the self-intersection points of the component C of D , and the edges joining two different vertices C_1 and C_2 of $\Gamma(D)$ are in bijection with the points of $C_1 \cap C_2$. The weight of C in $\Gamma(D)$ is the self-intersection index C^2 of the component C in X . The rational vertices in $\text{Ratio}(\Gamma(D))$ correspond to the rational components of D .

An NC-pair (X, D) is an SNC-pair if and only if $\Gamma(D)$ contains no loops and multiple edges. Clearly, any NC-pair is dominated by an SNC-pair. An NC-pair (X, D) is called *minimal* if the dual graph $\Gamma(D)$ is minimal.

The notions of inner and outer blowups of an NC-pair are consistent with the notions of inner and outer blowups of the dual graph $\Gamma(D)$, respectively.

Example 2.7.2. Recall that any non-complete normal algebraic surface Y admits an SNC-completion (X, D) , where $Y = X \setminus \text{supp } D$, and an NC-completion with a minimal dual graph.

For example, if $X = \mathbb{P}^2$ and D is a nodal cubic, then $\Gamma(D)$ is minimal and consists of a single vertex of weight 9 and a loop. So the pair (\mathbb{P}^2, D) is an NC-completion of the affine surface $Y = \mathbb{P}^2 \setminus D$ with a minimal dual graph. There exists a non-minimal SNC-completion (X, D_1) whose dual graph $\Gamma(D_1)$ is a cycle with three rational vertices and a cyclically ordered sequence of weights $((-2, -1, 4))$. Moreover, there exists a minimal such completion with a sequence of weights $((0, 0, -2, -2, -2, -2, -3))$. Thus, two minimal cyclic graphs with rational vertices and the cyclically ordered sequences of weights $((9))$ and $((0, 0, -2, -2, -2, -2, -3))$, respectively, are birationally equivalent.

Similarly, one can show that for any $a, b > 0$ the minimal linear graphs with sequences of weights

$$[[a]], \quad \underbrace{[-2, -2, \dots, -2, 0, 0]}_{a+1} \quad \text{and} \quad \underbrace{[-2, \dots, -2, -3]}_{a-1}, \underbrace{-2, \dots, -2, 0, 1]}_{b+1}$$

are birationally equivalent, cf. Proposition 5.0.8.

The following criterion of affineness is a particular case of the Goodman criterion [21, Theorem 1].

Lemma 2.7.3. *Let (X, D) is an NC-pair. The open surface $Y = X \setminus \text{supp } D$ is affine if and only if there is a sequence of blowups $\phi: X' \rightarrow X$ in the points of $\text{supp } D$ and infinitesimally near points such that $\text{supp } \phi^*(D)$ coincides with the support of an effective ample divisor on X .*

Remark 2.7.4. In particular, if $\text{supp } D$ coincides with the support of an effective ample divisor then Y is affine. Anyway, if Y is affine then $\text{supp } D$ is connected and cannot be contractible, see the next Lemma 2.7.5.

The assumption that the surface $Y = X \setminus \text{supp } D$ is affine imposes severe restrictions on the dual graph $\Gamma(D)$.

Lemma 2.7.5. *Let (X, D) be an NC-pair such that the surface $Y = X \setminus \text{supp } D$ is affine. Then the following hold.*

- (a) $\text{supp } D$ is connected and supports a nef² divisor.

²I.e. numerically effective.

- (b) The dual graph $\Gamma(D)$ is connected, the intersection form $I(\Gamma(D))$ is not negative definite, and D and $\Gamma(D)$ are not contractible.
- (c) If $D = C$ is a curve, then $C^2 > 0$ and the linear system $|C|$ is ample. If, moreover, C is smooth and rational then X is rational.

Proof. It is well known that $\text{supp } D$ is connected if Y is affine, see e.g. [21, p. 166, Corollary] or [22, Corollary II.6.2]. Hence $\Gamma(D)$ is connected. Consider a proper embedding $Y \hookrightarrow \mathbb{A}^n$ and the closure \bar{Y} of Y in $\mathbb{P}^n \supset \mathbb{A}^n$. The hyperplane at infinity in \mathbb{P}^n cuts out an ample effective divisor \bar{H} on \bar{Y} with $\text{supp } \bar{H} = \bar{Y} \setminus Y$. Let $\pi': (X', H'_{\text{red}}) \rightarrow (\bar{Y}, \bar{H}_{\text{red}})$ be a partial embedded resolution of singularities such that (X', H'_{red}) is an SNC-completion of Y and $H' = (\pi')^*(\bar{H})$ is an effective divisor on X' with $\text{supp } H' = X' \setminus Y$. By the Projection Formula, H' is nef, see [30, Example 1.4.4]. According to Lemma 2.6.1 there exists a commutative diagram

$$\begin{array}{ccc} & (\tilde{X}, \tilde{D}) & \\ \tilde{\pi} \swarrow & & \searrow \pi \\ (X', H'_{\text{red}}) & \dashrightarrow & (X, D) \\ & \Phi & \end{array}$$

where Φ is a birational map which extends id_Y and (\tilde{X}, \tilde{D}) is an NC-completion of Y . By the above argument the effective divisor $\tilde{H} = \pi^*(H')$ on \tilde{X} with $\text{supp } \tilde{H} = \text{supp } \tilde{D}$ is nef. The effective divisor $H = \pi_*(\tilde{H})$ on X with $\text{supp } H = \text{supp } D$ is nef too. Indeed, if C is a curve on X then, by the Projection Formula, $H \cdot C = \tilde{H} \cdot \pi^*(C) \geq 0$. In more detail, first we resolve the singularities of Y in X and \tilde{X} simultaneously, then apply the Projection Formula, and finally contract the exceptional divisors, which does not affect the resulting inequalities. This proves statement (a).

Since $H^2 \geq 0$ and $H \neq 0$, the intersection form $I(\Gamma(D))$ cannot be negative definite. By Lemma 2.4.4 the graph $\Gamma(D)$ is non-contractible, hence also D is, as stated in (b).

Assume now that $D = C$ is irreducible. Then $C^2 > 0$, see [19, Lemma 2], and the linear system $|C|$ is ample by the Nakai-Moishezon criterion. If C is smooth and rational then X is rational as well, see [19, Remarks 2 and 3] or [2, Proposition V.4.3]. This shows (c). \square

Remark 2.7.6. The assumption of nefness in (a) is a necessary condition for Y to be affine, but not sufficient. Indeed, the complement Y to a rational (0)-curve C on a smooth rational surface X is not affine because Y contains complete curves which are members of the linear pencil $|C|$, see e.g. [19]. Notice that C is a nef divisor on X .

Lemma 2.7.5 justifies the following convention.

2.7.7. Convention. We consider in the sequel only weighted graphs Γ with no contractible connected component.

Definition 2.7.8. Every decomposition of $\Phi \in \text{Bir}((X_1, D_1), (X_2, D_2))$ into a sequence of isomorphisms, blowups and blowdowns induces a birational transformation of dual graphs $\phi: \Gamma(D_1) \dashrightarrow \Gamma(D_2)$ called a *representation* of Φ .

Concerning the inverse correspondence, one has the following result.

Proposition 2.7.9 (cf. [11, Proposition 3.34]). *Given an NC-pair (X, D) and a birational transformation of the dual graph $\phi: \Gamma(D) \dashrightarrow \Gamma'$ consisting of a sequence of blowups and blowdowns one can find a new NC-pair (X', D') and a birational transformation of NC-pairs $\Phi: (X, D) \dashrightarrow (X', D')$ such that $\Phi|_{X \setminus \text{supp } D}: X \setminus \text{supp } D \rightarrow$*

$X' \setminus \text{supp } D'$ is an isomorphism and Φ is represented by ϕ ; in particular, $\Gamma(D') = \Gamma'$. If ϕ contains no outer blowups then Φ is uniquely defined up to isomorphism of NC-pairs.

Proof. The first statement clearly holds for a single blowup and a single blowdown of $\Gamma(D)$. By recursion, it holds in the general case. The uniqueness follows by recursion from the facts that to an inner blowup of $\Gamma(D)$ there corresponds the blowup of X at a uniquely defined node of D , and to a blowdown of a (-1) -vertex of $\Gamma(D)$ there corresponds the contraction of a uniquely defined (-1) -component of D . \square

Remark 2.7.10. Let (X, D) be an NC-pair, and let C be a smooth component of D which corresponds to a non-isolated at most linear (0) -vertex of $\Gamma(D)$. Blowing up a node of D on C and contracting the proper transform of C yields a birational transformation Φ of (X, D) called an *inner elementary transformation*. The corresponding birational transformation ϕ of $\Gamma(D)$ is also called *elementary*; it affects the weights of the neighbors of C in $\Gamma(D)$.

In the case where C is an end (0) -vertex of $\Gamma(D)$ and the center of blowup on C is a smooth point of D we have an *outer elementary transformation*; see (7) for an example.

Let now $D = C$ be a smooth rational curve and so $\Gamma(D)$ consists of a single isolated vertex C of weight $a = C^2$ with no loop. Take two distinct points P_1 and P_2 of C . Blowing up P_i produces an NC-pair (X_i, D_i) whose dual graph $\Gamma(D_i)$ has sequence of weights $[[-1, a-1]]$. Let $\sigma_i: X_i \rightarrow X$ be the blowdown of the exceptional curve. The pairs (X_1, D_1) and (X_2, D_2) are not isomorphic, in general, while the dual graphs $\Gamma(D_1)$ and $\Gamma(D_2)$ are. The corresponding birational transformation $\Phi = \sigma_2^{-1}\sigma_1: (X_1, D_1) \dashrightarrow (X_2, D_2)$ is not an isomorphism of pairs. It restricts to the identity on $Y = X \setminus \text{supp } D$. On the level of dual graphs, the birational map $\phi = \sigma_2^{-1}\sigma_1: \Gamma(D_1) \dashrightarrow \Gamma(D_2)$ in the induced representation of Φ fits in the commutative diagram

$$\begin{array}{ccccc}
 & & -1 & a-2 & -1 \\
 & & \bullet & \text{---} & \bullet \\
 \Gamma & & & & \\
 & p_1 & & & p_2 \\
 & \swarrow & & & \searrow \\
 \Gamma(D_1) & a-1 & \bullet & \text{---} & \bullet & a-1 & \Gamma(D_2) \\
 & & & \phi & & & \\
 & \sigma_1 & & & \sigma_2 & & \\
 & \swarrow & & & \searrow & & \\
 & & \bullet & a & & & \\
 & & \Gamma(D) & & & &
 \end{array}$$

where Γ is the dual graph of the pair (\tilde{X}, \tilde{D}) obtained from (X, D) by blowing up the points P_1 and P_2 on X . Varying the positions of P_1 and P_2 on C yields nontrivial deformations of the participating pairs. Thus, Φ cannot be reconstructed in general from its representation $\phi = p_2 \circ p_1^{-1}$ once outer blowups are involved.

We have the following geometric analog of Lemma 2.6.8.

Lemma 2.7.11. *Any birational transformation of NC-pairs $\Phi: (X_1, D_1) \dashrightarrow (X_2, D_2)$ fits in a commutative diagram (2) such that a representation ϕ of Φ fits into the corresponding relatively minimal diagram (3) of dual graphs.*

Proof. Starting with an arbitrary commutative diagram (2) we repeat the procedure in the proof of Lemma 2.6.8 on the geometric level. Namely, to each at most linear (-1) -vertex of $\Gamma(\tilde{D})$ there corresponds a (-1) -component C of \tilde{D} which meets transversally in at most two points the union of other components. If C is contracted under the both $P_i: X \rightarrow X_i$, $i = 1, 2$ then there are factorizations

$$P_i: \tilde{X} \xrightarrow{\sigma_C} \tilde{X}' \xrightarrow{P'_i} X_i,$$

where σ_C stands for the contraction of C in \tilde{X} , see [3, Proposition II.8]. This produces a new NC-pair (\tilde{X}', \tilde{D}') which still dominates the both (X_i, D_i) fitting in a new diagram (2) with the same $\Phi: (X_1, D_1) \dashrightarrow (X_2, D_2)$. A simultaneous resolution of singularities of X_1, X_2 and \tilde{X} does not affect our procedure. Hence we may assume that all these surfaces are smooth. We have $\rho(\tilde{X}') = \rho(\tilde{X}) - 1$ where ρ stands for the Picard rank, see e.g. [3, Proposition II.3]. The induction on the Picard rank shows that this procedure ends with a relatively minimal diagram. Finally we contract simultaneously the exceptional divisors and arrive at the same conclusion for the original surfaces. \square

3. THE GRAPH LEMMA AND BIRATIONAL RIGIDITY OF WEIGHTED GRAPHS

In this section we elaborate a criterion as to when $\text{Bir}(X, D) = \text{Inn}(X, D)$, see Theorem 3.2.5.

3.1. The Graph Lemma. The Graph Lemma 3.1.4 plays an important role when dealing with birational transformations of weighted graphs. It provides a criterion as to when every birational transformation corresponding to a relatively minimal diagram between two given weighted graphs is inner. We use the following terminology.

Definition 3.1.1. For a proper subset V of $\text{Vert}(\Gamma)$ we let $\Gamma \ominus V$ be the subgraph of Γ obtained by deleting V and all its incident edges. The connected components of $\Gamma \ominus \text{Br}(\Gamma)$ are called *segments*. We also call a *segment* a connected graph without branching vertices; such a graph coincides with its only segment.

A segment can be either linear or circular. A circular segment is a cycle in Γ which passes through no branching vertex; it is a connected component of Γ .

Definition 3.1.2. Let Γ be a minimal weighted graph. A circular segment C of Γ is called *admissible* if either the weights of its vertices are ≤ -2 , or it consists of a single vertex of weight ≤ 2 and a loop. A linear segment L of Γ is called *admissible* if the weights of its vertices are ≤ -2 . It is called

- (1) *extremal* if L contains one or two end vertices of Γ ;
- (2) *inner* if L contains no end vertices of Γ . So the end vertices of L are linked to branching vertices of Γ .

The following Graph Lemma is based on [13, Proposition A1] and [11, Lemma 2.4(b)]. For the reader's convenience we provide a proof. In the proof we use the following notion.

Definition 3.1.3. Let $p: \Gamma \rightarrow \Gamma'$ be a contraction. Recall that for a vertex $v \in \text{Vert}(\Gamma)$ we denote by p^v the maximal subsequence of blowdowns in p preserving v . We say that v is *p-rigid* if $p^v = p$ and the degree of v does not change under p^v .

Graph Lemma 3.1.4. Any relatively minimal diagram (3) with minimal weighted graphs Γ_1 and Γ_2 satisfies the following.

- (a) For the branching sets we have

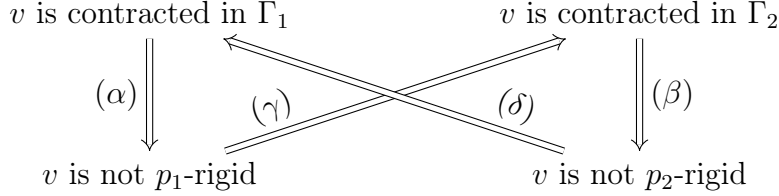
$$\text{Br}(\Gamma_1) = \text{Br}(\Gamma) \cap \text{Vert}(\Gamma_1) = \text{Br}(\Gamma) \cap \text{Vert}(\Gamma_2) = \text{Br}(\Gamma_2),$$

where $\text{Vert}(\Gamma_i)$ is considered to be a subset of $\text{Vert}(\Gamma)$, that is, we identify every vertex of Γ_i with its preimage under p_i in Γ . Furthermore, the degrees of the vertices in $B := \text{Br}(\Gamma_1) = \text{Br}(\Gamma_2)$ do not change under the contractions p_1 and p_2 .

- (b) The image under p_i of any connected component of $\Gamma \setminus B$ is a segment of Γ_i , $i = 1, 2$. This provides a one-to-one correspondence between the sets of circular, inner linear and extremal linear segments of Γ_1 and Γ_2 , respectively.

(c) If $\text{Br}(\Gamma) \neq B$, then both Γ_1 and Γ_2 contain non-admissible extremal linear segments. More precisely, for every $v \in \text{Br}(\Gamma) \setminus B$ there are two simple branches W_1 and W_2 of Γ at v such that p_1 contracts W_1 and v and sends W_2 into a non-admissible extremal linear segment of Γ_1 , and symmetrically for p_2 . Moreover, $\deg_\Gamma(v) = 3$, there are exactly 3 branches of Γ at v and all of them are simple.

Proof of Graph Lemma 3.1.4. (a) Given a branching vertex v in Γ , there are the following implications:



Indeed, implications (α) and (β) follow from Definition 3.1.3. By symmetry, it suffices to show (γ) . Assuming that $v \in \text{Br}(\Gamma)$ is not p_1 -rigid there is a simple branch W_1 of Γ at v contracted by p_1^v . The latter follows from Lemma 2.5.1 if p_1 contracts v and from Definition 3.1.3 if $p_1 = p_1^v$ contracts a branch W_1 at v and changes the degree of v . Indeed, assuming on the contrary that W_1 is not simple, the contraction of W_1 creates some loops at v without changing the degree of v , a contradiction.

Let v_1 be at most linear (-1) -vertex of W_1 contracted the first by p_1^v . Due to the relative minimality assumption, v_1 is not contracted by p_2 . On the other hand, $p_2(W_1) \subset \Gamma_2$ contains no at most linear (-1) -vertex due to the minimality of Γ_2 . Since both W_1 and $p_2(W_1)$ are contractible, see Lemma 2.4.4(c), the latter is possible only if v is contracted by p_2 , and so (γ) follows.

It follows from the above diagram that $\text{Br}(\Gamma)$ splits into two disjoint subsets, namely, the set B of vertices which are simultaneously p_1 - and p_2 -rigid and the set $\text{Br}(\Gamma) \setminus B$ of vertices contracted by both p_1 and p_2 . This implies (a).

(b) By (a), any connected component of $\Gamma \setminus B$ is contracted by p_i into a segment of Γ_i , $i = 1, 2$. This provides a bijective correspondence between the sets of segments of Γ_1 and Γ_2 . It is easily seen that this correspondence has desired properties.

(c) Clearly, every non-rational vertex of Γ belongs to B . Given a vertex $v \in \text{Br}(\Gamma) \setminus B$, v is rational and there are simple branches W_1 and W_2 of Γ at v contracted by p_1^v and p_2^v , respectively. Due to the relative minimality assumption, these branches are distinct. In fact, there are exactly 3 distinct branches W_1, W_2 and W_3 of Γ at v , since p_1^v contracts W_1 , does not contract W_2 and p_1 contracts v . By the same reason $\deg_\Gamma(v) = 3$, that is, all the three branches of Γ at v are simple. Since W_1 is contracted in Γ_1 , it contains no vertex of $B = \text{Br}(\Gamma_1) = \text{Br}(\Gamma_2)$; the latter also holds for W_2 . Hence $p_i(W_j) \cap \text{Br}(\Gamma_j) = \emptyset$ for $i \neq j$, $i, j \in \{1, 2\}$, that is, W_i is a tree whose image under p_j contains no branching point of Γ_j . It follows that $p_i(W_j)$ is contained in a segment, say, S_i of Γ_i . We claim that the S_i are linear, extremal and non-admissible.

Assume on the contrary that, say, S_1 is a circular segment of Γ_1 . Then $p_1^{-1}(S_1)$ is a connected component of Γ which contains v and all three branches W_1, W_2 and W_3 of Γ at v , where W_1 and W_2 are simple branches and trees and W_3 contains a cycle. Then $p_1(W_3) = S_1 \supset p_1(W_2) \neq \emptyset$, which is impossible. Thus, S_1 is linear and by symmetry also S_2 is. Clearly, both of these segments are extremal.

The contractible tree W_i contains an at most linear (-1) -vertex, say, v_i contracted in Γ_i , $i = 1, 2$. Since Γ is relatively minimal, v_i is not contracted by p_j ($j \neq i$). Its weight can only increase under the contraction p_j , and it must increase indeed as Γ_j

is minimal. Thus the extremal linear segment S_j has a vertex $p_j(v_i)$ of non-negative weight, hence is non-admissible, which proves (c). \square

Example 3.1.5. We present on Figure 3 a graph Γ which dominates two minimal linear graphs Γ_1 and Γ_2 . The birational morphism p_1 contracts the branch W_1 of Γ and the vertex v , while p_2 contracts the branch W_2 and v .

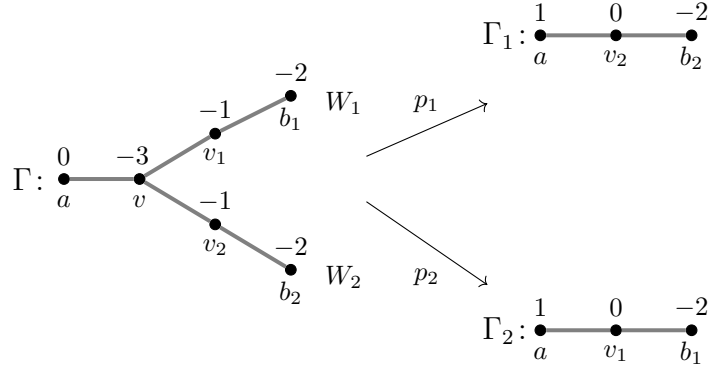


FIGURE 3. A weighted graph Γ with branches W_1 and W_2 at vertex v which dominates Γ_1 and Γ_2 via non-inner contractions p_1 and p_2 .

3.2. Birationally rigid graphs.

Definition 3.2.1. A minimal weighted graph Γ_1 is said to be *birationally rigid* if in any relatively minimal diagram (3) with a minimal graph Γ_2 no vertex of $\text{Vert}(\Gamma_1) \subset \text{Vert}(\Gamma)$ is contracted in Γ_2 . In particular, $p_2 \circ p_1^{-1}$ in (3) is an isomorphism. We say that Γ_1 is *admissible* if all its segments are admissible.

We have the following criterion of birational rigidity; cf. [13, Corollaries A.3 and A.4]. Notice that our birationally rigid graphs correspond to absolutely minimal graphs of [13].

Proposition 3.2.2. *A minimal weighted graph Γ_1 is birationally rigid if and only if it is admissible.*

Proof. Assume that all the segments of Γ_1 are admissible and, on the contrary, Γ_1 is not birationally rigid. Consider a relatively minimal diagram (3) with a minimal graph Γ_2 . Let $v \in \text{Vert}(\Gamma)$ be the (-1) -vertex blown down the first under the morphism $p_2: \Gamma \rightarrow \Gamma_2$. Due to relative minimality of diagram (3), v is not contracted under $p_1: \Gamma \rightarrow \Gamma_1$, that is, $v \in \text{Vert}(\Gamma_1) \subset \text{Vert}(\Gamma)$.

According to Graph Lemma 3.1.4(a), $v \notin B := \text{Br}(\Gamma_1) = \text{Br}(\Gamma_2)$, hence v lies on a segment S_1 of Γ_1 . Let $w_1(v)$ be the weight of v in Γ_1 and $w(v)$ be its weight in Γ . Clearly, $w(v) \leq w_1(v)$. If the admissible segment S_1 is either linear or circular and contains at least two vertices, then we have $-1 = w(v) \leq -2$, which is impossible. Let now S_1 be circular with a single vertex v of weight $w_1(v) \leq 2$, and let S be the preimage of S_1 in Γ . If S has at least two vertices, that is, the loop of S was blown up, then again $-1 = w(v) \leq -2$ hold, a contradiction. If S contains a single vertex v , then v is incident to a loop in Γ , hence cannot be contracted in Γ_2 , which once again gives a contradiction.

To show the converse assume that Γ_1 has a non-admissible segment S_1 . Both Γ_1 and S_1 are minimal, so they are not contractible. Being non-admissible S_1 has a vertex v of weight a , where $a \geq 0$ if S_1 is different from a circular segment with a single vertex and $a \geq 3$ otherwise. Blowing up Γ_1 successively we reduce the weight of v to -1 while keeping it at most linear. In more detail, first we blowup at an

edge $[u, v]$ incident to v in Γ_1 unless Γ_1 consists of a single isolated vertex v ; in the latter case we start with an outer blowup at v . Let e_1 be the (-1) -vertex created in the first step. In the second step, we blowup the edge $[e_1, v]$ creating a new vertex e_2 , etc. In the i th step we blowup the edge $[e_{i-1}, v]$ by adding a new vertex e_i . After $k > 0$ successive blowups, where $k = a + 1$ if S_1 is different from a circular segment with a single vertex and $k = a - 2$ otherwise, we obtain a weighted graph Γ with the vertices e_1, \dots, e_k appearing in that order under our procedure. For $i = 1, \dots, k - 1$ the weight of e_i in Γ equals -2 while e_k and v are at most linear (-1) -vertices of Γ . Notice that e_k is the unique (-1) -vertex of Γ blown down under the contraction $p_1: \Gamma \rightarrow \Gamma_1$ of e_k, \dots, e_1 in this order.

Starting with the blowdown of v in Γ we continue to blowdown the new appearing (-1) -vertices until we reach a minimal graph Γ_2 ; this yields a birational morphism $p_2: \Gamma \rightarrow \Gamma_2$.

By Lemma 2.4.4(c) the contractibility is a birational invariant. Hence Γ_2 is not empty. The graphs Γ, Γ_1 and Γ_2 along with the birational morphisms $p_i: \Gamma \rightarrow \Gamma_i$ form a diagram (3).

After the blowdown of v under p_2 the weight of e_k becomes 0 and remains ≥ 0 under the successive blowdowns which form p_2 . Thus, the only (-1) -vertex e_k of Γ blown down under p_1 is not blown down under p_2 . This shows that the resulting diagram (3) is relatively minimal. Since the vertex v of Γ_1 is contracted in Γ_2 , the graph Γ_1 is not birationally rigid, see Definition 3.2.1. \square

Summarizing we get the following proposition.

Proposition 3.2.3. *Assume we are given a relatively minimal diagram (3) with minimal weighted graphs Γ_1 and Γ_2 dominated by a graph Γ . Then the following hold.*

1. *The birational map $\phi' := p_2 \circ p_1^{-1}: \Gamma_1 \dashrightarrow \Gamma_2$ induces*
 - (a) *a bijection between the sets of branching vertices $\text{Br}(\Gamma_1) \simeq \text{Br}(\Gamma_2)$;*
 - (b) *a bijection between the sets of segments that sends admissible (extremal, respectively) linear segments of Γ_1 to admissible (extremal, respectively) linear segments of Γ_2 and circular segments to circular ones;*
 - (c) *isomorphisms between every pair of corresponding admissible linear (resp., admissible circular) segments;*
 - (d) *an inner birational transformation between every pair of corresponding non-admissible non-extremal linear segments;*
 - (e) *a birational transformation between every pair of corresponding non-admissible extremal linear segments.*
2. *If the Γ_i are circular graphs, then Γ in diagram (3) is also circular and the birational morphisms p_i are inner.*
3. *If all extremal linear segments in either Γ_1 or Γ_2 are admissible, then the birational transformation $\phi' = p_2 \circ p_1^{-1}: \Gamma_1 \dashrightarrow \Gamma_2$ is inner and induces a bijection between the sets of end vertices $\text{End}(\Gamma_1) \simeq \text{End}(\Gamma_2)$.*
4. *If ϕ' is inner and the Γ_i are linear graphs, then also Γ is a linear graph.*
5. *If ϕ' is inner and the Γ_i are linear graphs with two vertices, then $\phi': \Gamma_1 \xrightarrow{\simeq} \Gamma_2$ is an isomorphism of weighted graphs.*

Proof. Statement 1(a) follows from Graph Lemma 3.1.4(a). By Graph Lemma 3.1.4(b) we may restrict ϕ' to any segment of Γ_1 extended by its incident edges and adjacent branching vertices. Such a restriction includes all the blowups and blowdowns in ϕ' that happen on this segment, its incident edges and their successive images. It yields again a relatively minimal diagram (3) with minimal dominated graphs. Thus, 1(b) and 1(c) follow from Graph Lemma 3.1.4(b) and Proposition 3.2.2, respectively.

To prove 1(d) we have to show that a relatively minimal transformation between non-admissible non-extremal linear segments is inner with respect to the graphs Γ_i and their successive images. Assuming the contrary, the connected component of $\Gamma \setminus B$ corresponding to these segments contains a branching vertex $v \in \text{Br}(\Gamma) \setminus B$. Then by Graph Lemma 3.1.4(c) these segments are extremal, contrary to our assumption.

See [11, Lemma 2.7] for statement 2. Statement 3 is immediate from 1(c), 1(d) and statement 2. To show statement 4 it suffices to observe that an inner blowup cannot create a new branching point.

To show statement 5 suppose that the linear segment Γ in diagram (3) contains more than 2 vertices. Clearly, p_i sends the end vertices of Γ to the end vertices of Γ_i , cf. statement 3. Therefore, Γ contains a (-1) -vertex contracted in both Γ_1 and Γ_2 , which is impossible since our diagram (3) is relatively minimal. Hence Γ contains just 2 vertices, and so the p_i are isomorphisms. \square

Remark 3.2.4. Given a relatively minimal diagram (3), where Γ_1 and Γ_2 are minimal non-admissible extremal linear segments, the morphisms p_i do not need to be inner, in general, see Example 3.1.5 and [11, Examples 2.6 and 3.30].

Proposition 3.2.3 leads to the following theorem.

Theorem 3.2.5. *Let (X, D) be an NC-pair with minimal dual graph $\Gamma(D)$ such that all extremal linear segments of $\Gamma(D)$ are admissible. Then $\text{Bir}(X, D) = \text{Inn}(X, D)$.*

Proof. By Lemma 2.7.11 any $\Phi \in \text{Bir}(X, D)$ fits in a relatively minimal commutative diagram (2). Hence $\Phi = P_2 \circ P_1^{-1}$, where $P_i: (\tilde{X}, \tilde{D}) \rightarrow (X, D)$ for $i = 1, 2$ is a birational morphism of NC-pairs composed of a sequence of blowups of X in points of D and infinitesimally near points and an isomorphism. Let $p_i: \Gamma(\tilde{D}) \rightarrow \Gamma(D)$ be the birational morphism of dual graphs induced by P_i , $i = 1, 2$. Then the induced birational transformation of dual graphs $\phi = p_2 \circ p_1^{-1}: \Gamma(D) \dashrightarrow \Gamma(D)$ (see Definition 2.7.1) fits in a relatively minimal diagram (3). According to Proposition 3.2.3 ϕ is an inner birational transformation of weighted graphs. Hence Φ is an inner birational transformation of NC-pairs, i.e. $\Phi \in \text{Inn}(X, D)$. \square

4. \mathbb{G}_a -ACTIONS AND RULINGS BY AFFINE LINES

In this section we show that $\text{Bir}(X, D) = \text{Inn}(X, D)$ if and only if the surface $Y = X \setminus \text{supp } D$ admits no \mathbb{G}_a -action. We deduce that in the latter case the identity component $\text{Aut}^\circ(X)$ is an algebraic torus, see Theorem 4.3.4.

4.1. \mathbb{G}_a -actions. Recall that an \mathbb{A}^1 -fibration, or an *affine ruling*, on a normal affine surface Y is a morphism $Y \rightarrow B$ to a smooth curve with a general fiber isomorphic to the affine line \mathbb{A}^1 . The following lemma is well known, see e.g. [12, Lemma 1.6]; cf. the proof of Proposition 4.1.2.

Lemma 4.1.1. *Let Y be a normal affine surface. Given an \mathbb{A}^1 -fibration $\pi: Y \rightarrow B$ over a smooth affine curve B there exists an SNC-completion (X, D) with the following properties:*

- π extends to a \mathbb{P}^1 -fibration $\bar{\pi}: X \rightarrow \bar{B}$ over the smooth completion \bar{B} of B ;
- there is a unique component S of D with $S^2 = 0$ which is a section of $\bar{\pi}$;
- all the other components of D are smooth rational curves contained in fibers of $\bar{\pi}$;
- the dual graph $\Gamma(D)$ is a minimal tree;
- the (0) -vertices of $\Gamma(D)$ different from S belong to $\text{End}(\Gamma(D))$; these correspond to the components of D that are full fibers of $\bar{\pi}$.

For the following facts see e.g. [14, Lemma 1.6] or [12, Remark 1.7].

Proposition 4.1.2. *Let Y be a normal affine surface and let (X, D) be a minimal NC-completion of Y . The following conditions are equivalent:*

- (i) Y admits an effective \mathbb{G}_a -action;
- (ii) Y admits an \mathbb{A}^1 -fibration over a smooth affine curve;
- (iii) the dual graph $\Gamma(D)$ has a non-admissible extremal linear segment.

For the convenience of the reader we provide a proof.

Proof. Assuming (iii) $\Gamma(D)$ is birationally equivalent to a graph Γ' with a (0)-vertex v of degree 1, see e.g. [11, Examples 2.11]. By Proposition 2.7.9 we can replace the original NC-completion (X, D) by a new one (X', D') with $\Gamma(D') = \Gamma'$. The vertex v corresponds to a smooth rational component, say, C of D' with $C^2 = 0$. Resolving singularities of X' we obtain a smooth projective surface \tilde{X}' with exceptional divisor E and a (0)-curve C . Indeed, the singular points of X' being points of $Y = X' \setminus \text{Supp } D'$ the resolution does not affect the divisor D' . Such a curve C on \tilde{X}' is a reduced member of a linear pencil with no base point, see e.g. [2, Proposition V.4.3]. This pencil defines a \mathbb{P}^1 -fibration $\tilde{\pi}: \tilde{X}' \rightarrow \bar{B}$ over a smooth projective curve \bar{B} such that C is a reduced fiber of $\tilde{\pi}$. Since $E \cdot C = 0$, each connected component of E is properly contained in a fiber of $\tilde{\pi}$. The contraction of E yields a \mathbb{P}^1 -fibration $\bar{\pi}: X' \rightarrow \bar{B}$ with a reduced fiber C .

The surface $Y = X' \setminus \text{supp } D'$ being affine, $\text{supp } D'$ is connected (see Lemma 2.7.5(a)) and Y contains no complete curve, in particular, no entire fiber of $\bar{\pi}$. It follows that $\text{supp } D' \neq C$. Since $\deg(v) = 1$, C meets transversally just one other component S of D' . Since $C \cdot S = 1$, S is a section of $\bar{\pi}$. Hence $\bar{\pi}|_Y: Y \rightarrow B$ is an \mathbb{A}^1 -fibration over an affine curve $B \subset \bar{B} \setminus \{\bar{\pi}(C)\}$. This proves the implication (iii) \Rightarrow (ii).

Assume now that (ii) holds. By Lemma 4.1.1 the \mathbb{A}^1 -fibration $\pi: Y \rightarrow B$ over a smooth affine curve B admits an extension $\bar{\pi}: \bar{X} \rightarrow \bar{B}$ to an SNC-completion (\bar{X}, \bar{D}) of Y , where \bar{B} is a smooth completion of B . Places at infinity in general \mathbb{A}^1 -fibers of π lie on a section, say, S of $\bar{\pi}$. If $P \in \bar{B} \setminus B$, then $\bar{\pi}^{-1}(P) \subset \bar{D}$. Contracting subsequently (-1) -curves in the fiber $\bar{\pi}^{-1}(P)$ yields an NC-completion (\bar{X}', \bar{D}') of Y along with a \mathbb{P}^1 -fibration $\bar{\pi}': \bar{X}' \rightarrow \bar{B}$ with a reduced irreducible fiber $C = (\bar{\pi}')^{-1}(P)$. The component C of \bar{D}' corresponds to a (0)-vertex v of the dual graph $\Gamma(\bar{D}')$. We have $C \cdot (\bar{D}' - C) = C \cdot S = 1$. It follows that v is an end vertex of a non-admissible extremal linear segment of $\Gamma(\bar{D}')$. The latter segment corresponds to a non-admissible extremal linear segment of $\Gamma(D)$, see Proposition 3.2.3.1(b). This yields the equivalence (iii) \Leftrightarrow (ii).

Assume that (i) holds, that is, Y admits an effective \mathbb{G}_a -action. Let δ be the nonzero locally nilpotent derivation on $\mathcal{O}(Y)$ generating the given \mathbb{G}_a -action. The algebra of \mathbb{G}_a -invariants $\ker \delta$ is finitely generated, see [10, Lemma 1.1]. The embedding $\ker \delta \hookrightarrow \mathcal{O}(Y)$ yields an \mathbb{A}^1 -fibration $Y \rightarrow B$ along the orbits of the \mathbb{G}_a -action, where $B = \text{Spec}(\ker \delta)$ is a smooth affine curve. Thus, (ii) holds.

Conversely, given an \mathbb{A}^1 -fibration $\pi: Y \rightarrow B$ over a smooth affine curve B as in (ii) there exists as before an SNC-completion (X, D) of Y and an extension $\bar{\pi}: X \rightarrow \bar{B}$ to a \mathbb{P}^1 -fibration on X with a section $S \subset D$. Shrinking the base B suitably we obtain a principal Zariski open cylinder $U \simeq Z \times \mathbb{A}^1$ on Y with an affine base $Z \subset B$, where $U = \bar{\pi}^{-1}(Z) \setminus S$. By [27, Proposition 3.1.5] there is an effective \mathbb{G}_a -action on Y . This shows the equivalence (i) \Leftrightarrow (ii). \square

Remark 4.1.3. Under the assumptions of Proposition 4.1.2 assume that $\Gamma(D)$ has a non-admissible extremal linear segment, and let $(X', D'), S$ and C be as in the proof of the proposition. Then $\Gamma(D')$ is a tree with at most one non-rational vertex S , a root of $\Gamma(D')$. Indeed, $\Gamma(D')$ is connected and every component C' of D' except the section S is a component of a fiber of $\bar{\pi}$, because $C \cdot C' = 0$. Since the fibers of $\bar{\pi}$

are trees of rational curves, our claim follows. Since $\Gamma(D)$ and $\Gamma(D')$ are birationally equivalent, $\Gamma(D)$ also is a tree.

4.2. Degree function. Let (X, D) be an NC-pair with an affine open subset $Y = X \setminus \text{supp } D$. By Lemma 2.7.3, up to passing to a suitable new NC-pair which dominates (X, D) we may consider that there exists an ample divisor $H \subset X$ such that $\text{supp } H = \text{supp } D$. Replacing H by its suitable high multiple we also may consider that $X \subset \mathbb{P}^n$ and $H \subset X$ is a hyperplane section. Then $Y = X \setminus \text{supp } H \hookrightarrow \mathbb{A}^n = \mathbb{P}^n \setminus \text{supp } H$ is a proper embedding.

Definition 4.2.1 (cf. e.g. [5, (1.1)]). For $\Phi \in \text{Bir}(X, D)$ we define its degree as $\deg \Phi = H \cdot \Phi^*(H)$. For $\Phi \in \text{Aut}(Y)$ we understand $\deg \Phi$ as the degree of the extension $\hat{\Phi} \in \text{Bir}(X, D)$.

Remarks 4.2.2. 1. The above degree function coincides with the usual degree of an automorphism of an affine subset $Y \subset \mathbb{A}^n$. Indeed, let Φ be written in homogeneous coordinates of \mathbb{P}^n as

$$\Phi = (p_0 : \dots : p_n)|_X$$

where the p_i are homogeneous polynomials in $n + 1$ variables of the same degree d which do not belong simultaneously to the homogeneous ideal of X . Then $\deg \Phi = \min\{d\}$ where the minimum is taken over all such representations.

2. We have $\text{Bir}(X, D) = \varinjlim \text{Bir}(X, D)_{\leq d}$ where

$$\text{Bir}(X, D)_{\leq d} = \{\Phi \in \text{Bir}(X, D) \mid \max\{\deg \Phi, \deg \Phi^{-1}\} \leq d\}$$

is an affine algebraic variety and $\text{Bir}(X, D)_{\leq d} \subset \text{Bir}(X, D)_{\leq d+1}$ is a closed embedding. This yields an affine ind-structure on $\text{Bir}(X, D)$. It is easily seen that this structure coincides with the usual ind-structure on $\text{Aut}(Y)$ defined by the closed embedding $Y \subset \mathbb{A}^n$, see e.g. [18, Section 5.1] or [27, Section 2.1]. By definition of the Zariski ind-topology, a subgroup $G \subset \text{Bir}(X, D)$ is connected if any element $g \in G$ belongs to a connected algebraic subset of G which contains the identity. The identity component G° of a subgroup $G \subset \text{Bir}(X, D)$ is the maximal connected subgroup of G .

In this subsection we prove the following result.

Proposition 4.2.3. $\text{Inn}^\circ(X, D) = \text{Aut}^\circ(X, D)$ and any connected algebraic subgroup $G \subset \text{Inn}(X, D)$ is contained in $\text{Aut}^\circ(X, D)$.

We start with some preliminaries.

Lemma 4.2.4. Let X be a normal projective surface and H be an ample divisor on X . Then for any $n \in \mathbb{N}$ the number of effective divisor classes C in the Néron-Severi group $NS(X)$ with $H \cdot C \leq n$ is finite.

Proof. This follows from [30, Example 1.4.31]. □

Lemma 4.2.5. For any $d \geq 0$ there exists $N = N(d) \in \mathbb{N}$ such that each $\Phi \in \text{Bir}(X, D)_{\leq d}$ admits a representation with at most N blowups and N blowdowns.

Proof. By Lemma 4.2.4 the Cartier divisors $\Phi^*(H)$ with $\deg(\Phi) = H \cdot \Phi^*(H) \leq d$ fall in a finite number of classes in the Néron-Severi group $NS(X)$. Clearly, the latter holds as well for Φ running over $\text{Bir}(X, D)_{\leq d}$. Thus, the quantity

$$N(d) := \max\{(\Phi^*H)^2 \mid \Phi \in \text{Bir}(X, D)_{\leq d}\}$$

is finite. It follows by [3, Theorem II.7]³ that each $\Phi \in \text{Bir}(X, D)_{\leq d}$ fits in a commutative diagram (2) where $X_1 = X_2 = X$ and the morphism P_1 is composed of at

³In [3, Theorem II.7] the surface X is supposed to be smooth. However, since all the indeterminacy points of Φ are situated in the smooth part of X , the proof goes without changes in our setting as well.

most $N(d)$ blowdowns. Since every blowdown decreases the Picard rank by 1, P_1 and P_2 contain the same number of blowdowns. \square

Lemma 4.2.6. *Any irreducible algebraic subset A of $\text{Inn}(X, D)$ is contained in a left coset $h \circ \text{Aut}^{\natural}(X, D)$ for some $h \in \text{Inn}(X, D)$.*

Proof. Let $d \in \mathbb{N}$ be such that $A \subset \text{Bir}(X, D)_{\leq d}$. By Lemma 4.2.5 there exists a natural number $N = N(d)$ such that each $\Phi \in A$ admits a representation with at most N blowups and N blowdowns. Since $A \subset \text{Inn}(X, D)$, these blowups and blowdowns are inner, and there are in total a finite number of them when Φ runs over A . Recall that $\text{Aut}^{\natural}(X, D)$ is a normal algebraic subgroup of $\text{Inn}(X, D)$ closed in $\text{Bir}(X, D)$, see Lemma 2.2.3. Therefore, $\text{Inn}(X, D)$ is the union of the closed algebraic subvarieties $h \circ \text{Aut}^{\natural}(X, D)$ of $\text{Bir}(X, D)$ with $h \in \text{Inn}(X, D)$. By the preceding, A is covered by a finite number of these cosets. Since the ground field \mathbb{K} is infinite and A is irreducible, it is covered by one of them. \square

Corollary 4.2.7. *$\text{Inn}(X, D)$ is a closed ind-subgroup of $\text{Bir}(X, D)$.*

Proof. By Lemma 4.2.6 and its proof, for any $d \in \mathbb{N}$

$$\text{Inn}(X, D)_{\leq d} = \text{Inn}(X, D) \cap \text{Bir}(X, D)_{\leq d}$$

is contained in a finite union of left cosets $h_i \circ \text{Aut}^{\natural}(X, D)$, $i = 1, \dots, n$. The latter union is closed in $\text{Bir}(X, D)$, and also $\text{Bir}(X, D)_{\leq d}$ is. Hence their intersection $\text{Inn}(X, D)_{\leq d}$ is a closed algebraic subvariety of $\text{Bir}(X, D)_{\leq d}$. Now the claim follows. \square

Proof of Proposition 4.2.3. We have $\text{Inn}^{\circ}(X, D) = \varinjlim A_i$, where the $A_i \subset \text{Inn}(X, D)$ are closed irreducible algebraic subsets of the ind-group $\text{Inn}(X, D)$ containing the identity, see [18, Proposition 1.6.3]. By Lemma 4.2.6 for each i the image of A_i in the quotient group $\text{Inn}(X, D)/\text{Aut}^{\natural}(X, D)$ is the identity, i.e. $A_i \subset \text{Aut}^{\natural}(X, D)$ and moreover, $A_i \subset (\text{Aut}^{\natural}(X, D))^{\circ} = \text{Aut}^{\circ}(X, D)$, see Lemma 2.2.4. Then also $\text{Inn}^{\circ}(X, D) \subset \text{Aut}^{\circ}(X, D)$. For the converse inclusion see Lemma 2.2.3. This proves the first assertion. The second follows from Lemma 4.2.6. \square

4.3. Proof of the main theorem. The following lemma is well known.

Lemma 4.3.1 ([14, Corollary 1.2]). *If an affine variety Y of dimension ≥ 2 admits a nontrivial \mathbb{G}_a -action, then $\text{Aut}(Y)$ contains a connected infinite-dimensional commutative unipotent ind-subgroup.*

Proof. Indeed, let $H \subset \text{Aut}^{\circ}(Y)$ be the one-parameter unipotent subgroup which corresponds to the given \mathbb{G}_a -action. Then H can be written as $H = \exp(\mathbb{K}\partial)$, where ∂ is a nonzero locally nilpotent derivation of the algebra $\mathcal{O}(Y)$, see [16]. The ring of invariants $\ker \partial = \mathcal{O}(Y)^H$ is an infinite dimensional vector space, and $\exp((\ker \partial)\partial)$ is a desired subgroup of $\text{Aut}^{\circ}(Y)$. \square

Remark 4.3.2. The assumption that Y is affine is important in Lemma 4.3.1. For instance, the open surface $Y = \mathbb{P}^1 \times \mathbb{A}^1$ does admit a nontrivial \mathbb{G}_a -action. However, $\text{Aut}(Y)$ is an algebraic group; see [29] for a description of $\text{Aut}(Y)$.

Lemma 4.3.3 ([15, Lemma 3.1]). *Let Y be an affine variety over an algebraically closed field \mathbb{K} of characteristic zero. Assume that Y admits no \mathbb{G}_a -action. Then any two \mathbb{G}_m -actions on Y commute.*

For the reader's convenience we provide an argument.

Proof. Suppose there are two non-commuting \mathbb{G}_m -actions on Y . Let δ_1 and δ_2 be corresponding non-zero locally finite (semi-simple) derivations on $\mathcal{O}(Y)$. Consider the grading on $\mathcal{O}(Y)$ which corresponds to δ_1 , and a decomposition

$$(6) \quad \delta_2 = \sum_{i=k}^l \delta_2^{(i)}$$

into homogeneous derivations with respect to this grading, where the extremal components $\delta_2^{(k)}$ and $\delta_2^{(l)}$ are non-zero. Since δ_1 and δ_2 do not commute, either the highest index l in (6) is strictly positive, or the lowest index k is strictly negative. By [15, Lemma 3.1] the non-zero derivation $\delta_2^{(k)}$ in the first case and $\delta_2^{(l)}$ in the second is locally nilpotent, which contradicts our assumption. \square

The next theorem yields Theorem 1.0.3 from the Introduction.

Theorem 4.3.4. *Let (X, D) be a minimal NC-completion of a normal affine surface Y . Then the following are equivalent:*

- (a) *all the extremal linear segments of the dual graph $\Gamma(D)$ are admissible;*
- (b) *Y admits no effective \mathbb{G}_a -action;*
- (c) *$\text{Bir}(X, D) = \text{Inn}(X, D)$;*
- (d) *$\text{Aut}^\circ(Y)$ is an algebraic torus of rank $\leq \dim Y$.*

Proof. Properties (a) and (b) are equivalent by Proposition 4.1.2. The implication (a) \implies (c) follows from Theorem 3.2.5.

To deduce the implication (c) \implies (b) assume that (c) holds. If in contrary Y admits an effective \mathbb{G}_a -action, then $\text{Bir}(X, D) = \text{Aut}(Y)$ contains connected algebraic subgroups of arbitrary dimension, see Lemma 4.3.1. This contradicts the fact that every connected algebraic subgroup G of $\text{Bir}(X, D) = \text{Inn}(X, D)$ is contained in the algebraic group $\text{Aut}^\circ(X, D)$, see Proposition 4.2.3. Thus, the conditions (a)–(c) are equivalent.

The implication (d) \implies (b) is immediate. Suppose now that one of the equivalent conditions (a)–(c) is fulfilled. By (c) $\text{Aut}(Y) = \text{Bir}(X, D) = \text{Inn}(X, D)$. By Proposition 4.2.3 the closed connected ind-subgroup $\text{Aut}^\circ(Y) = \text{Inn}^\circ(X, D)$ of $\text{Bir}(X, D)$ coincides with its algebraic subgroup $\text{Aut}^\circ(X, D)$. Since $\text{Aut}^\circ(Y)$ contains no \mathbb{G}_a -subgroup, it is an algebraic torus. This proves the remaining implication (b) \implies (d). \square

Combining Theorem 4.3.4, Proposition 4.1.2 and its proof we arrive at the following corollary; cf. [11, Proposition 3.34].

Corollary 4.3.5. *For a normal affine surface Y the following are equivalent.*

- (i) *Y admits an effective \mathbb{G}_a -action;*
- (ii) *Y admits an \mathbb{A}^1 -fibration over a smooth affine curve;*
- (iii) *for every minimal NC-completion (X, D) of Y the dual graph $\Gamma(D)$ has a non-admissible extremal segment;*
- (iv) *there exists a minimal NC-completion (X, D) of Y such that $\Gamma(D)$ has a (0)-vertex of degree 1;*
- (v) *$\text{Bir}(X, D) \neq \text{Inn}(X, D)$ for every minimal NC-completion (X, D) of Y ;*
- (vi) *there exists a minimal NC-completion (X, D) of Y such that $\text{Bir}(X, D) \neq \text{Inn}(X, D)$.*

Example 4.3.6. As a simple application of Corollary 4.3.5, let us show that the affine surface $Y = \mathbb{P}^2 \setminus \text{supp } D$, where D is a nonzero reduced effective divisor on \mathbb{P}^2 with only nodes as singularities, is rigid if and only if $\deg(D) \geq 3$.

First of all, consider the case $\deg(D) \leq 2$, that is, $\text{supp } D$ is either a projective line, or a pair of lines, or a smooth conic. In all three cases there exists a pencil of

conics \mathcal{L} on \mathbb{P}^2 which includes D (resp. $2D$ in the former case) and has a unique base point. The restriction $\mathcal{L}|_Y$ yields an \mathbb{A}^1 -fibration over \mathbb{A}^1 . Thus, Y admits an effective \mathbb{G}_a -action and so is not rigid, see Corollary 4.3.5(i)–(ii).

Suppose further that $\deg(D) \geq 3$. Using the Bezout theorem and analyzing separately the cases where D has 1, 2 and at least 3 components, one can easily conclude that any component C of D corresponds either to a branching vertex of $\Gamma(D)$, or to a vertex sitting on a cycle. Therefore, $\Gamma(D)$ has no non-admissible extremal linear segment. By Corollary 4.3.5 this implies the rigidity.

As an alternative proof in the case $\deg(D) \geq 3$ one can use the fact that $\bar{k}(Y) = -\infty$ provided Y admits an effective \mathbb{G}_a -action, where \bar{k} stands for the logarithmic Kodaira dimension. Since D is nodal one has $K_{\mathbb{P}^2} + D = (\deg(D) - 3)H$, where $H \in \text{Pic}(\mathbb{P}^2)$ is the class of a line. Therefore, $\bar{k}(Y) = -\infty$ if and only if $\deg(D) \leq 2$.

The third alternative approach⁴, is as follows. Suppose Y admits an effective \mathbb{G}_a -action. Consider an SNC-completion (\bar{X}, \bar{D}) of Y constructed in the proof of Proposition 4.1.2. Since X is rational the dual graph $\Gamma(\bar{D})$ is a tree with only rational vertices, see Remark 4.1.3. The latter properties of $\Gamma(\bar{D})$ are preserved under birational transformations. Hence $\Gamma(D)$ is also a tree with only rational vertices. By the Bezout theorem this implies as before that $\deg(D) \leq 2$.

5. APPENDIX: MINIMAL MODELS OF WEIGHTED GRAPHS

In this section we classify the weighted graphs with a unique up to isomorphism minimal model, and show that any other weighted graph has an infinite number of non-isomorphic minimal models, see Proposition 5.0.8. We use the following notation from [11]. We let $[[w_1, \dots, w_n]]$ stand for the linear segment with a sequence of weights of its vertices w_1, \dots, w_n . Similarly, we let $((w_1, \dots, w_n))$ be the circular graph with a cyclically ordered sequence of weights w_1, \dots, w_n . We abbreviate by a_k the sequence $\underbrace{a, \dots, a}_k$.

The following corollary of Propositions 3.2.2 and 3.2.3.1(a)–(c) is immediate.

Corollary 5.0.1. *Let Γ be a birationally rigid minimal weighted graph. Then up to isomorphism Γ is a unique minimal graph in its class of birational equivalence.*

However, the sufficient condition of birational rigidity in Corollary 5.0.1 is not a necessary one. In the following proposition we provide examples of non-birationally rigid minimal weighted graphs with a unique minimal model. We use the following definitions.

Definition 5.0.2 (*Triangulation of a circular graph*). Recall that a simplicial 2-complex C is homeomorphic to a disc if and only if the incidence graph of triangles in C is a tree. Assume that all vertices of C lie on the boundary of the disc $\partial C = B$. Then the 1-complex B represents a circular graph; we assume that all its vertices are rational. To a vertex of degree k we prescribe the weight $-k + 1$. Under this setting we call C a *triangulation* of the weighted graph B , and we say that B is *triangulable*.

Definition 5.0.3. We say that an inner segment S of a weighted graph Γ is a *charm earring* if S consists of a single linear (0)-vertex adjacent to a branching vertex via two edges. We say that a minimal graph Γ is *admissible modulo charm earrings* if every non-admissible segment of Γ is a charm earring.

Proposition 5.0.4. *Let Γ_1 be either one of the graphs $[[0]]$, $((3))$, $((4))$, $((0, m))$ with $m \leq -2$ and $((0, 0))$, or a graph admissible modulo charm earrings. Then up to isomorphism Γ_1 is a unique minimal graph in its birational equivalence class.*

⁴We thank Shulim Kaliman for suggesting this approach.

Proof. Assume that Γ_2 is a minimal graph birationally equivalent to Γ_1 . By Lemma 2.6.8 there exists a relatively minimal diagram (3) with Γ that dominates Γ_1 and Γ_2 via birational morphisms p_1 resp. p_2 . We may suppose that p_1 is composed of $n \geq 1$ blowdowns and p_2 also is composed of blowdowns. Let p_1^{-1} add vertices v_1, \dots, v_n in this order. By the relative minimality assumption p_2 cannot contract any (-1) -vertex of Γ among v_1, \dots, v_n . Since $n \geq 1$ such a (-1) -vertex of Γ does exist.

Case 1: $\Gamma_1 = [[0]]$. Let $\text{Vert}(\Gamma_1) = \{v_0\}$. We prove by induction that for every $k = 1, \dots, n$ the following hold.

- (i_k) The first k blowups in p_1^{-1} are outer and yield the graph $[-1, -2_{k-1}, -1]$, with vertices v_0, \dots, v_k appearing in this order;
- (ii_k) the further blowups in p_1^{-1} do not change the weights of v_0, \dots, v_{k-1} ;
- (iii_k) the first k blowdowns in p_2 contract the vertices v_0, \dots, v_{k-1} of Γ in this order.

Since p_2 contracts no (-1) -vertex of Γ different from v_0 and $\Gamma \ominus \{v_0\}$ is not minimal, (i₁)–(iii₁) hold. Assume by induction that for some $k \in \{1, n-1\}$, (i_k)–(iii_k) hold. By (ii_k), the $(k+1)$ st blowup in p_1^{-1} must be the outer blowup at v_k , hence (i_{k+1}) holds. Denote by $\Gamma_{2,k}$ the image of Γ under the first k blowdowns in p_2 . By (iii_k), $\Gamma_{2,k} \ominus \{v_k\}$ coincides with $\Gamma \setminus \{v_0, \dots, v_k\}$ and contains the at most linear (-1) -vertex v_n . Hence this graph is not minimal and no of its (-1) -vertices is contracted by p_2 . Thus, v_k is the only vertex of $\Gamma_{2,k}$ that can be contracted by the next blowdown. Its weight is -1 in $\Gamma_{2,k}$ and -2 in Γ . Now (ii_{k+1}) and (iii_{k+1}) follow. By recursion (i_n) holds, i.e. $\Gamma = [-1, -2_{n-1}, -1]$. So $\Gamma_2 = [[0]]$ due to the minimality of Γ_2 .

Case 2: $\Gamma_1 = ((m, 0))$ with $m \leq -2$. Since Γ_1 is circular, Γ is also circular, and p_1, p_2 consist of inner blowdowns, see Proposition 3.2.3(2). Let $\text{Vert}(\Gamma_1) = \{u, v_0\}$, where v_0 is the (0) -vertex. The same recursion as in Case 1 shows that for $k = 1, \dots, n$ we have:

- (i_k) the first k blowups in p_1^{-1} yield the graph $((m-k, -1, -2_{k-1}, -1))$ with vertices u, v_0, \dots, v_k in this order;
- (ii_k) the further blowups in p_1^{-1} do not change the weights of v_0, \dots, v_{k-1} ;
- (iii_k) the first k blowdowns in p_2 contract vertices v_0, \dots, v_{k-1} .

The only additional observation that we need is the following: the weight of u in $\Gamma_{2,k-1}$ is at most $m-k+(k-1) \leq -3$, so u cannot be contracted on the next step. Thus, we conclude as before that $\Gamma_2 \cong ((m, 0))$.

Case 3: $\Gamma_1 = ((3))$. After the first blowup we get $\Gamma'_1 = ((-1, -1))$ with vertices v_0 and v_1 where v_1 is contracted in Γ_1 . The second blowup would change the weight of v_0 to -2 , which is not possible since v_0 must be contracted under p_2 . So $\Gamma_2 \cong ((3))$.

Case 4: $\Gamma_1 = ((4))$. After the first blowup we get $\Gamma'_1 = ((-1, 0))$. Letting now $m = -1$, the argument from Case 2 can be applied mutatis mutandis to Γ'_1 instead of Γ_1 .

Case 5: Γ_1 is admissible modulo charm earrings. By Proposition 3.2.3, $p_2 \circ p_1^{-1}: \Gamma_1 \dashrightarrow \Gamma_2$ induces a bijection between $\text{Br}(\Gamma_1)$ and $\text{Br}(\Gamma_2)$, an isomorphism between the corresponding admissible segments of Γ_1 and Γ_2 and a birational transformation between every charm earring S of Γ_1 and the corresponding segment of Γ_2 . Let v_0 be the (0) -vertex of S adjacent to a vertex $u \in \text{Br}(\Gamma_1)$. Since u cannot be contracted by p_2 , the argument of Case 2 can be applied to the circular subgraph of Γ_1 with vertices u and v_0 . This shows that $p_2 \circ p_1^{-1}$ sends every charm earring of Γ_1 isomorphically onto a charm earring of Γ_2 and induces an isomorphism $\Gamma_1 \cong \Gamma_2$.

Case 6: $\Gamma_1 = ((0, 0))$. We claim that the circular graph B which appears at any step of the birational transformation $\phi' = p_2 \circ p_1^{-1}$, except for the initial graph $\Gamma_1 = ((0, 0))$, is triangulable, see Definition 5.0.2. Indeed, the first blowup in p_1^{-1} yields the circular graph $((-1, -1, -1))$ which has a triangulation with a single

triangle. Assume a circular graph B is equipped with a triangulation C . Then the blowup of B at the edge $[v_i, v_{i+1}]$ with a new vertex u results in gluing along the side $[v_i, v_{i+1}]$ a new triangle to C with vertex u ; this adds a new leaf to the incidence tree of triangles. Conversely, a blowdown of a (-1) -vertex v of B is expressed in removing the unique triangle in C with vertex v . This is always possible, except for $B = ((-1, -1, -1))$.

So Γ_2 is either $((0, 0))$ or triangulable. In the latter case Γ_2 is not minimal. Indeed, the incidence tree of triangles associated with the triangulation C of Γ_2 contains a leaf that corresponds to a triangle with a vertex of weight -1 . \square

Remark 5.0.5. An alternative approach is based on the following observation. Suppose we are in one of the first 4 cases or in case 6. We claim that Γ in the relatively minimal diagram (3) contains exactly two (-1) -vertices unless $\Gamma = ((-1, -1, -1))$. Indeed, let a graph Γ_1 be as in Proposition 5.0.4 but not an admissible graph modulo charm earrings. It is easily seen that for the inertia indices of Γ_1 we have $i_0(\Gamma_1) + i_+(\Gamma_1) = 1$. Let v be a (-1) -vertex of Γ that belongs to the image of $\text{Vert}(\Gamma_1)$. We may suppose that Γ is not isomorphic to Γ_1 and is different from $((-1, -1, -1))$. Then v is the unique (-1) -vertex of Γ contracted under p_2 . Assume on the contrary that Γ contains two distinct (-1) -vertices u_1 and u_2 different from v . They are not adjacent, see Corollary 2.4.5(b), are not contained among the vertices of Γ_1 and are not contracted under p_2 . Since Γ_2 is minimal, the weights of u_1 and u_2 in Γ_2 are non-negative. One can show that u_1 and u_2 in Γ_2 cannot be adjacent. Then the intersection form $I(\Gamma_2)$ is semipositive definite on the vector subspace spanned by the mutually orthogonal vectors u_1 and u_2 . However, this contradicts the fact that $i_0(\Gamma_2) + i_+(\Gamma_2) = i_0(\Gamma_1) + i_+(\Gamma_1) = 1$, see Corollary 2.3.8.

Remark 5.0.6. A linear segment is called *standard* if it is one of the following:

$$[[0_{2k}, w_1, \dots, w_n]] \quad \text{and} \quad [[0_{2k+1}]]$$

where $k, n \geq 0$ and $w_i \leq -2 \forall i$. Similarly, a circular segment is called *standard* if it is one of the following:

$$((0_{2k}, w_1, \dots, w_n)), \quad ((0_l, w)) \quad \text{and} \quad ((0_{2k}, -1, -1))$$

where $k, l \geq 0, n > 0, w \leq 0$ and $w_i \leq -2 \forall i$, see [11, Definition 2.13]. Notice that for $w = 0$ the second circular graph above becomes $((0_{l+1}))$. Any minimal segment is birationally equivalent to a standard one, see [11, Theorem 2.15(b)]. Moreover, for a minimal linear segment L its birational equivalence class contains at most two standard graphs related by a *reversion*

$$[[0_{2k}, w_1, \dots, w_n]] \mapsto [[0_{2k}, w_n, \dots, w_1]].$$

For a minimal circular segment C , the standard graph is unique in the birational equivalence class up to a cyclic permutation of its nonzero weights and reversion, see [11, Corrigendum, Corollary 3.33].

For instance, the circular graphs with sequences $((-1, -1))$ and $((0, -1))$, respectively, are the unique standard graphs in the respective birational equivalence classes of $((3))$ and $((4))$, while $[[0]]$ is the unique standard graph in its birational equivalence class.

Lemma 5.0.7. *Let Γ be a (not necessarily minimal) non-contractible connected weighted graph. Assume that Γ has a (0) -vertex v which is either of degree 1 or of degree 2 with no incident loop and multiple edges. Then Γ admits an infinite number of non-isomorphic minimal models.*

Proof. By our assumption Γ contains an edge $[v, u]$ isomorphic to $[[0, c]]$ for some $c \in \mathbb{Z}$. Applying iteratively elementary transformations at v ,

$$(7) \quad [[0, c]] \rightsquigarrow [[-1, -1, c]] \rightsquigarrow [[0, c + 1]],$$

which is inner if $\deg(v) = 2$ and outer otherwise, see Remark 2.7.10, we obtain an infinite number of graphs $\{\Gamma_n\}$ from the birational equivalence class of Γ with the same number of vertices, where Γ_n has a vertex of weight n . Every Γ_n dominates a minimal graph Γ'_n . We claim that among the Γ'_n there is an infinite number of non-isomorphic minimal graphs. Indeed, the contraction $\Gamma_n \rightarrow \Gamma'_n$ consists of at most $N = \text{card}(\text{Vert}(\Gamma))$ blowdowns. Hence it drops the maximal weight of vertices in Γ at most by $4N$. Therefore, the range of maximal weights of all the Γ'_n is unbounded, which proves our claim. \square

Proposition 5.0.8. *Let Γ be a connected minimal weighted graph. Then the birational equivalence class of Γ contains an infinite number of non-isomorphic minimal models if and only if Γ is not admissible modulo charm earrings and is different from the graphs $[[0]]$, $((3))$, $((4))$, $((0, m))$ with $m \leq 0$. In the opposite case, up to isomorphism Γ is a unique minimal graph in its birational equivalence class.*

Proof. The second assertion follows immediately from Proposition 5.0.4. To show the first assertion we start with a non-admissible circular segment Γ with N vertices. If $N \geq 3$, then there is a linear vertex v of a non-negative weight a with two distinct neighbors. Performing a inner blowups near v we drop the weight of v to 0, and then Lemma 5.0.7 gives the result.

Let now $N = 2$. If Γ has a vertex of positive weight, then after an inner blowup we are reduced to the previous case. Otherwise, $\Gamma = ((0, m))$ with $m \leq 0$, which is excluded by our assumption. Finally, if $N = 1$ and $\Gamma = ((m))$ where $m \geq 5$ by our assumption, then an inner blowup yields the 2-cycle $((m - 4, -1))$ with a positive weight and returns us to a previous case.

Let now Γ be non-circular. Since by assumption Γ is not admissible with charm earrings, $\Gamma \ominus \text{Br}(\Gamma)$ contains a non-admissible linear segment with a vertex v of weight $a \geq 0$ where $a > 0$ if either $\Gamma = [[a]]$ or v has two incident edges $[v, u]$ with the same branching vertex $u \in \text{Br}(\Gamma)$. Performing a blowups near v we reduce the setup to the one of Lemma 5.0.7, which implies the assertion. \square

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